A dynamic IS-LM model with adaptive expectations

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Abstract

We analyze the stability of a discrete-time dynamic model with an IS-ML structure. We assume that the Aggregate Supply function is of Lucas type and the monetary policy rule is of Friedman type. The mechanism of expectations formation is assumed to be of adaptive type (Friedman-Cagan).

In its final form, the model contains two state-variables, namely money supply and expected inflation. From the mathematical point of view, it is an affine discrete-time system, whose stability properties are analyzed in the paper.

We deduce sufficient conditions concerning the „learning coefficient" involved in the Friedman-Cagan type of forecast equation, so that the model be stable.

1The model

We assume that the functions IS and LM are of Cobb-Douglas type :

$$\textbf{IS:}\ Y = A \cdot Y^{c} \cdot e^{-g \cdot r} \qquad (1)$$

$$\textbf{LM:}\ \frac{M}{P} = Y^{l_1} \cdot e^{-l_2\left(r+\pi^{e}\right)} \qquad (2)$$

The variables are the following :

- Y – GDP size;
- M – nominal money supply;
- P – price index ;
- r – real interest rate;
- π^e – expected inflation rate;
- A, c, g, l_1, l_2 – parameters

In logarithmic form, equations(1) şi (2) become:

$$y = a + c \cdot y - g \cdot r \qquad (3)$$

$$m = l_1 \cdot y - l_2\left(r + \pi^{e}\right) \qquad (4)$$

We used the following notations:

$$y = \ln(Y),\ m = \ln\left(\frac{M}{P}\right),\ a = \ln(A) \qquad (5)$$

The coordinates of the equilibrium point are:

$$\begin{cases} y^* = \dfrac{1}{1 - c + g \cdot \dfrac{l_1}{l_2}}\left(a + \dfrac{g}{l_2} m + g \cdot \pi^e\right) \\ r^* = \dfrac{1}{l_2}\left(l_1 \cdot y^* - m - l_2 \cdot \pi^e\right) \end{cases} \qquad (6)$$

We denote by k the multiplyer :

$$k = \frac{1}{1 - c + g \cdot \dfrac{l_1}{l_2}} \qquad (7)$$

The first relationship in (6) becomes :

$$y^* = k \cdot \left(a + \frac{g}{l_2} m + g \cdot \pi^e\right) \qquad (8)$$

Relation (8) expresses the fact that GDP (in logarithmic form) depends on the money supply (m), on the expected inflationi(π^e), as well as on the coefficient „a", which reflects fiscal policy and the level of net export.
We assume that the quantity of money supplied by the Central Bank is governed by a Friedman type monetary policy rule:

$$M_{t+1} = M_t \cdot e^{\mu};\ \mu = \text{constant} \qquad (9)$$

or

$$\ln\left(\frac{M_{t+1}}{M_t}\right) = \mu \qquad (10)$$

We assume that the aggregative supply (AS) is modeled by the following Phillips – Lucas type function :

$$y_t = \bar{y} + \frac{1}{\theta}\left(\pi_t - \pi_t^{\,e}\right) \qquad (11)$$

where $\bar{y}$ represents the „natural" size of GDP (in logarithmic form), and π_t - the size of inflation. The coefficient θ is given. From (11) it follows :

$$\pi_t = \pi_t^{\,e} + \theta \cdot \left(y_t - \bar{y}\right) \qquad (12)$$

As concerns inflation π_t, it is defined by :

$$P_{t+1} = P_t \cdot e^{\pi_t} \qquad (13)$$

or :

$$\pi_t = \ln\left(\frac{P_{t+1}}{P_t}\right) \qquad (14)$$

As concerns the mechanism by which economic agents form expectations of future inflation, we assume that it is of adaptive type (Friedman - Cagan):

$$\pi^e{}_{t+1} = \pi_t{}^e + \delta\left(\pi_t - \pi_t{}^e\right); 0 \le \delta \le 1 \qquad (15)$$

The learning parameter δ is given.

