A Continuous Review Inventory Model with Advance Policy Change and Obsolescence

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In this paper, we consider a continuous review inventory system of a slow moving item for which the demand rate drops to a lower level at a pre-determined time. Inventory system is controlled according to one-for-one replenishment policy with fixed lead time. Adaptation to the lower demand rate is achieved by changing the control policy in advance and letting the demand take away the excess stocks. We showed that the timing of the control policy change primarily determines the tradeoff between backordering penalties and obsolescence costs. We propose an approximate solution for the optimal time to shift to the new control policy minimizing the expected total cost during the transient period. We found that the advance policy change results in significant cost savings and our model yields near optimal expected total costs.

Keywords: inventory control; obsolescence; spare parts; advance policy change; excess stock; installed base.

1. Introduction

For many companies conducting service-centric operations, reducing spare parts inventories without jeopardizing the availability of the supported products is essential for their competitiveness. However, efficient management of expensive and slow moving parts inventories is notoriously difficult due to scarcity of historical data, fluctuations in demand rate and risk of obsolescence. Companies realizing these facts start keeping track of the changes in their own or customers' base of installed products (installed base) to trace customers and operating units more closely and to react to the changes in demand rate as early as possible. A recent study by Jalil et al. (2009) revealed that at IBM, tracking of the installed base for spare parts can lead to savings up to 58% in transportation and inventory holding costs.

When contextual information is combined with installed base tracking, the timing and the size of the shift in demand rate are either known in advance or can be estimated within a reasonable accuracy. In practice, such shifts typically occur when the size of the installed base at certain geographical location changes. For example, when a customer announces that its going to relocate its production equipments, after sales service provider anticipates a change in demand for parts between the locations. Similarly, when a customer decides to upgrade its machinery, the old generation equipments usually leave the installed base of the service provider or manufacturer as a result of discarding or salvaging.

When a sudden change in demand rate can be foreseen, timely adaptation of the base stock levels is crucial for optimal stock control. In such cases, upward jumps in demand rate can be handled relatively easily by giving advance or emergency replenishment orders to be delivered before the jump occurs. However, adaptation to the drop in demand rate is more difficult since running down of excess stocks depends on the demand process. For example, when a certain proportion of the installed base is relocated, service providers usually suffer from excess inventories remaining at the previous location. When relocation of spare parts with the installed base is not feasible, it becomes much more difficult to get rid of the excess stocks due to the diminished demand. Consequently, in many cases these excess stocks end up as obsolete stocks.

Generation upgrades may result in a similar problem as well. For example, when an airline announces the selling of their old generation aircrafts to the countries outside Europe, service providers of this airline expect a sudden drop in demand for relevant parts at their service locations in Europe. In such cases, if a prior action is not taken to adjust the base stock levels then the excess stocks might become obsolete. A striking example comes from an aerospace service provider with which the authors are familiar. At this service provider obsolete stocks constitute 5% of the total inventory carried which add up to more than $1,000,000 in stock value.

Even when the timing and the size of the drop is known exactly, when to change the inventory control policy to minimize obsolete stocks without staking availability remain as a challenging question. If the adaptation is too early before the drop occurs then the risk of backordering increases as a result of lower base stock level. Since availability is crucial for many companies operations, stockouts can be detrimental to their businesses. On the other hand, if the adaptation is too late or postponed after the drop then the costs associated with obsolescence increase. In this paper, we address this issue by focusing on a continuous review inventory system of a slow moving item for which the demand rate drops to a lower level at a pre-determined (announced) time. We assume that the inventory system is controlled according to one-for-one replenishment policy with fixed lead time. Adaptation to the lower demand rate is achieved by changing the control policy in advance and letting the demand process take away the excess stocks. Our goal is to find the optimal time for a policy change and to investigate its impacts on the costs incurred during the transient period.

Our work is related to the inventory management models considering obsolescence. Hadley and Within (1963) were early contributors in this area. They analyzed a finite horizon periodic review inventory system in which the mean demand rate may vary in every period and there is a finite number of possible obsolescence dates. Pierskalla (1969) studied a similar problem with independent and identically distributed demands and zero lead times.

Brown et al. (1964) offered a more general model for obsolescence in which the demand in each period is generated according to an underlying Markov chain and the state probabilities are updated in Bayesian fashion. Song and Zipkin (1996) also employed a similar Markovian submodel to reflect the processes leading to obsolescence by assuming that the current state of the process is completely observable. They found that the obsolescence has substantial effects on inventory costs and these effects cannot be remedied by simple parameter adjustments.

Besides the periodic review models Masters (1991), Jonglekar and Lee (1993), David and Mehrez (1995) considered the EOQ model in which the time to obsolescence is exponentially distributed. To the best of our knowledge, there are no studies in the literature considering obsolescence for a continuous review inventory system facing stochastic demand.

Another stream of literature that is related to our study consists of the so called excess stock disposal models. In these models the problem is to determine the economic retention quantity or time period given the excess stock of an inventory item. Earlier works by Simpson (1955), Mohan and Garg (1961) and Hart (1973) investigated the excess inventory disposal problem for deterministic demand case with the possibility of obsolescence. Stulman (1989) considered continuous review inventory system with stochastic demand but without obsolescence. Rosenfield (1989) investigated the similar problem for slow moving items by including perishability or obsolescence but without stockout penalties. In all of these studies it is assumed that the excess stocks are result of over purchasing or a drop in demand rate in the past. Therefore, the inventory level is found higher than the maximum level at time zero and the excess inventory is reduced by first disposing, and then letting the demand take away the retained quantity. Our model differs from this literature mainly by letting the demand take away the stocks before the excess occurs.

In this study, our contribution is threefold: First, we analyze the obsolescence problem for a continuous review inventory system facing stochastic demand for the first time. Our findings are consistent with the earlier works studying periodic review systems that the obsolescence has significant effects on operating costs and should be taken into account explicitly. We extend these findings by showing that for a continuous review inventory system advance policy change results in significant cost savings. Our numerical experiments revealed that if the control policy is not changed in advance then the transient period costs are on average doubled. Furthermore, we found that the timing of the control policy change primarily determines the tradeoff between backordering penalties and obsolescence costs.

Second, we provide the first practical formulas to tradeoff the risk of obsolescence and backordering specifically for expensive, slow moving items with high downtime costs. For this class of items, it is well known that the continuous review policies are preferred over periodic review ones since they require lower safety stocks for the same level of availability. Thus, our formulas can be used as a managerial guide in studying the impacts of advance policy change on operational costs and obsolete inventories.

Third, our model can be seen as the link between the two separate streams of inventory literature, the obsolescence models and the excess stock disposal models. The former does not include the continuous review models while the latter disregards the possibility of advance policy change.

The remainder of this paper is organized as follows: In section 2, we introduce the model and the transition control policy. In section 3, we give the expressions for the operating characteristics of the transient period and the objective function, and discuss their general behaviors. In section 4, we discuss the results of our numerical study. Finally, in section 5, we conclude and provide some future research paths. All proofs are provided in the online Appendix.

2. Model

We consider a single item, single location continuous review inventory system for slow moving items with nonstationary demand process and fixed lead times. It is assumed that the demand follows a Poisson process with rate λ_0 up to a pre-determined time point T after which the demand rate drops to a lower state λ_1 and stays there (*i.e.* $\lambda_0 > \lambda_1 \geq 0$). The inventory control policy is based on the $(S-1, S)$ policy which is commonly used for high cost low demand items (Hadley and Whitin 1963). According to this policy whenever a demand occurs a replenishment order is placed.

We denote the steady state optimal base stock levels for demand rates λ_0 and λ_1 with S_0 and S_1, respectively. They are calculated with the standard formulas given in Hadley and Whitin (1963). We assume that the shift in demand rate is downward (*i.e.* $S_0 > S_1 \geq 0$). In order to adapt to the new base stock level, we employ the following transition control policy based on the inventory position (the net inventory level plus the quantity on order):

Policy: Up to time $T-X$ a replenishment order of size one is placed whenever the inventory position drops to the reorder level S_0-1. After time $T-X$ a replenishment order of size one is placed whenever the inventory position drops to the reorder level S_1-1.

In other words, we use (S_0-1, S_0) policy until time $T-X$ and (S_1-1, S_1) policy thereafter. Observe that according to this control policy adaptation to the new base stock level is achieved by not giving N $(=S_0-S_1)$ consecutive orders starting at $X \geq 0$ time units earlier from time T. Hence, we let the demand take away N excess stocks starting from $T-X$. Our goal is to find the optimal time to initiate the excess stock removal process.

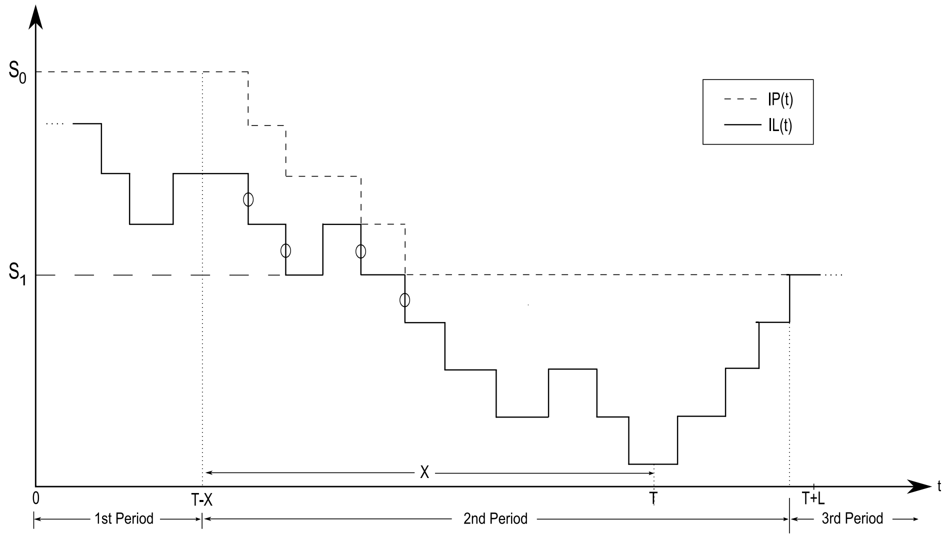
The rationale behind the proposed policy is that once the obsolescence date is known with certainty, early adaptation of base stock level should tradeoff the risk of backordering and obsolescence, and decrease the number of excess/obsolete stocks. We do not claim that the transition control policy is optimal. However, as we will demonstrate in our numerical experiments, it indeed leads to significant reduction on obsolescence costs compared to policy without an early adaptation $(X=0)$.

Figure 1 shows a possible realization of the net inventory level process $\{IL(t) : t \geq 0\}$ and the corresponding inventory position process $\{IP(t) : t \geq 0\}$. Note that the trajectories of these processes can be analyzed in three different periods. The first period starts at time zero and ends at time $T-X$. Since a replenishment order is placed upon each demand arrival the inventory position is fixed at S_0 during the first period. We assume that $T-X$ is long enough such that $IL(t)$ is in steady state. This is reasonable since life cycles of many products requiring parts replacements and service support are very long. For example, average useful life time of a commercial aircraft may last up to 30 years. Thus, the inventory system of a spare part supporting such product has enough time to reach to steady state before the obsolescence occurs.

