THREE-TYPES MODELS OF MULTIDIMENSIONAL SCREENING

Luigi Brighi
Università di Modena e Reggio Emilia*

Marcello D'Amato
Università di Salerno**

October 2005

Abstract: This paper analyzes the variety of optimal screening contracts in a relatively simple multidimensional framework *à la* Armstrong and Rochet (1999), when only three types of agents are present. It is shown, among other things, that the well known principle in optimal contract theory of 'no distortion at the top' does not carry over to the multidimensional case.

JEL Classification Number: D58, L51.

Keywords: Asymmmetric information, multidimensional screening, optimal contract.

* Dipartimento di Economia Politica, Università di Modena e Reggio Emilia, viale Berengario 51, 41100 Modena, Italy; e-mail: brighi@unimo.it

** Dipartimento di Scienze economiche, Università di Salerno, via Ponte Don Melillo 80144 Salerno, Italy; e-mail: damato@diima.unisa.it

Three-types models of multidimensional screening

1. Introduction

The theory of optimal screening contracts applies to a large variety of relevant topics such as nonlinear pricing, monopoly regulation, procurement, optimal taxation and auctions. Most of these applications are based on the simplifying assumption that agent's types can be 'ordered' by a single dimension of private information. This assumption greatly simplifies the analysis and leads to well defined results; for instance, the optimal contract prescribes that 'lower' types be distorted in order to extract the informational rent from the 'higher' types.

There are, however, many economic contexts where a single parameter of private information is not sufficient to model economic problems accurately. For example, a multiproduct monopolist may face consumers whose preferences are best described by a different taste parameter for each good. Clearly, when consumers preferences are not publicly observable, the nonlinear pricing problem of the monopolist can only be analysed by means of a multidimensional screening model. Another weakness of one-dimensional models is that the results they provide are not very robust. In fact, as shown in the recent theoretical literature,[1] many of the properties of optimal contracts can be lost when further dimensions of private information are introduced into the model. All this calls for the introduction of multidimensional models.

Unfortunately, the analysis of general multidimensional screening problems is far more complex as compared to the single dimensional case. There are certainly a few interesting results on the qualitative properties of the

[1] See Armstrong (1996), Rochet and Choné (1998), Armstrong and Rochet (1999) among others.

optimal contracts, but multidimensional analysis has not yet succeded in providing closed-form solutions to the general model. Research in this area proceeds then by the identification and the study of tractable special cases.[2]

In this paper we show the variety of optimal screening contracts that might arise in a simple multidimensional framework *à la* Armstrong and Rochet (1999), where only three types of agents are present. Even such a relatively simple context exhibits the typical difficulty characterizing all the multidimensional models, which is the absence of an exogenous pattern of binding incentive compatibility constraints. We will show that several patterns of binding constraints are possible and that they determine optimal contracts with quite unusual and counter-intuitive features.

Section 2 introduces the basic assumptions of a principal-agent model where the agent is engaged in two activities and has private information on two discrete technological parameters. Section 3 derives the optimal contract when the agent's type with 'high' technological efficiency in both activities is not present. In Section 4 the missing type is the one with 'low' efficiency in both activities, while Section 5 deals with the more standard case where one of the 'mixed' types is absent.

2. Basic assumptions

In a principal-agent framework the agent can undertake two kinds of activities denoted by A and B.[3] The levels of the two activities are given,

[2] For an excellent survey of the literature on multidimensional screening see Rochet and Stole (2003).

[3] The model presented below has been analysed by Spence (1980), Dana (1993) and Armstrong and Rochet (1999) among others.

respectively, by the positive real variables x^A and x^B. The agent has private information on technology in both activities. Specifically, we assume that technology may exhibit either High or Low efficiency, so that there can only be four types of agents denoted by LL, HL, LH and HH, where LL stands for low efficiency in both activities, HL means high efficiency in activity A and low efficiency in activity B and so on. The principal does not observe the realization of technology, but the probability of occurrence of each type, denoted by α_{ij} with $i, j = L, H$, is common knowledge.

