Eventology of random-fuzzy events

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Abstract

A brief introduction to the *eventology*, which has originated recently as a new line of probability theory. This line studies *eventological motion of random-fuzzy events* (*eventological motion of events* — motion of matter or motion of mind — changing the eventological distributions), introduces the mind as *an eventological distribution* to scientific and mathematical research, and absorbs the famous theory of fuzzy sets by Zadeh as a very particular approach, that still has not been grounded strictly from *eventological* point of view.

Eventology[1] [1], a new line of the probability theory, which studies *motion of random fuzzy events* – is underlain with two extremely essential observations, in my opinion. One of those observations was formulated laconically by the Nobel Prize winner Bertrand Russell in 1946 [14]: *«Matter is simply a convenient way of linking events in a sequence»*. And the second one was formulated by the author 55 years later in a similar way intentionally: *«Mind is simply a convenient way of linking events in a sequence»*. The second observation is unlikely to provoke grave objections inasmuch as it is so natural, in my opinion. However Russell came to his fundamental observation, which can be found on the last pages of his book [14], resting upon the results of the greatest discoveries in physics; as for an eventologist, only experience of his own mind has to be acknowledged as a rest. Thus, let's emphasize it once again: *«matter and mind are simply two convenient ways of linking events in a sequence»*.

The idea of eventological grounds of Zadeh's theory of fuzzy sets [18, 19] was advanced in [1]. The theory of fuzzy sets, which was proposed by Lotfi Zadeh in 1965, has already become classical and widely applicable; though it bears obvious and generally marked analogies to the probability theory in general and

[1] *Eventology* (the term introduced by author: from lat. *eventum* — an event; *eventus* – an outcome, luck, an occurrence, destiny + *logy*) — a theory of random-fuzzy events

to the theory of random events in particular, it still has not had any common theoretical grounds with them. Though numerous Zadeh's following use fuzzy sets in various fields indiscriminately, they have only to notice some uncertainty of one or another kind, and then it is not difficult to discover that the theory by Zadeh is more successful in use, here and then where and when *fuzziness is generated by presence of man and his mind.* One of the justifications of eventology is at the bottom of this statement. Eventology, *the theory of random fuzzy events* [1], which originated within the probability theory, purposes the only aim — *eventological description of motion of mind.* It's the theory, which, unexpectedly for casual observer, pretends to propose an original and rather natural mathematical language for discussion of common theoretical grounds of this kind.

Brief survey on development of Zadeh' theory of fuzzy sets

At present transition from a usual characteristic function of a set (an indicator of a set) to a function of membership, which was first performed by Lotfi Zadeh[2] in 1965 [18], looks dictated by the natural development course of mathematics. Indeed, three-valued logic by Jan Lukasiewicz[3] [10], multi-valued logic by E.Post[4] [13], as well as infinite-valued logic had been already formulated by that time. These logics conveyed an idea, that the law of the excluded middle — first introduced by L.Brower[5] — is insufficient for cognition. There had been methods of the probability theory, which made it possible to work with distribution functions, and quantum physics with it's uncertainty principle. At last American

[2] *Zadeh Lotfi (born 1921)* — American mathematician. Founder of the fuzzy set theory (1965), fuzzy logic (1973) and soft computing (1991).

[3] *Lukasiewicz Jan (1878—1956)* — Polish mathematician and logician, who formulated the first system of multi-valued logic (three-valued logic by Lukasiewicz) in 1920 and then formulated the modal logic system on it's basis.

[4] *Post Emil Leon (1897—1954)* — American mathematician and logician, who generalized classical two-valued calculus of propositions into multi-valued calculus.

[5] *Brower Luitzen Egbert Jan (1881-1966)* — Dutch mathematician. His basic works are devoted to topology. Since 1904 he criticized so-called pure proof of existence, which was based on the logic principle of excluded middle. In the long run his critics initiated a line of mathematical foundations — mathematical intuitionism.

philosopher M.Black[6] [8], who studied the phenomenon of fuzziness, had introduced so-called «consistency profiles», which were direct predecessors of membership functions.

But this transition is unlikely to be just the result of conscious work of L.Zadeh, who rested on some analogies to approaches listed above, which seem rather pellucid now. Most likely it's a creation of his unconscious, his intuition, which was inspired not by the success of multi-valued logic, the probability theory and quantum physics, but by the aspiration to bring together mathematical language and fuzzy language of mind, the only possible language that mind created. It was incapable of creating something clearer and, moreover, had not the faintest wish to do it. This was the way things came out, and it can be confirmed by the naked pleasure with which L.Zadeh bestowed his «linguistic variable» [25] into a very core of the theory of fuzzy sets.

At the same time Zadeh is «in for a penny» since 1965, when he suggested using the function of membership instead of the indicator, but he is still not «in for a pound» as he hasn't given an answer where do these functions and operations over them come from. The result of such understatement s a «zoo» of operations over fuzzy sets, which have wantoned excessively within the theory owing to «over-zealous» Zadeh's following.

Recently Moscow school of artificial intelligence, fuzzy sets and soft computing has got in real earnest to the answer on this fundamental question — to acknowledgment of fuzziness, based on humanization of models; and only humanization of models is worth making fuss of fuzzy mathematics. However the decisive step has not been taken yet. The acknowledgement of «humanized» fuzziness has not gone beyond philosophic non-mathematical reasonings, which even came up to raising the wantoned «zoo» of operations over fuzzy sets to the mark of definition of fuzziness itself. But this is the path leading nowhere.

Now only eventology of fuzzy events is capable of taking the decisive step — of introducing mind to mathematical research openly and «intrepidly». And only eventology revealed a creator of fuzziness as mind, which as well as matter

[6] *Black Max (1909—1988)* — American philosopher and logician, who has basic works on philosophy of language, philosophy of mathematics, science and art.

is «simply a convenient way of linking events in a sequence». Eventology is in possession of rather powerful mathematical apparatus of the theory of random events [1, 2], and I hope it will have time and chance to imagine mathematically the way mind creates fuzziness in surroundings of matter, which generates randomness.

Basic definitions and denotations of Zadeh theory

Let's state the foundations of the *theory of fuzzy sets by Zadeh* by the example of the term of *fuzzy event by Zadeh*. Thus will have an opportunity to compare the classical theory of fuzzy sets with a new *eventological theory of fuzzy events*.

Fuzzy event by Zadeh $\widetilde{A}$ in the space of elementary events Ω is a set of pairs $\Big(\mathcal{Z}_{\widetilde{A}}(\omega),\ \omega\Big)$, $\omega \in \Omega$, where $\mathcal{Z}_{\widetilde{A}} : \Omega \to [0,1]$ is a *function of membership degree*[7] *of an elementary event* ω *to a fuzzy event* $\widetilde{A}$, or in other notations: $\mathcal{Z}_{\widetilde{A}}(\omega) = \mathcal{Z}\left(\omega \in \widetilde{A}\right)$, $\omega \in \Omega$. *A height of a fuzzy event by Zadeh* $\widetilde{A}$ is the maximum of the functions of membership $\text{height}(\widetilde{A}) = \max_{\omega\in\Omega} \mathcal{Z}_{\widetilde{A}}(\omega)$. If height of a fuzzy event by Zadeh is equal to 1, than it is *high*, otherwise it is — *low*. *A depth of a fuzzy event by Zadeh* $\widetilde{A}$ is the minimum of the function of membership $\text{Deep}(\widetilde{A}) = \min_{\omega\in\Omega} \mathcal{Z}_{\widetilde{A}}(\omega)$. If the depth of a fuzzy event by Zadeh is equal to zero, it is *deep*, otherwise it is *shallow*. *A carrier* of a fuzzy event by Zadeh $\widetilde{A}$ is an event $\text{supp}(\widetilde{A}) = \{\omega \in \Omega :\ \mathcal{Z}_{\widetilde{A}}(\omega) > 0\}$, the elementary events of which have nonzero degrees of membership. *An impossible* fuzzy event by Zadeh is the event $\tilde{\emptyset}$, the carrier of which is an impossible event: $\text{supp}(\tilde{\emptyset}) = \emptyset$. *A core* of a fuzzy event by Zadeh is an $\widetilde{A}$ event $\text{core}(\widetilde{A}) \subseteq \Omega$, elementary events of which have individual degrees of membership $\text{core}(\widetilde{A}) = \{\omega \in \Omega :\ \mathcal{Z}_{\widetilde{A}}(\omega) = 1\}$. *A certain* fuzzy event by Zadeh is the event $\tilde{\Omega}$, the core of which is the ceratin event: $\text{core}(\tilde{\Omega}) = \Omega$. *An event-cut (or an event-level)* of a fuzzy event by Zadeh

[7]Traditionally the theory of fuzzy sets uses Greek μ to designate the function of membership degree; the origin of μ is connected with the first letter of the English word *membership*. Unfortunately, this letter plays a main role in eventology of a fuzzy events: μ designates individual mind, whereas a set of minds taken together is designated by $\mathfrak{M}$. Eventologists have chosen the letters μ, $\mathfrak{M}$, M and $\mathbf{M}$ due to the English word *mind*. Therefore we had to offer up the tradition of the theory by Zadeh to avoid misunderstanding; we suggested using the letter $\mathcal{Z}$ instead of μ to designate the function of membership degree introduced by Zadeh, since $\mathcal{Z}$ coincides with the name of the founder. Thus we tender sincere apology to the followers of the theory: please believe, we have not found another way out.

is $\widetilde{A}$ an event $A_\alpha \subseteq \Omega$, elementary events of which have degrees of membership, that are not less than α: $A_\alpha = \{\omega \in \Omega : \ \alpha \leqslant \mathcal{Z}_{\widetilde{A}}(\omega)\}, \quad \alpha \in [0,1]$. Fuzzy events by Zadeh are equal, when their functions of membership coincide in Ω : $\widetilde{A} = \widetilde{B} \iff \mathcal{Z}_{\widetilde{A}}(\omega) = \mathcal{Z}_{\widetilde{B}}(\omega),\ \omega \in \Omega$.

Let's adduce basic operations on fuzzy events, introduced by Zadeh [18, 19] as well. *A complement* of a fuzzy event is $\widetilde{A}$ a fuzzy event $\widetilde{A}^c$ with a function of membership $\mathcal{Z}_{\widetilde{A}^c}(\omega) = 1 - \mathcal{Z}_{\widetilde{A}}(\omega)$, $\omega \in \Omega$. *An intersection* of fuzzy events $\widetilde{A}$ and $\widetilde{B}$ is a fuzzy event $\widetilde{A} \cap \widetilde{B}$ with a function of membership $\mathcal{Z}_{\widetilde{A}\cap\widetilde{B}}(\omega) = \min\left\{\mathcal{Z}_{\widetilde{A}}(\omega),\ \mathcal{Z}_{\widetilde{B}}(\omega)\right\}$, $\omega \in \Omega$. *An union* of fuzzy events $\widetilde{A}$ and $\widetilde{B}$ is a fuzzy event $\widetilde{A} \cup \widetilde{B}$ with function of membership $\mathcal{Z}_{\widetilde{A}\cup\widetilde{B}}(\omega) = \max\left\{\mathcal{Z}_{\widetilde{A}}(\omega),\ \mathcal{Z}_{\widetilde{B}}(\omega)\right\}$, $\omega \in \Omega$.

In addition to classical operations the binary operations, that are generalizing the former, are introduced in the theory of fuzzy sets by Zadeh. These binary operations $\beta : [0,1] \times [0,1] \to [0,1]$ meet four axioms (of monotonicity, associativity, commutativity and boundary condition) and are called *triangular norms* and *triangular conorms*[8].

Triangular norm is a binary operation $T : [0,1] \times [0,1] \to [0,1]$, which meets 4 axioms:

1. $b \leqslant c \ \rightarrow \ T(a,b) \leqslant T(a,c)$ и $T(b,a) \leqslant T(c,a)$ (*monotonicity*),
2. $T(T(a,b),c) = T(a,T(b,c))$ (*associativity*),
3. $T(a,b) = T(b,a)$ (*commutativity*),
4. $T(a,1) = T(1,a) = a$ (*boundary condition*).

Triangular conorm is a binary operation $S : [0,1] \times [0,1] \to [0,1]$, which meets 4 axioms:

1. $b \leqslant c \ \rightarrow \ S(a,b) \leqslant S(a,c)$ и $S(b,a) \leqslant S(c,a)$ (*monotonicity*),
2. $S(S(a,b),c) = S(a,S(b,c))$ (*associativity*),
3. $S(a,b) = S(b,a)$ (*commutativity*),

[8] The *triangular norms* and *triangular conorms* first were introduced in 1951 in probability geometry by K.Menger [12]. The geometry considered inequality of a triangle, in which position of points was determined by functions distribution of probabilities. Further triangular norms and conorms were investigated by B.Schweizer and A.Sklar in detail [16].

4. $S(a, 0) = S(0, a) = a$ (*boundary condition*).

T a b l e.
«The zoo» of binary operations
in the theory of fuzzy sets by Zadeh

Triangular norm $T(a, b)$	**Triangular conorm $S(a, b)$**	**Comment**
$\min\{a, b\}$	$\max\{a, b\}$	by Zadeh
ab	$a + b - ab$	probabilistic
$\max\{0, a + b - 1\}$	$\min\{1, a + b\}$	by Lukasiewicz
$\log_\lambda \left\{1 + \dfrac{(\lambda^a - 1)(\lambda^b - 1)}{\lambda - 1}\right\}$	$1 - \log_\lambda \left\{1 + \dfrac{(\lambda^{1-a} - 1)(\lambda^{1-b} - 1)}{\lambda - 1}\right\}$	log-operation $\lambda > 0$, $\lambda \neq 1$
$\begin{cases} T_Z, & \lambda = 0, \\ T_0, & \lambda = 1, \\ T_L, & \lambda = +\infty, \\ T^\lambda_{\log}, & \text{otherwize.} \end{cases}$	$\begin{cases} S_Z, & \lambda = 0, \\ S_0, & \lambda = 1, \\ S_L, & \lambda = +\infty, \\ S^\lambda_{\log}, & \text{otherwise.} \end{cases}$	by Frank
$ab\, \boldsymbol{\delta}\Big(0,\ (1 - a)(1 - b)\Big)$	$1 - (1 - a)(1 - b)\, \boldsymbol{\delta}\Big(1,\ 1 - ab\Big)$	delta-operation
$\dfrac{ab}{\lambda + (1 - \lambda)(a + b - ab)}$	$\dfrac{a + b - (2 - \lambda)ab}{1 - (1 - \lambda)ab}$	$\lambda > 0$
$\left(1 + \sqrt[\lambda]{\left(\frac{1}{a} - 1\right)^\lambda + \left(\frac{1}{b} - 1\right)^\lambda}\right)^{-1}$	$\left(1 + \sqrt[-\lambda]{\left(\frac{1}{a} - 1\right)^{-\lambda} + \left(\frac{1}{b} - 1\right)^{-\lambda}}\right)^{-1}$	$\lambda > 0$
$1 - \sqrt[\lambda]{(1 - a)^\lambda + (1 - b)^\lambda + (1 - a)^\lambda (1 - b)^\lambda}$	$\sqrt[\lambda]{a^\lambda + b^\lambda + a^\lambda b^\lambda}$	$\lambda > 0$
$\dfrac{ab}{\max\{a, b, \lambda\}}$	$\dfrac{a + b - ab - \min\{a, b, 1 - \lambda\}}{\max\{1 - a, 1 - b, \lambda\}}$	$\lambda \in [0, 1]$
$\max\left\{0,\ 1 - \sqrt[\lambda]{(1 - a)^\lambda + (1 - b)^\lambda}\right\}$	$\min\left\{1,\ \sqrt[\lambda]{a^\lambda + b^\lambda}\right\}$	$\lambda \geqslant 1$
$\max\left\{0,\ \dfrac{a + b - 1 + \lambda ab}{1 + \lambda}\right\}$	$\min\left\{1,\ a + b + \frac{\lambda}{1+\lambda} ab\right\}$	$\lambda \geqslant -1$

Triangular norm T and triangular conorm S are said to be *complemental* binary operations, if $T(a, b) + S(1 - a, 1 - b) = 1$ for $(a, b) \in [0, 1] \times [0, 1]$. In the theory by Zadeh three pairs of additional binary operations enjoy the widest popularity: 1) *Intersection and union by Zadeh*: $T_Z(a, b) = \min\{a, b\}$, $S_Z(a, b) = \max\{a, b\}$. 2) *Intersection and union by Lukasiewicz*: $T_L(a, b) = \max\{0, a + b - 1\}$, $S_L(a, b) = \min\{1, a + b\}$. 3) *Probabilistic intersection and union*: $T_0(a, b) = ab$, $S_0(a, b) = a + b - ab$. Complemental binary operations of triangular norm and conorm used in the theory by Zadeh are arranged in the table below.

Eventological modification of Zadeh operations

Binary operations $\beta : [0,1]\times[0,1] \to [0,1]$, which meet four axioms (of monotony, associativity, commutativity and boundary condition) and are called to be triangular norms and triangular conorms, are used in the theory of fuzzy sets as generalization of the classical operations on fuzzy sets, introduces by Zadeh.

From the view point of eventology it's rather sufficient for the theory of fuzzy events by Zadeh to use a narrow class of binary operations over fuzzy sets, which are determined as triangular norms and conorms and meet one more extra axiom: *inequalities by Fréchet* [3]. Due to this special denotations are used for these operations: *bounds of intersection* and *bounds of union* accordingly. Let's draw to strict definitions.

Bound of intersection is a binary operation $T : [0,1] \times [0,1] \to [0,1]$,, which meets five axioms:

1. $b \leqslant c \ \to \ T(a,b) \leqslant T(a,c)$ и $T(b,a) \leqslant T(c,a)$ (*monotonicity*),
2. $T(T(a,b),c) = T(a,T(b,c))$ (*associativity*),
3. $T(a,b) = T(b,a)$ (*commutativity*),
4. $T(a,1) = T(1,a) = a$ (*bound condition*),
5. $\max\{0,\ a+b-1\} \leqslant T(a,b) \leqslant \min\{a,b\}$ (*inequalities by Fréchet*).

Bound of union is a binary operation $S : [0,1] \times [0,1] \to [0,1]$, which meets five axioms:

1. $b \leqslant c \ \to \ S(a,b) \leqslant S(a,c)$ и $S(b,a) \leqslant S(c,a)$ (*monotonicity*),
2. $S(S(a,b),c) = S(a,S(b,c))$ (*associativity*),
3. $S(a,b) = S(b,a)$ (*commutativity*),
4. $S(a,0) = S(0,a) = a$ (*bound condition*).
5. $\max\{a,b\} \leqslant S(a,b) \leqslant \min\{1,\ a+b\}$ (*inequalities by Fréchet*).

Bound of intersection T and bound of union S are called to be complementary binary operations, if $T(a,b) + S(1-a, 1-b) = 1$ for $(a,b) \in [0,1] \times [0,1]$.

Following eventological modifications meet three most popular pairs of binary operations on fuzzy sets in theory by Zadeh: three pairs of complementary bound of intersections and bound of union of fuzzy events, which have got new eventological appellations and definitions accordingly; all together they are called to be *operation by Fréchet*[9] 1) *Right bound of intersection and left bound of union by Fréchet (intersection and union by Zadeh)*: $T_r(a,b) = \min\{a,b\}$, $S_l(a,b) = \max\{a,b\}$. 2) *Left bound of intersection and right bound of union by Fréchet (intersection and union by Lukasiewicz)*: $T_l(a,b) = \max\{0, a+b-1\}$, $S_r(a,b) = \min\{1, a+b\}$. 3) *Probability independent intersection and union*: $T_0(a,b) = ab$, $S_0(a,b) = a+b-ab$.

T a b l e.
Eventological generalization
of pair of complementary binary operations
from the theory of fuzzy sets by Zadeh

	Bound of intersection $\boldsymbol{T(a,b)}$	**Bound of union** $\boldsymbol{S(a,b) = 1 - T(1-a, 1-b)}$	**Name**
Z	$\min\{a,b\}$	$\max\{a,b\}$	*right bound of intersection and left bound of union by Fréchet* (intersection and union by Zadeh)
0	ab	$a+b-ab$	*probability independent bounds of intersection and union*
L	$\max\{0, a+b-1\}$	$\min\{1, a+b\}$	*left bound of intersection and right bound of union by Fréchet* (intersection and union by Lukasiewicz)
F	$\begin{cases} (1-\varphi)T_0 + \varphi T_Z, & \varphi \in [0,1], \\ (1+\varphi)T_0 - \varphi T_L, & \varphi \in [-1,0]. \end{cases}$	$\begin{cases} (1-\varphi)S_0 + \varphi S_Z, & \varphi \in [0,1], \\ (1+\varphi)S_0 - \varphi S_L, & \varphi \in [-1,0]. \end{cases}$	*bound of intersection and union by Fréchet* (φ — coefficient of Fréchet-correlation)

Let's refresh the way correlation of random events by Fréchet is defined, and in order to facilitate usage of this — a bit hard — term — let's introduce a more compact synonym: Fréchet-correlation of fuzzy events. Let $X \subseteq \mathfrak{X} \subseteq \mathfrak{F}$ — an arbitrary subset of the set of selected events $\mathfrak{X}$, that were selected from algebra

[9] It necessary to specify that operations by Fréchet (as well as operations by Zadeh) are carried out not on the very fuzzy events (in the eventological meaning), but only on their functions of membership. Due to the theory by Zadeh – fuzzy event is no more than a function of membership, defined in space of elementary events.