The second period begins at time $T-X$ and the excess stocks are removed by not giving replenishment orders for N consecutive demands. Hence, the inventory position decreases by one at every demand arrival until it hits the target base stock level S_1. In Figure 1, examples of stock removal instances are marked by circles on the inventory level process. If the inventory position process hits S_1 before time T then the replenishment orders are placed again whenever a demand occurs. Thus, the end of the second period is the random time point greater or equal to T at which the inventory position is equal to S_1 and all outstanding orders given before time T have arrived (see Figure 1).

Note that, the second period is the transient period in which the inventory system adapts

Figure 1: Possible realization of $IL(t)$ and $IP(t)$ with stock removals



itself to the anticipated obsolescence. Since all orders given before time T are replenished before the second period ends, the third period can be seen as a separate inventory system with demand rate $\lambda_1 \geq 0$. If λ_1 is positive then we assume that the net inventory level process during the third period can be described by the stationary process. In many practical situations relocation of installed base or generation upgrades might result in such partial obsolescence situations where the demand is severely diminished but not necessarily vanished. In that case, the third period is similar to the first one but the system is controlled according to $(S_1 - 1, S_1)$ policy. Clearly, in case of full obsolescence ($\lambda_1 = 0$) there is no third period.

Our main goal is to find optimal X minimizing the total expected cost incurred in the second (transient) period. As we will demonstrate in the numerical section, the transient period costs are significant since they include the costs related with obsolescence. Unless a prior action is taken, partial obsolescence ($\lambda_1 > 0$) results in excess stock situations whereas full obsolescence ($\lambda_1 = 0$) results in obsolete stocks. As discussed earlier, for many slow movers the costs due to obsolescence are very high under both scenarios. Hence, in the sequel, we only focus on the analysis of the transient period since savings over obsolescence costs can be achieved only within this period.

Since fixed costs are irrelevant for optimization under one-for-one replenishment policy, we only consider holding and backordering costs incurred per unit per time, denoted by h and π respectively. In addition to that the unit obsolescence/relocation cost c_o is incurred per remaining on hand inventory after time T if full obsolescence occurs ($\lambda_1 = 0$).

In the next section, we explain the transient analysis of the net inventory level process, give the expressions for the operating characteristics of the second period and state the optimization problem.

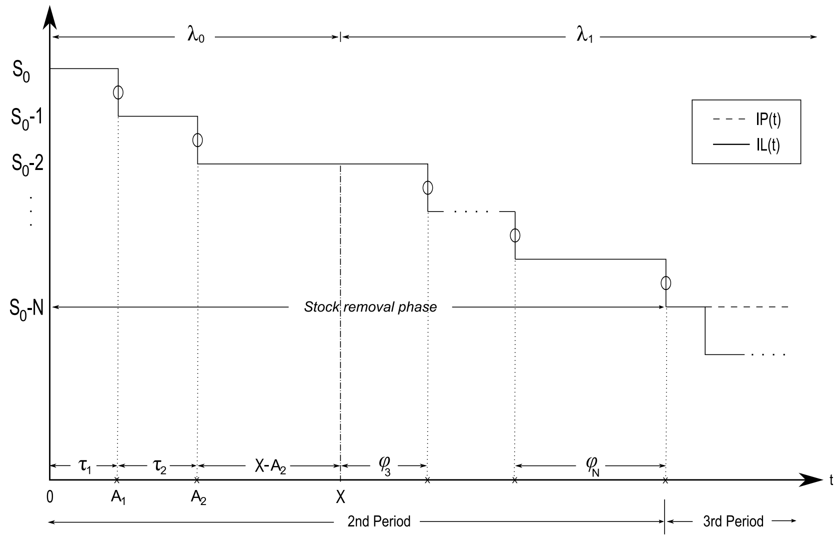
3. Operating Characteristics of The Second Period

Our model differs from the standard inventory models due to removal of excess stocks and nonhomogenous demand process. These differences necessitate the transient analysis of the net inventory level process. Unfortunately, outstanding orders before time $T-X$ complicate the analysis beyond tractability. Since the complication results from outstanding orders, conditioning on the net inventory level at time $T-X$ or its expectation does not yield closed form expressions for the operating characteristics. In order to overcome this analytical difficulty and provide good approximations for operating characteristics that can be calculated easily, we assume that the net inventory level is equal to S_0 at time $T-X$. We can justify this assumption by appealing to the characteristics of the problem. For slow movers, the base stock levels are usually not very high due to low demand rates and high opportunity costs. On the other hand, due to high backordering penalties the net inventory level process mostly stays in the positive half-plane. Therefore, the average net inventory level at any time is not very far from S_0. Indeed, for all the instances used in our numerical experiments, which are generated to reflect real life scenarios, the average S_0 is found to be 3.24 with maximum of 10. For the same instances, the average difference between S_0 and $E(IL(T-X))$ is found to be 1.1 with maximum of 5. Consequently, we observed that our approximate model performs quite satisfactorily compared to simulation. The effects of our assumptions will be discussed in more detail under section 3.4.1. Moreover, as we will demonstrate in the numerical section, the optimal X found by using our approximate formulas yields near optimal expected total costs. Hence, we conclude that this assumption does not change the main implications of our study.

The analysis of the net inventory level process is independent of the time axis due to Poisson demand arrivals. Therefore, in the sequel, we shift the beginning of the second period from $T-X$ to 0 for the sake of clarity. Let $\tau_i,\ i=1,\ldots,N$ denote the interarrival times between not replenished demand instances when the arrival rate is λ_0. We refer to $A_k := \sum_{i=1}^{k} \tau_i$ as the arrival time of the kth demand before the drop occurs.

Figure 2 shows a realization in which the new base stock level S_1 $(= S_0 - N)$ is hit by

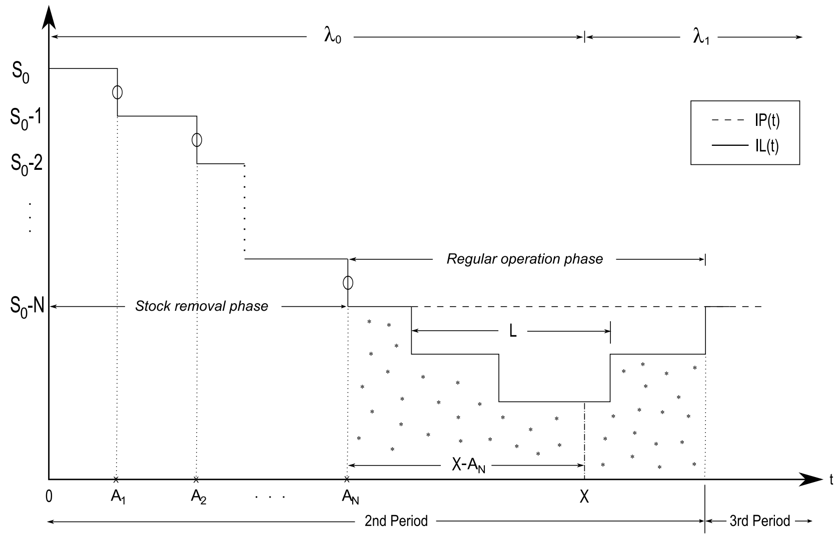
Figure 2: Possible realization of $IL(t)$ and $IP(t)$ during 2nd period ($A_N > X$)



the net inventory level process after the drop in demand rate occurs at time X. Observe that, as a result of our assumption about the outstanding orders ($IL(T - X) = S_0$), the net inventory level process is tantamount to the inventory position process until the $N+1$st demand arrives. In the figure, $\varphi_j,\ j = 3, \ldots, N$ denote the interarrival times between not replenished demands arriving after time X. Hence, φ_js are exponentially distributed with mean λ_1. Note that, in Figure 2, the second period ends immediately after the arrival of the Nth demand since the inventory position is equal to S_1 and there are no outstanding orders before time X. On the other hand, if S_1 is reached before time X then replenishment orders are placed again for every demand arriving thereafter. A realization of this scenario can be seen in Figure 3. Observe that, in Figure 3, the second period ends at the moment the last order given between A_N and X is replenished.

From Figure 2 and Figure 3, it is clear that the net inventory level process in the second period can be analyzed in two different phases. The first one is the *stock removal phase.* This is the time period in which the excess stocks are taken away by the demand. Thus, the stock removal phase starts at the beginning of the second period and ends when the Nth demand arrives. The second one is the *regular operation phase.* This is the time period in which the replenishment orders are placed again upon every demand arrival since all of the excess stocks are removed before the drop in demand rate occurs. Hence, the regular operation phase starts at A_N and ends when the second period ends (see Figure 3). Note that the regular operation phase of the second period exists if and only if the Nth excess

Figure 3: Possible realization of $IL(t)$ and $IP(t)$ during 2nd period ($A_N \leq X$)



stock is removed before time X ($i.e.$ $A_N \leq X$).

We begin our analysis with the calculation of the expected total inventory carried during the second period denoted by $E[OH]$. Observe that the random variable OH depends on the arrival time of the kth demand during the stock removal phase, and therefore it can be calculated by conditioning on $A_k,\ k = 1, \ldots, N$. If the arrival time of the first demand A_1 is greater than X then the second period ends at the moment Nth demand arrives. Thus, OH is equal to the inventory carried until time X ($= S_0X$) plus another random variable $OH_1^{'}$ ($= \sum_{i=1}^{N}(S_0 - i + 1)\varphi_i$) representing the inventory carried from time X until the second period ends. Note that if the stock removal phase extends after time X then the trajectory of $IL(t)$ should be analyzed separately for the periods before and after time X due to different demand rates. Hence, the need for an additional random variable $OH_1^{'}$. On the other hand, if A_1 is less than or equal to X then OH is equal to the inventory carried until the first demand arrives ($= S_0\tau_1$) plus another random variable OH_2. Essentially, OH_2 is similar to OH but it depends on A_2 and the new inventory level $S_0 - 1$. Put more formally,

$$OH \quad = \quad \begin{cases} S_0X + OH_1^{'} & \text{if} \quad A_1 > X \\ S_0\tau_1 + OH_2 & \text{if} \quad A_1 \leq X \end{cases} \tag{1}$$

If we continue in this fashion for $k = 2, 3, \ldots, N$ when $N \geq 2$ then we come up with the following recursive equations to calculate the total inventory carried during the second period:

$$OH_k \quad = \quad \begin{cases} (S_0 - k + 1)\,(X - A_{k-1}) + OH_k^{'} & \text{if} \quad A_{k-1} \leq X,\ A_k > X \\ (S_0 - k + 1)\tau_k + OH_{k+1} & \text{if} \quad A_k \leq X \\ \quad 0 & \text{o.w.} \end{cases} \tag{2}$$

where

$$OH_k' \quad = \quad \sum_{i=k}^{N}(S_0 - i + 1)\varphi_i, \quad k = 1, \ldots, N. \tag{3}$$

represents the inventory carried from time X until the end of the stock removal phase when $N - k + 1$ stocks are yet to be removed.