The agent's utility is separable in the two activity levels and quasi-linear in income. Denoting by T a payment from the agent to the principal, the ij-type agent's utility is given by

$$u_i^A(x^A) + u_j^B(x^B) - T$$

We assume that the function $u_i^k(\cdot)$ is differentiable, monotone and strictly concave with $u_i^k(0) = 0$, $k = A, B$ and $i = H, L$. By

$$\delta^k(x) = u_H^k(x) - u_L^k(x)$$

we denote the *incremental utility* in activity k. We assume that $\delta^k(x) \geq 0$ and that $\delta^k(\cdot)$ is increasing, i.e. $\delta^{k\prime}(x) > 0$. Therefore, the 'single-crossing' property is satisfied in each activity, i.e. the marginal utility in activity k is monotone in type realizations.

The principal welfare, which is assumed to be additively separable in activity levels and quasi-linear in income, may also depend on agent's utility and is given by

$$v_i^A(x^A) + v_j^B(x^B) + T + \beta[u_i^A(x^A) + u_j^B(x^B) - T]$$

where $v_i^k(\cdot)$ is an increasing and concave function. The parameter $0 \leq \beta \leq 1$ is the weight the principal places on the agent's welfare. For example, in a regulatory context such as Baron and Myerson (1982) a low level of β represents a strong distributive concern of the regulation authority (the principal) in favor of consumer welfare rather than monopoly profits.

The principal offers the agent a menu of contracts specifying, for each type ij, the activity levels, x_{ij}^A and x_{ij}^B, in return for a payment, T_{ij}. The contracts $(x_{ij}^A, x_{ij}^B, T_{ij})$ are implementable if they satisfy the incentive compatibility (IC) constraints,

$$u_i^A(x_{ij}^A) + u_j^B(x_{ij}^B) - T_{ij} \geq u_i^A(x_{i'j'}^A) + u_j^B(x_{i'j'}^B) - T_{i'j'}$$

for all pairs ij and $i'j'$. Moreover, individual rationality (IR) requires that each type obtains a level of utility not less than an outside option level which is normalized to zero; therefore the implementable contracts must also satisfy the IR constraints $u_i^A(x_{ij}^A) + u_j^B(x_{ij}^B) - T_{ij} \geq 0$, for all i, j.

Under truthful revelation, the contract grants to type ij the rent

$$R_{ij} = u_i^A(x_{ij}^A) + u_j^B(x_{ij}^B) - T_{ij}$$

so that the principal welfare in state ij can be more succintly written as follows

$$w_i^A(x_{ij}^A) + w_j^B(x_{ij}^B) - (1-\beta)R_{ij}$$

where $w_i^k(x) = v_i^k(x) + u_i^k(x)$ is total surplus from activity k. Also, the IC constraints can be rewritten as

$$R_{ij} \geq R_{i'j'} + u_i^A(x_{i'j'}^A) - u_{i'}^A(x_{i'j'}^A) + u_j^B(x_{i'j'}^B) - u_{j'}^B(x_{i'j'}^B)$$

for all pairs ij and $i'j'$ and the IR constraints as $R_{ij} \geq 0$. For convenience we shall consider the menu of contracts $(x_{ij}^A, x_{ij}^B, R_{ij})$ instead of $(x_{ij}^A, x_{ij}^B, T_{ij})$.

By $\bar{x}_i^k$ we denote the first-best level of activity k, i.e. the level of x^k which maximizes total surplus $w_i^k(x^k)$. We assume that first-best levels are increasing in the type realizations, i.e. $\bar{x}_H^k > \bar{x}_L^k$, for $k = A, B$.

For future reference let us introduce the following function

$$x^k(\zeta) := \text{argmax}_{x \geq 0} \ \ w_L^k(x) - \zeta \delta^k(x) \tag{1}$$

where ζ is a real variable. We assume that ${w_L^k}'(x)$ goes to infinity as x goes to 0, so that $x^k(\zeta)$ is never negative. It is easily seen that $x^k(\zeta)$ is a decreasing function with $x^k(0) = \bar{x}_L^k$. To further simplify notation let

$$\hat{\delta}^k(\zeta) := \delta^k(x^k(\zeta)) \tag{2}$$

so that $\hat{\delta}^k(\zeta)$ is a decreasing function.