$\mathfrak{F}$ of probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Fréchet-correlation of the subset of events is a value

$$\mathsf{Kor}_X = \begin{cases} \mathsf{Kov}_X / F^+_X, & \mathsf{Kov}_X \leqslant 0, \\ \\ \mathsf{Kov}_X / F^-_X, & \mathsf{Kov}_X \leqslant 0, \end{cases}$$

where $-F^-_X = -\min\left\{\prod_{x \in X} \mathbf{P}(x),\ \prod_{x \in X} \mathbf{P}(x) + \sum_{x \in X} \mathbf{P}(x^c) - 1\right\}$ — *left bound by Fréchet* and $F^+_X = -\min_{x \in X} \mathbf{P}(x) - \prod_{x \in X} \mathbf{P}(x)$ — *right bound by Fréchet* for arity covariation $\mathsf{Kov}_X = \mathbf{P}\left(\bigcap_{x \in X} x\right) - \prod_{x \in X} \mathbf{P}(x)$ of the same subset of events $X \subseteq \mathfrak{X}$. Thus, if for arbitrary $X \subseteq \mathfrak{X}$ $-F^-_X \leqslant \mathsf{Kov}_X \leqslant F^+_X$, then values of arity Fréchet-correlation of arbitrary subset of events always belong to the interval $[-1, 1]$: $-1 \leqslant \mathsf{Kor}_X \leqslant 1$. It follows directly from the definition of Fréchet-correlation of two events $x \in \mathfrak{X}$ and $y \in \mathfrak{X}$ that if their probabilities are fixed: $a = \mathbf{P}(x), \quad b = \mathbf{P}(y)$, then left and right Fréchet-bounds of pair covariation of any two events with the same probabilities are equal $-F^-_{xy} = \max\{0,\ a + b - 1\} - ab, \quad F^+_{xy} = \min\{a, b\} - ab$. Complementary binary operations of intersection and union by Fréchet are generalizing three of the most popular pairs of complementary operations since

$$T^\varphi_F = \begin{cases} T_Z, & \varphi = 1, \\ T_0, & \varphi = 0, \\ T_L, & \varphi = -1, \end{cases} \qquad S^\varphi_F = \begin{cases} S_L, & \varphi = 1, \\ S_0, & \varphi = 0, \\ S_Z, & \varphi = -1. \end{cases}$$

Popularity of these three pairs of complementary binary operations in the theory of fuzzy events by Zadeh is explained eventologically by the fact that every pair corresponds to an according eventological structure of dependence of events, which is *non-intersected* ($\varphi = -1$), *independent* ($\varphi = 0$) or *embedded* ($\varphi = +1$). Intersection and union by Fréchet for arbitrary parameter $\varphi \in [-1, 1]$ accord to the eventological structures of dependence of events with a coefficient of pair correlation by Fréchet, that is equal to φ.

Lemma (about the meaning of parameter in operations by Fréchet). *Parameter φ in operations by Fréchet $T^\varphi_F(a, b)$ and $S^\varphi_F(a, b)$ has the meaning of correlation by Fréchet $\varphi = \mathsf{Kor}_{xy}$ of events x and y with probabilities a and b accordingly.*

Proof. Let's adduce the proof for the operation of intersection by Fréchet only, since it's is quite the same for the operation of union by Fréchet. Let's take two events. Their probabilities are equal to . And the probability of intersection is determined by the operation of intersection by Fréchet having the parameter

$$\mathbf{P}(x \cap y) = \begin{cases} (1-\varphi)T_0(a,b) + \varphi T_Z(a,b), & \varphi \in [0,1], \\ (1+\varphi)T_0(a,b) - \varphi T_L(a,b), & \varphi \in [-1,0]. \end{cases}$$

Let's pay some attention to the fact that left and right bounds by Fréchet for covariations of events are shown, through operations T_Z, T_0 and T_L according to formulae $-F^-_{xy} = \max\{0,\ a+b-1\} - ab = T_L(a,b) - T_0(a,b)$, $F^+_{xy} = \min\{a,b\} - ab = T_Z(a,b) - T_0(a,b)$. By the definition, their pair covariation is equal to $\mathsf{Kov}_{xy} = \mathbf{P}(x\cap y) - \mathbf{P}(x)\mathbf{P}(y) = \mathbf{P}(x\cap y) - ab = \mathbf{P}(x\cap y) - T_0(a,b)$. Let's compute Fréchet-correlation of events in two situations 1) Let $\varphi \in [0,1]$ then

$$\mathsf{Kor}_{xy} = \frac{\mathsf{Kov}_{xy}}{F^+_{xy}} = \frac{\mathbf{P}(x\cap y) - T_0(a,b)}{T_Z(a,b) - T_0(a,b)} = \frac{(1-\varphi)T_0(a,b) + \varphi T_Z(a,b) - T_0(a,b)}{T_Z(a,b) - T_0(a,b)} = \varphi.$$

2) Let $\varphi \in [-1,0]$, then

$$\mathsf{Kor}_{xy} = \frac{\mathsf{Kov}_{xy}}{F^-_{xy}} = \frac{\mathbf{P}(x\cap y) - T_0(a,b)}{-T_L(a,b) + T_0(a,b)} = \frac{(1+\varphi)T_0(a,b) - \varphi T_L(a,b) - T_0(a,b)}{-T_L(a,b) + T_0(a,b)} = \varphi.$$

Lemma is proven.

In the theory of fuzzy sets by Zadeh a fuzzy event is understood only as a function of membership, which is determined in space; which takes on numerical values from unit interval. Thus this, which is called binary operations on two fuzzy events in the theory, is a binary operation on values of two functions of membership actually, i.e. maps of kind $\beta : [0,1]\times[0,1] \to [0,1]$, which confronts each pair of numbers (a,b) from $[0,1]\times[0,1]$ with a number: $\beta : (a,b) \to \beta(a,b)$. With the help of every map β it is possible to confront any pair of functions of membership $f : \Omega \to [0,1]$, $g : \Omega \to [0,1]$, with a third function of membership $h : \Omega \to [0,1]$, which is a result of binary operation «β» and is determined by formula $h(\omega) = \beta\Big(f(\omega),\ g(\omega)\Big)$ for any $\omega \in \Omega$.

Generalized Fréchet operations

Generalized binary and arity Fréchet intersections and unions in Zadeh theory notation

Generalized binary operations by Fréchet. Let parameter φ depend on a and b, i.e. to be arbitrary field $\varphi : [0,1] \times [0,1] \to [-1,1]$ having values $[-1,1]$, then will get generalized binary operations of *intersection and union by Fréchet*:

$$T_F^{\Phi}(a,b) = \begin{cases} (1-\varphi_{ab})T_0(a,b) + \varphi_{ab} T_Z(a,b), & \varphi_{ab} \in [0,1], \\ \\ (1+\varphi_{ab})T_0(a,b) - \varphi_{ab} T_L(a,b), & \varphi_{ab} \in [-1,0], \end{cases}$$

$$S_F^{\Phi}(a,b) = \begin{cases} (1-\varphi_{ab})S_0(a,b) + \varphi_{ab} T_L(a,b), & \varphi_{ab} \in [0,1], \\ \\ (1+\varphi_{ab})S_0(a,b) - \varphi_{ab} S_Z(a,b), & \varphi_{ab} \in [-1,0], \end{cases}$$

each of them is determined by it's own *field of pair Fréchet-correlations* $\Phi = \Big\{\varphi_{ab},\ (a,b) \in [0,1] \times [0,1]\Big\}$.

Generalized arity operations by Fréchet. Let φ depend on an arbitrary subset of parameters $A = \{a, b, \ldots, c\} \subseteq \mathcal{A} = \{a, b,\ \ldots, z\}$, i.e. to be arbitrary fields $\varphi_A : \underbrace{[0,1] \times \ldots \times [0,1]}_{|A|} \to [-1,1]$ having values from $[-1,1]$, then will get *generalized arity operations of intersection and union by Fréchet*:

$$T_F^{\Phi}(A) = \begin{cases} (1-\varphi_A)T_0(A) + \varphi_A T_Z(A), & \varphi_A \in [0,1], \\ \\ (1+\varphi_A)T_0(A) - \varphi_A T_L(A), & \varphi_A \in [-1,0], \end{cases}$$

$$S_F^{\Phi}(A) = \begin{cases} (1-\varphi_A)S_0(A) + \varphi_A T_L(A), & \varphi_A \in [0,1], \\ \\ (1+\varphi_A)S_0(A) - \varphi_A S_Z(A), & \varphi_A \in [-1,0], \end{cases}$$

each of them is determined by it's own *field of arity Fréchet-correlations* $\Phi = \Big\{\varphi_A,\ A \subseteq \mathcal{A}\Big\}$.

Fuzzy events in eventology

Now we have to consent undoubtedly that there is nothing else, but events in the real world (matter) and ideal world (mind). The consent will allow us to glance over the approaching top of eventology[10], called «fuzzy events», freshly and naively[11]: *«matter and mind are only two convenient ways of linking events into a sequence»*. Mind observes existence of matter as *a sequences of events* and exists as *a sequence of events*. Any *particular matter*, as well as any *particular mind* is determined by this or that consequence of events every time, in other words, by this or that *set of events*, and consequently, this or that *fuzzy event*.

Random and fuzzy experiments

A presentation of standard course of the probability theory begins from a notion of *random experiment*. The classical examples of random experiment are «tossing up a coin» and «rolling a die» (see Fig. 1).

A presentation of eventology begins from a notion of *fuzzy experiment*. The example — «creating a heap of grains» — is as well classical (see Fig. 2).

The fundamentally difference between *the fuzzy experiment* and *the random experiment* is direct participation of a set of individual minds $\mathfrak{M}$ in the former. The classical random experiment is required only *a coin* or *a die* and *an arena* and *nothing else*. Outcomes of *the random experiment do not depend on* individual mind's opinion. The following is required for the fuzzy experiment:

- *a funnel* with *grain* and with *a bolt*, which can be *closed or opened*, thus enabling grain to pour out of the funnel onto *a plane arena*.

- *individual minds* $\mu \in \mathfrak{M}$, arranged around the arena, are observing a grain, which has fallen onto the arena after the bolt was opened during some time.

[10] If eventology is a top of the theory of probability, then fuzzy event is a unique and forbidding until the very last moment top of eventology. Fuzzy event makes eventologists climb and reach for it from wherever they are.

[11] Any event is always random and it has a probability. A certain event — is an event occurring with unit probability only. Thus customary term «random event» is obviously superfluous and can be considered as a tautology.

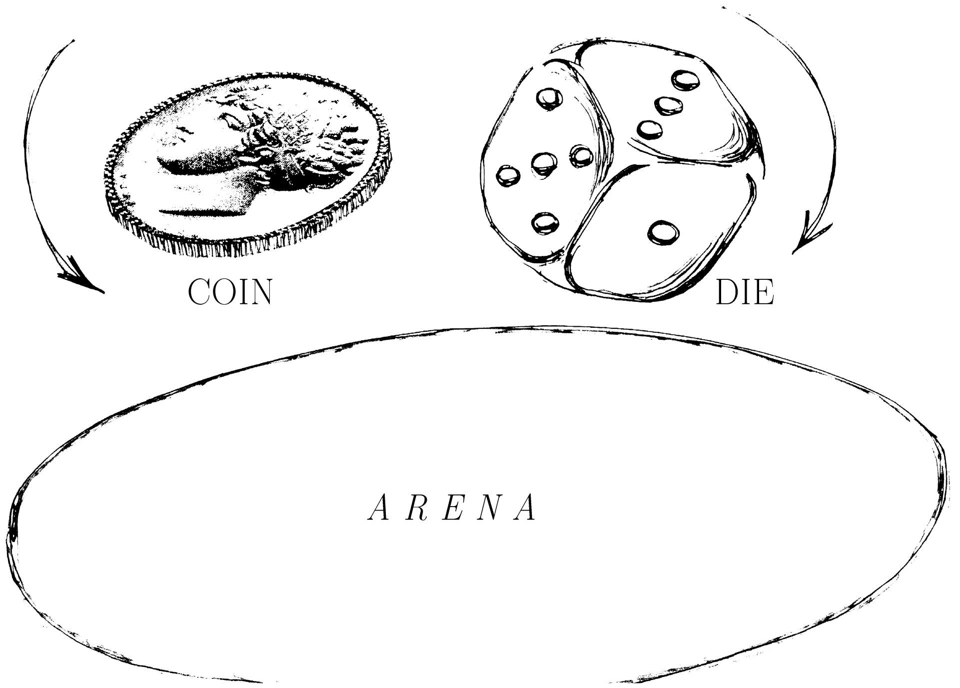

Рис. 1: Random experiments «tossing a coin» and «rolling a die»

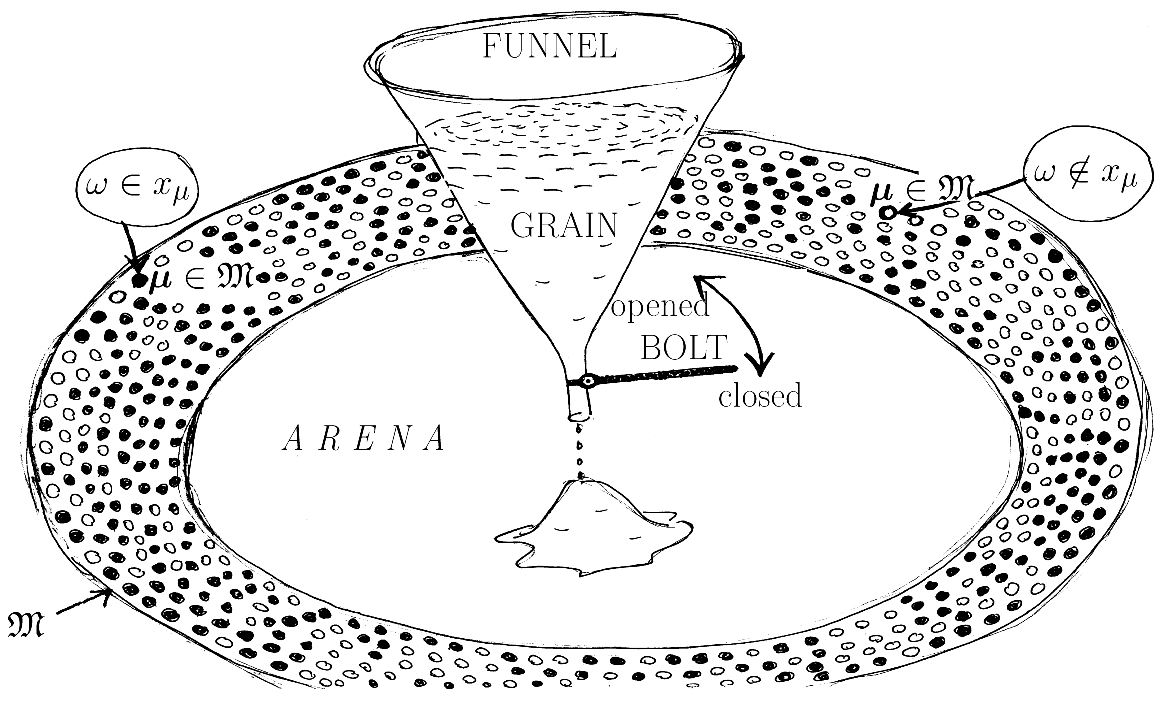

Рис. 2: Fuzzy experiment «creating a heap of grain»

Every individual mind — a participant of the fuzzy experiment — has his own opinion *if the fallen grains are to be considered a heap or to be not.* All of their opinions form *a fuzzy event* as an outcome of the fuzzy experiment.

Let's designate

$$x = \text{«a heap of grain»}$$

as *a name of fuzzy event «a heap of grain»*, which occurs fuzzily in result of the fuzzy experiment, and let x_μ — be a usual event, which occurs, when the individual mind μ considers that «a heap of grain» has fallen. Eventology denotes *a fuzzy event «x with wave»*:

$$\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}$$

as a set of usual events x_μ, when $\mu \in \mathfrak{M}$.

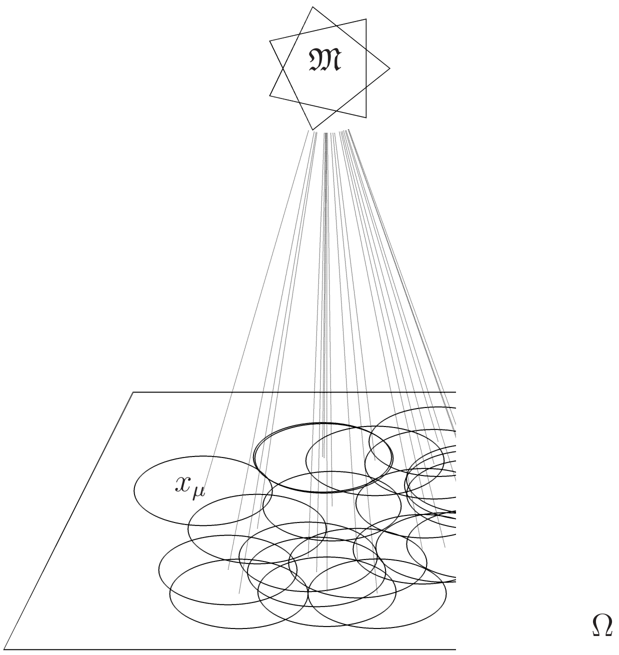

Рис. 3: An $\mathfrak{M}$-fuzzy event $\tilde{x} = \{x_\mu :\ \mu \in \mathfrak{M}\}$ as «stratification» of a private matter x by a mind $\mathfrak{M}$. A symbolic image of the mind $\mathfrak{M}$ (heptagon), «shining» through private minds $\mu \in \mathfrak{M}$, as if through the tools of observation upon the private matter x, which it observes («perceives») as the fuzzy event $\tilde{x} = x_{\mathfrak{M}}$. At the same time every private mind μ selects («elucidates») it's own layer of the private matter x, observing («perceiving») the matter as the event $x_\mu \subseteq \Omega$

Definitions and designations

Let $(\Omega, \mathcal{F}, \mathbf{P})$ — be a probability space[12], while $\mathfrak{X}$ and $\mathfrak{M}$ — are two finite sets of names. The elements $x \in \mathfrak{X}$ are names of events and elements $\mu \in \mathfrak{M}$ are names

[12] Probability space $(\Omega, \mathcal{F}, \mathbf{P})$ describes outcomes of the universal experiment, denoted as «being», while every $\omega \in \Omega$ — is a state of «being» in the point of time.

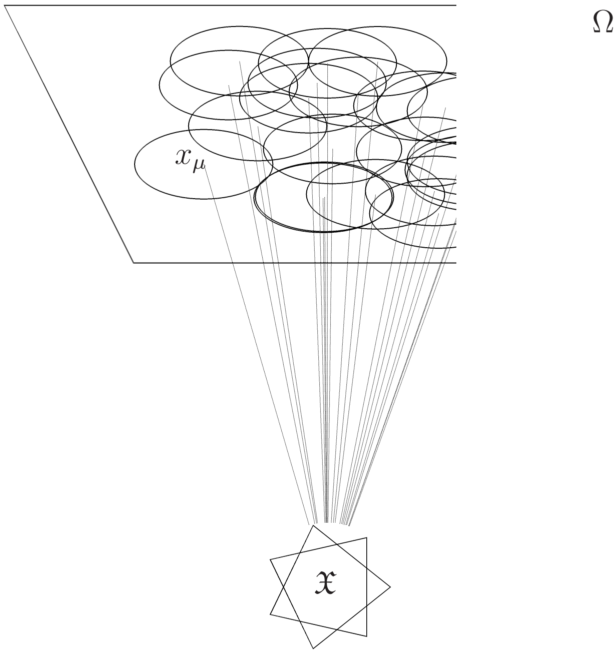

Рис. 4: An $\mathfrak{X}$-fuzzy event $\breve{\mu} = \{x_\mu : \ x \in \mathfrak{X}\}$ as «stratification» of a private mind μ by a matter $\mathfrak{X}$. The symbolic image of the matter $\mathfrak{X}$ (heptagon), observed («perceived») by the private mind μ as by the fuzzy event $\breve{\mu} = \mathfrak{X}_\mu$. The private mind μ observes the matter $\mathfrak{X}$; in every private matter $x \in \mathfrak{X}$ it selects («elucidates») it's own layer, which it observes as an event $x_\mu \subseteq \Omega$

of individual minds.