Using relation (12), relation (15) becomes:

$$\pi^e{}_{t+1} = \pi_t{}^e + \delta \cdot \theta\left(y_t - \bar{y}\right) \qquad (16)$$

Using relation (8):

$$y_t = y_t{}^* = k \cdot \left(a + \frac{g}{l_2} m_t + g \cdot \pi_t{}^e\right) \qquad (17)$$

the dynamic equation (16) for expected inflation becomes:

$$\pi^e{}_{t+1} = \pi_t{}^e + \delta \cdot \theta\left[k \cdot \left(a + \frac{g}{l_2} m_t + g \cdot \pi_t{}^e\right) - \bar{y}\right], \quad \text{or:}$$

$$\pi^e{}_{t+1} = \delta \cdot \theta \cdot k \cdot \frac{g}{l_2} m_t + \left(1 + \delta \cdot \theta \cdot k \cdot g\right)\pi_t{}^e + \delta \cdot \theta \cdot \left(k \cdot a - \bar{y}\right) \qquad (18)$$

We denote**:**

$$\alpha = \theta \cdot k \cdot \frac{g}{l_2};\ h = \theta \cdot \left(k \cdot a - \bar{y}\right) \qquad (19)$$

The dynamic equation (18) becomes:

$$\pi^e{}_{t+1} = \delta \cdot \alpha m_t + \left(1 + \delta \cdot \alpha \cdot l_2\right)\pi_t{}^e + \delta \cdot h \qquad \textbf{(20)}$$

As concerns money supply, we have :

$$m_{t+1} - m_t = \ln\frac{M_{t+1}}{P_{t+1}} - \ln\frac{M_t}{P_t} = \ln\frac{M_{t+1}}{M_t} - \ln\frac{P_{t+1}}{P_t}$$

Using monetary policy rule (10), as well as the definition of inflation (14), we obtain:

$$m_{t+1} - m_t = \mu - \pi_t \text{ or } m_{t+1} = m_t + \mu - \pi_t \qquad \textbf{(21)}$$

Using relations (12) and (17), it follows:

$$m_{t+1} = m_t - \pi_t^{\ e} - \theta \cdot \left[k \cdot \left(a + \frac{g}{l_2} m_t + g \cdot \pi_t^{\ e} \right) - \bar{y} \right] + \mu \quad \text{or}$$

$$m_{t+1} = \left(1 - \theta \cdot k \cdot \frac{g}{l_2} \right) m_t - (1 + \theta \cdot k \cdot g) \pi_t^{\ e} - \theta \cdot [k \cdot a - \bar{y}] + \mu .$$

Using notations (19), we obtain:

$$m_{t+1} = (1 - \alpha) m_t - (1 + \alpha \cdot l_2) \pi_t^{\ e} + \mu - h \qquad \textbf{(22)}$$

The dynamics of the system (the dynamicIS-LM model), in the variant proposed on the basis of the adopted assumptions, is given by relations (20) and (22).

$$\begin{cases} m_{t+1} = (1 - \alpha) m_t - (1 + \alpha \cdot l_2) \pi_t^{\ e} + \mu - h \\ \pi^e{}_{t+1} = \delta \cdot \alpha m_t + (1 + \delta \cdot \alpha \cdot l_2) \pi_t^{\ e} + \delta \cdot h \end{cases} \qquad \textbf{(23)}$$

2. Analysis of the dynamics

In order to analyze the dynamics of system (23), we shall calculate the stationary trajectories :

$$\begin{aligned} & m_{t+1} = m_t = m = \text{constant}, \forall t \in N \\ & \pi^e{}_{\text{t+1}} = \pi_t^{\ e} = \pi^e = \text{constant}, \forall t \in N \end{aligned} \qquad \textbf{(24)}$$

$$m = -\frac{1 + \alpha \cdot l_2}{\alpha} \pi^e + \frac{1}{\alpha} (\mu - h) \text{ for } (\text{m}_{\text{t+1}} = m_t = m) \qquad (25)$$

$$m = -l_2 \cdot \pi^e - \frac{1}{\alpha} h \ \text{ for } \left(\pi^{\text{e}}{}_{t+1} = \pi_t^{\ e} = \pi^e \right) \qquad \textbf{(26)}$$