Finally, notice that, $x^k(\zeta)$ can be interpreted as the activity level of the low type in a single-dimensional problem where only activity k is taken into account. For example, substituting $k = A$ and $\zeta = (1 - \beta)[\alpha_{HL}/\alpha_{LL}]$ in (1) gives the optimal contract level of activity A for the low efficiency type in a single-dimension, single-activity screening problem where only types HL and LL are present. Accordingly, the amount

$$\hat{\delta}^A \left((1 - \beta) \frac{\alpha_{HL}}{\alpha_{LL}} \right)$$

will be the informational rent of the high efficiency type, R_{HL}.

3. The model without the high efficiency type

Let us consider the first model with only three types and specifically the case where the most efficient agent in both the activities is missing, i.e. the case where $\alpha_{HH} = 0$. Therefore, we have two 'mixed' types, HL and LH, and the non specialized type of agent, LL.

The principal expected welfare is given by

$$\begin{aligned} L = \alpha_{LL}[w_L^A(x_{LL}^A) + w_L^B(x_{LL}^B)] + \\ + \alpha_{HL}[w_H^A(x_{HL}^A) + w_L^B(x_{HL}^B) - (1 - \beta)R_{HL}] + \\ + \alpha_{LH}[w_L^A(x_{LH}^A) + w_H^B(x_{LH}^B) - (1 - \beta)R_{LH}] \end{aligned}$$

The screening problem consists in finding a menu of contracts $(x^A_{ij}, x^B_{ij}, R_{ij})$ maximizing L subject to the individual rationality (IR) and incentive compatibility (IC) constraints. It is not difficult to see that the optimal contract satisfies $R_{LL} = 0$, $x^A_{HL} > x^A_{LL}$, $x^B_{LH} > x^B_{LL}$ and that the IC constraints of type LL always hold with a strict inequality; therefore the only potentially IC binding constraints are those of types HL and LH, i.e.

$$R_{HL} \geq R_{LH} + \delta^A(x^A_{LH}) - \delta^B(x^B_{LH}) \tag{3}$$

$$R_{HL} \geq \delta^A(x^A_{LL}) \tag{4}$$

$$R_{LH} \geq R_{HL} - \delta^A(x^A_{HL}) + \delta^B(x^B_{HL}) \tag{5}$$

$$R_{LH} \geq \delta^B(x^B_{LL}) \tag{6}$$

From the analysis of the first order conditions we see that there are three different forms of the optimal contract. Each of these cases corresponds to a specific pattern of binding IC constraints and is determined by the particular specification of utility functions and the distribution of types.

Let us define

$$M = \hat{\delta}^A\left(\frac{(1-\beta)\alpha_{HL}}{\alpha_{LL}}\right) - \hat{\delta}^B\left(\frac{(1-\beta)\alpha_{LH}}{\alpha_{LL}}\right)$$

The magnitude M, which can be computed directly from the data of the problem, turns out to be crucial in determining the actual pattern of binding incentive constraints and therefore the shape of the optimal contract. The economic interpretation of M is not difficult to grasp; for example, let's take activity A. As we know from Section 2, when the 'off-diagonal' IC constraints are neglected, $\hat{\delta}^A((1-\beta)\alpha_{HL}/\alpha_{LL})$ is the minimal informational rent that prevents the agent specialized in activity A from mimicking the low efficiency type LL. A similar interpretation holds for $\hat{\delta}^B((1-\beta)\alpha_{LH}/\alpha_{LL})$, therefore,