The recursive structure of equations (1) and (2) gives the positive area under the net inventory level process depending on whether the kth excess stock is removed before time X or not. For example, if all excess stock is not removed before time X (i.e. $A_k > X$ for some k) then equations (1) and (2) give the area under a similar scenario depicted in Figure 2. Otherwise, they give the area similar to the one shown in Figure 3.

Note that the equations (1)-(3) mainly generate the expressions for the total inventory carried during the stock removal phase. Since no orders are given in this phase, the equations are independent of the lead time. The total inventory carried in the regular operation phase is represented implicitly in those equations with the random variable OH_{N+1}. The shaded region in Figure 3 shows a possible realization of OH_{N+1}. We will analyze the regular operation phase in detail in the sequel.

Let $p(n; \lambda) = e^{-\lambda}\lambda^n/n!$, $n = 0, 1, 2, \ldots$ be the pdf of Poisson distribution with parameter $\lambda \geq 0$ and denote its cdf with $P(n; \lambda)$. Also, let $\mathbf{1}(\cdot)$ denote the indicator function. Taking expectations of (1) and (2), and exploiting the recurrence structure, we find $E[OH]$ as follows:

$$E[OH] \quad = \quad \mathcal{F}(X) + E[OH_{N+1}\mathbf{1}(A_N \leq X)] \tag{4}$$

where

$$\mathcal{F}(X) := \lambda_0^{-1} N \left[S_0 - \frac{N-1}{2} \right] + \frac{\lambda_0 - \lambda_1}{\lambda_0 \lambda_1} \sum_{i=0}^{N-1} (S_0 - i) P(i; \lambda_0 X), \quad \lambda_1 > 0 \tag{5}$$

In equation (4), $\mathcal{F}(X)$ represents the expected inventory carried during the stock removal phase whereas $E[OH_{N+1}\mathbf{1}(A_N \leq X)]$ is the expected inventory carried during the regular operation phase. Note that, OH_{N+1} exists only if the new base stock level is reached before X (*i.e.* $A_N \leq X$).

So far we derived the closed form expressions only for the expected inventory carried during the stock removal phase. In the sequel, we provide an exact transient analysis of the inventory level process during the regular operation phase and derive the expressions for the operating characteristics of this phase.

3.1 Analysis of the Regular Operation Phase

We want to compute the expected on hand carried and the expected time weighted backorders incurred in the regular phase which starts at time $A_N (\leq X)$ and lasts until the end of the second period. To compute these operating characteristics we represent the inventory level process in terms of the demand process. Since we are only interested in the time period after A_N, in the sequel, we shift the time axis from A_N to 0 for clarity. Hence, the drop in demand rate occurs at $X - A_N$ time units after the regular operation phase begins (see Figure 3). Thus, for $t \geq 0$ the inventory level $IL(t)$ conditional on A_N can be given as:

$$IL(t)|A_N = \begin{cases} S_1 - D(t) & \text{if} \quad t \leq L \\ S_1 - (D(t) - D(t-L)) & \text{if} \quad t > L \end{cases} \tag{6}$$

where $\{D(t) : t \geq 0\}$ is a nonhomogenous Poisson process with intensity function $\Lambda(t)$: $[0, \infty) \to [0, \infty)$ given by

$$\Lambda(t) = \int_0^t \lambda(z)dz \tag{7}$$

with arrival rate

$$\lambda(z) = \begin{cases} \lambda_0 & \text{if} \quad z \leq X - A_N \\ \lambda_1 & \text{if} \quad z > X - A_N \end{cases} \tag{8}$$

Substituting (8) in (7) yields

$$\Lambda(t) = \begin{cases} \lambda_0 t & \text{if} \quad t \leq X - A_N \\ (\lambda_0 - \lambda_1)(X - A_N) + \lambda_1 t & \text{if} \quad t > X - A_N \end{cases} \tag{9}$$

Equation (6) is the representation of the net inventory level at any time point based on the demand up to time t, the lead time demand and inventory position. Recall that the inventory position remains constant at the level S_1 during the regular operation phase since an order is placed each time there is a demand. Therefore, if $t \leq L$ then $IL(t)$ is equal to the inventory position minus the total demand up to time t. Whereas, if $t > L$ then $IL(t)$ is equal to the inventory position minus the lead time demand.

The end of the regular operation phase is a random time point depending on the inventory level at time X. For example, if the net inventory level at time X is equal to $S_0 - N$ then there are no outstanding orders and the regular operation phase ends. Otherwise, it ends when all outstanding orders given between $X - L$ and X are replenished up to time $X + L$. However, dealing with the random end time complicates the analysis beyond tractability. Hence, we assume that the regular operation phase always ends at time $X + L$. This approximation simply results in the overestimation of the expected total cost due to extended calculation period but does not change the optimal X drastically since the shift in the expected total cost is mainly upwards.

We start with the computation of $E[OH_{N+1}\mathbf{1}(A_N \le X)]$ by conditioning on A_N such that,

$$E[OH_{N+1}\mathbf{1}(A_N \le X)] = \int_0^X E[OH_{N+1}\mathbf{1}(A_N \le X)|A_N = s] f_{A_N}(s)\, ds \tag{10}$$

where

$$f_{A_N}(s) = \frac{\lambda_0 e^{-\lambda_0 s}(\lambda_0 s)^{N-1}}{(N-1)!}, \qquad s \ge 0.$$

is the pdf of the Erlang distribution with parameters N and λ_0.

We are interested in expected on hand carried from time A_N until $X + L$. Since the time axis is shifted, the expected inventory carried during this period is the positive area under the expected trajectory of the net inventory level process from 0 to $X - A_N + L$. Thus, for a given A_N this area can be computed as follows:

$$E[OH_{N+1}\mathbf{1}(A_N \le X)|A_N = s] = \int_0^{X-s+L} E\left[(IL(t))^+|A_N = s\right] dt \tag{11}$$

From (6),

$$\begin{aligned} E\left[(IL(t))^+|A_N = s\right] &= \sum_{n=0}^{S_1-1}(S_1 - n)P(D(t) = n)\mathbf{1}(t \le L) \\ &+ \sum_{n=0}^{S_1-1}(S_1 - n)P(D(t) - D(t-L) = n)\mathbf{1}(t > L) \end{aligned} \tag{12}$$

Substituting (12) in (11) yields

$$\begin{aligned} E[OH_{N+1}\mathbf{1}(A_N \le X)|A_N = s] &= \sum_{n=0}^{S_1-1}(S_1 - n)\left[\int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!}\, dt \right. \\ &+ \left. \int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t) - \Lambda(t-L))^n}{n!}\, dt\right] \end{aligned} \tag{13}$$

and using the result in (10) gives that,

$$\begin{aligned} E[OH_{N+1}\mathbf{1}(A_N \le X)] &= \sum_{n=0}^{S_1-1}(S_1 - n)\int_0^X \left[\int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!}\, dt \right. \\ &+ \left. \int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t) - \Lambda(t-L))^n}{n!}\, dt\right] f_{A_N}(s)\, ds \end{aligned} \tag{14}$$

We define the following functions,

$$b_N(r;n,\rho) := \binom{r+n-1}{n-1}\rho^n(1-\rho)^r$$

and

$$\xi(r,n) := np(r+n;\lambda_0 X)\binom{r+n}{n}\sum_{k=0}^{r}\binom{r}{k}\frac{(-1)^k}{n+k}\left(\frac{X-L}{X}\right)^{n+k}$$

where $r \in \{0,1,2,...\}$, $n \in \{1,2,...\}$ and $\rho \in \mathbb{R}$. Moreover, we let $\bar{P}(n;\lambda) := 1 - P(n-1;\lambda)$ denote the complementary cdf of Poisson distribution.

The integrals in equation (14) can be calculated with respect to the relationship between X and L. Thus, for $\lambda_1 > 0$, the expected on-hand inventory carried during the regular operation phase is found as follows:

$$E[OH_{N+1}\mathbf{1}(A_N \le X)] \;=\; \begin{cases} \sum_{n=0}^{S_1-1}(S_1-n)\left[f(n)+g_1(n)\right] & \text{if} \quad L \le X \\ \sum_{n=0}^{S_1-1}(S_1-n)\left[f(n)-g_2(n)\right] & \text{if} \quad L > X \end{cases} \tag{15}$$

where

$$f(n) \;=\; \left[\frac{1}{\lambda_0}+\frac{P(n;\lambda_1 L)}{\lambda_0-\lambda_1}\right]\bar{P}(N;\lambda_0 X)+\frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1}\left[P(N+n;\lambda_0 X)-P(N-1;\lambda_0 X)\right] \tag{16}$$

$$\begin{aligned} g_1(n) \;=\;& p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L))-\lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right] \\ &- \frac{2\lambda_0-\lambda_1}{\lambda_0(\lambda_0-\lambda_1)}P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L))-\frac{(\lambda_0-\lambda_1)}{\lambda_0\lambda_1}\sum_{i=0}^{n}\xi(i,N) \\ &- \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\sum_{k=0}^{i}p\left(i-k;\lambda_1 L+(\lambda_0-\lambda_1)X\right)b_N\!\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\delta(k) \end{aligned} \tag{17}$$

with $\delta(k) = P(N+k-1;\lambda_1(X-L)) - P(N+k-1;\lambda_1 X)$.

$$g_2(n) \;=\; \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\sum_{k=0}^{i}p(i-k;\lambda_1 L+(\lambda_0-\lambda_1)X)b_N\!\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\bar{P}(N+k;\lambda_1 X) \tag{18}$$

The expected time weighted backorders incurred during the regular operation phase can be calculated essentially the same way as described above. Hence, we skip the analysis for brevity and directly give the result:

$$E[BO] \;=\; \begin{cases} \sum_{n=S_1}^{\infty}(n-S_1)\left[f(n)+g_1(n)\right] & \text{if} \quad L \le X \\ \sum_{n=S_1}^{\infty}(n-S_1)\left[f(n)-g_2(n)\right] & \text{if} \quad L > X \end{cases} \tag{19}$$

3.2 Objective Function

We can now obtain the expected total cost incurred in the second period by using the operating characteristics derived above. The general structure of the expected total cost incurred in the second period can be given as follows:

$$TC(X) = hE[OH] + \pi E[BO] \tag{20}$$

Using equations (4), (15) and (19) in (20) and defining,

$$c(x) := \begin{cases} hx & \text{if} \quad x > 0 \\ -\pi x & \text{if} \quad x \leq 0 \end{cases} \tag{21}$$

we obtain that,

$$TC(X) \quad = \quad h\mathcal{F}(X) + \begin{cases} \sum_{n=0}^{\infty} c(S_1 - n)\,[f(n) + g_1(n)] & \text{if} \quad L \leq X \\ \sum_{n=0}^{\infty} c(S_1 - n)\,[f(n) - g_2(n)] & \text{if} \quad L > X \end{cases} \tag{22}$$

Our goal is to find the optimal time for policy change that minimizes the expected total cost incurred during the second period. Thus, the optimization problem can be stated as,

$$\min_{X \geq 0} TC(X) \tag{23}$$

Despite the complicated appearance of equation (22) the optimal solution to problem (23) can be found easily. This is because the equations (16)-(18) are mainly composed of elementary probability functions and some combinatorial expressions. For the dimensions that we are interested in all of the functions can be calculated easily with a general purpose programming language. Besides, as we will discuss in more detail in section 3.4, $TC(X)$ is observed to be unimodal in X. Hence, X^* can be searched very efficiently with standard nonlinear optimization methods.