$\overset{(\sim)}{\mathfrak{X}}$

$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\tilde{x}$	$\dots$	x_λ	$\dots$	x_μ	$\dots$	x_ν	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\tilde{y}$	$\dots$	y_λ	$\dots$	y_μ	$\dots$	y_ν	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\tilde{z}$	$\dots$	z_λ	$\dots$	z_μ	$\dots$	z_ν	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
	$\dots$	$\breve{\lambda}$	$\dots$	$\breve{\mu}$	$\dots$	$\breve{\nu}$	$\dots$

$\overset{(\sim)}{\mathfrak{M}}$

Рис. 5: *Matrix of selected events* $\mathfrak{X}_{\mathfrak{M}} = \{x_\mu, \ x \in \mathfrak{X}, \ \mu \in \mathfrak{M}\}$; it's sets of «line» elements $x_{\mathfrak{M}} = \{x_\mu, \ \mu \in \mathfrak{M}\}$ determines *fuzzy events* $\tilde{x} = x_{\mathfrak{M}}$, which form the set $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x}, \ x \in \mathfrak{X}\}$; while it's sets of «column» elements $\mathfrak{X}_\mu = \{x_\mu, \ x \in \mathfrak{X}\}$ determines *fuzzy events* $\breve{\mu} = \mathfrak{X}_\mu$, which form the set $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu}, \ \mu \in \mathfrak{M}\}$.

Matrix of selected events

Let's define *the matrix of selected random events* with the help of two finite sets $\mathfrak{X}$ and $\mathfrak{M}$ - as the set of events $\mathfrak{X}_{\mathfrak{M}} = \{x_\mu,\ x \in \mathfrak{X},\ \mu \in \mathfrak{M}\}$, where $x_\mu \in \mathcal{F}$ are fuzzy events measurable relative to algebra. Thus every pair $(x, \mu) \in \mathfrak{X} \times \mathfrak{M}$ determines a random event $x_\mu \subseteq \Omega$.

Let's introduce some «natural» designations for arbitrary subsets of events, which form the matrix, in other words, for arbitrary «submatricies» of the «matrix» $X_W = \{x_\mu, x \in X,\ \mu \in W\} \subseteq \mathfrak{X}_{\mathfrak{M}}$, where $X \subseteq \mathfrak{X},\ W \subseteq \mathfrak{M}$. The «lines» of events $x_{\mathfrak{M}} = \{x_\mu,\ \mu \in \mathfrak{M}\} \subseteq \mathfrak{X}_{\mathfrak{M}}, \quad x \in \mathfrak{X}$, and «columns» of events $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\} \subseteq \mathfrak{X}_{\mathfrak{M}}, \quad \mu \in \mathfrak{M}$, as well as «sublines» of events $x_W = \{x_\mu, \mu \in W\} \subseteq x_{\mathfrak{M}}$, and «subcolumns» of events $X_\mu = \{x_\mu, x \in X\} \subseteq \mathfrak{X}_\mu$ belong to the most important «submatricies».

«Set-of-Events making» operators

Now let's define *set-of-events making* operators[13]. Here and there, on various pretexts *even a short experience in construction of principles of random events theory*[14] resurrects one and the same designation: $\tilde{x}$, which appears under various circumstances, but possesses one and the same character. Lively imagination can even take the designation for a result of an operator «$\sim$» forcing on a $x \in \mathfrak{X}$. The operator «$\sim$» forcing on x *makes a set of events*:

$$\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}, \tag{1}$$

The set of events is formed by events $x_\mu \subseteq \Omega$; Those-Which-Have-Names $\mu \in \mathfrak{M}$ link the occurrence of the events with Those-Which-Has-A-Name $x \in \mathfrak{X}$. Other operators «$\backsim$» (similar to the operator «$\sim$» in everything, but a set, generating it) may attempt to force on a $\mu \in \mathfrak{M}$ to *make a set of events*:

$$\breve{\mu} = \{x_\mu,\ x \in \mathfrak{X}\}, \tag{2}$$

[13] We are going to define inverse *set-of-events collapse* operators in the subsequent publications. *Collapse* (from Latin *collapsus* – «fallen») — *breakdown, demolition* or *reduction, disappearance*

[14] *Eventology — the theory of fuzzy random events —* is to play a part of a mathematical tool of mind motion study; study of the mind, which brings about fuzziness in everything it touches upon or meditates on. *«Mind appears there and then, where and when an ability of a fuzzy choice appears»*. Or even clearer: *When an ability of a fuzzy choice appears, mind appears at once.*

The set of events is formed by events $x_\mu \subseteq \Omega$; Those-Which-Has-A-Name $\mu \in \mathfrak{M}$ links the occurrence of the events with Those-Which-Have-Names $x \in \mathfrak{X}$. Sets of events from the ratios (1) and (2) are considerably different: the Set-Of-Events (1) is generated by the set $\mathfrak{M}$, while the Set-of-Events (2) — by the set $\mathfrak{X}$. Speaking matrix of selected events $\mathfrak{X}_\mathfrak{M}$ terminology: the Set-Of-Events (1) corresponds with the «line» $x_\mathfrak{M} = \tilde{x}$, while the Set-of-Events (2) — with the «column» $\mathfrak{X}_\mu = \breve{\mu}$ of the matrix. Let's emphasize the fact, that the result of applying the operator «$\sim$» to $x \in \mathfrak{X}$ is designated as $\tilde{x}$, while the result of applying the operator «$\backsim$» to $\mu \in \mathfrak{M}$ is designated as $\breve{\mu}$. The operator «$\sim$» is denoted as $\mathfrak{M}$-*operator*, while the operator «$\backsim$» — as *Set-Of-Events Making* $\mathfrak{X}$-*operator*.

«Set-Of-Events making» operators by Minkowski

Minkowski principle allows to turn the operators «$\sim$» and «$\backsim$» into the «Set-Of-Events making» operators by Minkowski, which are designated as «$(\sim)$» and «$(\backsim$ $)$» correspondingly. The essence of such a conversion is a standard-way spreading of operator forcing from the elements $x \in \mathfrak{X}$ or $\mu \in \mathfrak{M}$ to the sets of this elements $\mathfrak{X}$ or $\mathfrak{M}$, to their arbitrary subsets $X \subseteq \mathfrak{X}$ or $W \subseteq \mathfrak{M}$: $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$, $\overset{(\sim)}{X} = \{\tilde{x},\ x \in X\}$, и $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$, $\overset{(\backsim)}{W} = \{\breve{\mu},\ \mu \in W\}$.

Designations' «pun»

Introducing of the operators «$\sim$», «$\backsim$» and «$(\sim)$», «$(\backsim)$», which appear to be «parallel» designations to some subsets of the matrix of selected events $\mathfrak{X}_\mathfrak{M}$, provides us with an opportunity to «pun» the designations. Particularly the «pun» helps to test technical efficiency of the whole newly introduced notation of the theory of fuzzy random events. The fact that a subset of a matrix $\mathfrak{X}_\mathfrak{M}$ has two designations is a trite «microstatement»; it's proof follows from the definitions of the operators. Let's consider basic «microstatements».

Microstatement 1. $\widetilde{\mathfrak{X}} = \overset{(\sim)}{\mathfrak{M}}$. ***Proof.*** $\widetilde{\mathfrak{X}} = \{\mathfrak{X}_\mu,\ \mu \in \mathfrak{M}\} = \{\breve{\mu},\ \mu \in \mathfrak{M}\} = \overset{(\sim)}{\mathfrak{M}}$, because for every fixed $\mu \in \mathfrak{M}$ the corresponding «column» of the matrix $\mathfrak{X}_\mathfrak{M}$

has two designations $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\} = \breve{\mu}$ by definition of $\breve{\mu}$ — the operator «$\backsim$» forcing on $\mu \in \mathfrak{M}$.

Microstatement 2. $\widetilde{\mathfrak{M}} = \overset{(\sim)}{\mathfrak{X}}$. ***Proof.*** $\widetilde{\mathfrak{M}} = \{x_{\mathfrak{M}},\ x \in \mathfrak{X}\} = \{\tilde{x},\ x \in \mathfrak{X}\} = \overset{(\sim)}{\mathfrak{X}}$, because for every fixed $x \in \mathfrak{X}$ the corresponding «line» of the matrix $\mathfrak{X}_{\mathfrak{M}}$ has two designations $x_{\mathfrak{M}} = \{x_\mu,\ \mu \in \mathfrak{M}\} = \tilde{x}$ by definition of $\tilde{x}$ — the operator «$\sim$» forcing on $x \in \mathfrak{X}$.

Microstatement 3. $X \subseteq \mathfrak{X} \implies \overset{(\sim)}{X} \subseteq \overset{(\sim)}{\mathfrak{X}}$ and $W \subseteq \mathfrak{M} \implies \overset{(\backsim)}{W} \subseteq \overset{(\backsim)}{\mathfrak{M}}$. ***Proof.*** If $X \subseteq \mathfrak{X}$, then $\overset{(\sim)}{X} = \{\tilde{x},\ x \in X\} \subseteq \{\tilde{x},\ x \in \mathfrak{X}\} = \overset{(\sim)}{\mathfrak{X}}$. If $W \subseteq \mathfrak{M}$, then $\overset{(\backsim)}{W} = \{\breve{\mu},\ \mu \in W\} \subseteq \{\breve{\mu},\ \mu \in \mathfrak{M}\} = \overset{(\backsim)}{\mathfrak{M}}$.

Microstatement 4. $X \subseteq \mathfrak{X} \implies \widetilde{X}\ (\subseteq)\ \widetilde{\mathfrak{X}}$ and $W \subseteq \mathfrak{M} \implies \breve{W}(\subseteq)\ \widetilde{\mathfrak{M}}$. ***Proof.*** If $X \subseteq \mathfrak{X}$, then $\widetilde{X} = \{X_\mu,\ \mu \in \mathfrak{M}\}\ (\subseteq)\ \{\mathfrak{X}_\mu,\ \mu \in \mathfrak{M}\} = \widetilde{\mathfrak{X}}$, because $X_\mu = \{x_\mu,\ x \in X\} \subseteq \{x_\mu,\ x \in \mathfrak{X}\} = \mathfrak{X}_\mu$ for every $\mu \in \mathfrak{M}$. If $W \subseteq \mathfrak{M}$, then $\breve{W} = \{x_W,\ x \in \mathfrak{X}\}\ (\subseteq)\ \{x_{\mathfrak{M}},\ x \in \mathfrak{X}\} = \widetilde{\mathfrak{M}}$, because $x_W = \{x_\mu,\ \mu \in W\} \subseteq \{x_\mu,\ \mu \in \mathfrak{M}\} = x_{\mathfrak{M}}$ for every $x \in \mathfrak{X}$.

Types of terrace-events (t-events)

Two types of terrace-events (t-events) — subsets Ω — appear in eventological description of fuzzy events.

The first type terrace-events (ftt-events)

Under fixed $\mu \in \mathfrak{M}$ set of events[15] $\breve{\mu} = \{x_\mu,\ x \in \mathfrak{X}\}$, generates the *first type terrace-events (ftt-events)*

$$\mathrm{ter}_\mu(X) = \bigcap_{x \in X} x_\mu \bigcap_{x \in X^c} x_\mu^c, \quad X \subseteq \mathfrak{X},$$

where $x_\mu^c = \Omega - x_\mu$, $\quad X^c = \mathfrak{X} - X$ — are event supplement $x_\mu \subseteq \Omega$ up to Ω and set supplement $X \subseteq \mathfrak{X}$ up to $\mathfrak{X}$. Ftt-events for every fixed $\mu \in \mathfrak{M}$ are pairwise disjoint: $\mathrm{ter}_\mu(X) \cap \mathrm{ter}_\mu(X') = \emptyset \iff X \neq X'$ they form partition of elementary events' space $\Omega = \sum_{X \subseteq \mathfrak{X}} \mathrm{ter}_\mu(X)$.

[15] $\breve{\mu}$ — $\mathfrak{X}$-*fuzzy event (individual mind).*

The second type terrace-events (stt-events)

The second type terrace-events (stt-events) — are t-events

$$\mathrm{ter}_x(W) = \bigcap_{\mu\in W} x_\mu \bigcap_{\mu\in W^c} x^c_\mu, \quad W \subseteq \mathfrak{M},$$

generated under fixed $x \in \mathfrak{X}$ set of events[16] $\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}$, where $x^c_\mu = \Omega - x_\mu$, $\quad W^c = \mathfrak{M} - W$ — are event supplement $x_\mu \subseteq \Omega$ up to Ω And set supplement $W \subseteq \mathfrak{M}$ up to $\mathfrak{M}$. Stt-events for every fixed $x \in \mathfrak{X}$ are pairwise disjoint: $\mathrm{ter}_x(W) \cap \mathrm{ter}_x(W') = \emptyset \quad \Longleftrightarrow \quad W \neq W'$ they form partition of elementary events' space $\Omega = \sum_{W\subseteq\mathfrak{M}} \mathrm{ter}_x(W)$.

Types of fuzzy terrace-events

Eventology of fuzzy random events also deals with fuzzy t-events. There are two types of fuzzy t-events: they are distinguished by sets, which generate the «fuzziness» of t-events. Ftt-events' «fuzziness» is generated by the set $\mathfrak{M}$, stt-events «fuzziness» – by the set $\mathfrak{X}$.

$\mathfrak{M}$-*Fuzzy first type terrace-events*

$\mathfrak{M}$-*Fuzzy first type terrace-events* ($\mathfrak{M}$-*Fuzzy ftt-events*) $\mathrm{ter}_\mu(X)$, correspondingly generated by sets of events $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}$ — fuzzy events (minds) $\breve{\mu} = \mathfrak{X}_\mu$; all together, when $\mu \in \mathfrak{M}$, form the set of events

$$\tilde{\mathrm{ter}}(X) = \Big\{ \mathrm{ter}_\mu(X),\ \mu \in \mathfrak{M} \Big\}, \quad X \subseteq \mathfrak{X}.$$

which is denoted as *fuzzy ftt-event* or $\mathfrak{M}$-*fuzzy t-event*. Fuzzy ftt-events are pairwise disjoint by Minkowski $\tilde{\mathrm{ter}}(X)\ (\cap)\ \tilde{\mathrm{ter}}(X') = \breve{\emptyset}_{\mathfrak{M}} \iff X \neq X'$; they form *partition by Minkowski* of authentic $\mathfrak{M}$-fuzzy event $\tilde{\Omega} = (\sum)_{X\subseteq\mathfrak{X}} \tilde{\mathrm{ter}}(X)$, because $\Omega = \sum_{X\subseteq\mathfrak{X}} \mathrm{ter}_\mu(X)$ for every $\mu \in \mathfrak{M}$ partition of authentic event Ω usual t-events are generated. Set-operations by Minkowski make it possible to deduce the following formula for $\mathfrak{M}$-fuzzy t-events in a standard way:

$$\tilde{\mathrm{ter}}(X) = \Big(\bigcap_{x\in X}\Big)\, \tilde{x} \Big(\bigcap_{x\in X^c}\Big)\, \tilde{x}^{(c)},$$

[16] $\tilde{x}$ — $\mathfrak{M}$-*fuzzy event (partial matter).*

where $X \subseteq \mathfrak{X}$ — is a usual subset $\mathfrak{X}$, $X^c = \mathfrak{X} - X$ — is a usual supplement up to $\mathfrak{X}$, while $\tilde{x}^{(c)} = \widetilde{\Omega}\ (-)\ \tilde{x}$ — is a supplement $\tilde{x}$ by Minkowski up to $\widetilde{\Omega}$, while $(\bigcap)$ — is an intersection of sets of events by Minkowski. $\mathfrak{M}$-Fuzzy t-events can be also viewed as t-events, generated by set of fuzzy events $\overset{(\sim)}{\mathfrak{X}}$; thus another, though a correct one, but rather bulky formula can be deduced:

$$\text{ter}(\overset{(\sim)}{X}) = \underset{\tilde{x} \in \overset{(\sim)}{X}}{\left(\bigcap\right)} \tilde{x} \underset{\tilde{x} \in (\overset{(\sim)}{X})^c}{\left(\bigcap\right)} \tilde{x}^{(c)} = \tilde{\text{ter}}(X),$$

where $\overset{(\sim)}{X} \subseteq \overset{(\sim)}{\mathfrak{X}}$ — is a usual subset $\overset{(\sim)}{\mathfrak{X}}$, $(\overset{(\sim)}{X})^c = \overset{(\sim)}{\mathfrak{X}} - \overset{(\sim)}{X}$ — is a usual supplement up to $\overset{(\sim)}{\mathfrak{X}}$.

$\mathfrak{X}$-Fuzzy the second type terrace-events

$\mathfrak{X}$-Fuzzy the second type terrace-events ($\mathfrak{X}$-Fuzzy stt-events) $\text{ter}_\mu(X)$, correspondingly generated by sets of events $x_{\mathfrak{M}} = \{x_\mu,\ \mu \in \mathfrak{M}\}$ — fuzzy events $\tilde{x} = x_{\mathfrak{M}}$; all together, when $x \in \mathfrak{X}$, form the set of events

$$\breve{\text{ter}}(W) = \left\{ \text{ter}_x(W),\ x \in \mathfrak{X} \right\}, \quad W \subseteq \mathfrak{M}.$$

which is denoted as *fuzzy stt-event* or *$\mathfrak{X}$-fuzzy t-event.* Fuzzy stt-events are pairwise disjoint by Minkowski $\breve{\text{ter}}(W)\ (\cap)\ \breve{\text{ter}}(W') = \breve{\emptyset}_{\mathfrak{X}} \iff W \neq W'$; they form *partition by Minkowski* of authentic $\mathfrak{X}$-fuzzy event $\breve{\Omega} = (\sum)_{W \subseteq \mathfrak{M}} \breve{\text{ter}}(W)$, because $\Omega = \sum_{W \subseteq \mathfrak{M}} \text{ter}_x(W)$ for every $x \in \mathfrak{X}$ partition of authentic event Ω usual t-events are generated. Set-operations by Minkowski make it possible to deduce the following formula for $\mathfrak{X}$-fuzzy t-events in a standard way:

$$\breve{\text{ter}}(W) = \underset{\mu \in W}{\left(\bigcap\right)} \breve{\mu} \underset{\mu \in W^c}{\left(\bigcap\right)} \breve{\mu}^{(c)},$$

where $W \subseteq \mathfrak{M}$ — is a usual subset $\mathfrak{M}$, $W^c = \mathfrak{M} - W$ — is a usual supplement up to $\mathfrak{M}$, while $\breve{\mu}^{(c)} = \breve{\Omega}\ (-)\ \breve{\mu}$ — is a supplement $\breve{\mu}$ by Minkowski up to $\breve{\Omega}$, while $(\bigcap)$ — is an intersection of sets of events by Minkowski. $\mathfrak{X}$-Fuzzy t-events can be also viewed as t-event s, generated by set of fuzzy events $\overset{(\sim)}{\mathfrak{M}}$; thus another, though a correct one, but rather bulky formula can be deduced:

$$\mathrm{ter}(\overset{(\backsim)}{W}) = \Big(\bigcap_{\breve{\mu}\in\overset{(\backsim)}{W}}\Big)\,\tilde{x}\,\Big(\bigcap_{\breve{\mu}\in(\overset{(\backsim)}{W})^c}\Big)\,\breve{\mu}^{(c)} = \breve{\mathrm{ter}}(W),$$

where $\overset{(\backsim)}{W} \subseteq \overset{(\sim)}{\mathfrak{M}}$ — is a usual subset $\overset{(\sim)}{\mathfrak{M}}$, $(\overset{(\backsim)}{W})^c = \overset{(\sim)}{\mathfrak{M}} - \overset{(\backsim)}{W}$ — is a usual supplement up to $\overset{(\sim)}{\mathfrak{M}}$.