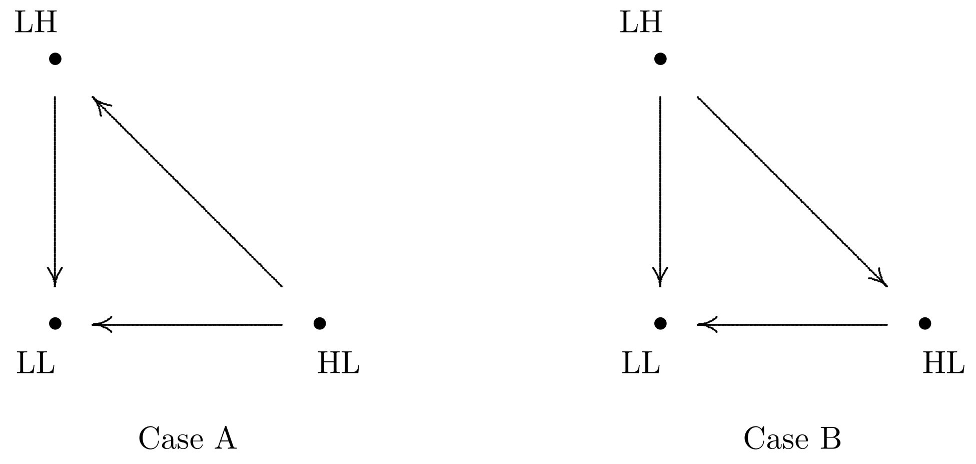

Fig. 1. Patterns of binding constraints

M gives the relative profitability in terms of potential rents between the mixed type specialized in A, HL, and the mixed type specialized in B, LH.

Case A.

The 'off-diagonal' IC constraint of type HL with respect to LH, (3), is binding. This case occurs whenever the following condition holds

$$\delta^A(\bar{x}_L^A) - \delta^B(\bar{x}_H^B) > M \tag{7}$$

The left hand side is the 'net' incremental utility of type HL with respect to type LH in both lines of activities when the former mimicks the latter and the activity levels are fixed at LH first-best. If this term is greater than M then the type HL has a stronger incentive in mimicking the type LH rather than the type LL.

Case B.

The 'off-diagonal' IC constraint of type LH with respect to HL, (5), is binding. This case is the mirror image to case A and occurs whenever the following condition holds

$$\delta^A(\bar{x}_H^A) - \delta^B(\bar{x}_L^B) < M \tag{8}$$

Case C.

Only 'downward' IC constraints are binding, i.e. (4) and (6). This case occurs whenever both (7) and (8) are violated, i.e.

$$\delta^A(\bar{x}_L^A) - \delta^B(\bar{x}_H^B) \leq M \leq \delta^A(\bar{x}_H^A) - \delta^B(\bar{x}_L^B) \tag{9}$$

The three patterns of incentive constraints are depicted in figures 1 and 2, where a solid line pointing from type ij to type $i'j'$ means that the incentive constraint that ij not be tempted to chose the $i'j'$ contract is binding.

The main features of the optimal contract in each of the three cases are summarized in the following

Proposition 1.

Let $\alpha_{HH} = 0$. In all the three cases, A, B and C, the low efficiency type, LL, does not earn any rent, i.e. $R_{LL} = 0$, and has below first-best levels in both activities, i.e. $x_{LL}^k < \bar{x}_L^k$, for $k = A, B$. Moreover, in all the three cases the mixed types, HL and LH, earn strictly positive rents, i.e. $R_{HL} > 0$ and $R_{LH} > 0$.

In Case A, the mixed type specialized in activity A, HL, has efficient levels in both activities, i.e. $x_{HL}^A = \bar{x}_H^A$ and $x_{HL}^B = \bar{x}_L^B$ the levels of the agent specialized in activity B, LH, are distorted away from first-best levels. Specifically, the activity levels of type LH are above first-best in the specialized activity and below first-best in the non specialized activity, i.e. $x_{LH}^A < \bar{x}_L^A$ and $x_{LH}^B > \bar{x}_H^B$.

Case B is the mirror image of Case A.

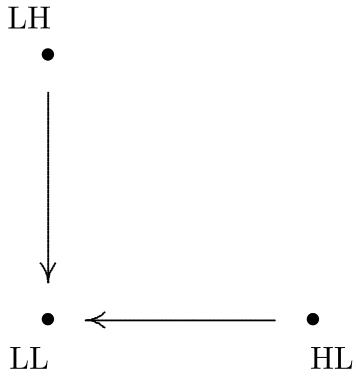

Fig. 2. Pattern of binding constraints – Case C.

In Case C, both the mixed types, HL and LH, have first-best levels in both activities, i.e. $x^A_{HL} = \bar{x}^A_H$, $x^B_{HL} = \bar{x}^B_L$, $x^A_{LH} = \bar{x}^A_L$ *and* $x^B_{LH} = \bar{x}^B_H$.

The proof is in Appendix A.