3.3 Full Obsolescence Case ($\lambda_1 = 0$)

So far we have considered an inventory system facing obsolescence in which the demand drops to a lower level but does not vanish ($\lambda_1 > 0$). However, in some practical cases the demand might disappear after a certain time point and the remaining stocks are either sold in secondary markets or sent to locations where the demand is still healthy. Although the analysis of the net inventory level process for full obsolescence case is essentially the same as described in the previous section, the operating characteristics and the objective function have to be slightly modified.

When $\lambda_1 = 0$ the number of excess stocks to be removed is equal to S_0, and therefore the inventory is carried only during the stock removal phase. Hence, the term $E[OH_{N+1}\mathbf{1}(A_N \leq X)]$ drops from the equation (4). Similarly, in equation (5) the term representing the expected inventory carried after the drop $(=\lambda_1^{-1}\sum_{i=0}^{N-1}(S_0 - i)P(i;\lambda_0 X))$ becomes irrelevant since under full obsolescence the stock removal can only be possible before time X. Thus, the expected total inventory carried during the second period can be given as:

$$E[OH] = \lambda_0^{-1}\left[\frac{S_0(S_0+1)}{2} - \sum_{i=0}^{S_0-1}(S_0 - i)P(i;\lambda_0 X)\right] \tag{24}$$

If full obsolescence occurs before all of the excess stocks are removed then the remaining on hand inventory is usually salvaged (disposed) or relocated. In that case the obsolescence cost c_o is incurred per unit of remaining inventory at the end of the second period. In case of salvaging c_o can be interpreted as the overage cost of the well known newsboy problem. Otherwise, it can be seen as the cost of transporting per unit of remaining inventory to a location where the demand is healthier. Since S_0 items should be removed before time X the expected number of remaining stock at the end of the second period can be given by the following expression:

$$E[RS] = \sum_{i=0}^{S_0-1}(S_0 - i)p(i;\lambda_0 X) \tag{25}$$

where $p(i;\lambda_0 X)$ is the probability that i items are demanded from the beginning of the second period until the obsolescence occurs. Note that $E[RS]$ is not affected by our assumption that there are no outstanding orders at the beginning of the second period since the number of stocks removed before time X only depends on the demand arrival process but not the net inventory level process. Moreover, it can be easily shown that $E[RS]$ is convex in X.

The analysis of the regular operation phase is similar to the one with positive λ_1. However, under full obsolescence there are no inventory carried during the regular operation phase since the base stock level is zero. Thus, the expression for the expected time weighted backorders incurred during this phase is found as:

$$E[BO] = \begin{cases} \sum_{n=0}^{\infty} n\,[f(n)+g(n)] & \text{if} \quad L \leq X \\ \sum_{n=0}^{\infty} n f(n) & \text{if} \quad L > X \end{cases} \tag{26}$$

where

$$f(n) = \lambda_0^{-1}\left[2\bar{P}(N+n+1;\lambda_0 X) + Np(N+n+1;\lambda_0 X)\right]$$

$$- (X-L)p(N+n;\lambda_0 X) \tag{27}$$

$$g(n) = p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L)) - \lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right]$$

$$- 2\lambda_0^{-1}\left[P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L)) - \sum_{i=0}^{n}\xi(i,N)\right]$$

$$+ (X-L)\xi(n,N) - \lambda_0^{-1}N\xi(n,N+1) \tag{28}$$

Therefore, the expected total cost incurred during the second period under full obsolescence can be given as,

$$TC(X) = hE[OH] + c_o E[RS] + \pi E[BO] \tag{29}$$

In our numerical experiments, we observed that equation (29) is unimodal in X. Hence, the optimal solution of $TC(X)$ can be found very easily for the full obsolescence case as well.

3.4 General Behavior of Objective Function and Operating Characteristics

In this section, we investigate the general behavior of the objective function and the operating characteristics. For comparison purposes we conducted 5000 simulations of the demand arrival process for given λ_0 and λ_1 pair. Then for a given X value, the operating characteristics and the objective function are found by averaging the values calculated at each of the simulated trajectories. In the sequel, we use subscript 's' to denote the simulated values for clarity. Figures 4 - 5 illustrate the general behavior of the objective function and the operating characteristics. In the figures simulated values are given along with their 95% confidence intervals.

Throughout our numerical study we observed that the expected total cost function is unimodal in X (see Figure 4). The intuition behind this behavior can be explained as follows: If X is too short then the inventory system does not have enough time to remove all of the excess stocks (N) before the drop in demand rate occurs. Therefore, the remaining excess stocks either increase the holding costs since the natural attrition of these stocks takes longer due to diminished demand or they result in obsolescence cost -in case of full obsolescence- due to disposal or relocation. In both cases the system incurs extra holding cost or obsolescence cost for not removing all of the excess stocks before the drop. Hence, we observe a decrease in expected total cost function as X diverges from zero.

On the other hand, if X is too long then all of the excess stocks are removed too early and the inventory system returns to its regular operation mode before the drop in demand rate occurs. Consequently, the system operates under a lower base stock level S_1 in order to satisfy the demand until the drop occurs and incurs more backordering costs. Figure 5b shows how the expected backorders increase in X. Therefore, there exists an optimal X value balancing the obsolescence related costs (extra holding cost, obsolescence/relocation cost) with the cost of backordering.

Figure 4: Behavior of Objective Functions ($\lambda_0 = 10,\ \lambda_1 = 2,\ L = 0.15,\ \pi = 10,\ N = 2$)

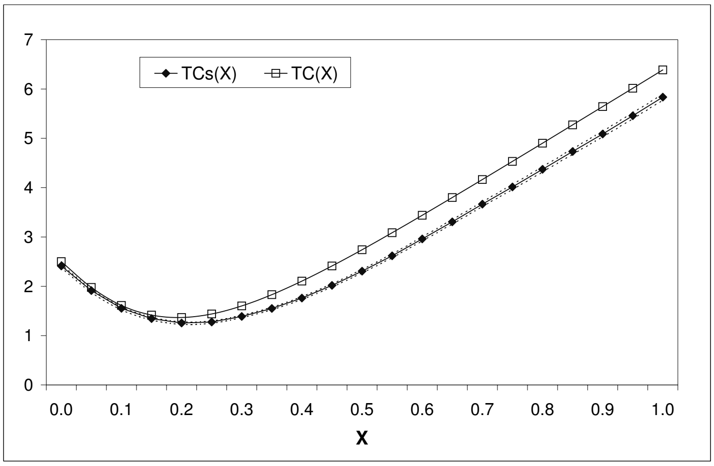
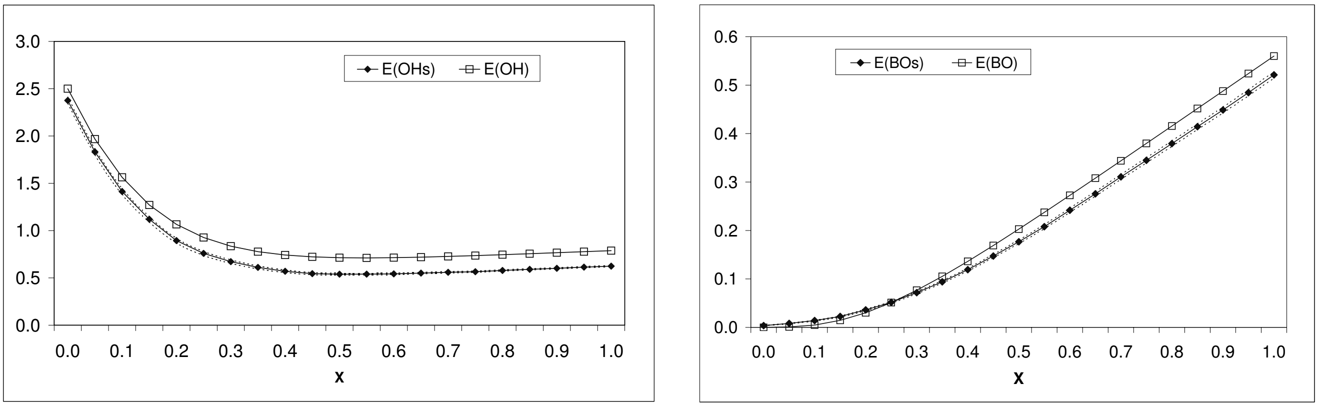


Figure 5a presents an example of the rapid decrease in the expected on hand as X diverges from zero when λ_1 is positive. For the full obsolescence case, however, the behavior of the expected on hand is different. The inventory is carried only in the stock removal phase and for small X values it usually ends before all of the excess stocks are removed. Therefore, when $\lambda_1 = 0$ the expected on hand is generally increasing in X until it converges to a constant (the expected positive area under the net inventory level process when the stock removal ends before the drop occurs). Although the inventory system tends to carry less stock as X decreases, the expected total cost keeps on increasing due to the increase in the expected number of remaining stocks.

Figure 5: Behavior of Operating Characteristics ($\lambda_0 = 10,\ \lambda_1 = 2,\ L = 0.15,\ \pi = 10, N = 2$)

(a) Expected On-Hand Carried (b) Expected Backorder



3.4.1 Comparison with Simulation

Our two assumptions about the initial inventory level and the end of the second period result in different sample paths of the net inventory level process for our model and simulation in periods $[T-X, T-X+L]$ and $[T, T+L]$. When λ_1 is positive $E\,[OH]$ always overestimates $E\,[OH_s]$ due to higher on hand inventory level between $T-X$ and $T-X+L$, and the extended calculation period. This can be observed in Figure 5a. On the other hand, when $\lambda_1 = 0$, $E\,[OH_s]$ is larger for X values near zero since the outstanding orders at time $T-X$ are likely to arrive after time T and therefore, in simulations the second period is likely to be longer compared to our model.

When λ_1 is positive expected backorders are underestimated by $E\,[BO]$ as long as the initial inventory level S_0 is high enough to cover the demand before time T. However, as X gets larger, the system returns to its regular operation mode earlier and $E\,[BO]$ begins to overestimate $E\,[BO_s]$ due to the extended calculation period. For example, in Figure 5b, we observe that $E\,[BO]$ starts to overestimate $E\,[BO_s]$ for the X values greater than 0.25. Furthermore, we found that the performance of $E\,[BO]$ is much better for the full obsolescence case. Because when the stock removal phase ends before T, the sample path differences between simulation and our model are only from $T-X$ until $T-X+L$.