Fuzzy elementary events

Let's consider three types of fuzzy events. The fuzzy events are denoted as *fuzzy elementary events (fuzzy e-events)* and formed from fragments of one or another partitions of space of elementary events Ω (e-events, generated by three types of events' sets): by «lines»

$$x_{\mathfrak{M}} = \{x_\mu,\ \mu \in \mathfrak{M}\}, \quad x \in \mathfrak{X},$$

by «columns»

$$\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}, \quad \mu \in \mathfrak{M},$$

of a «matrix» of selected events, and by the whole «matrix»

$$\mathfrak{X}_{\mathfrak{M}} = \{x_\mu,\ x \in \mathfrak{X},\ \mu \in \mathfrak{M}\}.$$

Let's note, that «lines» $x_{\mathfrak{M}}$ and «columns» $\mathfrak{X}_\mu$ have one more denotation: «lines» are denoted as $\mathfrak{M}$-*fuzzy events* $\tilde{x} = x_{\mathfrak{M}}$, while «columns» are denoted as $\mathfrak{X}$-*fuzzy events* $\breve{\mu} = \mathfrak{X}_\mu$. From the view point of the formal eventology, the «matrix» of selected events, as the set of events $\mathfrak{X}_{\mathfrak{M}}$, might be also considered as the only one $(\mathfrak{X}, \mathfrak{M})$-*fuzzy event*,

$$\{\underset{\sim}{\smile}\} = \{x_\mu,\ x \in \mathfrak{X},\ \mu \in \mathfrak{M}\} = \mathfrak{X}_{\mathfrak{M}},$$

which encloses all $\mathfrak{M}$-fuzzy events and all $\mathfrak{X}$-fuzzy events, as it's subsets[17]:

$$\{\underset{\sim}{\smile}\} = \sum_{x\in\mathfrak{X}} \tilde{x} = \sum_{\mu\in\mathfrak{M}} \breve{\mu}.$$

[17]Here the sign «$\sum$» designates a set-operation of union of non-intersected sets of events.

Let's imagine a subset of the fuzzy event $\{\eqsim\}$, which consists of events $x_\mu \in \mathfrak{X}_{\mathfrak{M}}$; the events occur, when the e-event $\omega \in \Omega$ occurs. Let's introduce a special designation for the subset:

$$\{\eqsim\}(\omega) = \{x_\mu \in \mathfrak{X}_{\mathfrak{M}} : \ \omega \in x_\mu\} \subseteq \{\eqsim\}.$$

Let's also introduce a special designation for the subsets of fuzzy events $\tilde{x}$ and $\breve{\mu}$, which occur, when the e-event $\omega \in \Omega$ does. Let's designate for every $x \in \mathfrak{X}$:

$$\tilde{x}(\omega) = \{x_\mu \in \tilde{x} : \ \omega \in x_\mu\} \subseteq \tilde{x},$$

$$\tilde{x}^c(\omega) = \{x_\mu \in \tilde{x} : \ \omega \in x_\mu^c\} = \tilde{x} - \tilde{x}(\omega) \subseteq \tilde{x}$$

— are corresponding subsets $\tilde{x}$. Let's designate for every $\mu \in \mathfrak{M}$

$$\breve{\mu}(\omega) = \{x_\mu \in \breve{\mu} : \ \omega \in x_\mu\} \subseteq \breve{\mu},$$

$$\breve{\mu}^c(\omega) = \{x_\mu \in \breve{\mu} : \ \omega \in x_\mu^c\} = \breve{\mu} - \breve{\mu}(\omega) \subseteq \breve{\mu}$$

— are corresponding subsets $\breve{\mu}$. Let's also introduce special «parallel» designations for the subsets of μ- and x-names of events x_μ, which belong to fuzzy events $\tilde{x}$ and $\breve{\mu}$ correspondingly. The fuzzy events occur, when the e-event $\omega \in \Omega$ occurs. Let's designate for every $x \in \mathfrak{X}$:

$$W_x(\omega) = \{\mu \in \mathfrak{M} : \ \omega \in x_\mu\} \subseteq \mathfrak{M},$$

$$W_x^c(\omega) = \{\mu \in \mathfrak{M} : \ \omega \in x_\mu^c\} = \mathfrak{M} - W_x u(\omega) \subseteq \mathfrak{M}$$

— are corresponding subsets of μ-names. Let's designate for every $\mu \in \mathfrak{M}$

$$X_\mu(\omega) = \{x \in \mathfrak{X} : \ \omega \in x_\mu\} \subseteq \mathfrak{X},$$

$$X_\mu^c(\omega) = \{x \in \mathfrak{X} : \ \omega \in x_\mu^c\} = \mathfrak{X} - X_\mu(\omega) \subseteq \mathfrak{X}.$$

— are corresponding subsets of x-names. Following notions are true for every $\omega \in \Omega$ in the newly introduced designations:

$$\tilde{x}(\omega) = \{x_\mu, \ \mu \in W_x(\omega)\}, \qquad \breve{\mu}(\omega) = \{x_\mu, \ x \in X_\mu(\omega)\}.$$

1) «Quasi-fuzzy» e-event. Set of events $\mathfrak{X}_{\mathfrak{M}} = \{x_\mu, \ x \in \mathfrak{X}, \ \mu \in \mathfrak{M}\}$, or the fuzzy event $\{\eqsim\} = \mathfrak{X}_{\mathfrak{M}}$ (which is quiet the same), partition the space of e-events

Ω into terrace-events, that have been «numbered» by it's arbitrary subsets $X_W \subseteq \mathfrak{X}_{\mathfrak{M}}$. The terrace-events are determined with corresponding terrace formulas:

$$\mathrm{ter}(X_W) = \bigcap_{x_\mu \in X_W} x_\mu \bigcap_{x_\mu \in X_W^c} x_\mu^c.$$

If $\omega \in \Omega$ is fixed and designated as

$$X_W(\omega) = \{x_\mu \in \mathfrak{X}_{\mathfrak{M}} : \ \omega \in x_\mu\} = \{\breve{\sim}\}(\omega)$$

— an only one from the subsets, which has been formed from the events, occurring simultaneously, when $\omega \in \Omega$ occurs; then, when $\omega \in \Omega$ occurs, the fuzzy event $\{\breve{\sim}\}$ cannot «comprehend» that *the* e-event $\omega \in \Omega$ has occurred, but — only *an* e-event, which belongs to a terrace-event[18] $\mathrm{ter}(X_W(\omega))$. The given terrace-event is denoted as $\{\breve{\sim}\}$-*e-event* and has two special designations:

$$\omega_{\{\breve{\sim}\}} = \mathrm{ter}(X_W(\omega)) = \omega_{\mathfrak{X}_{\mathfrak{M}}},$$

The monoplet — so called *«quasi-fuzzy» e-event* - formally gets generated by the only one terrace-event:

$$\breve{\tilde{\omega}} = \left\{\omega_{\{\breve{\sim}\}}\right\} = \left\{\omega_{\mathfrak{X}_{\mathfrak{M}}}\right\}.$$

2) $\mathfrak{X}$-fuzzy events. Every set of events $x_{\mathfrak{M}} = \{x_\mu, \ \mu \in \mathfrak{M}\}, \ x \in \mathfrak{X}$, or the $\mathfrak{M}$-fuzzy event $\tilde{x} = x_{\mathfrak{M}}$ splits the space of e-events Ω into terrace-events, that have been «numbered» by the subsets of μ-names $W_x \subseteq \mathfrak{M}$. The terrace-events are determined with corresponding terrace formulas:

$$\mathrm{ter}_x(W_x) = \bigcap_{\mu \in W_x} x_\mu \bigcap_{\mu \in W_x^c} x_\mu^c, \quad x \in \mathfrak{X}.$$

If $\omega \in \Omega$ is fixed and designated as

$$W_x(\omega) = \{\mu \in \mathfrak{M} : \ \omega \in x_\mu\} \subseteq \mathfrak{M}$$

— an only one from the subsets, which has been formed from the μ-names of events $x_\mu \in \tilde{x}$, occurring simultaneously, when $\omega \in \Omega$ occurs; then, when $\omega \in \Omega$ occurs, the fuzzy event $\tilde{x}$ cannot «comprehend» that *the* e-event $\omega \in \Omega$ has occurred, but — only *an* e-event, which belongs to a terrace-event[19] $\mathrm{ter}_x(W_x(\omega))$.

[18] $\{\breve{\sim}\}$-e-events $\omega_{\mathfrak{X}_{\mathfrak{M}}}$ restrict «resolvability» of fuzzy event $\{\breve{\sim}\}$ to distinguish elementary events $\omega \in \Omega$ from each other: elementary events, which come upon one $\{\breve{\sim}\}$-e-event $\omega_{\mathfrak{X}_{\mathfrak{M}}}$, turn out to be equivalent for the fuzzy event $\{\breve{\sim}\}$.

[19] x-e-events ω_x restrict «resolvability» of fuzzy event $\tilde{x}$ to distinguish elementary events $\omega \in \Omega$ from each other: elementary events, which come upon one x-e-event ω_x, turn out to be equivalent for the fuzzy event $\tilde{x}$.

The given terrace-event is denoted as x*-e-event* and has two special designations:

$$\omega_x = \mathrm{ter}_x(W_x(\omega)),$$

taken all together they form a so-called $\mathfrak{X}$*-fuzzy e-event*

$$\breve{\omega} = \{\omega_x,\ x \in \mathfrak{X}\}.$$

3) $\mathfrak{M}$-fuzzy e-events. Every set of events $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\},\ \mu \in \mathfrak{M}$, or the $\mathfrak{X}$-fuzzy event $\breve{\mu} = \mathfrak{X}_\mu$, splits the space of e-events Ω into terrace-events, that have been «numbered» by the subsets x-имен $X_\mu \subseteq \mathfrak{X}$. The terrace-events are determined with corresponding terrace formulas:

$$\mathrm{ter}_\mu(X_\mu) = \bigcap_{x \in X_\mu} x_\mu \bigcap_{x \in X_\mu^c} x_\mu^c, \quad \mu \in \mathfrak{M}.$$

If $\omega \in \Omega$ is fixed and designated as

$$X_\mu(\omega) = \{x \in \mathfrak{X} :\ \omega \in x_\mu\} \subseteq \mathfrak{X}$$

— an only one from the subsets, which has been formed from the x-names of events $x_\mu \in \breve{\mu}$, occurring simultaneously, when $\omega \in \Omega$ occurs; then, when $\omega \in \Omega$ occurs, the fuzzy event $\breve{\mu}$ cannot «comprehend» that *the* e-event $\omega \in \Omega$ has occurred, but — only *an* e-event, which belongs to a terrace-event[20] $\mathrm{ter}_\mu(X_\mu(\omega))$. The given terrace-event is denoted as μ*-e-event* and has a special designation:

$$\omega_\mu = \mathrm{ter}_\mu(X_\mu(\omega)),$$

taken all together they form $\mathfrak{M}$*-fuzzy e-event*

$$\tilde{\omega} = \{\omega_\mu,\ \mu \in \mathfrak{M}\}.$$

Fuzzy e-events theorem. *Let* $\breve{\breve{\omega}} = \left\{\omega_{\{\breve{\breve{\ }}\}}\right\} = \left\{\omega_{\mathfrak{X}_\mathfrak{M}}\right\}$ *— be a quasi-fuzzy e-event,* $\tilde{\omega} = \{\omega_\mu,\ \mu \in \mathfrak{M}\}$ *— be* $\mathfrak{M}$*-fuzzy e-event,* $\breve{\omega} = \{\omega_x,\ x \in \mathfrak{X}\}$ *— be* $\mathfrak{X}$*-fuzzy e-event. Then*

$$\omega_{\mathfrak{X}_\mathfrak{M}} = \bigcap_{\mu \in \mathfrak{M}} \omega_\mu = \bigcap_{x \in \mathfrak{X}} \omega_x,$$

[20] μ-e-events ω_μ restrict «resolvability» of fuzzy event $\breve{\mu}$ to distinguish elementary events $\omega \in \Omega$ from each other: elementary events, which come upon one μ-e-event ω_μ turn out to be equivalent for the fuzzy event $\breve{\mu}$

or otherwise:

$$\omega_{\{\tilde{\smile}\}} = \bigcap_{\omega_\mu \in \tilde{\omega}} \omega_\mu = \bigcap_{\omega_x \in \breve{\omega}} \omega_x.$$

Proof follows from the definitions of fuzzy e-events evidently.

The «shorthand» of Kolmogorov axiomatics of definition of probability in eventology of fuzzy events

Let's give a list of basic definitions of axiomatics by Kolmogorov in a «shorthand» style. These definitions have been transferred into eventology of fuzzy events by analogy. We shall not delve deeply into properties of new eventological concepts, because they are completely similar to classical concepts by Kolmogorov, with one exception: the place of set-operations over events has been given to *set-operations over fuzzy events by Minkowski* — over sets of usual events. For example, *algebra of fuzzy events* is defined as a set of fuzzy events, which are closed relative to set-operations by Minkowski, and which contains impossible fuzzy event. *Probability, determined in algebra of fuzzy events,* has an usual property of additivity, but only relative to set-operations by Minkowski.

In the «matrix» of selected events $\mathfrak{X}_{\mathfrak{M}} = \{x_\mu,\ \mu \in \mathfrak{M},\ x \in \mathfrak{X}\}$ we distinguish fuzzy events of two kinds: «lines» $\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}$ are called to be *$\mathfrak{M}$-fuzzy events*, and «columns» $\breve{\mu} = \{x_\mu,\ X \in \mathfrak{X}\}$ – *$\mathfrak{X}$-fuzzy events (minds).* Although Kolmogorov axiomatics definitions for fuzzy events of these two kinds are completely similar, we shall give both for a reader could compare and pay attention to details.

«Shorthand» of basic definitions of $\mathfrak{M}$-fuzzy events

Definition 1-$\mathfrak{M}$. *$\mathfrak{M}$-fuzzy elementary event (e-event)* – is a $|\mathfrak{M}|$-set $\tilde{\omega} = \{\omega_\mu,\ \mu \in \mathfrak{M}\}$, which consists of $\mathcal{F}$-measurable events $\omega_\mu \subseteq \Omega$, and each of them occurs, when $\omega \in \Omega$ occurs, and is one of terrace-events, generated by the set of events $\breve{\mu} = \{x_\mu,\ x \in \mathfrak{X}\}$:

$$\omega_\mu = \mathrm{ter}_\mu(X_\mu(\omega)),$$

in which $\omega \in \Omega$, that has occurred, falls.

Definition 2-$\mathfrak{M}$. *Space of elementary $\mathfrak{M}$-fuzzy events* – is an $|\mathfrak{M}|$-set $\tilde{\Omega} = \{\Omega,\ \mu \in \mathfrak{M}\} = \{\underbrace{\Omega, \ldots, \Omega}_{|\mathfrak{M}|}\}$, which consists of the same space of elementary events Ω. ***Definition 3-$\mathfrak{M}$.*** *$\mathfrak{M}$-fuzzy event* — is an $|\mathfrak{M}|$-set, which consists of $\tilde{\mathfrak{F}}$-measured, events: $\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}$, where $x_\mu \subseteq \Omega$. ***Definition 4-$\mathfrak{M}$.*** *Certain $\mathfrak{M}$-fuzzy event* – is an $|\mathfrak{M}|$-set $\tilde{\Omega} = \{\Omega,\ \mu \in \mathfrak{M}\} = \{\underbrace{\Omega, \ldots, \Omega}_{|\mathfrak{M}|}\}$, which consists from the same certain events Ω. ***Definition 5-$\mathfrak{M}$.*** *Impossible $\mathfrak{M}$-fuzzy event* – is an $|\mathfrak{M}|$-set $\tilde{\emptyset} = \{\emptyset,\ \mu \in \mathfrak{M}\} = \{\underbrace{\emptyset, \ldots, \emptyset}_{|\mathfrak{M}|}\}$, which consists of the same impossible event $\emptyset$. ***Definition 6-$\mathfrak{M}$.*** *Algebra of $\mathfrak{M}$-of fuzzy events* – is an $|\mathfrak{M}|$-set $\tilde{\mathfrak{F}} = \{\mathfrak{F}_\mu,\ \mu \in \mathfrak{M}\}$, which consists of the algebras of usual events. ***Definition 7-$\mathfrak{M}$.*** *Measurable space of $\mathfrak{M}$-fuzzy events* — is an $|\mathfrak{M}|$-set $(\tilde{\Omega}, \tilde{\mathfrak{F}}) = \{(\Omega, \mathfrak{F}_\mu),\ \mu \in \mathfrak{M}\}$, which consists of measured spaces of events.***Definition 8-$\mathfrak{M}$.*** *Probability of $\tilde{\mathfrak{F}}$-measurable $\mathfrak{M}$-fuzzy event* $\tilde{x}$ $(\in)$ $\tilde{\mathfrak{F}}$ is determined by the formula:

$$\mathbf{P}(\tilde{x}) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{P}(x_\mu).$$

Definition 9-$\mathfrak{M}$. *Probabilistic space of $\mathfrak{M}$-fuzzy events* — is an $|\mathfrak{M}|$-set

$$(\tilde{\Omega}, \tilde{\mathfrak{F}}, \mathbf{P}) = \{(\Omega, \mathfrak{F}_\mu, \mathbf{P}),\ \mu \in \mathfrak{M}\},$$

which consists of probabilistic spaces of usual events, distinguished by algebras only $\mathfrak{F}_\mu,\ \mu \in \mathfrak{M}$.

«Shorthand» of basic definitions of $\mathfrak{X}$-fuzzy events

Definition 1-$\mathfrak{X}$. *$\mathfrak{X}$-fuzzy elementary event (э-event)* – is a $|\mathfrak{X}|$-set $\breve{\omega} = \{\omega_x,\ x \in \mathfrak{X}\}$, consists of, $\mathfrak{F}$-measurable events $\omega_x \subseteq \Omega$, and each of them occurs, when $\omega \in \Omega$ occurs, and is one of terrace-events, generated by the set of events $\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}$:

$$\omega_x = \mathrm{ter}_x(W_x(\omega)),$$

in which gets come $\omega \in \Omega$.

Definition 2-$\mathfrak{X}$. *Space of elementary $\mathfrak{X}$-fuzzy events* – is an $|\mathfrak{X}|$-set $\breve{\Omega} = \{\Omega,\ x \in \mathfrak{X}\} = \{\underbrace{\Omega, \ldots, \Omega}_{|\mathfrak{X}|}\}$, which consists of the same, spaces of elementary events Ω. ***Definition 3-$\mathfrak{X}$.*** *$\mathfrak{X}$-fuzzy event* — is an $|\mathfrak{X}|$-set, which consists of $\breve{\mathcal{F}}$-measurable events: $\breve{\mu} = \{x_\mu,\ x \in \mathfrak{X}\}$, where $x_\mu \subseteq \Omega$. ***Definition 4-$\mathfrak{X}$.*** *Certain $\mathfrak{X}$-fuzzy event* — is an $|\mathfrak{X}|$-set $\breve{\Omega} = \{\Omega,\ x \in \mathfrak{X}\} = \{\underbrace{\Omega, \dots, \Omega}_{|\mathfrak{X}|}\}$,which consists of the same ceratin, events Ω. ***Definition 5-$\mathfrak{X}$.*** *Impossible $\mathfrak{X}$-fuzzy event* – is an $|\mathfrak{X}|$-set $\breve{\emptyset} = \{\emptyset,\ X \in \mathfrak{X}\} = \{\underbrace{\emptyset, \dots, \emptyset}_{|\mathfrak{X}|}\}$ which consists of the same impossible event $\emptyset$. ***Definition 6-$\mathfrak{X}$.*** *Algebra of $\mathfrak{X}$-fuzzy Events* — is an $|\mathfrak{X}|$-set $\breve{\mathcal{F}} = \{\mathcal{F}_x,\ x \in \mathfrak{X}\}$, which consists of algebras of usual events. ***Definition 7-$\mathfrak{X}$.*** *Measurable space of $\mathfrak{X}$-fuzzy events* – is an $|\mathfrak{X}|$-set $(\breve{\Omega}, \breve{\mathcal{F}}) = \{(\Omega, \mathcal{F}_\mu),\ x \in \mathfrak{X}\}$, which consists of measurable spaces of events. ***Definition 8-$\mathfrak{X}$.*** *Probability of $\breve{\mathcal{F}}$-measurable $\mathfrak{X}$-fuzzy event $\breve{\mu}$* $(\in)$ $\breve{\mathcal{F}}$ is determined by the formula:

$$\mathbf{P}(\breve{\mu}) = \frac{1}{|\mathfrak{X}|} \sum_{x \in \mathfrak{X}} \mathbf{P}(x_\mu).$$

Definition 9-$\mathfrak{X}$. *Probability space of $\mathfrak{X}$-fuzzy events* — is an $|\mathfrak{X}|$-set

$$(\breve{\Omega}, \breve{\mathcal{F}}, \mathbf{P}) = \{(\Omega, \mathcal{F}_x, \mathbf{P}),\ x \in \mathfrak{X}\},$$

which consists of probability spaces of usual events, distinguished by algebras only $\mathcal{F}_x,\ x \in \mathfrak{X}$.