The equilibrium (steady-state) point, i.e. the solution of the system formed of equations (23) and (24), is :

$$\begin{cases} \hat{\pi}^e = \mu \\ \hat{m} = -l_2 \cdot \mu - \frac{1}{\alpha} h \end{cases} \qquad \textbf{(27)}$$

Taking into account notations (19), it follows :

$$\begin{cases} \hat{\pi}^e = \mu \\ \hat{m} = \frac{l_2}{g} \left(\frac{1}{k} \bar{y} - a - g \cdot \mu \right) \end{cases} \qquad \textbf{(28)}$$

Using relations (28), we obtain from (8) the equilibrium value for the GDP :

$$\hat{Y}^* = \bar{Y} \qquad \textbf{(29)}$$

Relation (29) shows that, in equilibrium, the size of GDP will coincide with its „natural" value.

At the same time, the first relation in (28) shows that , at equilibrium, expected inflation coincides with the growth rate of money supply.

To check the stability of the dynamic IS-LM system, we shall solve the system (23). This is an affine system of difference equations. Taking into account (27), it is easy to prove that system (23) can be rewritten as:

$$\begin{cases} m_{t+1} - \hat{m} = (1-\alpha)(m_t - \hat{m}) - (1+\alpha \cdot l_2)(\pi_t{}^e - \pi^e) \\ \pi^e{}_{t+1} - \pi^e = \delta \cdot \alpha(m_t - \hat{m}) + (1+\delta \cdot \alpha \cdot l_2)(\pi_t{}^e - \hat{\pi}) \end{cases} \quad \textbf{(30)}$$

We denote:

$$\begin{cases} m_t - \hat{m} = u_t \\ \pi_t{}^e - \pi^e = z_t \end{cases} \quad \textbf{(31)}$$

The variables u_t and z_t represent the deviations from the respective equilibrium values.

With notations (31), system (30) can be written as :

$$\begin{cases} u_{t+1} = (1-\alpha) \cdot u_t - (1+\alpha \cdot l_2) \cdot z_t \\ z_{t+1} = \delta \cdot \alpha \cdot u_t + (1+\delta \cdot \alpha \cdot l_2) \cdot z_t \end{cases} \quad \textbf{(32)}$$

The difference equations system (32) has the advantage of being linear.

In order to write system (32) in matrix form, we denote:

$$X_t = \begin{pmatrix} u_t \\ z_t \end{pmatrix}; \mathrm{M} = \begin{pmatrix} 1-\alpha & -(1+\alpha \cdot \mathrm{l}_2) \\ \delta \cdot \alpha & 1+\delta \cdot \alpha \cdot \mathrm{l}_2 \end{pmatrix} \quad \textbf{(33)}$$

System (32) becomes:

$$X_{t+1} = M \cdot X_t \quad \textbf{(34)}$$

From (34), we obtain, successively:

$X_1 = M \cdot X_0; \ldots X_n = M^n \cdot X_0$

Hence, the general solution of equation (34) is:

$$X_t = M^t \cdot X_0 \quad \textbf{(35)}$$

It is more convenient to represent matrix *M* in Jordan form:

$$M = K \cdot \Lambda \cdot K^{-1} \quad \textbf{(36)}$$

where Λ is the diagonal matrix:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \textbf{(37)}$$

λ_1 and λ_2 being the eigenvalues of matrix *M*, and *K* is the matrix whose columns are the eigenvectors of matrix *M*[1].