For positive λ_1 and X values small enough we observe that the percent difference between $TC(X)$ and $TC_s(X)$ is relatively low since the overestimation of $E\,[OH_s]$ is compensated by the underestimation of $E\,[BO_s]$. Moreover, $TC(X)$ underestimates $TC_s(X)$ as long as the real backordering cost is larger than the overestimated quantity in holding cost (*i.e.* $\pi E\,[BO_s] > h[E\,[OH] - E\,[OH_s]]$). Otherwise, $TC(X)$ is larger than $TC_s(X)$ as a result of overestimation in holding costs. Similar intuitive results are observed for the full obsolescence case as well. Finally, for 256 experiment instances, we found that the average absolute percent difference between $TC(X)$ and $TC_s(X)$ is approximately 11% for positive λ_1 while it is only 1.25% when $\lambda_1 = 0$ as a result of increased accuracy in $E[BO]$ and the exact calculation of $E[RS]$.

4. Numerical Study

In this section, we first investigate the changes in optimal policy parameter and expected total cost function under different parameter sets. Then, we identify the performance of our model and its impact on expected total costs by comparing it with simulation optimization. Finally, we close the section with a discussion about the value of advance policy change. In the sequel, we use '$*$' to indicate optimality and denote the optimal X value found by simulation optimization with X_s^*.

Throughout the numerical study we assume that simulation is representative of underlying real world model. Thus, we compare $TC_s(X^*)$ with $TC_s^*(X_s^*)$ to measure the impact of operating under X^*. As a simulation optimization technique, we employ response surface methodology as described in Myers and Montgomery (1995).

The experiment instances used in our numerical study is generated with the following parameter set: $\lambda_0 \in \{0.5, 0.7, 1, 5, 7, 10\}$ per year, $\lambda_1 \in \{0, 0.2, 2\}$ per year, $h = 1$ per unit per year, $\pi \in \{5, 15, 25, 50, 75, 150, 300\}$ per unit per year, $c_o \in \{5, 10\}$ per unit, $L \in \{0.15, 0.25, 0.50, 0.75, 1\}$ years. In total, we generate 281 instances for which the average number of excess stocks to be removed is approximately 3 units. Some of the results from the numerical study are tabulated in Table 1-2.

4.1 General Behavior of Optimal Policy Parameters and Total Cost Functions

As can be seen from Table 1, when λ_1 is positive, we do not always observe a monotonic behavior in optimal X values and expected total costs due to discrete jumps in S_0 or S_1 as L or π increases. However, when all other parameters are constant if an increase in L or π does not effect S_0 and S_1 then the optimal X decreases to reduce the risk of backordering.

For the full obsolescence case, we observe similar non-monotonic behavior in optimal values with respect to the changes in L or π. However, optimal X and $TC_s(\cdot)$ are monoton-

Table 1: Performance of X^* and Value of Advance Policy Change When $\lambda_1 > 0$

$h = 1$

λ_0	λ_1	π	L	S_0	N	X_s^*	X^*	$\Delta_x\%$	$TC_s^*(X_s^*)$	$TC_s(X^*)$	$\Delta_c\%$	$TC_s(0)$	$\Delta_o\%$
0.5	0.2	50	0.50	2	1	1.80	1.64	-8.94	7.66	7.73	0.88	10.17	31.61
			0.75	2	1	1.29	1.11	-13.77	8.35	8.51	1.89	10.19	19.73
			1.00	2	1	0.96	0.80	-16.60	9.06	9.25	2.07	10.33	11.69
		300	0.50	2	0	-	-	-	-	-	-	-	-
			0.75	3	1	1.65	1.45	-11.88	11.73	11.80	0.65	15.22	28.96
			1.00	3	1	1.24	1.05	-15.43	12.52	12.62	0.81	15.28	21.10
1	0.2	50	0.50	2	1	1.02	0.87	-15.02	6.88	6.88	0.01	10.07	46.34
			0.75	3	2	1.71	1.58	-7.24	14.47	14.53	0.40	24.88	71.21
			1.00	3	2	1.40	1.31	-6.07	16.22	16.41	1.12	24.94	52.03
		300	0.50	3	1	1.16	1.06	-8.32	9.20	9.24	0.42	15.08	63.23
			0.75	4	2	1.74	1.65	-4.78	19.86	20.12	1.28	34.84	73.17
			1.00	5	3	2.26	2.19	-2.98	32.32	32.77	1.40	59.46	81.45
5	2	5	0.05	1	1	0.18	0.15	-13.36	0.39	0.40	1.60	0.50	24.38
			0.15	2	1	0.24	0.19	-20.66	0.70	0.71	0.74	0.96	35.62
			0.25	2	1	0.16	0.12	-23.52	0.84	0.86	2.11	0.98	13.95
		50	0.05	2	1	0.19	0.16	-13.33	0.78	0.78	1.25	0.99	26.33
			0.15	3	1	0.18	0.14	-23.67	1.15	1.19	3.42	1.48	23.79
			0.25	4	2	0.29	0.25	-14.17	2.51	2.55	1.75	3.39	32.75
10	2	5	0.05	1	1	0.12	0.09	-21.22	0.34	0.34	0.19	0.51	47.96
			0.15	3	2	0.29	0.25	-14.01	1.02	1.04	1.30	2.42	133.47
			0.25	4	3	0.35	0.30	-13.07	1.81	1.82	0.27	4.28	135.56
		50	0.05	2	1	0.11	0.09	-20.35	0.68	0.69	1.69	1.02	46.98
			0.15	4	2	0.21	0.18	-12.23	1.88	1.89	0.51	3.44	82.05
			0.25	6	4	0.33	0.31	-6.55	4.27	4.33	1.22	8.79	103.16

Table 2: Performance of X^* and Value of Advance Policy Change When $\lambda_1 = 0$

$L = 0.25,\ h = 1$

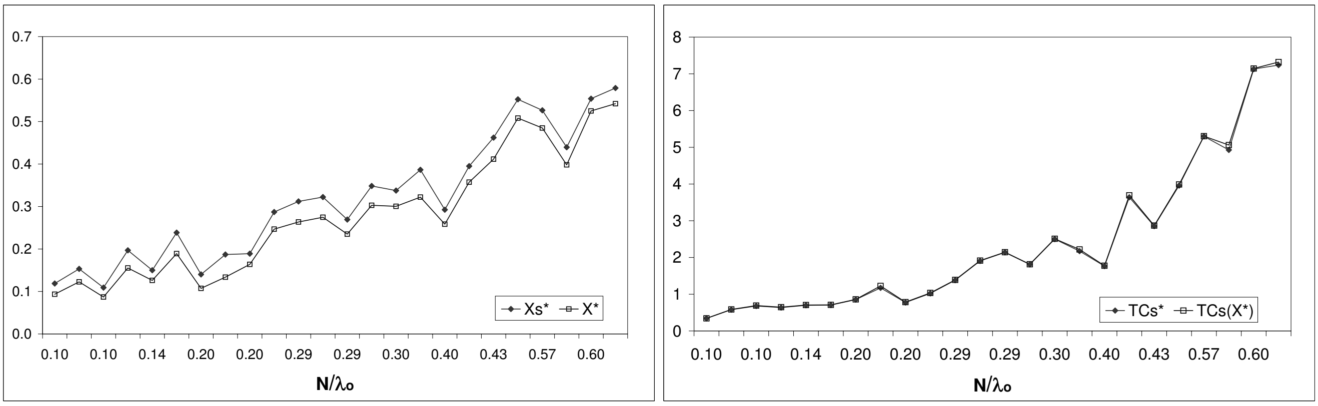
λ_0	π	c_o	N	X_s^*	X^*	$\Delta_x\%$	$TC_s^*(X_s^*)$	$TC_s(X^*)$	$\Delta_c\%$	$TC_s(0)$	$\Delta_o\%$
0.5	50	5	1	0.44	0.43	-2.34	4.73	4.74	0.16	5.03	6.17
		10	1	1.00	0.99	-1.25	8.22	8.22	0.02	10.03	21.99
	300	5	2	0.36	0.40	9.40	9.88	9.89	0.11	10.02	1.35
		10	2	0.93	0.94	1.27	18.11	18.14	0.16	20.02	10.36
1	50	5	2	0.86	0.87	1.19	8.26	8.28	0.28	10.03	21.16
		10	2	1.40	1.40	-0.51	13.31	13.31	0.04	20.03	50.50
	300	5	2	0.29	0.32	11.00	9.44	9.48	0.44	10.06	6.17
		10	2	0.50	0.53	6.05	17.40	17.40	0.03	20.06	15.30
5	5	5	2	0.59	0.59	-0.78	3.37	3.38	0.45	10.14	199.65
		10	2	0.75	0.75	-0.65	4.32	4.34	0.40	20.14	363.86
	50	5	4	0.52	0.52	0.33	11.77	11.82	0.47	20.30	71.73
		10	4	0.67	0.67	0.07	18.33	18.35	0.12	40.30	119.65
10	5	5	4	0.56	0.55	-1.99	4.74	4.74	0.15	20.38	329.45
		10	4	0.66	0.65	-1.79	5.88	5.90	0.36	40.38	584.32
	50	5	6	0.43	0.44	1.09	14.74	14.76	0.12	30.68	107.88
		10	6	0.53	0.53	-0.14	22.75	22.79	0.19	60.68	166.20

ically increasing in c_o since N is independent of the obsolescence cost. Thus, as c_o increases optimal X values also increase to reduce the number of remaining stocks and the expected total costs increase as a result of higher obsolescence penalty (see Table 2).

An important indicant for the behavior of the optimal X is the ratio N/λ_0, average time needed to remove Nth excess stock before the drop occurs. In general, we observe that optimal X values and corresponding expected total costs are increasing in N/λ_0. This is to be expected since as the ratio increases more time is needed to complete the stock removal process before the drop occurs. Hence, the system adjusts itself accordingly. On the other hand, the increase in optimal values is not monotonic. This is because the ratio is only a measure of the stock removal process but not the regular operation phase. In other words, the inventory system might incur backordering cost once the stock removal is completed. Hence, the optimal values are not monotonically increasing in N/λ_0. Figure 6 illustrates the change in optimal values as N/λ_0 increases for the instances with λ_0 varying from 5 to 10 and N varying from 1 to 6.

Figure 6: Change in Optimal Values wrt N/λ_0

(a) Optimal X Values (b) Optimal Expected Total Costs



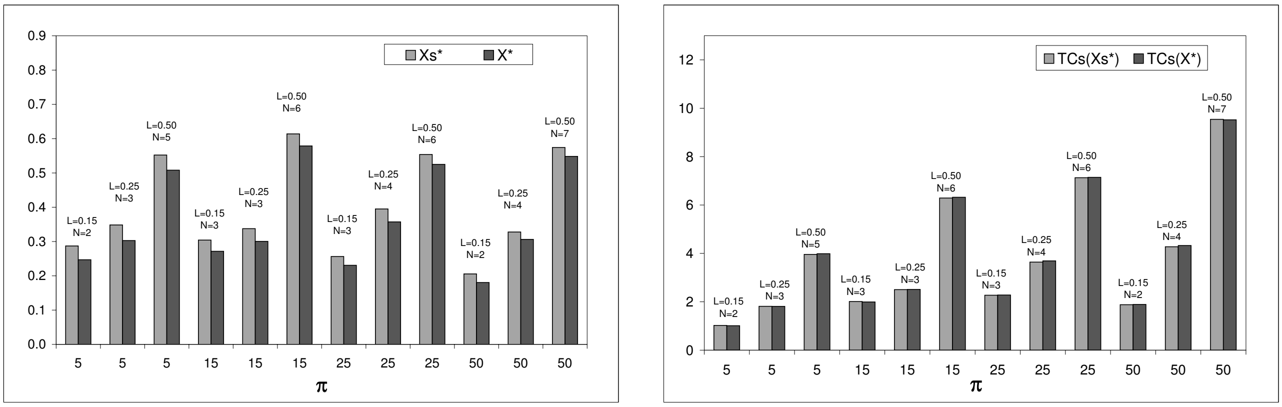
4.2 Overall Performance of X^*

Next, we compare the performance of X^* *vis à vis* X_s^*. For comparison purposes we use percent error which gives the percentage deviation from the optimal values found by simulation optimization. Hence, we define $\Delta_x\% = \frac{X^*-X_s^*}{X_s^*} \times 100$ as the percent deviation from X_s^* whereas $\Delta_c\% = \frac{TC_s(X^*)-TC_s^*(X_s^*)}{TC_s^*(X_s^*)} \times 100$ is defined as the percent deviation from the optimal expected total cost $TC_s^*(X_s^*)$ as a result of using X^* instead of X_s^*. Figure 7 illustrates a comparison of optimal X values and corresponding expected total costs.