Classical probability

Let's consider probability space $(\Omega, \mathcal{F}, \mathbf{P}^{\cdot})$, where Ω — finite space of elementary events — outcomes «of a random experiment», $\mathbf{P}^{\cdot}$ — is *classical probability*, determined for every $\mathcal{F}$-measurable event $x \in \mathcal{F}$ as «the ratio of number of outcomes favorable x, to general number of outcomes in Ω», in other words, as

$$\mathbf{P}^{\cdot}(x) = \frac{|x|}{|\Omega|}$$

— *ratio of power*[21] *of events x to power of* Ω. As far as classical probability is determined by the ratio of powers, it is always determined for any event from

[21] If it is Ω more than finite - it is not required yet. But if the necessity in the infinite Ω occurs, than *classical probability* on the algebra of it's events can be defined as geometrical probability, i.e. as the ratio of a certain *uniform measure* (for example, Lebesgue' measure) of an event x and whole Ω, where uniform measure is understood as the measure invariant relative to one-onely maps of $\mathcal{F}$-measurable events.

algebra of finite space of e-events. It doesn't require any additional assumptions or any other information for the definition, except the information about an event. As soon as the event is determined, then — at once — it's classical probability is also determined.

Let's look at classical probability $\mathbf{P}^{\because}$ as at one more characteristic of an event. Let's denote probability space $(\Omega, \mathcal{F}, \mathbf{P}^{\because})$ as the *classical probability space.* Moreover, we are going to mention the term *classical* among the names of all characteristics of $\mathcal{F}$-measurable events and functions, which are determined on classical probability space. For example, classical probability of an event $x \in \mathcal{F}$ — it nothing else but *classical expectation*[22] of it's indicator:

$$\mathbf{P}^{\because}(x) = \mathbf{E}_{\mathbf{P}^{\because}} \mathbf{1}_x = \sum_{\omega \in \Omega} \mathbf{1}_x(\omega) \mathbf{P}^{\because}(\omega) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \mathbf{1}_x(\omega) = \frac{|x|}{|\Omega|}.$$

Introduction of a one more «parallel» characteristics of an event leads to the fact, that two definitions are added to each kind of fuzzy events of Kolmogorov's axiomatics «shorthand». ***Definition 8-$\mathfrak{M}/\mathbf{P}^{\because}$.*** *Classical probability of $\tilde{\mathcal{F}}$-measurable $\mathfrak{M}$-fuzzy event $\tilde{x}$* $(\in)$ $\tilde{\mathcal{F}}$ is defined by the formula:

$$\mathbf{P}^{\because}(\tilde{x}) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{P}^{\because}(x_\mu).$$

definition 9-$\mathfrak{M}/\mathbf{P}^{\because}$. *Classical probability space $\mathfrak{M}$-of fuzzy events* — is a $\mathfrak{M}|$-set

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbf{P}^{\because}) = \{(\Omega, \mathcal{F}_\mu, \mathbf{P}^{\because}), \ \mu \in \mathfrak{M}\},$$

which consists of membership spaces of usual events, that differ only with algebras of events, $\mathcal{F}_\mu, \ \mu \in \mathfrak{M}$. ***Definition 8-$\mathfrak{X}/\mathbf{P}^{\because}$.*** *Classical probability of $\breve{\mathcal{F}}$-measurable $\mathfrak{X}$-fuzzy event $\breve{\mu}$* $(\in)$ $\breve{\mathcal{F}}$ is determined by the formula:

$$\mathbf{P}^{\because}(\breve{\mu}) = \frac{1}{|\mathfrak{X}|} \sum_{x \in \mathfrak{X}} \mathbf{P}^{\because}(x_\mu).$$

Definition 9-$\mathfrak{X}/\mathbf{P}^{\because}$. *Classical probability space of $\mathfrak{X}$-fuzzy events* — is a $|\mathfrak{X}|$-set

$$(\breve{\Omega}, \breve{\mathcal{F}}, \mathbf{P}^{\because}) = \{(\Omega, \mathcal{F}_x, \mathbf{P}^{\because}), \ x \in \mathfrak{X}\},$$

which consists of classical probability spaces of usual events, that differ only with algebras of events $\mathcal{F}_x, \ x \in \mathfrak{X}$.

[22] *Classical expectation* is expectation relative to classical probability.

Eventological membership function is an indicator of fuzzy event

Let's examine a «matrix» of selected «usual» events

$$\mathfrak{X}_{\mathfrak{M}} = \{x_\mu, \ \mu \in \mathfrak{M}, \ x \in \mathfrak{X}\}.$$

Each event $x_\mu \in \mathfrak{X}_{\mathfrak{M}}$, as to a subset Ω, one-to-onely corresponds to it's own indicator

$$\mathbf{1}_{x_\mu}(\omega) = \begin{cases} 1, & \omega \in x_\mu, \\ 0, & \text{otherwise}, \end{cases}$$

– $\mathfrak{F}$-measurable function on Ω, which, defining *membership* of each e-event $\omega \in \Omega$ to event x_μ by it's values, completely defines the event x_μ. As far as the issue concerns «usual» event x_μ, then for every e-event $\omega \in \Omega$ only two outcomes are probable: ω belongs x_μ, or – doesn't belong, in complete one-to-onely correspondence to values of it's indicator $\mathbf{1}_{x_\mu}(\omega)$, which equals 1, when $\omega \in X_\mu$, otherwise — it equals zero. Reasonably, it is possible to consider values of indicator of the event x_μ as degrees of membership of e-events to the event, and indicator, as the function on Ω, may be naturally denoted as *as function of membership degree* of e-events to «usual» event x_μ.

Eventologically $\mathfrak{M}$-fuzzy event $\tilde{x} = \{x_\mu, \ \mu \in \mathfrak{M}\}$ and $\mathfrak{X}$-fuzzy event $\tilde{\mu} = \{x_\mu, \ x \in \mathfrak{X}\}$ are determined as suitable sets of $\mathfrak{F}$-measurable events — of subsets of space of e-events Ω, selected from initial «matrix» of «usual» events $\mathfrak{X}_{\mathfrak{M}}$. Let's examine arithmetic means of indicators of «usual» events, which form sets of events $\tilde{x}$ and $\tilde{\mu}$, let's introduce special designations for these means:

$$\mathbf{1}_{\tilde{x}}(\omega) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{1}_{x_\mu}(\omega), \quad x \in \mathfrak{X},$$

$$\mathbf{1}_{\tilde{\mu}}(\omega) = \frac{1}{|\mathfrak{X}|} \sum_{x \in \mathfrak{X}} \mathbf{1}_{x_\mu}(\omega), \quad \mu \in \mathfrak{M},$$

Let's denote them as *indicator of $\mathfrak{M}$-fuzzy event $\tilde{x}$* and *indicator of $\mathfrak{X}$-fuzzy event $\tilde{\mu}$* accordingly and as ***eventological membership functions (E-membership functions)*** also.

Theorem (representation of E-membership functions of fuzzy events). *The E-membership functions of* $\mathfrak{M}$*-fussy event* $\widetilde{x}$ *and of* $\mathfrak{X}$*-fuzzy event* $\widetilde{\mu}$ *can be represented as*:

$$\mathbf{1}_{\widetilde{x}}(\omega) = |W_x(\omega)|/|\mathfrak{M}|, \quad \mathbf{1}_{\widetilde{\mu}}(\omega) = |X_\mu(\omega)|/|\mathfrak{X}|, \quad \omega \in \Omega,$$

where

$$W_x(\omega) = \{\mu \in \mathfrak{M},\ \omega \in x_\mu\}, \quad X_\mu(\omega) = \{x \in \mathfrak{X},\ \omega \in x_\mu\}.$$

Proof is obvious.

Remark (the indicator of a fuzzy event does not determine it). There is one-to-one map between «usual» events and their indicators, which means that either event determines the indicator or indicator determines event. This doesn't concern indicators of fuzzy events. Though, certainly, the $\mathfrak{M}$-fuzzy event $\widetilde{x}$ completely determines it's indicator $\mathbf{1}_{\widetilde{x}}$, as function on Ω, but inverse statement is not correct. If the indicator $\mathbf{1}_{\widetilde{x}}$ is defined on Ω, then it is impossible to restore the set of events $\widetilde{x}$ by the values, i.e. fuzzy event $\widetilde{x}$ of the indicator. It is possible only to tell, that the indicator of each fuzzy event serves the indicator of the whole set of fuzzy events, the indicators of which coincide. This, certainly, concerns indicators of $\mathfrak{X}$-fuzzy events as well. As to the sole fuzzy event

$$\{\widetilde{\widetilde{\ }}\} = \{x_\mu,\ \mu \in \mathfrak{M}, x \in \mathfrak{X}\},$$

determined by whole «matrix » of selected events $\mathfrak{X}_{\mathfrak{M}}$, it's indicator is defined similarly, as arithmetic mean of indicators of « usual » events forming this set:

$$\mathbf{1}_{\{\widetilde{\widetilde{\ }}\}}(\omega) = \frac{1}{|\mathfrak{X}||\mathfrak{M}|} \sum_{x \in \mathfrak{X}} \sum_{\mu \in \mathfrak{M}} \mathbf{1}_{x_\mu}(\omega).$$

This indicator has all advantages and drawbacks of all the considered indicators of $\mathfrak{M}$-fuzzy and $\mathfrak{X}$-fuzzy events. As well as these indicators, the indicator $\mathbf{1}_{\{\widetilde{\widetilde{\ }}\}}$ doesn't define fuzzy event $\{\widetilde{\widetilde{\ }}\}$ one-to-one, and serves the indicator for the whole set of various «matrixes» of selected events having coinciding indicators. Certainly, analogical theorem similar to the theorem about introduction of indicators of fuzzy events (page. 29) is true for the indicator of fuzzy events $\{\widetilde{\widetilde{\ }}\}$.

Theorem (two representations of indicator of fuzzy event $\{\approx\}$). *The indicator of fuzzy event* $\{\approx\}$ *can be represented in two equivalent types*:

$$\mathbf{1}_{\{\approx\}}(\omega) = \frac{1}{|\mathfrak{M}||\mathfrak{X}|} \sum_{x \in \mathfrak{X}} |W_x(\omega)| = \frac{1}{|\mathfrak{M}||\mathfrak{X}|} \sum_{\mu \in \mathfrak{M}} |X_\mu(\omega)|, \quad \omega \in \Omega.$$

There are no difficulties in proof, which follows, for example, from the theorem of fuzzy e-events (page. 24).

Value interpretation of introduced indicators of fuzzy events is rather easy. Every value, as the function on Ω, takes on e-event's $\omega \in \Omega$ value, which equals to a peer share of those «usual» events from the sets. The set determines these fuzzy events, which occur when the e-event $\omega \in \Omega$ occurs. Such obvious interpretation justifies one more denotation of the considered indicators of fuzzy events: all of three indicators are also called *eventological functions of membership degree* of e-events to fuzzy event. Newly introduced functions on Ω do not differ from *functions of membership degree*, which are determined in the theory of fuzzy sets by Zadeh. The one and only, but still basic, difference are the formulas, determining these functions and linking them with indicators of «usual» events:

$$\mathbf{1}_{\tilde{x}}(\omega) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{1}_{x_\mu}(\omega), \quad x \in \mathfrak{X},$$

$$\mathbf{1}_{\tilde{\mu}}(\omega) = \frac{1}{|\mathfrak{X}|} \sum_{x \in \mathfrak{X}} \mathbf{1}_{x_\mu}(\omega), \quad \mu \in \mathfrak{M},$$

$$\mathbf{1}_{\{\approx\}}(\omega) = \frac{1}{|\mathfrak{X}||\mathfrak{M}|} \sum_{x \in \mathfrak{X}} \sum_{\mu \in \mathfrak{M}} \mathbf{1}_{x_\mu}(\omega).$$

There has been no such formulas in the classical theory of fuzzy sets by Zadeh. These formulas have enabled to prove basic eventological theorem of fuzzy events, with the help of which eventological function of membership degree of any set-operation over the set of fuzzy events is determined uniquely. Owing to the absence of the similar theorem in the theory of fuzzy sets by Zadeh, the diverse variety of variants of function of degree of membership is used for the one and the same set-operation. Actually, a variant of function of degree of membership for this or that set-operation is selected for each task. The suggested eventological theory of fuzzy events is devoid of the drawback.

Besides eventological theory points out the fact that the reason of function of membership degree variant plurality in the theory of fuzzy sets by Zadeh are the dependence structures of «usual» events, from which fuzzy events are formed as sets of «usual» events. Finally, the dependence structures of events do determine a type of eventological function of membership degree. The plurality of variants in the classical theory is explained by the necessity to lean not only over function of membership degree — which do not contain any information about fuzzy events — but over a whole E-distribution of set of events – which form the fuzzy event — and on an E-distribution of set of fuzzy events as well. Such approach is widely accepted by new eventological theory of fuzzy events.

Eventological distribution of a set of fuzzy events

Either the set of $\mathfrak{M}$-fuzzy events or the set of $\mathfrak{X}$-fuzzy events are determined by *the eventological distribution (E-distribution).*

Let's consider only a detailed definition of eventological distribution of sets of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\} \subseteq \tilde{\mathfrak{F}}$, which split the space of elementary $\mathfrak{M}$-fuzzy events $\tilde{\Omega} = \sum_{X \subseteq \mathfrak{X}} \tilde{\text{ter}}(X)$ into fuzzy ftt-events

$$\tilde{\text{ter}}(X) = \left(\bigcap_{x \in X}\right) \tilde{x} \left(\bigcap_{x \in X^c}\right) \tilde{x}^{(c)}, \quad X \subseteq \mathfrak{X},$$

The fuzzy ftt-events are nothing, but $\mathfrak{M}$-fuzzy events

$$\tilde{\text{ter}}(X) = \{\text{ter}_\mu(X),\ \mu \in \mathfrak{M}\},$$

where $\text{ter}_\mu(X),\ \mu \in \mathfrak{M}$ — are usual terrace-events, subsets Ω of the first type.

Eventological distribution (E-distribution) of the sets of $\mathfrak{M}$*-fuzzy events* $\overset{(\sim)}{\mathfrak{X}}$ is denoted as a set of probabilities of $\mathfrak{M}$-fuzzy ftt-events

$$\tilde{p}(X) = \mathbf{P}\left(\tilde{\text{ter}}(X)\right), \quad X \subseteq \mathfrak{X},$$

where

$$\mathbf{P}\left(\tilde{\text{ter}}(X)\right) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{P}\left(\text{ter}_\mu(X)\right) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} p_\mu(X),$$

while $p_\mu(X),\ X \subseteq \mathfrak{X}$ is an eventological distribution of sets of usual events $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}$, which is nothing, but a fuzzy event (mind) $\breve{\mu} = \mathfrak{X}_\mu$. E-distribution of sets of $\mathfrak{M}$-fuzzy events has all the properties of E-distributions of sets of usual events. For example, it's obvious that$\sum_{X\subseteq\mathfrak{X}} \tilde{p}(X) = 1$. Thus the E-distribution $\tilde{p}(X),\ X \subseteq \mathfrak{X}$ can be treated as the E-distribution $p(X),\ X \subseteq \mathfrak{X}$ of sets of usual events, of subsets Ω. Similarly *eventological distribution (E-distribution) of sets of $\mathfrak{X}$-fuzzy events* $\overset{(\sim)}{\mathfrak{M}}$ is denoted as a set of probabilities of $\mathfrak{X}$-fuzzy stt-events:

$$\breve{p}(W) = \mathbf{P}\left(\breve{\text{ter}}(W)\right), \quad W \subseteq \mathfrak{M},$$

where

$$\mathbf{P}\left(\breve{\text{ter}}(W)\right) = \frac{1}{|\mathfrak{X}|}\sum_{x\in\mathfrak{X}} \mathbf{P}\left(\text{ter}_x(W)\right) = \frac{1}{|\mathfrak{X}|}\sum_{x\in\mathfrak{X}} p_x(W),$$

while $p_x(W),\ W \subseteq \mathfrak{M}$ is an eventological distribution of sets of usual events $x_\mathfrak{M} = \{x_\mu,\ \mu \in \mathfrak{M}\}$, which is nothing, but a fuzzy event $\tilde{x} = x_\mathfrak{M}$.

Mobius-Minkowski' inverse formulae

As well as sets of usual events, sets of fuzzy events of any of the two types may be equivalently determined by E-distributions of different kind. Among the E-distributions six are the most common; they correspond with similar E-distributions of sets of usual events: $p(X),\ p_X,\ p^X,\ u(X),\ u_X$ и $u^X,\ X \subseteq \mathfrak{X}$.

As well as various E-distributions of sets of usual events are pairwise linked by *Mobius-Minkowski' inverse formulae*, various E-distributions of sets of fuzzy events are pairwise linked by the so-called *Mobius-Minkowski' inverse formulae*, which are naturally similar to classical *Mobius-Minkowski' inverse formulae.*

On the surface Mobius-Minkowski' inverse formulae are not different from their classical analogues. For example, a pair of formulae, which links E-distributions of sets of fuzzy events $\tilde{p}(X),\ X \subseteq \mathfrak{X}$ и $\tilde{p}_X,\ X \subseteq \mathfrak{X}$, is expressed like this:

$$\tilde{p}_X = \sum_{X\subseteq Y} \tilde{p}(Y), \quad \tilde{p}(X) = \sum_{X\subseteq Y} (-1)^{|Y|-|X|}\tilde{p}_Y, \quad X \subseteq \mathfrak{X},$$

while a pair of formulae, which links E-distributions $\breve{p}(W),\ W \subseteq \mathfrak{M}$ and $\breve{p}_W,\ W \subseteq$

$\mathfrak{M}$, is expressed like this:

$$\breve{p}_W = \sum_{W \subseteq V} \breve{p}(V), \quad \breve{p}(W) = \sum_{W \subseteq V} (-1)^{|V|-|W|} \breve{p}_V, \quad W \subseteq \mathfrak{M},$$

These formulae are similar to the corresponding couple of classical Mobius inverse formulae, which links E-distributions of sets of usual events $p(X)$, $X \subseteq \mathfrak{X}$ and p_X, $X \subseteq \mathfrak{X}$.

Let's emphasize the fact, that *Mobius-Minkowski' inverse formulae* — are the Mobius inverse formulae, but — intended for functions. The functions, which are determined at sets' structures, partially ordered not relative to inclusion relation of sets, but — relative to inclusion relation by Minkowski. Indeed, fuzzy terrace-events of both types are expressed in another equivalent way:

$$\widetilde{\mathrm{ter}}(X) = \mathrm{ter}(\overset{(\sim)}{X}), \quad \breve{\mathrm{ter}}(W) = \mathrm{ter}(\overset{(\smile)}{W}),$$

have been «numbered» neither by the subsets $X \subseteq \mathfrak{X}$ nor $W \subseteq \mathfrak{M}$, but by corresponding subsets of $\mathfrak{M}$-fuzzy or $\mathfrak{X}$-fuzzy events: $\overset{(\sim)}{X} = \{\tilde{x},\ x \in X\}$ or $\overset{(\smile)}{W} = \{\breve{\mu},\ \mu \in W\}$. Hence an equivalent expression for probabilities of fuzzy terrace-events is deduced:

$$p(\overset{(\sim)}{X}) = \tilde{p}(X), \quad p(\overset{(\smile)}{W}) = \tilde{p}(W),$$

the Mobius-Minkowski' inverse formulae remain correct, although they turn rather bulky:

$$p_{\overset{(\sim)}{X}} = \sum_{\overset{(\sim)}{X} (\subseteq) \overset{(\sim)}{Y}} p(\overset{(\sim)}{Y}), \quad p(\overset{(\sim)}{X}) = \sum_{\overset{(\sim)}{X} (\subseteq) \overset{(\sim)}{Y}} (-1)^{|\overset{(\sim)}{Y}|-|\overset{(\sim)}{X}|} p_{\overset{(\sim)}{Y}}, \quad \overset{(\sim)}{X} \ (\subseteq)\ \overset{(\sim)}{\mathfrak{X}},$$

$$p_{\overset{(\smile)}{W}} = \sum_{\overset{(\smile)}{W} (\subseteq) \overset{(\smile)}{V}} p(\overset{(\smile)}{V}), \quad p(\overset{(\smile)}{W}) = \sum_{\overset{(\smile)}{W} (\subseteq) \overset{(\smile)}{V}} (-1)^{|\overset{(\smile)}{V}|-|\overset{(\smile)}{W}|} p_{\overset{(\smile)}{V}}, \quad \overset{(\smile)}{W} \ (\subseteq)\ \overset{(\smile)}{\mathfrak{M}},$$

but still they clearly indicate, that the considered structure is partially ordered relative to inclusion by Minkowski.

Dependence of fuzzy events

It's possible to tell either about fuzzy events $\tilde{x}$ $(\subseteq)$ $\tilde{\Omega}$ or $\breve{\mu}$ $(\subseteq)$ $\breve{\Omega}$ about events - subsets Ω, that they are *independent, «pulling» and «pushing» each other relative* *to probability* $\mathbf{P}$ at probability spaces of fuzzy events $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \mathbf{P})$ and $(\overset{\frown}{\breve{\Omega}}, \overset{\frown}{\breve{\mathfrak{F}}}, \mathbf{P})$ correspondingly. Since one more numerical function — the *classical probability* $\mathbf{P}^{:}$ — is «automatically» determined at the algebra of fuzzy events, then the same is true of events, *«independent», «pulling» and «pushing» each other relative to classical probability* $\mathbf{P}^{:}$ at classical probability spaces $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \mathbf{P}^{:})$ and $(\overset{\frown}{\breve{\Omega}}, \overset{\frown}{\breve{\mathfrak{F}}}, \mathbf{P}^{:})$ correspondingly.