The characteristic equation of matrix *M* is:

$$\det(M - \lambda \cdot I_2) = 0 \qquad \textbf{(38)}$$

i.e:

$$\begin{vmatrix} 1-\alpha-\lambda & -(1+\alpha \cdot l_2) \\ \delta \cdot \alpha & 1+\delta \cdot \alpha \cdot l_2 - \lambda \end{vmatrix} = 0$$

We obtain:

$$\lambda^2 - (1-\alpha+1+\delta \cdot \alpha \cdot l_2)\lambda + \delta \cdot \alpha(1+\alpha \cdot l_2) + (1-\alpha)(1+\delta \cdot \alpha \cdot l_2) = 0$$

or:

$$\lambda^2 - (2-\alpha+\delta \cdot \alpha \cdot l_2)\lambda + 1 - \alpha + \delta \cdot \alpha + \delta \cdot \alpha \cdot l_2 = 0 \qquad \textbf{(39)}$$

Substituting the solutions λ_1 and λ_2 of ecuation (39) and taking into account (36), the solution (35) of system (34) becomes:

$$X_t = \left[K \cdot \Lambda \cdot K^{-1}\right]^t \cdot X_0 \qquad \textbf{(40)}$$

It is easy to see that:

$$\left[K \cdot \Lambda \cdot K^{-1}\right]^t = K \cdot \Lambda^t \cdot K^{-1}$$

In fact, the above relationship represents one of the advantages of the Jordan decomposition of a matrix.Taking into account that:

$$\Lambda^t = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^t = \begin{bmatrix} \lambda_1{}^t & 0 \\ 0 & \lambda_2{}^t \end{bmatrix}$$

solution (40) becomes:

$$X_t = K \cdot \begin{bmatrix} \lambda_1{}^t & 0 \\ 0 & \lambda_2{}^t \end{bmatrix} \cdot K^{-1} \cdot X_0 \qquad \textbf{(41)}$$

We denote:

$$\begin{pmatrix} m_t \\ \pi_t{}^e \end{pmatrix} = \Psi_t \qquad \textbf{(42)}$$

Taking into account (31) and the first relation in (33), it follows:

$$\Psi_t = \hat{\Psi} + X_t \qquad \textbf{(43)}$$

Obviously, the equilibrium solution is:

$$\hat{\Psi} = \begin{pmatrix} \hat{m} \\ \hat{\pi}^e \end{pmatrix} \qquad \textbf{(44)}$$

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[1] See the mathematical note at the end of the section

Taking into account (41), solution (43) can be written as:

$$\begin{pmatrix} m_t \\ \pi_t^{\,e} \end{pmatrix} = \begin{pmatrix} \hat{m} \\ \hat{\pi}^e \end{pmatrix} + K \cdot \begin{bmatrix} \lambda_1^{\,t} & 0 \\ 0 & \lambda_2^{\,t} \end{bmatrix} \cdot K^{-1} \cdot X_0 \text{ or}$$

$$\begin{pmatrix} m_t \\ \pi_t^{\,e} \end{pmatrix} = \begin{pmatrix} \hat{m} \\ \hat{\pi}^e \end{pmatrix} + K \cdot \begin{bmatrix} \lambda_1^{\,t} & 0 \\ 0 & \lambda_2^{\,t} \end{bmatrix} \cdot K^{-1} \cdot \begin{pmatrix} m_0 - \hat{m} \\ \pi_0^{\,e} - \hat{\pi}^e \end{pmatrix} \qquad \textbf{(45)}$$

The dynamic IS-LM system will be stable if şi and only if

$$|\lambda_1| < 1 \text{ şi } |\lambda_2| < 1 \qquad \textbf{(46)}$$

In this case

$$\begin{pmatrix} m_t \\ \pi_t^{\,e} \end{pmatrix} \longrightarrow \begin{pmatrix} \hat{m} \\ \hat{\pi}^e \end{pmatrix}$$

It is to be checked if the solutions of equation (39) satisfy condition (46)

The discriminant of equation (39) is:

$$\Delta = \alpha\left[(\alpha - 4\delta) + \delta \cdot \alpha \cdot l_2(\delta \cdot l_2 - 2)\right]$$

If $\delta > \frac{1}{4}\alpha$, then the discriminant will be negative, hence λ_1 and λ_2 will be complex conjugates.