We observed that the expected total cost is quite robust to the changes in X_s^*. For example, for the instances considered in Figure 7 we found that X^* underestimates X_s^* on average by 9.21%. For the same instances, however, the average deviation from the optimal expected total cost is only 0.67%. This robust behavior of the expected total cost function can be seen more clearly in Figure 6. Moreover, we found that X^* might underestimate or

Figure 7: Performance of X^* ($\lambda_0 = 10,\ \lambda_1 = 2,\ \overline{N} = 4$)

(a) $\overline{X_s^*} = 0.40,\ |\overline{\Delta_x}|\% = 9.21\%$ (b) $\overline{TC_s^*} = 3.86,\ \overline{\Delta_c}\% = 0.67\%$



overestimate X_s^* depending on the interplay between the extra costs resulting from our two main assumptions and the costs related with obsolescence. This can be best observed in Table 2.

For all instances with positive λ_1 (89 instances out of 281), the mean absolute deviation from X_s^* is found to be 11.87%. For the same instances we found that using X^* instead of X_s^* results in a deviation from the optimal expected total cost on average 1.03% and maximum 4.42%. For the full obsolescence case, we found that the mean absolute deviation from X_s^* is 5.42% and the average deviation from the optimal expected total costs is 0.56% with a maximum of 3.44%. Thus, we conclude that X^* performs satisfactorily and it gives near optimal results for expected total costs. For more detailed results we refer the reader to Tables 1-2.

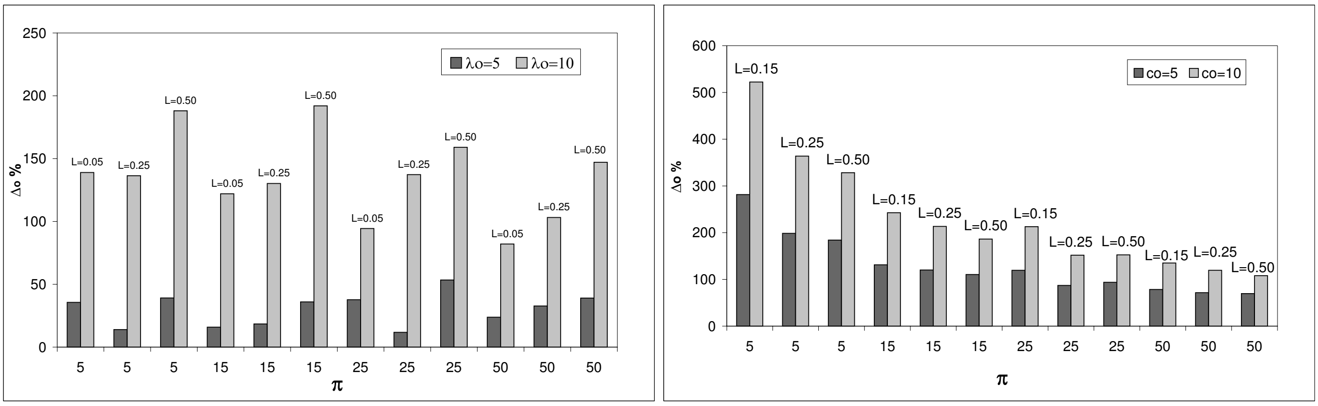
4.3 Value of Advance Policy Change

Next, we discuss the value of changing the control policy to initiate the stock removal process before the drop in demand rate occurs. To this extend, we compare the expected total cost incurred by changing the policy X^* time units earlier before the drop occurs with the expected total cost incurred by changing it immediately after the drop occurs ($X = 0$). For comparison purposes we use percent deviation in expected total cost functions defined as $\Delta_o\% = \frac{TC_s(0) - TC_s(X^*)}{TC_s(X^*)} \times 100$. Figure 8 illustrates the changes in $\Delta_o\%$ for different λ_0 and λ_1 values.

We found that the impact of advance policy change on costs is significant. For example, in Figure 8a when $\lambda_0 = 5$ the average $TC_s(X^*)$ is found to be 2.11. For these instances,

Figure 8: Value of Advance Policy Change ($\Delta_o\%$)

(a) $\lambda_0 = 5, 10,\ \lambda_1 = 2$ (b) $\lambda_0 = 5,\ \lambda_1 = 0$



waiting until the drop occurs increases the expected total costs on average by 30%. The increase in total cost is due to the increase in holding costs since the natural attrition of the remaining excess stocks takes longer once the drop occurs. Moreover, we found that when all other parameters are constant, the cost of postponing the policy change increases very rapidly in λ_0 (see Table 1). This can be seen clearly from Figure 8a; when λ_0 increases from 5 to 10 the average percent deviation due to postponement increases from 30% to 136%.

Our observations for the full obsolescence case are similar. However, when $\lambda_1 = 0$ the increase in total cost is mainly due to the obsolescence/relocation cost charged per remaining excess stock. For the instances given in Figure 8b, we found that when $\lambda_0 = 5$, the average $\Delta_o\%$ increases from 129.07% to 228.12% as c_o doubles. Moreover, we observed that $\Delta_o\%$ is decreasing in π. Because as π increases the cost of obsolescence becomes relatively cheaper compared to backordering. These behaviors can be seen in more detail in Table 2.

We close our discussion about the value of advance policy change by giving the summaries about $\Delta_o\%$. Over all the numerical experiments conducted, we found that when λ_1 is positive, changing the control policy after the drop occurs increases the expected total costs on average by 60%. In case of full obsolescence, we found that the expected total costs are on average more than doubled as a result of not taking an early action ($\overline{\Delta_o}\% = 133.04\%$). These findings show us that employing an advance policy change in face of pre-determined obsolescence results in important savings.

5. Conclusion

In this paper, we considered a continuous review inventory system of a slow moving item in which the demand rate drops to a lower level at a pre-determined (announced) time in the future. Adaptation to the new demand rate is achieved by changing the control policy before the drop occurs, and therefore letting the demand process to take away the excess stocks. We focused on the behavior of the net inventory level process during the transient period and proposed an approximate solution for the optimal time to shift to the new control policy minimizing the expected total cost incurred during this period. We found that the advance policy change results in significant cost savings and our model yields near optimal solutions for the expected total costs.

The key contribution of this paper lies in the analysis of a continuous review inventory system facing stochastic demands in the context of obsolescence for the first time. Earlier works on obsolescence were focused on periodic review models. The main insights from these works were that the obsolescence has a substantial impact on optimal policies and it should be incorporated into inventory control models explicitly. In our study, we extend these findings for a continuous review system and show that the advance policy change in the face of pre-determined obsolescence results in significant cost savings. Our numerical experiments reveal that for slow movers the timing of the control policy change primarily determines the tradeoff between backordering penalties and obsolescence costs.

The practical importance of our model comes from its consideration of expensive, slow moving items with high downtime costs for which continuous review policies are preferred over periodic review ones due to lower safety stock requirements. For this class of items, efficient management of inventories is notoriously difficult. Not surprisingly, inventory managers of many companies in after sales service industry are recurrently facing the problem of obsolete or excess inventories of such items. Knowing when to change the control policy is the key to reduce obsolete inventories without jeopardizing the availability. If the change is too early then the risk of backordering is too high and the stockouts can be detrimental to companies' operations. On the other hand, if the change is too late then the risk of obsolescence is too high and obsolete stocks lay as dead weight on the books which in return reduces the competitiveness of companies. Our model can be used to study the impact of the timing of policy change on operational costs and to identify the optimum time that balances the tradeoff between the risk of obsolescence and backordering.

While developing our model, we employed a couple of assumptions to keep the analysis in the boundaries of mathematical tractability. Although some of these assumptions limit the generality, the model offers an increased understanding of the transient behavior of inventory systems and the impacts of advance policy change on operational costs. Given the scarcity of research on continuous review systems facing obsolescence, we consider that our model bears a reasonable balance between realism and tractability for the insights obtained. Therefore, it can stand as a building block for more complicated and realistic models.

There are a couple of directions for future research. It would be useful to extend the model with demand rate decreasing by time. Such model would be more suitable for the products at the the end of their life cycles. Another possibility is to incorporate the uncertainty into the timing of the obsolescence or into the size of the drop in demand rate. These extensions would yield interesting insights about the timing of a policy change. Extending the model for a general class of continuous review control policies seems particularly worthwhile because for many products, the prospect of obsolescence has increased drastically due to the rapid changes in consumer taste and technological innovations.