Probability dependence

Probability dependence of $\mathfrak{M}$-fuzzy events. Two $\mathfrak{M}$-fuzzy events $\tilde{x}$ and $\tilde{y}$ are denoted as *pairwise independent relative to probability* $\mathbf{P}$, if for every $\mu \in \mathfrak{M}$ a couple of events x_μ and y_μ is independent. Otherwise, if two $\mathfrak{M}$-fuzzy events $\tilde{x}$ and $\tilde{y}$ are *pairwise independent relative to probability* $\mathbf{P}$, then

$$\mathbf{P}(\tilde{x}(\cap)\tilde{y}) = \mathbf{P}^i(\tilde{x}(\cap)\tilde{y}),$$

where

$$\mathbf{P}^i(\tilde{x}(\cap)\tilde{y}) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{P}(x_\mu)\mathbf{P}(y_\mu).$$

Obviously, the reversed statement is true. The set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{X} = \{\tilde{x},\ x \in X\}$ is denoted as $|X|$-*arity independent relative to probability* $\mathbf{P}$, if for every $\mu \in \mathfrak{M}$ the set of events $X_\mu = \{x_\mu,\ x \in X\}$ is $|X|$-arity independent. In other words, if set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{X} = \{\tilde{x},\ x \in X\}$ is $|X|$-*arity independent relative to probability* $\mathbf{P}$, then

$$\mathbf{P}\left(\left(\bigcap_{x \in X}\right) \tilde{x}\right) = \mathbf{P}^i\left(\left(\bigcap_{x \in X}\right) \tilde{x}\right),$$

where

$$\mathbf{P}^i\left(\left(\bigcap_{x \in X}\right) \tilde{x}\right) = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \prod_{x \in X} \mathbf{P}(x_\mu).$$

The inverse statement is false. The set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$ is denoted as *independent in total relative to probability* $\mathbf{P}$, if for every $\mu \in \mathfrak{M}$ the sets of events $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}$ are independent in total. In other words, if the set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$ is *independent совокупности relative to probability* $\mathbf{P}$, then all the subsets of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{X} \subseteq are \overset{(\sim)}{\mathfrak{X}} |X|$-arity independent.

Probability dependence of $\mathfrak{X}$-fuzzy events. Two $\mathfrak{X}$-fuzzy events $\breve{\mu}$ и $\breve{\nu}$ are denoted as *pairwise independent relative to probability* $\mathbf{P}$, if for every $x \in \mathfrak{X}$ the couple of events x_μ and y_ν is independent. In other words, of two $\mathfrak{X}$-fuzzy events $\breve{\mu}$ and $\breve{\nu}$ *pairwise independent relative to probability* $\mathbf{P}$, then

$$\mathbf{P}(\breve{\mu}(\cap)\breve{\nu}) = \mathbf{P}^i(\breve{\mu}(\cap)\breve{\nu}),$$

where

$$\mathbf{P}^i(\breve{\mu}(\cap)\breve{\nu}) = \frac{1}{|\mathfrak{M}|} \sum_{x \in \mathfrak{X}} \mathbf{P}(x_\mu)\mathbf{P}(y_\mu).$$

Obviously, the reversed statement is false. The set of $\mathfrak{M}$-fuzzy events $\overset{(\smile)}{W} = \{\breve{\mu},\ \mu \in W\}$ is denoted as $|W|$-*arity independent relative to probability* $\mathbf{P}$, if for every $x \in \mathfrak{X}$ the set of events $W_x = \{x_\mu,\ \mu \in W\}$ is $|W|$-arity independent. In other words, if the set of $\mathfrak{X}$-fuzzy events $\overset{(\smile)}{W} = \{\breve{\mu},\ \mu \in W\}$ is $|W|$-*arity independent relative to probability* $\mathbf{P}$, then

$$\mathbf{P}\left(\bigcap_{\mu \in W} \breve{\mu} \right) = \mathbf{P}^i\left(\bigcap_{\mu \in W} \breve{\mu} \right),$$

where

$$\mathbf{P}^i\left(\bigcap_{\mu \in W} \breve{\mu} \right) = \frac{1}{|\mathfrak{X}|} \sum_{x \in \mathfrak{X}} \prod_{\mu \in W} \mathbf{P}(x_\mu).$$

The reversed statement is false. The set of $\mathfrak{X}$-fuzzy events $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$ is denoted as *independent in total relative to probability* $\mathbf{P}$, if for every $x \in \mathfrak{X}$ the sets of events $\mathfrak{M}_x = \{x_\mu,\ \mu \in \mathfrak{M}\}$ are independent in total. In other words, if the set of $\mathfrak{X}$-fuzzy events $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$ is *independent в совокупности relative to probability* $\mathbf{P}$, then all the subsets $\mathfrak{X}$-fuzzy events $\overset{(\smile)}{W} \subseteq \overset{(\sim)}{\mathfrak{M}}$ are $|W|$-arity independent.

Classical probability dependence

Classical probability dependence of $\mathfrak{M}$-fuzzy events. Two $\mathfrak{M}$-fuzzy events $\tilde{x}$ and $\tilde{y}$ are denoted as *pairwise independent relative to classical probability* $\mathbf{P}^{:}$, if for every $\mu \in \mathfrak{M}$ a pair of events x_μ and y_μ is independent. In other words, if two $\mathfrak{M}$-fuzzy events $\tilde{x}$ and $\tilde{y}$ are *pairwise independent relative to classical probability* $\mathbf{P}^{:}$, then

$$\mathbf{P}^{:}(\tilde{x}(\cap)\tilde{y}) = \mathbf{P}^{:i}(\tilde{x}(\cap)\tilde{y}),$$

where

$$\mathbf{P}^{:i}(\tilde{x}(\cap)\tilde{y}) = \frac{1}{|\mathfrak{M}|}\sum_{\mu\in\mathfrak{M}} \mathbf{P}^{:}(x_\mu)\mathbf{P}^{:}(y_\mu).$$

Obviously, the reversed statement is false. The set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{X} = \{\tilde{x},\ x \in X\}$ is denoted as $|X|$*-arity independent relative to classical probability* $\mathbf{P}^{:}$, if for every $\mu \in \mathfrak{M}$ the set of events $X_\mu = \{x_\mu,\ x \in X\}$ is $|X|$-arity independent. In other words, if the set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{X} = \{\tilde{x},\ x \in X\}$ is $|X|$*-arity independent relative to classical probability* $\mathbf{P}^{:}$, then

$$\mathbf{P}^{:}\left(\left(\bigcap_{x\in X}\right)\tilde{x}\right) = \mathbf{P}^{:i}\left(\left(\bigcap_{x\in X}\right)\tilde{x}\right),$$

where

$$\mathbf{P}^{:i}\left(\left(\bigcap_{x\in X}\right)\tilde{x}\right) = \frac{1}{|\mathfrak{M}|}\sum_{\mu\in\mathfrak{M}}\prod_{x\in X} \mathbf{P}^{:}(x_\mu).$$

The reversed statement is true. The set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$ is denoted as *independent in total relative to classical probability* $\mathbf{P}^{:}$, if for every $\mu \in \mathfrak{M}$ the sets of events $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}$ are independent in total. In other words, if the set of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$ is *independent in total relative to classical probability* $\mathbf{P}^{:}$, then all the subsets of $\mathfrak{M}$-fuzzy events $\overset{(\sim)}{X} \subseteq \overset{(\sim)}{\mathfrak{X}}$ are $|X|$-arity independent.

Classical probability dependence of $\mathfrak{X}$-fuzzy events. Two $\mathfrak{X}$-fuzzy events $\breve{\mu}$ and $\breve{\nu}$ are denoted as *pairwise independent relative to classical probability* $\mathbf{P}^{:}$, if for every $x \in \mathfrak{X}$ pair of events x_μ and y_ν is independent. In other words, if two $\mathfrak{X}$-fuzzy events $\breve{\mu}$ and $\breve{\nu}$ are *pairwise independent relative to classical probability* $\mathbf{P}^{:}$, then

$$\mathbf{P}^{:}(\breve{\mu}(\cap)\breve{\nu}) = \mathbf{P}^{:i}(\breve{\mu}(\cap)\breve{\nu}),$$

where

$$\mathbf{P}^{:i}(\breve{\mu}(\cap)\breve{\nu}) = \frac{1}{|\mathfrak{M}|}\sum_{x\in\mathfrak{X}} \mathbf{P}^{:}(x_\mu)\mathbf{P}^{:}(y_\mu).$$

Obviously, the reversed statement is true. The set of $\mathfrak{M}$-fuzzy events $\overset{(\smile)}{W} = \{\breve{\mu},\ \mu \in W\}$ is denoted as $|W|$-*arity independent relative to classical probability* $\mathbf{P}^{:}$, if for every $x \in \mathfrak{X}$ the set of events $W_x = \{x_\mu,\ \mu \in W\}$ is $|W|$-arity independent. In other words, if the set of $\mathfrak{X}$-fuzzy events $\overset{(\smile)}{W} = \{\breve{\mu},\ \mu \in W\}$ is $|W|$-*arity independent relative to classical probability* $\mathbf{P}^{:}$, then

$$\mathbf{P}^{:}\left(\underset{\mu\in W}{(\cap)}\breve{\mu}\right) = \mathbf{P}^{:i}\left(\underset{\mu\in W}{(\cap)}\breve{\mu}\right),$$

where

$$\mathbf{P}^{:i}\left(\underset{\mu\in W}{(\cap)}\breve{\mu}\right) = \frac{1}{|\mathfrak{X}|}\sum_{x\in\mathfrak{X}}\prod_{\mu\in W}\mathbf{P}^{:}(x_\mu).$$

The reversed statement is false. The set of $\mathfrak{X}$-fuzzy events $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$ is denoted as *independent in total relative to classical probability* $\mathbf{P}^{:}$, if for every $x \in \mathfrak{X}$ the sets of events $\mathfrak{M}_x = \{x_\mu,\ \mu \in \mathfrak{M}\}$ are independent in total. In other words, if the set of $\mathfrak{X}$-fuzzy events $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$ is *independent in total relative to classical probability* $\mathbf{P}^{:}$, then all the subsets $\mathfrak{X}$-fuzzy events $\overset{(\smile)}{W} \subseteq \overset{(\sim)}{\mathfrak{M}}$ are $|W|$-arity independent.

Dependence measures: covariation and classical covariation of sets of fuzzy events

Any random events — subsets Ω — become fuzzy, when a mind interferes into them in one way or another. Thus, as soon as mind is included into the subject of investigation of the mathematical theory [23], the theory is to deal with not just random events, but with *random fuzzy events*. Hence the theory gets an opportunity to measure the relativity of the events, primarily *relative to probability*, *relative to classical probability* and *relative to their combinations* — the fields, which yet have not arrested attention of the research workers, applying mathematical theories to study events. *Covariation of events*, introduced in [2] for a set of events, is a measure of dependence of «usual»[24] events[25] $X \subseteq \mathfrak{X}$ as difference

$$\mathsf{Kov}_X = \mathbf{P}\left(\bigcap_{x \in X} x\right) - \mathbf{P}^i\left(\bigcap_{x \in X} x\right)$$

of the probabilities of intersection in the common state and independent state, where

$$\mathbf{P}^i\left(\bigcap_{x \in X} x\right) = \prod_{x \in X} \mathbf{P}(x)$$

— is the probability of intersection of the set of events $X \subseteq \mathfrak{X}$ in the state, when the events from the set X are independent relative to the probability $\mathbf{P}$. Since the probability for $\mathfrak{M}$-fuzzy and $\mathfrak{X}$-fuzzy events has been already determined, then so-called *arity covariations of sets of fuzzy events* can also be determined relative to probability $\mathbf{P}$ in the same way:

$$\widetilde{\mathsf{Kov}}_X = \mathbf{P}\left(\left(\bigcap_{x \in X}\right) \tilde{x}\right) - \mathbf{P}^i\left(\left(\bigcap_{x \in X}\right) \tilde{x}\right), \quad X \subseteq \mathfrak{X},$$

$$\breve{\mathsf{Kov}}_W = \mathbf{P}\left(\left(\bigcap_{\mu \in W}\right) \breve{\mu}\right) - \mathbf{P}^i\left(\left(\bigcap_{\mu \in W}\right) \breve{\mu}\right), \quad W \subseteq \mathfrak{M}.$$

and relative to classical probability $\mathbf{P}^{:}$:

$$\widetilde{\mathsf{Kov}}^{:}_X = \mathbf{P}^{:}\left(\left(\bigcap_{x \in X}\right) \tilde{x}\right) - \mathbf{P}^{:i}\left(\left(\bigcap_{x \in X}\right) \tilde{x}\right), \quad X \subseteq \mathfrak{X},$$

[23](eventology: it happens, when *the finite set* $\mathfrak{M}$ is determined and included into the range of basic concepts as *the set of individual minds*)

[24]The adjective «usual» is rather poor in expressing and defining the concept here; but still it proposes itself emphatically, when speaking about random events — subsets Ω, which are the traditional subject of investigation of Probability Theory by Kolmogorov; besides, no allusion to a mind — observing, participating, interfering and even managing events — has been made in the definition of the subsets.

[25]Covariation of the set of events $X \subseteq \mathfrak{X}$ is also denoted as $|X|$-*arity covariation*.

$$\breve{\mathsf{Kov}}^{\cdot}_W = \mathbf{P}^{\cdot}\left(\left(\bigcap_{\mu\in W}\right)\breve{\mu}\right) - \mathbf{P}^{:i}\left(\left(\bigcap_{\mu\in W}\right)\breve{\mu}\right), \quad W \subseteq \mathfrak{M}.$$

Detailed properties and statements, concerning *arity covariations of sets of fuzzy events* are considered in [4].

Set-operations over a set of events

And now some words about *arbitrary set-operations*[26] on the set of events $\mathfrak{X} \subseteq \mathcal{F}$, selected from algebra $\mathcal{F}$ of probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{O} \subseteq 2^{\mathfrak{X}}$ — any totality of subsets of the set $\mathfrak{X}$, where $2^{\mathfrak{X}}$ — set of all subsets of the set $\mathfrak{X}$.

$$\mathrm{ter}(X) = \bigcap_{x\in X} x \bigcap_{x\in X^c} x^c, \ X \subseteq \mathfrak{X}$$

is understood as events-terrace, which form partition of elementary events $\Omega = \sum_{X\in 2^{\mathfrak{X}}} \mathrm{ter}(X)$. *Arbitrary set-operation over the set of events* $\mathfrak{X}$ is denoted as $|\mathfrak{X}|$-ary set-operation $\mathcal{O} \ \underbrace{\mathcal{F}\times\ldots\times\mathcal{F}}_{|\mathfrak{X}|} \to \mathcal{F}$, generated by any totality of subsets $\mathcal{O} \subseteq 2^{\mathfrak{X}}$ as $\mathcal{F}$ a measurable event

$$\mathcal{O}_{x\in\mathfrak{X}}\, x = \sum_{X\in\mathcal{O}} \mathrm{ter}(X) \qquad (\star)$$

— non-intersecting union of events-terrace $\mathrm{ter}(X)$ over subsets X from the generating totality $\mathcal{O}$. Every totality of subsets of events $\mathcal{O} \subseteq 2^{\mathfrak{X}}$ one-onely corresponds with set-operation over the set of events $\mathfrak{X}$. For convenience such set-operation is designated as $\mathcal{O}$ — as well as the totality of subsets of events, which generates it. Thus general number of various set-operations, which can be carried out on the set of events $\mathfrak{X}$, is equal to the number of subsets of the set $2^{\mathfrak{X}}$, i.e. $2^{2^{|\mathfrak{X}|}}$. There are two constants among arbitrary set-operations determined by the formula $(\star)$: $\emptyset$ and Ω, and also there are popular set-operations — intersection, union and

[26] *set-operation* is a brief synonym of the term «set theoretical operation»

symmetrical difference:

$$\mathop{\mathcal{O}}_{x\in\mathfrak{X}} x = \begin{cases} \emptyset, & \mathcal{O}=\emptyset, \\ \bigcap\limits_{x\in\mathfrak{X}} x, & \mathcal{O}=\{\mathfrak{X}\}, \\ \mathop{\triangle}\limits_{x\in\mathfrak{X}} x, & \mathcal{O}=\{X\in 2^{\mathfrak{X}}:\ |X|=1 \pmod 2\}, \\ \bigcup\limits_{x\in\mathfrak{X}} x, & \mathcal{O}=2^{\mathfrak{X}}-\{\emptyset\}, \\ \Omega, & \mathcal{O}=2^{\mathfrak{X}}. \end{cases}$$

Moreover,

$$\mathop{\mathcal{O}}_{x\in\mathfrak{X}} x = \begin{cases} y, & \mathcal{O}=\{X\in 2^{\mathfrak{X}}:\ y\in X\}, \\ y\cap z, & \mathcal{O}=\{X\in 2^{\mathfrak{X}}:\ \{y,z\}\subseteq X\}, \\ \mathrm{ter}(X), & \mathcal{O}=\{X\}, \\ \mathrm{ter}(X)+\mathrm{ter}(Y), & \mathcal{O}=\{X,Y\}. \end{cases}$$

Set-operations by Minkowski over a set of fuzzy events

Set-operations over the set of events $\mathfrak{X}\subseteq\mathcal{F}$, that have been selected from algebra $\mathcal{F}$ of probability space $(\Omega,\mathcal{F},\mathbf{P})$, correspond with *set-operations by Minkowski*[27] *on the set of fuzzy events* $\overset{(\sim)}{\mathfrak{X}}=\{\tilde{x},\ x\in\mathfrak{X}\}$, that are generated by the set $\mathfrak{X}$.

The set of fuzzy events $\overset{(\sim)}{\mathfrak{X}}$ is selected from algebra of fuzzy events $\breve{\mathcal{F}}_{\mathfrak{M}}$ of probability space $(\breve{\Omega}_{\mathfrak{M}},\breve{\mathcal{F}}_{\mathfrak{M}},\mathbf{P})$, where probability $\mathbf{P}$ of a fuzzy event $\tilde{x}=\{x_\mu,\ \mu\in\mathfrak{M}\}\in\breve{\mathcal{F}}_{\mathfrak{M}}$ is induced on $\breve{\mathcal{F}}_{\mathfrak{M}}$ by probability $\mathbf{P}$ by the formula $\mathbf{P}(\tilde{x})=\frac{1}{|\mathfrak{M}|}\sum_{\mu\in\mathfrak{M}}\mathbf{P}(x_\mu)$. Let's remind, that intersection of two fuzzy events $\tilde{x},\ \tilde{y}\in\overset{(\sim)}{\mathfrak{X}}$ by Minkowski is determined as a fuzzy event $\tilde{x}(\cap)\tilde{y}=\{x_\mu\cap Y_\mu,\ \mu\in\mathfrak{M}\}$.

Let $\mathcal{O}\subseteq 2^{\mathfrak{X}}$ be an arbitrary set of subsets of the set $\mathfrak{X}$, where $2^{\mathfrak{X}}$ — the set of all subsets of the set $\mathfrak{X}$. Under $\tilde{\mathrm{ter}}(X)=(\bigcap)_{x\in X}\tilde{x}\,(\bigcap)_{x\in X^c}\tilde{x}^c,\ X\subseteq\mathfrak{X}$ it is understood the fuzzy events-terraces $\tilde{\mathrm{ter}}(X)=\Big\{\mathrm{ter}_\mu(X),\ \mu\in\mathfrak{M}\Big\}$, where for every $\mu\in\mathfrak{M}$ events $\mathrm{ter}_\mu(X)=\bigcap_{x\in X}x_\mu\bigcap_{x\in X^c}x^c_\mu,\ X\subseteq\mathfrak{X}$ — are

[27] *Minkowski Hermann (1864 - 1909)* — German mathematician and physicist, who was Russian by birth. He has basic works on geometry, theory of numbers, mathematical physics and theory of relativity.

usual events-terraces, which form partition of space of elementary events $\Omega = \sum_{X \in 2^{\mathfrak{X}}} \mathrm{ter}_\mu(X),\ \mu \in \mathfrak{M}$.

Aforesaid opinion is equivalent to the fact that fuzzy terrace-events do not pairwise intersect *by Minkowski* $\tilde{\mathrm{ter}}(X)\ (\cap)\ \tilde{\mathrm{ter}}(Y) = \tilde{\emptyset} \iff X \neq Y$, and form *partition by Minkowski* of fuzzy certain event $\tilde{\Omega} = (\sum)_{X \in 2^{\mathfrak{X}}} \tilde{\mathrm{ter}}(X)$.