In this case :

$$|\lambda_1| = |\lambda_2| = \sqrt{1 - \alpha + \delta \cdot \alpha + \delta \cdot \alpha \cdot l_2}$$

$$|\lambda_1| < 1 \text{ if } \delta < \frac{1}{1 + l_2}$$

In conclusion, it follows that the stability of the dynamic IS-LM model depinds in a decisive way on the learning parameter δ, which characterizes the manner in which economic agents form expectations of future inflation (relation 15).

If the learning parameter satisfies

$$\frac{1}{4}\alpha < \delta < \frac{1}{1 + l_2}, \qquad \textbf{(47)}$$

i.e.:

$$\frac{1}{4}\theta \cdot k \cdot \frac{g}{l_2} < \delta < \frac{1}{1 + l_2} \qquad \textbf{(48)}$$

then the dynamic IS-LM system is stable. Relation (48) can be written as:

$$\frac{1}{4}\theta \cdot \frac{\frac{g}{l_2}}{1 - c + g\frac{l_1}{l_2}} < \delta < \frac{1}{1 + l_2} \qquad \textbf{(49)}$$

We stress that relations (47), (48) and (49) represent sufficient conditions for stability, but not necessary conditions .

3. Phase portrait of the trajectories

In order to represent graphically the dynamics of the system, we represent in the phase plane (π^e, m) stationary trajectories (25) and (26).

The stationary trajectory for the real money supply is:

$$m_{t+1} = m_t = m, \forall t \in N, \qquad \text{or}$$

$$m = -\frac{1+\alpha \cdot l_2}{\alpha}\pi^e + \frac{1}{\alpha}(\mu - h) \qquad \textbf{(50)}$$

The graph of the line (50) is represented in Figure 1.

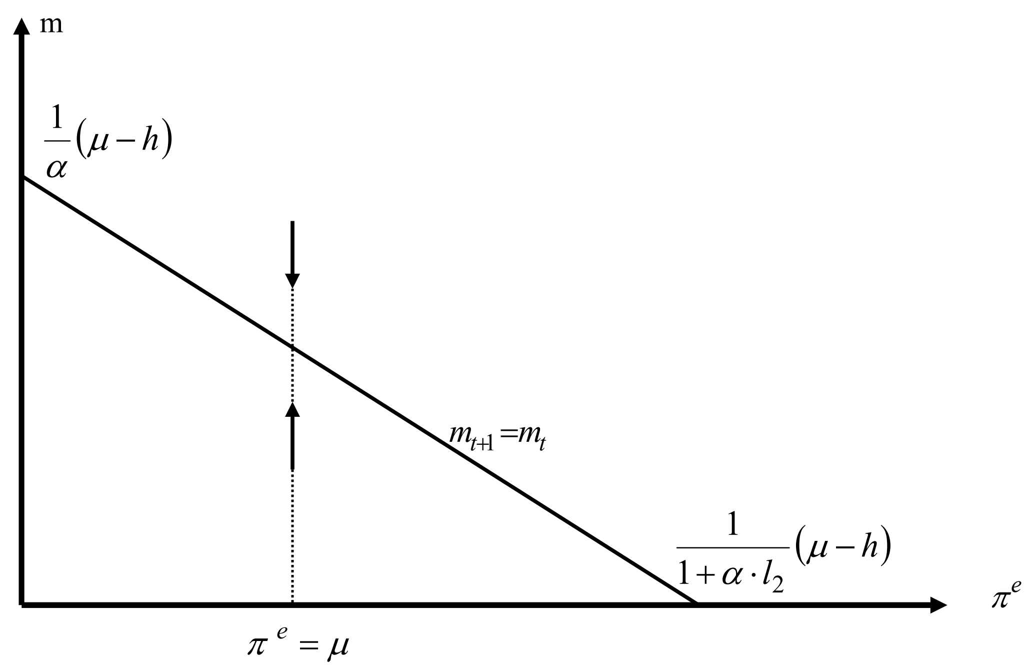

Figure 1

The line $m_{t+1} = m_t$ in Figure 1 divides the plane in two half-planes. For the points with coordnates (π^e, m) in one of the half-planes $m_{t+1} > m_t$, and for the points in the other half-plane $m_{t+1} < m_t$.