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Biographies

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Appendix for "A Continuous Review Inventory Model with Advance Policy Change and Obsolescence"

Proof of Equation (4). Taking the expectation of equation (2) yields that for $k = 2, \ldots, N$ and $N \geq 2$,

$$
\begin{aligned}
E[OH_k] &= \left[(S_0 - k + 1)X + E[OH'_k]\right] P\left(A_{k-1} \leq X,\ A_k > X\right) \\
&+ (S_0 - k + 1)[E\left[\tau_k \mathbf{1}(A_k \leq X)\right] - E\left[A_{k-1}\mathbf{1}(A_{k-1} \leq X,\ A_k > X)\right]] \\
&+ E[OH_{k+1}\mathbf{1}(A_k \leq X)] \qquad (30)
\end{aligned}
$$

Observe that the event $\{A_{k-1} \leq X,\ A_k > X\}$ implies that there are exactly $k-1$ demands during the period of length X. Since the total demand during this period is Poisson distributed with rate λ_0 we obtain that,

$$P\left(A_{k-1} \leq X,\ A_k > X\right) = p(k-1; \lambda_0 X), \quad k = 2, 3, \ldots \qquad (31)$$

Denote,

$$\varepsilon'_k := E\left[\tau_k \mathbf{1}(A_k \leq X)\right], \quad k = 2, 3, \ldots$$

and observe that $A_k = A_{k-1} + \tau_k$. Hence, by conditioning on A_{k-1} and after some algebra we get,

$$\varepsilon'_k = \int_0^X \int_0^{X-s} t f_{\tau_k}(t)\, f_{A_{k-1}}(s)\, dt\, ds = \lambda_0^{-1}[1 - P(k-1; \lambda_0 X)] \qquad (32)$$

Denote,

$$\varepsilon''_k := E\left[A_{k-1}\mathbf{1}(A_{k-1} \leq X,\ A_k > X)\right], \quad k = 2, 3, \ldots$$

Similarly, conditioning on A_{k-1} yields,

$$\varepsilon''_k = \int_0^X s P(\tau_k > X - s)\, f_{A_{k-1}}(s) ds = \lambda_0^{-1}(k-1)p(k; \lambda_0 X) \qquad (33)$$

Thus, the difference between ε'_k and ε''_k is found as follows:

$$\varepsilon'_k - \varepsilon''_k = \lambda_0^{-1}\left[1 - P(k-1; \lambda_0 X)\right] - Xp(k-1; \lambda_0 X), \quad k = 2, 3, \ldots \qquad (34)$$

Also, note that

$$E[OH_k'] = \lambda_1^{-1} \sum_{i=k}^{N} (S_0 - i + 1), \quad k = 1, \ldots, N. \tag{35}$$

Therefore, substituting (31), (34) and (35) in (30), and making necessary simplifications yields that for $k = 2, \ldots, N$ and $N \geq 2$,

$$\begin{aligned} E[OH_k] &= \lambda_1^{-1} p(k-1; \lambda_0 X) \sum_{i=k}^{N} (S_0 - i + 1) \\ &+ \lambda_0^{-1} (S_0 - k + 1) \left[1 - P(k-1; \lambda_0 X)\right] \\ &+ E[OH_{k+1} \mathbf{1}(A_k \leq X)] \end{aligned} \tag{36}$$

Observe that $E[OH_k] = E[OH_k \mathbf{1}(A_{k-1} \leq X)]$ since $E[OH_k \mathbf{1}(A_{k-1} > X)] = 0$. Thus, by exploiting the recursive structure of (36) and after some algebra we obtain that for $N \geq 1$,

$$\begin{aligned} E[OH_2 \mathbf{1}(A_1 \leq X)] &= \lambda_1^{-1} \sum_{i=1}^{N-1} (S_0 - i) \sum_{j=1}^{i} p(j; \lambda_0 X) + \lambda_0^{-1} (N-1) \left[S_0 - \frac{N}{2}\right] \\ &- \lambda_0^{-1} \sum_{i=1}^{N-1} (S_0 - i) P(i; \lambda_0 X) + E[OH_{N+1} \mathbf{1}(A_N \leq X)] \end{aligned} \tag{37}$$

with the convention that $\sum_{i=k}^{N}() = 0$ for $N < k$. Now, taking the expectation of equation (1) gives,

$$E[OH] = \left[S_0 X + E[OH_1']\right] P(A_1 > X) + S_0 E\left[A_1 \mathbf{1}(A_1 \leq X)\right] + E[OH_2 \mathbf{1}(A_1 \leq X)] \tag{38}$$

Thus, using (35) and (37) in (38), and rearranging the terms yield the expected on-hand as given by equation (4). □

Before starting the analysis of $E[OH_{N+1} \mathbf{1}(A_N \leq X)]$ we need the following lemma which is important for our derivations.

Lemma 1. *Let $f_E(t)$ be the pdf of Erlang distribution with parameters $\alpha \in \{1, 2, ...\}$, $\beta > 0$ and define*

$$I = \int_a^b p(r; \lambda t + \gamma) f_E(t) dt. \tag{39}$$

(i) If $\lambda \neq -\beta$ then

$$I = \sum_{k=0}^{r} p(r-k;\gamma) b_N\left(k;\alpha,\frac{\beta}{\lambda+\beta}\right)\delta(k) \tag{40}$$

where

$$\delta(k) = P(\alpha+k-1;(\lambda+\beta)a) - P(\alpha+k-1;(\lambda+\beta)b)$$

(ii) If $\lambda = -\beta$ then

$$I = \frac{\beta^\alpha}{(\alpha-1)!}\sum_{k=0}^{r} p(r-k;\gamma)\frac{(-\beta)^k\left(b^{\alpha+k}-a^{\alpha+k}\right)}{k!(\alpha+k)} \tag{41}$$

Proof of Lemma 1.

(i) From (39) we have,

$$I \;=\; \frac{e^{-\gamma}\beta^\alpha}{r!(\alpha-1)!}\int_a^b (\lambda t+\gamma)^r t^{\alpha-1} e^{-(\lambda+\beta)t} dt$$

Using binomial theorem and after some algebra,

$$\begin{aligned} I &= \frac{e^{-\gamma}\beta^\alpha}{r!(\alpha-1)!}\int_a^b \sum_{k=0}^{r}\binom{r}{k}\gamma^{r-k}(\lambda t)^k t^{\alpha-1}e^{-(\lambda+\beta)t}dt \\ &= \sum_{k=0}^{r}\frac{e^{-\gamma}\gamma^{r-k}}{(r-k)!}\binom{k+\alpha-1}{\alpha-1}\frac{\beta^\alpha}{(\lambda+\beta)^{\alpha-1}}\left(\frac{\lambda}{\lambda+\beta}\right)^k\int_a^b p(\alpha+k-1;(\lambda+\beta)t)dt \end{aligned} \tag{42}$$

It can be easily shown that for any $\lambda \neq 0$ the following holds,

$$\int_a^b p(n;\lambda t+\gamma)dt = \frac{1}{\lambda}[P(n;\lambda a+\gamma) - P(n;\lambda b+\gamma)] \tag{43}$$

Hence, applying (43) to the integral on the right-hand side of (42) yields,

$$\begin{aligned} I &= \sum_{k=0}^{r} p(r-k;\gamma)\binom{k+\alpha-1}{\alpha-1}\left(\frac{\beta}{\lambda+\beta}\right)^\alpha\left(\frac{\lambda}{\lambda+\beta}\right)^k[P(\alpha+k-1;(\lambda+\beta)a) \\ &- P(\alpha+k-1;(\lambda+\beta)b)] \\ &= \sum_{k=0}^{r} p(r-k;\gamma) b_N\left(k;\alpha,\frac{\beta}{\lambda+\beta}\right)[P(\alpha+k-1;(\lambda+\beta)a) - P(\alpha+k-1;(\lambda+\beta)b)]\,. \end{aligned}$$

(ii) When $\lambda = -\beta$ (39) can be simplified as,

$$I = \frac{e^{-\gamma}\beta^{\alpha}}{r!(\alpha-1)!}\int_a^b (-\beta t+\gamma)^r t^{\alpha-1}dt$$

and the result follows from the binomial theorem:

$$\begin{aligned} I &= \frac{e^{-\gamma}\beta^{\alpha}}{r!(\alpha-1)!}\sum_{k=0}^{r}\binom{r}{k}\gamma^{r-k}(-\beta)^k\int_a^b t^{\alpha+k-1}dt \\ &= \frac{\beta^{\alpha}}{(\alpha-1)!}\sum_{k=0}^{r} p(r-k;\gamma)\frac{(-\beta)^k\left(b^{\alpha+k}-a^{\alpha+k}\right)}{k!(\alpha+k)} \quad \square \end{aligned}$$

Proof of Equation (15).

Denote,

$$E[OH_{N+1}\mathbf{1}(A_N \le X)] = \sum_{n=0}^{S_1-1}(S_1-n)\,[K_1+K_2]$$

with

$$K_1 := \int_0^X\int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!} f_{A_N}(s)\,dt\,ds \tag{44}$$

$$K_2 := \int_0^X\int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t)-\Lambda(t-L))^n}{n!} f_{A_N}(s)\,dt\,ds \tag{45}$$

(i) If $L \le X$ then using the definition of $\Lambda(t)$ we can partition the integrals in (44) and (45) as follows:

$$\begin{aligned} K_1 &= \int_0^{X-L}\int_0^L p(n;\lambda_0 t)\,dt f_{A_N}(s)\,ds \\ &+ \int_{X-L}^X\left[\int_0^{X-s} p(n;\lambda_0 t)\,dt + \int_{X-s}^L p(n;\eta_1(t,s))\,dt\right] f_{A_N}(s)\,ds \end{aligned} \tag{46}$$

$$\begin{aligned} K_2 &= \int_0^{X-L}\left[\int_L^{X-s} p(n;\lambda_0 L)\,dt + \int_{X-s}^{X-s+L} p(n;\eta_2(t,s))\,dt\right] f_{A_N}(s)\,ds \\ &+ \int_{X-L}^X\int_L^{X-s+L} p(n;\eta_2(t,s))\,dt f_{A_N}(s)\,ds \end{aligned} \tag{47}$$

where $\eta_1(t,s) := \lambda_1 t + (\lambda_0-\lambda_1)(X-s)$ and $\eta_2(t,s) := \eta_1(t,s) - \lambda_0(t-L)$. From identity

(43) we obtain that,

$$K_1 = \int_0^{X-L} \frac{1}{\lambda_0}[1 - P(n;\lambda_0 L)] f_{A_N}(s)\, ds + \int_{X-L}^{X} \left[\frac{1}{\lambda_0}[1 - P(n;\lambda_0(X-s))] + \frac{1}{\lambda_1}[P(n;\lambda_0(X-s)) - P(n;\upsilon(s))]\right] f_{A_N}(s)\, ds \quad (48)$$

$$K_2 = \int_0^{X-L} \left[p(n;\lambda_0 L)(X-s-L) - \frac{1}{\lambda_0-\lambda_1}[P(n;\lambda_0 L) - P(n;\lambda_1 L)]\right] f_{A_N}(s)\, ds + \int_{X-L}^{X} \frac{1}{\lambda_0-\lambda_1}[P(n;\lambda_1 L) - P(n;\upsilon(s))] f_{A_N}(s)\, ds \quad (49)$$

with $\upsilon(s) := \lambda_1 L + (\lambda_0 - \lambda_1)(X-s)$. Summing K_1 and K_2 and rearranging the terms yields,

$$\begin{aligned} K_1 + K_2 &= \left[\frac{1}{\lambda_0} - \frac{2\lambda_0-\lambda_1}{\lambda_0(\lambda_0-\lambda_1)} P(n;\lambda_0 L) + p(n;\lambda_0 L)(X-L) + \frac{P(n;\lambda_1 L)}{\lambda_0-\lambda_1}\right] \int_0^{X-L} f_{A_N}(s)\, ds - p(n;\lambda_0 L) \int_0^{X-L} s f_{A_N}(s)\, ds \\ &+ \left[\frac{1}{\lambda_0} + \frac{P(n;\lambda_1 L)}{\lambda_0-\lambda_1}\right] \int_{X-L}^{X} f_{A_N}(s)\, ds \\ &+ \frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1} \int_{X-L}^{X} P(n;\lambda_0(X-s)) f_{A_N}(s)\, ds \\ &- \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)} \int_{X-L}^{X} P(n;\upsilon(s)) f_{A_N}(s)\, ds \end{aligned} \quad (50)$$