Any set-operation by Minkowski on the set of fuzzy events $\overset{(\sim)}{\mathfrak{X}}$ is called to be a $|\mathfrak{X}|$-ary set-operation by Minkowski $\left(\mathcal{O}\right) : \underbrace{\breve{\mathfrak{F}}_{\mathfrak{M}} \times \ldots \times \breve{\mathfrak{F}}_{\mathfrak{M}}}_{|\mathfrak{X}|} \to \breve{\mathfrak{F}}_{\mathfrak{M}}$, which is generated by any of whole of subsets $\mathcal{O} \subseteq 2^{\mathfrak{X}}$ as a $\breve{\mathfrak{F}}_{\mathfrak{M}}$-measurable fuzzy event

$$\underset{x \in \mathfrak{X}}{\left(\mathcal{O}\right)} \tilde{x} = \underset{X \in \mathcal{O}}{\left(\sum\right)} \tilde{\mathrm{ter}}(X) \tag{$\star\star$}$$

— *non-intersecting union by Minkowski* of fuzzy terrace-events $\tilde{\mathrm{ter}}(X)$ over subsets X from generating set $\mathcal{O}$. Let's notice, that set-operation by Minkowski might be determined by an equivalent way as a fuzzy event

$$\underset{x \in \mathfrak{X}}{\left(\mathcal{O}\right)} \tilde{x} = \left\{ \mathop{\mathcal{O}}_{x \in \mathfrak{X}} x_\mu,\ \mu \in \mathfrak{M} \right\} = \left\{ \sum_{X \in \mathcal{O}} \mathrm{ter}_\mu(X),\ \mu \in \mathfrak{M} \right\}, \tag{$\star\star'$}$$

constituted of set-operations $\mathcal{O}(\mathfrak{X}_\mu) = \mathop{\mathcal{O}}_{x \in \mathfrak{X}} x_\mu, \quad \mu \in \mathfrak{M}$ on the according sets of usual events $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}, \quad \mu \in \mathfrak{M}$.

Every set of subsets of events $\mathcal{O} \subseteq 2^{\mathfrak{X}}$ one-to-onely corresponds with set-operation $(\mathcal{O})$ on the set of fuzzy events $\overset{(\sim)}{\mathfrak{X}}$. Thus general number of various set-operations, which can be carried out on the set of fuzzy events $\mathfrak{X}$, is equal to the number of subsets of the set $2^{\mathfrak{X}}$, i.e. $2^{2^{|\mathfrak{X}|}}$. Among any set-operations, which are determined by the formula $(\star\star)$, there are two constants: $\breve{\emptyset}_{\mathfrak{M}} = \{\underbrace{\emptyset, \ldots \emptyset}_{|\mathfrak{M}|}\}$, and $\breve{\Omega}_{\mathfrak{M}} = \{\underbrace{\Omega, \ldots \Omega}_{|\mathfrak{M}|}\}$ and certainly popular set-operations — intersection, union and symmetrical difference by Minkowski — as well:

$$\left(\underset{x\in\mathfrak{X}}{\mathcal{O}}\right)\tilde{x}=\begin{cases}\breve{\emptyset}_{\mathfrak{M}}, & \mathcal{O}=\emptyset,\\ \left(\bigcap\limits_{x\in\mathfrak{X}}\right)\tilde{x}, & \mathcal{O}=\{\mathfrak{X}\},\\ \left(\underset{x\in\mathfrak{X}}{\triangle}\right)\tilde{x}, & \mathcal{O}=\{X\in 2^{\mathfrak{X}}:\ |X|=1\ (\mathrm{mod}\ 2)\},\\ \left(\bigcup\limits_{x\in\mathfrak{X}}\right)\tilde{x}, & \mathcal{O}=2^{\mathfrak{X}}-\{\emptyset\},\\ \breve{\Omega}_{\mathfrak{M}}, & \mathcal{O}=2^{\mathfrak{X}}.\end{cases}$$

Moreover,

$$\left(\underset{x\in\mathfrak{X}}{\mathcal{O}}\right)\tilde{x}=\begin{cases}\tilde{y}, & \mathcal{O}=\{X\in 2^{\mathfrak{X}}:\ y\in X\},\\ \tilde{y}(\cap)\tilde{z}, & \mathcal{O}=\{X\in 2^{\mathfrak{X}}:\ \{y,z\}\subseteq X\},\\ \widetilde{\mathrm{ter}}(X), & \mathcal{O}=\{X\},\\ \widetilde{\mathrm{ter}}(X)+\widetilde{\mathrm{ter}}(Y), & \mathcal{O}=\{X,Y\}.\end{cases}$$

Indicators of set operations

As soon as the result of any set-operation $\mathcal{O}$ over the set of events $\mathfrak{X}$ — is an $\mathcal{F}$-measurable event

$$\mathcal{O}(\mathfrak{X})=\underset{x\in\mathfrak{X}}{\mathcal{O}}\,x\in\mathcal{F},$$

then we can talk about it's indicator

$$\mathbf{1}_{\mathcal{O}(\mathfrak{X})}(\omega)=\begin{cases}1, & \omega\in\mathcal{O}(\mathfrak{X}),\\ 0, & \text{otherwise.}\end{cases}$$

By the definition of set-operation

$$\mathcal{O}(\mathfrak{X})=\underset{x\in\mathfrak{X}}{\mathcal{O}}\,x=\sum_{X\in\mathcal{O}}\mathrm{ter}(X).$$

As events-terraces $\mathrm{ter}(X)$ do not intersect,

$$\mathbf{1}_{\mathcal{O}(\mathfrak{X})}(\omega)=\sum_{X\in\mathcal{O}}\mathbf{1}_{\mathrm{ter}(X)}(\omega)$$

— the indicator of the set-operation $\mathcal{O}(\mathfrak{X})$ is equal to the sum of indicators of those terrace-events, which meet subsets of events $X \in \mathcal{O}$. But the indicator of event-terrace $\operatorname{ter}(X)$ is obviously equal to

$$\mathbf{1}_{\operatorname{ter}(X)}(\omega) = \prod_{x \in X} \mathbf{1}_x(\omega) \prod_{x \in X^c} \mathbf{1}_{x^c}(\omega) = \prod_{x \in X} \mathbf{1}_x(\omega) \prod_{x \in X^c} (1 - \mathbf{1}_x(\omega)).$$

As a result we derive a formula

$$\mathbf{1}_{\mathcal{O}(\mathfrak{X})}(\omega) = \sum_{X \in \mathcal{O}} \left(\prod_{x \in X} \mathbf{1}_x(\omega) \prod_{x \in X^c} (1 - \mathbf{1}_x(\omega)) \right),$$

which links the indicator of set-operation over the set of events with indicators of events from this set.

The basic eventological fuzzy events theorem

Basic eventological fuzzy events theorem proposes the formula for calculating an indicator of an arbitrary set-operation over a set of fuzzy events; the indicator plays the same role in the eventology of fuzzy events, as *membership function* does in the theory of fuzzy sets by Zadeh, thus the indicator is it's eventological generalization and is denoted as *eventological membership function.* Let's view the formulation and the proof of the theorem from the point, which won't show us the shortest way, but will guide us to the very sense of the theorem and it's method of proof along the path of detail and clarity.

Let's consider $(\Omega, \mathcal{F}, \mathbf{P})$ — probability space, and $\mathfrak{X}_{\mathfrak{M}} = \{x_\mu,\ x \in \mathfrak{X},\ \mu \in \mathfrak{M}\}$ — *matrix of selected $\mathcal{F}$-measurable events* $x_\mu \subseteq \Omega$, «lines» of which $x_{\mathfrak{M}} = \{x_\mu,\ \mu \in \mathfrak{M}\}$ define *fuzzy events* $\tilde{x} = x_{\mathfrak{M}}$, generating the set $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$, while «columns» $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}$ — *fuzzy minds* $\breve{\mu} = \mathfrak{X}_\mu$, generating the set $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$.

1. Partition of space of elementary events by a fuzzy event. Let's revive the fact, that any finite set of measurable events partitions space of elementary events into terrace-events, generated by the set. The same holds true for every fuzzy

$\overset{(\sim)}{\mathfrak{X}}$							
...	...	...	...	...	...	...	...
$\tilde{x}$	...	x_λ	...	x_μ	...	x_ν	...
...	...	...	...	...	...	...	...
$\tilde{y}$	...	y_λ	...	y_μ	...	y_ν	...
...	...	...	...	...	...	...	...
$\tilde{z}$	...	z_λ	...	z_μ	...	z_ν	...
...	...	...	...	...	...	...	...
$\overset{(\sim)}{\mathfrak{M}}$	...	$\breve{\lambda}$	...	$\breve{\mu}$	...	$\breve{\nu}$	...

Рис. 6: *Matrix of selected events* $\mathfrak{X}_{\mathfrak{M}} = \{x_\mu,\ x \in \mathfrak{X},\ \mu \in \mathfrak{M}\}$, formed from $\mathcal{F}$-measurable events $x_\mu \subseteq \Omega$, «lines» of the matrix $x_{\mathfrak{M}} = \{x_\mu,\ \mu \in \mathfrak{M}\}$ define *fuzzy events* $\tilde{x} = x_{\mathfrak{M}}$, generating the set $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$, while «columns» $\mathfrak{X}_\mu = \{x_\mu,\ x \in \mathfrak{X}\}$ — *fuzzy minds* $\breve{\mu} = \mathfrak{X}_\mu$, generating the set $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$

event, which is a finite set of events by definition.

$$\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}.$$

If W_x is designated as — arbitrary subset $\mathfrak{M}$, $W_x^c = \mathfrak{M} - W_x$ — as it's supplement, while

$$\mathrm{ter}_x(W_x) = \bigcap_{\mu \in W_x} x_\mu \bigcap_{\mu \in W_x^c} x_\mu^c$$

— terrace-events, generated by a fuzzy event $\tilde{x}$, then $\Omega = \sum_{W_x \subseteq \mathfrak{M}} \mathrm{ter}_x(W_x)$ — every fuzzy event $\tilde{x} \in \overset{(\sim)}{\mathfrak{X}}$ partitions space of elementary events Ω into corresponding terrace-events $\mathrm{ter}_x(W_x),\ W_x \subseteq \mathfrak{M}$, generated by the fuzzy event.

2. Partition of space of elementary events by a set of fuzzy events. It's evident, that a set of fuzzy events $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x},\ x \in \mathfrak{X}\}$, which is of special interest, partitions Ω into even smaller so called *terrace-subevents*:

$$\Omega = \sum_{\breve{W} \in \underbrace{2^{\mathfrak{M}} \times \ldots \times 2^{\mathfrak{M}}}_{|\mathfrak{X}|}} \mathrm{ter}(\breve{W}),$$

each of which has a shape of terrace-events' intersection:

$$\mathrm{ter}(\breve{W}) = \bigcap_{x \in \mathfrak{X}} \mathrm{ter}_x(W_x).$$

The symbol «$\sim$» of «stratification over $\mathfrak{X}$» operator is used for designating

$$\breve{W} = \{W_x,\ x \in \mathfrak{X}\} \in \underbrace{2^{\mathfrak{M}} \times \ldots \times 2^{\mathfrak{M}}}_{|\mathfrak{X}|}$$

— an arbitrary $|\mathfrak{X}|$-set of subsets $\mathfrak{M}$.

3. Arbitrary set-operation over a set of fuzzy events. Arbitrary set-operation

$$\left(\mathcal{O}\right) : \underbrace{\breve{\mathfrak{F}}_{\mathfrak{M}} \times \ldots \times \breve{\mathfrak{F}}_{\mathfrak{M}}}_{|\mathfrak{X}|} \rightarrow \breve{\mathfrak{F}}_{\mathfrak{M}}$$

over set of $\mathfrak{F}$-measurable fuzzy events $\overset{(\sim)}{\mathfrak{X}}$ is defined by corresponding arbitrary totality of subsets $\mathcal{O} \subseteq 2^{\mathfrak{X}}$ as fuzzy event

$$\underset{x \in \mathfrak{X}}{\left(\mathcal{O}\right)} \tilde{x} = \left(\sum_{X \in \mathcal{O}}\right) \tilde{\mathrm{ter}}(X) = \left\{ \underset{x \in \mathfrak{X}}{\mathcal{O}}\, x_\mu \ : \mu \in \mathfrak{M} \right\},$$

where

$$\underset{x \in \mathfrak{X}}{\mathcal{O}}\, x_\mu = \sum_{X \in \mathcal{O}} \mathrm{ter}_\mu(X) = \sum_{X \in \mathcal{O}} \bigcap_{x \in X} x_\mu \bigcap_{x \in X^c} x_\mu^c$$

— is a corresponding arbitrary set-operation over a set of $\mathfrak{F}$-measurable events $\breve{\mu} = \{x_\mu,\ x \in \mathfrak{X}\}$, defined by the totality $\mathcal{O} \subseteq 2^{\mathfrak{X}}$.

4. Indicator of arbitrary set-operation over a set of fuzzy — piecewise constant function on Ω. It's evident, that an indicator of any set-operation $\left(\mathcal{O}\right)$ over a set of fuzzy events $\overset{(\sim)}{\mathfrak{X}}$ — is a piecewise constant function on Ω. If the function changes, it changes from one terrace-subevent to another, remaining within every terrace-subevent $\mathrm{ter}(\breve{W}) = \bigcap_{x \in \mathfrak{X}} \mathrm{ter}_x(W_x)$ constant for each ω.

5. Inducing set-operations. If every $x \in \mathfrak{X}$ is confronted with an arbitrary subset $W_x \subseteq \mathfrak{M}$, then all together they form a $|\mathfrak{X}|$-set of arbitrary subsets $\mathfrak{M}$: $\breve{W} = \{W_x,\ x \in \mathfrak{X}\}$. In such a way does the operator $\sim :\ W \ \rightarrow\ \breve{W}$ operate; it stratifies the name «W — arbitrary subset $\mathfrak{M}$» with the help of a set $\mathfrak{X}$. Thus there are two equinumerous sets $|\mathfrak{X}| = |\breve{W}|$, with a one-one correspondence fixed between them: $x \ \rightarrow\ W_x, \quad x \in \mathfrak{X}$, или $x \ \leftarrow\ W_x, \quad W_x \in \breve{W}$. Speaking freely, elements $\breve{W}$ have been «numbered» by elements $\mathfrak{X}$. The subsets Ω serve as elements of the first set, while the subsets $\mathfrak{M}$ —- as elements of the second set.

It's obvious, that an arbitrary operation $\mathcal{O}$ over sets $\mathfrak{X}$ of subsets Ω, defined by a totality of subsets $\mathcal{O} \subseteq 2^{\mathfrak{X}}$, can be performed over a set $\breve{W}$ of subsets $\mathfrak{M}$, thus *inducing a new set-operation over a set of subsets* $\mathfrak{M}$ with the analogous formula

$$\mathcal{O}_{x \in \mathfrak{X}} W_x = \sum_{X \subseteq \mathcal{O}} \mathrm{ter}_{\breve{W}}(X) \subseteq \mathfrak{M}, \qquad (*)$$

where for $X \subseteq \mathfrak{X}$

$$\mathrm{ter}_{\breve{W}}(X) = \bigcap_{x \in X} W_x \bigcap_{x \in X^c} W_x^c \subseteq \mathfrak{M}$$

— is a terrace-subset $\mathfrak{M}$, *induced* by a terrace-set Ω:

$$\mathrm{ter}_{\mathfrak{X}}(X) = \mathrm{ter}(X) = \bigcap_{x \in X} x \bigcap_{x \in X^c} x^c \subseteq \Omega.$$

Hence an arbitrary set-operation over one set ($\mathfrak{X}$) *induces a new set-operation over another set of subsets* ($\breve{W}$) with the formula ($*$), *if a one-one correspondence has been fixed between them, i.e. if the subsets of the second set have been «numbered» by the subsets of the first set.*

All the designations for deducing the formula of an indicator of an arbitrary set-operation over sets of $\mathfrak{M}$-fuzzy events are ready. So,

Theorem (the formula of an indicator of an arbitrary set-operation over sets of $\mathfrak{M}$-fuzzy events) is true.

Let

$$(\mathcal{O})(\mathfrak{X}) = \left(\mathcal{O}_{x \in \mathfrak{X}}\right) \tilde{x} = \left\{ \mathcal{O}_{x \in \mathfrak{X}} x_\mu, \ \mu \in \mathfrak{M} \right\}$$

— is an arbitrary set-operation over a set of $\mathfrak{M}$*-fuzzy events* $\overset{(\sim)}{\mathfrak{X}} = \{\tilde{x}, \ x \in \mathfrak{X}\}$, *which is defined by a totality of subsets* $\mathcal{O} \subseteq 2^{\mathfrak{X}}$. *Then the indicator is calculated under the formula:*

$$\mathbf{1}_{(\mathcal{O})(\mathfrak{X})}(\omega) = \frac{1}{|\mathfrak{M}|} \left| \mathcal{O}_{x \in \mathfrak{X}} W_x(\omega) \right|, \quad \omega \in \Omega,$$

where

$$W_x(\omega) = \{\mu \in \mathfrak{M}, \ \omega \in x_\mu\}.$$

6. Dual collections of subsets, generated by an arbitrary subset of a matrix of selected events $\mathfrak{X}_{\mathfrak{M}}$. Let's select an arbitrary subset of elements $X_W \subseteq \mathfrak{X}_{\mathfrak{M}}$ from the matrix of selected events $\mathfrak{X}_{\mathfrak{M}}$ and represent the subset in two ways: as a union of the elements, forming the «columns»; then as a union — forming the «lines» of the matrix. Let's note, that a whole matrix of selected events can be represented in two ways: as a union of «columns», which are the sets $\breve{\mu} = \{x_\mu,\ x \in \mathfrak{X}\}$, and as a union of «lines», which are the sets $\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}$:

$$\mathfrak{X}_{\mathfrak{M}} = \bigcup_{\mu \in \mathfrak{M}} \breve{\mu} = \bigcup_{x \in \mathfrak{X}} \tilde{x}.$$

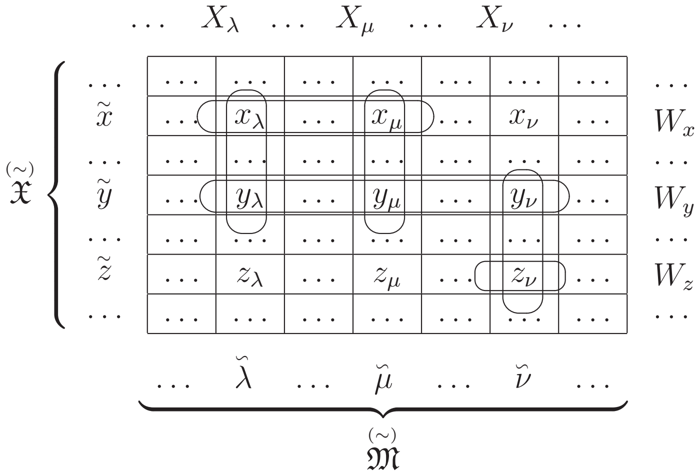

Рис. 7: *Dual collections of sets* $X_\mu = \{x : \mu \in W_x\},\ \mu \in \mathfrak{M}$ sets $\mathfrak{X}$ and subsets $W_x = \{\mu : x \in X_\mu\},\ x \in \mathfrak{X}$ sets $\mathfrak{M}$, generated by subset $X_W = \bigcup_{\mu \in \mathfrak{M}} \bigcup_{x \in X_\mu} x_\mu = \bigcup_{x \in \mathfrak{X}} \bigcup_{\mu \in W_x} x_\mu$ of elements of matrix of selected events $\mathfrak{X}_{\mathfrak{M}} = \{x_\mu,\ x \in \mathfrak{X},\ \mu \in \mathfrak{M}\}$

An arbitrary subset of elements of the matrix can be also represented in this two ways:

$$X_W = \bigcup_{\mu \in \mathfrak{M}} \big(X_W \cap \breve{\mu}\big) = \bigcup_{x \in \mathfrak{X}} \big(X_W \cap \tilde{x}\big).$$

Let's note, that

$$X_W \cap \breve{\mu} = \bigcup_{x \in X_\mu} x_\mu, \quad X_W \cap \tilde{x} = \bigcup_{\mu \in W_x} x_\mu,$$

where

$$X_\mu = \{x : x_\mu \in X_W \cap \breve{\mu}\} \subseteq \mathfrak{X}, \quad \mu \in \mathfrak{M},$$

$$W_x = \{\mu : x_\mu \in X_W \cap \tilde{x}\} \subseteq \mathfrak{M}, \quad x \in \mathfrak{X}.$$

Collections of subsets $\{X_\mu,\ \mu \in \mathfrak{M}\}$ и $\{W_x,\ x \in \mathfrak{X}\}$, defined by the subset $\mathfrak{X}_{\mathfrak{M}}$, have evident inverse correspondence $X_\mu = \{x : \mu \in W_x\} \subseteq \mathfrak{X}, \quad \mu \in \mathfrak{M}$ и $W_x = \{\mu : x \in X_\mu\} \subseteq \mathfrak{M}, \quad x \in \mathfrak{X}$.