In order to identify the sense of motion in each of the half-panes, we rewrite the first equation in (23) as :

$$m_{t+1} - m_t = -\alpha\left(m_t + \frac{1+\alpha \cdot l_2}{\alpha}\pi_t^{\,e} - \frac{1}{\alpha}(\mu - h)\right)$$

From the above relation it can be seen that , if

$$m_t > -\frac{1+\alpha \cdot l_2}{\alpha}\pi_t^{\ e} + \frac{1}{\alpha}(\mu - h)$$

then $\mathrm{m}_{t+1} - m_t < 0$

It follows that the sense of motion in the two half-planes is given by the arrows in Figure 1.

Analogously, we consider the line $\pi_{t+1}^{\ e} = \pi_t^{\ e}$, or

$$m = -l_2 \cdot \pi^e - \frac{1}{\alpha}h \qquad \textbf{(51)}$$

The graph of the stationary trajectory (51), as well as the sense of motion in the two half-planes defined by it are represented in Figure 2.

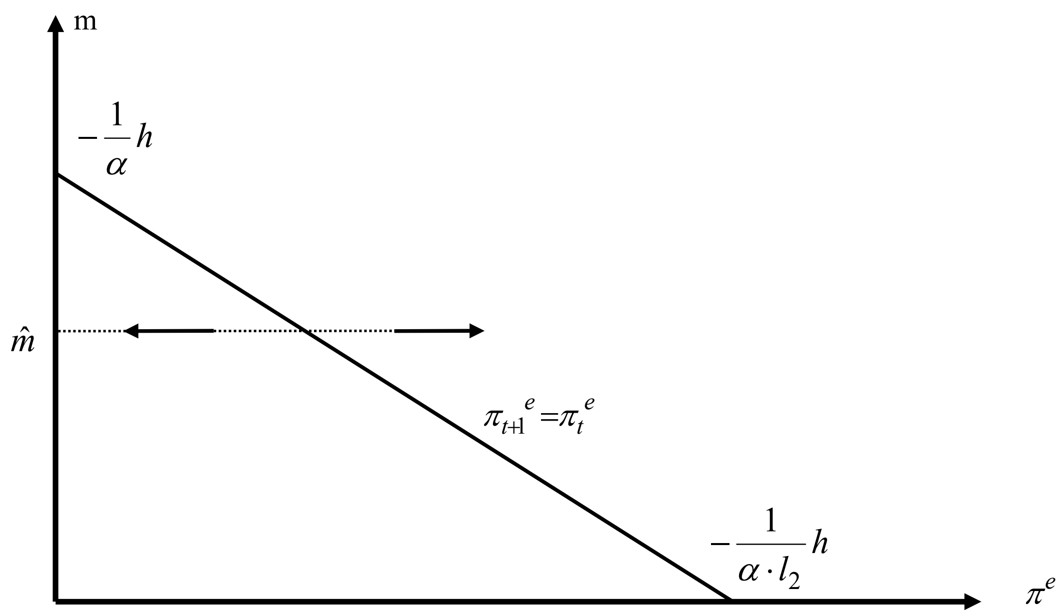

Figure 2

The sense of motion presented in Figure 2 results clearly from the second equation (23), which can be written as:

$$\pi_{t+1}^{\ e} - \pi_t^{\ e} = \delta \cdot \alpha \left(m_t + l_2 \cdot \pi_t^{\ e} + \frac{1}{\alpha}h \right)$$

Representing both stationary trajectories in the same reference system, the phase space will be divided into four quadrants. The sense of motion in each of these quadrants is presented in Figure 3.