Note that,

$$\int_{X-L}^{X} P(n;\lambda_0(X-s)) f_{A_N}(s)\, ds = \sum_{i=0}^{n} \int_{X-L}^{X} p(i;\lambda_0(X-s)) f_{A_N}(s)\, ds \quad (51)$$

$$\int_{X-L}^{X} P(n;\upsilon(s)) f_{A_N}(s)\, ds = \sum_{i=0}^{n} \int_{X-L}^{X} p(i;\upsilon(s)) f_{A_N}(s)\, ds \quad (52)$$

and denote,

$$I_1 := \int_{X-L}^{X} p(i;\lambda_0(X-s)) f_{A_N}(s)\, ds \quad (53)$$

$$I_2 := \int_{X-L}^{X} p(i;\upsilon(s)) f_{A_N}(s)\, ds \quad (54)$$

Using part (ii) of Lemma 1 and after some algebraic manipulations we obtain that,

$$\begin{aligned} I_1 &= \frac{{\lambda_0}^N}{(N-1)!}\sum_{k=0}^{i} p(i-k;\lambda_0 X)\frac{(-\lambda_0)^k\left(X^{N+k}-(X-L)^{N+k}\right)}{k!(N+k)} \\ &= p(N+i;\lambda_0 X)\left[1-N\binom{N+i}{N}\sum_{k=0}^{i}\binom{i}{k}\frac{(-1)^k}{N+k}\left(\frac{X-L}{X}\right)^{N+k}\right] \\ &= p(N+i;\lambda_0 X)-\xi(i,N) \end{aligned} \tag{55}$$

Thus, substituting (55) in (51) yields,

$$\int_{X-L}^{X} P(n;\lambda_0(X-s))f_{A_N}(s)\,ds = P(N+n;\lambda_0 X)-P(N-1;\lambda_0 X)-\sum_{i=0}^{n}\xi(i,N) \tag{56}$$

Similarly, from part (i) of Lemma 1 we found that

$$I_2 = \sum_{k=0}^{i} p(i-k;\lambda_1 L+(\lambda_0-\lambda_1)X)b_N\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\delta(k) \tag{57}$$

with

$$\delta(k)=P(N+k-1;\lambda_1(X-L))-P(N+k-1;\lambda_1 X)$$

Substituting (57) in (52) gives,

$$\int_{X-L}^{X} P(n;\upsilon(s))f_{A_N}(s)\,ds = \sum_{i=0}^{n}\sum_{k=0}^{i} p(i-k;\lambda_1 L+(\lambda_0-\lambda_1)X)b_N\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\delta(k) \tag{58}$$

Therefore, employing equations (56) and (58) in (50), and using the following identities

$$\int_0^x f_{A_N}(s)\,ds = \bar{P}(N;\lambda_0 x) \tag{59}$$

$$\int_0^x s f_{A_N}(s)\,ds = \lambda_0^{-1}N\bar{P}(N+1;\lambda_0 x) \tag{60}$$

yield that,

$$E[OH_{N+1}\mathbf{1}(A_N\le X)] = \sum_{n=0}^{S_1-1}(S_1-n)\left[f(n)+g_1(n)\right].$$

with

$$f(n) = \left[\frac{1}{\lambda_0} + \frac{P(n;\lambda_1 L)}{\lambda_0-\lambda_1}\right]\bar{P}(N;\lambda_0 X) + \frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1}\left[P(N+n;\lambda_0 X) - P(N-1;\lambda_0 X)\right]$$

$$\begin{aligned} g_1(n) &= p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L)) - \lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right] \\ &- \frac{2\lambda_0-\lambda_1}{\lambda_0(\lambda_0-\lambda_1)}P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L)) - \frac{(\lambda_0-\lambda_1)}{\lambda_0\lambda_1}\sum_{i=0}^{n}\xi(i,N) \\ &- \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\sum_{k=0}^{i} p\left(i-k;\lambda_1 L+(\lambda_0-\lambda_1)X\right) b_N\!\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\delta(k) \end{aligned}$$

(ii) If $L > X$ then by the definition of $\Lambda(t)$ the integrals in (44) and (45) can be partitioned as follows:

$$K_1 = \int_0^X \left[\int_0^{X-s} p(n;\lambda_0 t)\,dt + \int_{X-s}^{L} p(n;\eta_1(t,s))\,dt\right] f_{A_N}(s)\,ds \qquad (61)$$

$$K_2 = \int_0^X \int_L^{X-s+L} p(n;\eta_2(t,s))\,dt f_{A_N}(s)\,ds \qquad (62)$$

Using the identity (43) in K_1 and K_2, and summing the results yield that,

$$\begin{aligned} K_1+K_2 &= \left[\frac{1}{\lambda_0} + \frac{P(n;\lambda_1 L)}{\lambda_0-\lambda_1}\right]\bar{P}(N;\lambda_0 X) + \frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1}\sum_{i=0}^{n}\int_0^X p(i;\lambda_0(X-s))f_{A_N}(s)\,ds \\ &- \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\int_0^X p(i;\upsilon(s))f_{A_N}(s)\,ds \qquad (63) \end{aligned}$$

Denote,

$$I_3 := \int_0^X p(i;\lambda_0(X-s))f_{A_N}(s)\,ds$$

$$I_4 := \int_0^X p(i;\upsilon(s))f_{A_N}(s)\,ds$$

By employing part (ii) of Lemma 1 in I_3 we obtain that,

$$I_3 = \frac{{\lambda_0}^N}{(N-1)!}\sum_{k=0}^{i} p(i-k;\lambda_0 X)\frac{(-\lambda_0)^k X^{N+k}}{k!(N+k)} = p(N+i;\lambda_0 X) \qquad (64)$$

Similarly, from part (i) of Lemma 1 we have,

$$I_4 = \sum_{k=0}^{i} p(i-k;\lambda_1 L + (\lambda_0 - \lambda_1)X) b_N\Big(k; N, \frac{\lambda_0}{\lambda_1}\Big)\, \bar{P}(N+k;\lambda_1 X) \tag{65}$$

Therefore, employing (64) and (65) in (63) yields that,

$$E[OH_{N+1}\mathbf{1}(A_N \le X)] = \sum_{n=0}^{S_1-1} (S_1 - n)\,[f(n) - g_2(n)]$$

with

$$g_2(n) = \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)} \sum_{i=0}^{n}\sum_{k=0}^{i} p(i-k;\lambda_1 L + (\lambda_0 - \lambda_1)X) b_N\Big(k; N, \frac{\lambda_0}{\lambda_1}\Big)\, \bar{P}(N+k;\lambda_1 X) \quad \square$$

Proof of Equation (26).

Note that the integral expression for the expected time weighted backorders can be found similar to the expected on hand carried during the regular operation phase as described in section 3.1. Thus, for $\lambda_1 = 0$ we found that,

$$\begin{aligned} E[BO] &= \sum_{n=0}^{\infty} n \int_0^X \Bigg[\int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!}\,dt \\ &+ \int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t)-\Lambda(t-L))^n}{n!}\,dt\Bigg] f_{A_N}(s)\,ds \\ &= \sum_{n=0}^{\infty} n\,[K_1 + K_2] \end{aligned}$$

(i) If $L \le X$ then from the definition of $\Lambda(t)$ the integral expressions K_1 and K_2 can be partitioned as in equations (46) and (47) with $\lambda_1 = 0$. Using the identity (43) in K_1 and K_2 and summing the results yield that,

$$\begin{aligned} K_1 + K_2 &= \big[2\lambda_0^{-1}[1 - P(n;\lambda_0 L)] + p(n;\lambda_0 L)(X-L)\big] \int_0^{X-L} f_{A_N}(s)\,ds \\ &- p(n;\lambda_0 L)\int_0^{X-L} s f_{A_N}(s)\,ds + 2\lambda_0^{-1}\int_{X-L}^{X} [1 - P(n;\lambda_0(X-s))] f_{A_N}(s)\,ds \\ &- (X-L)\int_{X-L}^{X} p(n;\lambda_0(X-s)) f_{A_N}(s)\,ds + \int_{X-L}^{X} p(n;\lambda_0(X-s)) s f_{A_N}(s)\,ds \end{aligned} \tag{66}$$

Using the identities (59) and (60) in (66), and simplifying gives that,

$$\begin{aligned}
K_1 + K_2 &= 2\lambda_0^{-1}\left[\bar{P}(N;\lambda_0 X) - P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L))\right] \\
&- p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L)) - \lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right] \\
&- 2\lambda_0^{-1}\int_{X-L}^{X} P(n;\lambda_0(X-s))f_{A_N}(s)\,ds - (X-L)\int_{X-L}^{X} p(n;\lambda_0(X-s))f_{A_N}(s)\,ds \\
&+ \int_{X-L}^{X} p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds \qquad (67)
\end{aligned}$$

Observe that,

$$\int_{X-L}^{X} p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}N\int_{X-L}^{X} p(n;\lambda_0(X-s))f_{A_{N+1}}(s)\,ds \qquad (68)$$

where $f_{A_{N+1}}$ is the pdf of Erlang distribution with parameters λ_0 and $N+1$. Thus, by applying part (ii) of Lemma 1 to the right-hand side of (68) we obtain that,

$$\int_{X-L}^{X} p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}N\left[p(N+1+n;\lambda_0 X) - \xi(n,N+1)\right] \qquad (69)$$

Therefore, employing the results (55), (56) and (69) in (67), and making necessary simplifications yield the expected backorder as follows:

$$E[BO] = \sum_{n=0}^{\infty} n\left[f(n) + g(n)\right]$$

with

$$f(n) = \lambda_0^{-1}\left[2\bar{P}(N+n+1;\lambda_0 X) + Np(N+n+1;\lambda_0 X)\right] - (X-L)p(N+n;\lambda_0 X) \qquad (70)$$

$$\begin{aligned}
g(n) &= p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L)) - \lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right] \\
&- 2\lambda_0^{-1}\left[P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L)) - \sum_{i=0}^{n}\xi(i,N)\right] \\
&+ (X-L)\xi(n,N) - \lambda_0^{-1}N\xi(n,N+1)
\end{aligned}$$

(ii) If $L > X$ then from the definition of $\Lambda(t)$ the integral expressions K_1 and K_2 can be partitioned as in equations (48) and (49) with $\lambda_1 = 0$. Using the identity (43) in K_1 and K_2, and summing the results yield that,

$$\begin{aligned} K_1 + K_2 &= 2\lambda_0^{-1}\left[\bar{P}(N;\lambda_0 X) - \sum_{i=0}^{n}\int_0^X p(i;\lambda_0(X-s))f_{A_N}(s)\,ds\right] \\ &- (X-L)\int_0^X p(n;\lambda_0(X-s))f_{A_N}(s)\,ds + \int_0^X p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds \end{aligned} \quad (71)$$

Observe that

$$\int_0^X p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}N\int_0^X p(n;\lambda_0(X-s))f_{A_{N+1}}(s)\,ds \quad (72)$$

Thus, using part *(ii)* of Lemma 1 in (72) yields that,

$$\int_0^X p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}Np(N+1+n;\lambda_0 X) \quad (73)$$

Finally, using (64) and (73) in (71), and after some simplifications we obtain the expected backorder as follows:

$$E[BO] = \sum_{n=0}^{\infty} nf(n)$$

with $f(n)$ as given in (70). □