Now everything is ready for deducing the formula of an indicator of an arbitrary set-operation over a set $\mathfrak{X}$- of fuzzy events. So, the dual

Theorem (the formula of an indicator of an arbitrary set-operation over sets of $\mathfrak{X}$-fuzzy events) is true.

Let

$$(\mathcal{O})(\mathfrak{M}) = \left(\underset{\mu\in\mathfrak{M}}{\mathcal{O}}\right)\breve{\mu} = \left\{\underset{\mu\in\mathfrak{M}}{\mathcal{O}} x_\mu,\ x \in \mathfrak{X}\right\}$$

— as an arbitrary set-operation over a set of $\mathfrak{X}$*-fuzzy events* $\overset{(\sim)}{\mathfrak{M}} = \{\breve{\mu},\ \mu \in \mathfrak{M}\}$*, defined by a totality of subsets* $\mathcal{O} \subseteq 2^{\mathfrak{M}}$*. Then the indicator is calculated under the formula:*

$$\mathbf{1}_{(\mathcal{O})(\mathfrak{M})}(\omega) = \frac{1}{|\mathfrak{X}|}\left|\underset{\mu\in\mathfrak{M}}{\mathcal{O}} X_\mu(\omega)\right|, \quad \omega \in \Omega,$$

where

$$X_\mu(\omega) = \{x \in \mathfrak{X},\ \omega \in x_\mu\}.$$

Corollary of the basic eventological theorem

From basic eventological theorem on fuzzy events with evidence not demanding extra proofs there follows one and very important property of indicator of set-operation over a set of fuzzy events. Otherwise this indicator is called to be *eventological function of membership degree* of e-events to the fuzzy event — result of the set-operation. This indicator is an only eventologically correct generalization of «usual» function of membership degree for set-operations over fuzzy events used in many empirical variants within the theory of fuzzy sets by Zadeh (Table, page 6).

«Cosmetic» eventological modification of formulae for operation of functions of membership degree from classical theory of fuzzy events by Zadeh offered in this paper (page 7) reduces all the «zoo» of operations empirically used in this theory to the one and general *Fréchet-operation*, depending on the parameter interpreted as a *Fréchet-correlation* of fuzzy events (Table, page 8). *Fréchet-operation* introduced by eventologists profitably differs from a set of empirical variants of classical operation with an important property. It always satisfies *Fréchet-inequalities*, which often are broken by some classical operations of Zadeh theory (for example, delta-operation).

The basic eventological theorem goes essentially farther than «cosmetic» modification of classical formulae of Zadeh and solves the problem of choosing formulae for set-operations over fuzzy events finally. *It offers a one and only one general eventologically correct formula for indicator of any set-operation over any set of fuzzy events. In other words it offers the formula for eventological function of membership degree of e-event to result of the set-operation.* The basic theorem has a general character since as any set-operation usual operations: union, intersection, symmetrical difference and any others set-operations over a set of fuzzy events can be used. As a very important corollary from basic eventological theorem the following remarkable fact serves: eventological functions of membership degree of e-event to intersection and union of a set of fuzzy events always satisfy to *Fréchet-inequalities*.

Let's write Fréchet-inequalities for a function of membership degree of e-events to intersection and union of events in two variants. Firstly, for a set-operation over two fuzzy events, and then — over any set of fuzzy events. Let $\tilde{x}$ и $\tilde{y}$ be two any fuzzy events, $x, y \in \mathfrak{X}$, $X \subseteq \mathfrak{X}$ be any subset of $\mathfrak{X}$, and $\omega \in \Omega$ be any e-event from the space of e-events Ω.

1) Intersection and union of two fuzzy events $\tilde{x}$ and $\tilde{y}$:

$$\max\left\{0,\ \mathbf{1}_{\tilde{x}}(\omega) + \mathbf{1}_{\tilde{y}}(\omega) - 1\right\} \leqslant \mathbf{1}_{\tilde{x}\,(\cap)\,\tilde{y}}(\omega) \leqslant \min\left\{\mathbf{1}_{\tilde{x}}(\omega),\ \mathbf{1}_{\tilde{y}}(\omega)\right\},$$

$$\max\left\{\mathbf{1}_{\tilde{x}}(\omega),\ \mathbf{1}_{\tilde{y}}(\omega)\right\} \leqslant \mathbf{1}_{\tilde{x}\,(\cup)\,\tilde{y}}(\omega) \leqslant \min\left\{1,\ \mathbf{1}_{\tilde{x}}(\omega) + \mathbf{1}_{\tilde{y}}(\omega)\right\}.$$

2) Intersection and union of a set of fuzzy events $\{\tilde{x},\ x \in X\}$:

$$\max\left\{0,\ 1 - \sum_{x \in X}\left(1 - \mathbf{1}_{\tilde{x}}(\omega)\right)\right\} \leqslant \mathbf{1}_{(\bigcap)_{x\in X}\tilde{x}}(\omega) \leqslant \min_{x \in X} \mathbf{1}_{\tilde{x}}(\omega),$$

$$\max_{x\in X} \mathbf{1}_{\widetilde{x}}(\omega) \leqslant \mathbf{1}_{(\bigcup)_{x\in X}\widetilde{x}}(\omega) \leqslant \min\left\{1,\ \sum_{x\in X} \mathbf{1}_{\widetilde{x}}(\omega)\right\}.$$

Random fuzzy events

An $\mathfrak{M}$-fuzzy event

$$\widetilde{x} = \{x_\mu \in \mathcal{F} :\ \mu \in \mathfrak{M}\}, \quad x \in \mathfrak{X},$$

— is a set of $|\mathfrak{M}|$ $\mathcal{F}$-measurable events $x_\mu \subseteq \Omega$. With occurrence of an e-event $\omega \in \Omega$ some of the events from $\widetilde{x}$ occur, and some do not occur — in accordance with e-event ω, which falls to them or not. The events from $\widetilde{x}$, which occur with occurrence of e-event $\omega \in \Omega$, make the subset of events

$$\widetilde{K}_x(\omega) = \{x_\mu \in \widetilde{x} :\ \omega \in x_\mu\} \subseteq \widetilde{x}.$$

This is one of possible values of the random set of events

$$\widetilde{K}_x : (\Omega, \mathcal{F}, \mathbf{P}) \to \left(2^{\widetilde{x}}, 2^{2^{\widetilde{x}}}\right),$$

which is called to be *a random $\mathfrak{M}$-fuzzy event. Random $\mathfrak{X}$-fuzzy event* is determined in the same way:

$$\breve{K}_\mu : (\Omega, \mathcal{F}, \mathbf{P}) \to \left(2^{\breve{\mu}}, 2^{2^{\breve{\mu}}}\right),$$

having as values the subsets of fuzzy event:

$$\breve{K}_\mu(\omega) = \{x_\mu \in \breve{\mu} :\ \omega \in x_\mu\} \subseteq \breve{\mu},$$

where

$$\breve{\mu} = \{x_\mu \in \mathcal{F} :\ x \in \mathfrak{X}\}, \quad \mu \in \mathfrak{M},$$

are $\mathfrak{X}$-fuzzy events, sets of $|\mathfrak{X}|$ $\mathcal{F}$-measurable events $x_\mu \subseteq \Omega$.

Power of a fuzzy event

An $\mathfrak{M}$-fuzzy event $\widetilde{x} = \{x_\mu \in \mathcal{F} :\ \mu \in \mathfrak{M}\}$ — is a set, consisting of $|\mathfrak{M}|$ events $x_\mu \subseteq \Omega$. An integer $|\mathfrak{M}|$, equal to the power of set of events by which it is determined, is called *power of $\mathfrak{M}$-fuzzy event $\widetilde{x}$*.

$$|\widetilde{x}|_{\mathfrak{M}} = \Big|\{x_\mu \in \mathcal{F} :\ \mu \in \mathfrak{M}\}\Big| = |\mathfrak{M}|.$$

Power of random $\mathfrak{M}$ *- fuzzy event* $\widetilde{K}_x$ is determined as the random variable $|\widetilde{K}_x|_{\mathfrak{M}}$. Under $\omega \in \Omega$ the random variable values are

$$|\widetilde{K}_x(\omega)|_{\mathfrak{M}} = \Big|\{x_\mu \in \tilde{x} : \ \omega \in x_\mu\}\Big|$$

— powers of corresponding values of random set of events $\widetilde{K}_x$ and belonging to the set $\{0, 1, \dots, |\mathfrak{M}|\}$. The mean power of random $\mathfrak{M}$-fuzzy event $\widetilde{K}_x$ is determined as the expectation of r.v. $|\widetilde{K}_x|_{\mathfrak{M}}$ and can be computed under the famous theorem by Robbins

$$\mathbf{E}|\widetilde{K}_x|_{\mathfrak{M}} = \sum_{\mu \in \mathfrak{M}} \mathbf{P}(x_\mu). \tag{$*$}$$

The probability of $\mathfrak{M}$-fuzzy event $\tilde{x}$ is determined as expectation of it's indicator:

$$\mathbf{P}(\tilde{x}) = \mathbf{E}\mathbf{1}_{\tilde{x}} = \sum_{\omega \in \Omega} \mathbf{1}_{\tilde{x}}(\omega)\mathbf{P}(\omega) = \sum_{\omega \in \Omega} \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{1}_{x_\mu}(\omega)\mathbf{P}(\omega) =$$

$$= \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{E}\mathbf{1}_{x_\mu} = \frac{1}{|\mathfrak{M}|} \sum_{\mu \in \mathfrak{M}} \mathbf{P}(x_\mu).$$

It follows from $(*)$ that the probability of $\mathfrak{M}$ fuzzy event $\tilde{x}$ is linked with the average power of a random $\mathfrak{M}$ fuzzy event by formula:

$$\mathbf{P}(\tilde{x}) = \frac{\mathbf{E}|\widetilde{K}_x|_{\mathfrak{M}}}{|\tilde{x}|_{\mathfrak{M}}} = \frac{\mathbf{E}|\widetilde{K}_x|_{\mathfrak{M}}}{|\mathfrak{M}|}, \quad x \in \mathfrak{X},$$

as a ratio of an mean number of occurring events to a general number of events in $\tilde{x}$, $x \in \mathfrak{X}$. *power of* $\mathfrak{X}$ *fuzzy event* is determined analogically:

$$|\breve{\mu}|_{\mathfrak{X}} = \Big|\{x_\mu \in \mathfrak{F} : \ x \in \mathfrak{X}\}\Big| = |\mathfrak{X}|$$

and power $|\breve{K}_\mu|_{\mathfrak{X}}$ of a random $\mathfrak{X}$-fuzzy event $\breve{K}_\mu$ takes values

$$|\breve{K}_\mu(\omega)|_{\mathfrak{X}} = \Big|\{x_\mu \in \breve{\mu} : \ \omega \in x_\mu\}\Big|$$

from $\{0, 1, \dots, |\mathfrak{X}|\}$. The mean power of random $\mathfrak{X}$-fuzzy event $\breve{K}_\mu$ is determined as the expectation of r.v. $|\breve{K}_\mu|_{\mathfrak{X}}$ and can be computed under the theorem by Robbins

$$\mathbf{E}|\breve{K}_\mu|_{\mathfrak{X}} = \sum_{x \in \mathfrak{X}} \mathbf{P}(x_\mu). \tag{$**$}$$

The probability of $\mathfrak{X}$-fuzzy event $\breve{\mu}$ is linked to the mean power of a random $\mathfrak{X}$-fuzzy event $\breve{\mu}$ by formula:

$$\mathbf{P}(\breve{\mu}) = \frac{\mathbf{E}|\breve{K}_\mu|_{\mathfrak{X}}}{|\breve{\mu}|_{\mathfrak{X}}} = \frac{\mathbf{E}|\breve{K}_\mu|_{\mathfrak{X}}}{|\mathfrak{X}|}, \quad \mu \in \mathfrak{M},$$

as a ratio of an average number of occurring events to a general number of events in $\breve{\mu}$, $\mu \in \mathfrak{M}$.

Eventological set means of fuzzy events

Eventological set-means (E-set-means) of fuzzy events are usual events, determined as cutting their indicators of appointed levels. *Eventological set-median (E-set-median)* of $\mathfrak{M}$-fuzzy event $\tilde{x}$ and $\mathfrak{X}$-fuzzy event $\breve{\mu}$ are called to be events:

$$\operatorname{Med}\tilde{x} = \{\omega \in \Omega : \ \mathbf{1}_{\tilde{x}}(\omega) \geqslant 1/2\} \subseteq \Omega, \quad x \in \mathfrak{X},$$

$$\operatorname{Med}\breve{\mu} = \{\omega \in \Omega : \ \mathbf{1}_{\breve{\mu}}(\omega) \geqslant 1/2\} \subseteq \Omega, \quad \mu \in \mathfrak{M}.$$

Eventological set-expectation (E-set-expectation) of $\mathfrak{M}$- fuzzy event $\tilde{x}$ and $\mathfrak{X}$-fuzzy event $\breve{\mu}$ are called to be events:

$$\mathcal{E}\tilde{x} = \{\omega \in \Omega : \ \mathbf{1}_{\tilde{x}}(\omega) \geqslant \mathbf{P}(\tilde{x})\} \subseteq \Omega, \quad x \in \mathfrak{X},$$

$$\mathcal{E}\breve{\mu} = \{\omega \in \Omega : \ \mathbf{1}_{\breve{\mu}}(\omega) \geqslant \mathbf{P}(\breve{\mu})\} \subseteq \Omega, \quad \mu \in \mathfrak{M}.$$

It is not difficult to check that E-set-means $\mathfrak{F}$ are measurable, i.e.

$$\operatorname{Med}\tilde{x} \in \mathfrak{F}, \quad \operatorname{Med}\breve{\mu} \in \mathfrak{F}, \quad \mathcal{E}\tilde{x} \in \mathfrak{F}, \quad \mathcal{E}\breve{\mu} \in \mathfrak{F}.$$

That is evident due to the fact that indicators of fuzzy events are constant within events-terraces, which are generated by them. From $\mathfrak{F}$-measurability of E-set-means two facts follow. Firstly, for any E-set-mean as a $\mathfrak{F}$-measurable event its probability is defined:

$$\mathbf{P}(\operatorname{Med}\tilde{x}), \quad \mathbf{P}(\operatorname{Med}\breve{\mu}), \quad \mathbf{P}(\mathcal{E}\tilde{x}), \quad \mathbf{P}(\mathcal{E}\breve{\mu}).$$

Secondly, every E-set-mean is a result of a rather definite generalized set-operation upon the sets of events $\tilde{x}$ and $\breve{\mu}$ accordingly:

$$\operatorname{Med}\tilde{x} = \operatorname*{Med}_{\mu \in \mathfrak{M}}{}_x \ x_\mu = \sum_{W_x \in \operatorname{Med}_x} \operatorname{ter}_x(W_x),$$

$$\operatorname{Med}\breve{\mu} = \underset{x\in\mathfrak{X}}{\operatorname{Med}_{\mu}}\ x_\mu = \sum_{X_\mu \in \operatorname{Med}_\mu} \operatorname{ter}_\mu(X_\mu),$$

$$\mathcal{E}\tilde{x} = \underset{\mu\in\mathfrak{M}}{\mathcal{E}_{x}}\ x_\mu = \sum_{W_x \in \mathcal{E}_x} \operatorname{ter}_x(W_x),$$

$$\mathcal{E}\breve{\mu} = \underset{x\in\mathfrak{X}}{\mathcal{E}_{\mu}}\ x_\mu = \sum_{X_\mu \in \mathcal{E}_\mu} \operatorname{ter}_\mu(X_\mu),$$

where $\operatorname{Med}_x, \mathcal{E}_x \subseteq 2^{\tilde{x}}$, $\operatorname{Med}_\mu, \mathcal{E}_\mu \subseteq 2^{\breve{\mu}}$ are subsets of sets $2^{\tilde{x}}$ and $2^{\breve{\mu}}$ of all subsets of sets $\tilde{x}$ and $\breve{\mu}$ accordingly, determining generalized set-operations Med_x, Med_μ, $\mathcal{E}_x$ and $\mathcal{E}_\mu$.

Instead of conclusion

Eventology studies *motion of random fuzzy events,* which are regarded as *dynamics of eventological distributions*; but still it is rather far from being a complete theory. Stable foundations of eventology have been laid, essential assertions of fuzzy events, such as basic eventological theorem, have been proved. It has already become obvious that *eventology of fuzzy events*, being a more general theory, includes the famous theory of fuzzy sets by Zadeh, since the theory of fuzzy sets made a good showing, but was particular and insufficiently grounded. However a lot of theoretical and practical questions in eventology still remain vague. Thus we suggest several remarks instead of a conclusion. Actually these remarks contain more questions, than answers. But I hope that in foreseeable future the answers will be found and new eventological problems will be set in our following works. Eventology is advancing; before our eyes it headily becomes a *universal eventological language*, which can be successfully used for discussing and solving either problems of *matter* or problems of *mind.*

Two types of fuzziness

Eventology deals with two types of fuzzy events, which differ from each other in sets that generate «fuzziness». The first type of «fuzziness» is generated by the set $\mathfrak{M}$, and the second type — by the set $\mathfrak{X}$. The eventological interpretation of these «fuzzinesses» differs, but still has not been cleared up completely. Possibly «fuzziness» of an event $\tilde{x} = \{x_\mu,\ \mu \in \mathfrak{M}\}$ is generated by the set $\mathfrak{M}$ and measured with «membership»; different individual minds $\mu \in \mathfrak{M}$ have dissimilar notions of occurrence of every selected event $x \in \mathfrak{X}$ — and «fuzziness» of an event $\tilde{x}$ might be explained thus. «Fuzziness» of an event $\breve{\mu} = \{x_\mu,\ x \in \mathfrak{X}\}$ is generated by the set $\mathfrak{X}$ and measured with «probability»; this «fuzziness» follows from ability of every individual mind $\mu \in \mathfrak{M}$ (such is defined by it's own sequence of events, which contains the set of events $\breve{\mu}$) to make a probabilistic choice within it's own sequence of events. «Fuzziness of the second type», which is aroused by the set $\mathfrak{X}$ in behaviour of an individual mind μ, may turn out to be «probabilistic fuzziness», named «randomness». Thus «fuzziness» is generated by the set $\mathfrak{M}$ — by mind — and is measured with «membership»; «randomness» is generated by the set $\mathfrak{X}$ — by matter — and measured with «probability». Such paradoxical and, in my opinion, true analogy looks a bit strange, that's why it requires additional reflection and justification. In the meantime let's content ourselves with distinguishing two type of fuzziness and pondering upon an $\mathfrak{M}$-fuzzy event $\tilde{x}$ and an $\mathfrak{X}$-fuzzy event $\breve{\mu}$.

Mind and events

Mind is able *to be conscious* of occurring of an event, *to understand* the fact it occurred, *to perceive* an event only in the case when mind remembers a name of an event or names it, if an event had no name. Nameless event remains beyond the bounds of consciousness inside the unconscious, until mind calls it into being from unconscious depths and names it. For mind to be conscious of occurring of an event, to understand the fact it occurred, to perceive an event means *to exist, to be*. Events as well exist, whilst they are understood and perceived by mind, whilst mind is conscious of them. Mind and events can not exist without each other. *To be mind* is to perceive events, *to be events* is to be perceived by mind.

Membership and existence

In the theory by Zadeh a fuzzy event is equated to function of membership degree, which is given on Ω, in other words, to some random variable having values from segment $[0, 1]$. Membership degree is a key concept of the theory of fuzzy sets. However, in the probability theory membership of e-event $\omega \in \Omega$ to one or another event means occurrence of the latter. Thus when the case in point is membership degree of e-event to fuzzy event, it's more correct to speak about occurrence degree of fuzzy event at the time of occurrence of e-event $\omega \in \Omega$. In this connection, when speaking about eventology of fuzzy events instead of the key concept of the theory of fuzzy sets — «membership» — it is more pertinent to introduce equivalent, but absolutely new and, in my opinion, «revolutionary» term. There are some contenders for the term: «realization», «understanding», «perception», «existence», «being» – for us not to speak about membership degree of e-event to a fuzzy event, but about «degree of realization, understanding, perception» of a fuzzy event by mind, or about «degree of *existence of mind* within a fuzzy event», or about «degree of *existence of a fuzzy event* within the mind».

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