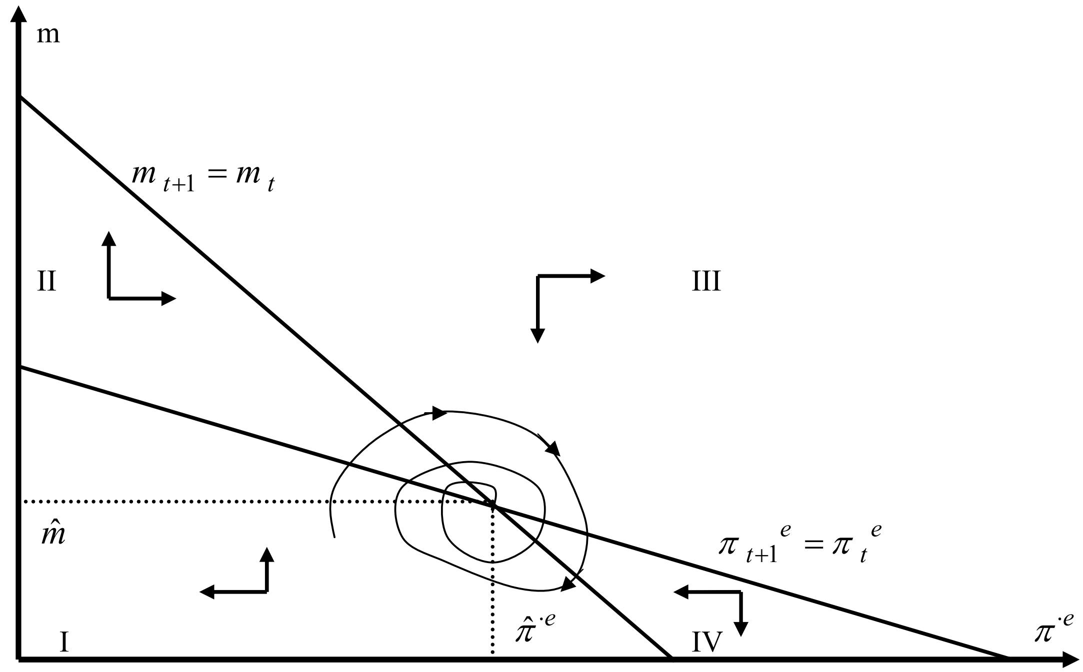

Figure 3

References

Azariadis, C. (1993) *Intertemporal Macroeconomics*, Oxford, Blackwell.

Blanchard, O. and Fischer, S. (1989) *Lectures on Macroeconomics*, Massachusetts Institute of Technology Press.

Cagan, P. (1956) ”The monetary Dynamics of Hyperinflation”. In *Studies in the Quantity Theory of Money*, ed. M. Freedman, University of Chicago Press.

Casares, M. and Bennett Mc Callum (2000), ”An Optimiying IS-LM Framework with Endogenous Investment”. NBER Working paper no. 7908.

De Cesare, L. and Mario Sportelli (2005) ”A dynamic IS-LM model with delayed taxation revenue,” *Chaos, Solitons & Fractals*, 233-244, july 2005.

Henri, U. and B. Venturi (2007) ”Stability and bifurcation in IS-LM economic models,” *International Review of Economics*, vol. 54, no. 1, march 2007.

King, R. (2000) ”The New IS-LM Model: Language, Logic and Limit” *Federal Reserve Bank of Richmond Economic Quarterly*.

Mc Callum, B. and Edward Nelson (2000) ”An Optimizing IS-LM Specification for Monetary Policy and Business Cycle Analysis, ” NBER working paper 5875.

Romer, D. (2000) ”Keynesian Macroeconomics without the LM Curve”, NBER Working Paper 7461.