In order to give a rough idea of the circumstances under which the various cases apply let us consider the special situation of *perfect simmetry* of activities and types, that is we assume that incremental utility is the same in each activity, i.e. $\delta^A(x) = \delta^B(x)$, and mixed types have the same probability, i.e. $\alpha_{HL} = \alpha_{LH}$. It can be easily verified that under perfect simmetry (7) and (8) are violated so that only case C can occur. Therefore, cases A and B are possible only when relevant asimmetries are present in activities as well as types.

There are two remarks about Proposition 1. First, Case C exhibits a non standard feature as compared to the analysis of one-dimensional models. Indeed, the optimal contract implements first-best levels of two types rather than only one. Specifically, the 'no distortion at the top' rule applies here to both the mixed types.

The second remark is concerned with the kind of distorsions required by the optimal contract. In cases A and B one of the mixed types has a peculiar pattern of activity levels, i.e. the agent over-produces with respect to the first-best levels in the most efficient activity and under-produces in the less efficient activity. As already noticed in the literature on multi-dimensional screening, this feature of the optimal contract is at odds with respect to the models with one-dimensional private information.

4. The model without the low efficiency type

Let us consider a model with only three types where the agent with low efficiency in both activities is missing, i.e. the case where $\alpha_{LL} = 0$. Therefore, we have two 'mixed' types, HL and LH, and the most efficient type of agent, HH.

The principal expected welfare is given by

$$L = \alpha_{HH}[w_H^A(x_{HH}^A) + w_H^B(x_{HH}^B) - (1-\beta)R_{HH}]+$$
$$+ \alpha_{HL}[w_H^A(x_{HL}^A) + w_L^B(x_{HL}^B) - (1-\beta)R_{HL}]+$$
$$+ \alpha_{LH}[w_L^A(x_{LH}^A) + w_H^B(x_{LH}^B) - (1-\beta)R_{LH}]$$

Clearly, the IR constraints of type HH is always met so that the potentially IR and IC binding constraints are the following

$$R_{HH} \geq R_{HL} + \delta^B(x_{HL}^B) \tag{10}$$
$$R_{HH} \geq R_{LH} + \delta^A(x_{LH}^A) \tag{11}$$
$$R_{HL} \geq 0 \tag{12}$$
$$R_{HL} \geq R_{HH} - \delta^B(x_{HH}^B) \tag{13}$$
$$R_{HL} \geq R_{LH} + \delta^A(x_{LH}^A) - \delta^B(x_{LH}^B) \tag{14}$$
$$R_{LH} \geq 0 \tag{15}$$

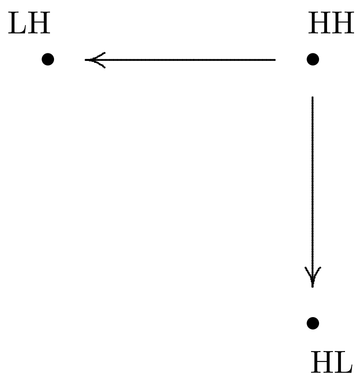

Fig. 3. Binding constraints – Case A.

$$R_{LH} \geq R_{HH} - \delta^A(x^A_{HH}) \tag{16}$$

$$R_{LH} \geq R_{HL} + \delta^B(x^B_{HL}) - \delta^A(x^A_{HL}) \tag{17}$$

For the analysis of the optimal contract we can identify three main cases. Each of these cases corresponds to a specific pattern of binding constraints as depicted in figures 3 and 4.

Case A.

Both the IC constraints, (10) and (11), of the high efficiency type, HH, are binding simultaneously, as depicted in figure 3. This case occurs when both the following conditions are satisfied,

$$\delta^A(\bar{x}^A_L) > \hat{\delta}^B\left(\frac{(1-\beta)\alpha_{HH}}{\alpha_{HL}}\right) \tag{18}$$

$$\delta^B(\bar{x}^B_L) > \hat{\delta}^A\left(\frac{(1-\beta)\alpha_{HH}}{\alpha_{LH}}\right) \tag{19}$$

Case B.

This case is characterized by the fact that condition (18) is violated, i.e.

$$\delta^A(\bar{x}^A_L) \leq \hat{\delta}^B\left(\frac{(1-\beta)\alpha_{HH}}{\alpha_{HL}}\right)$$

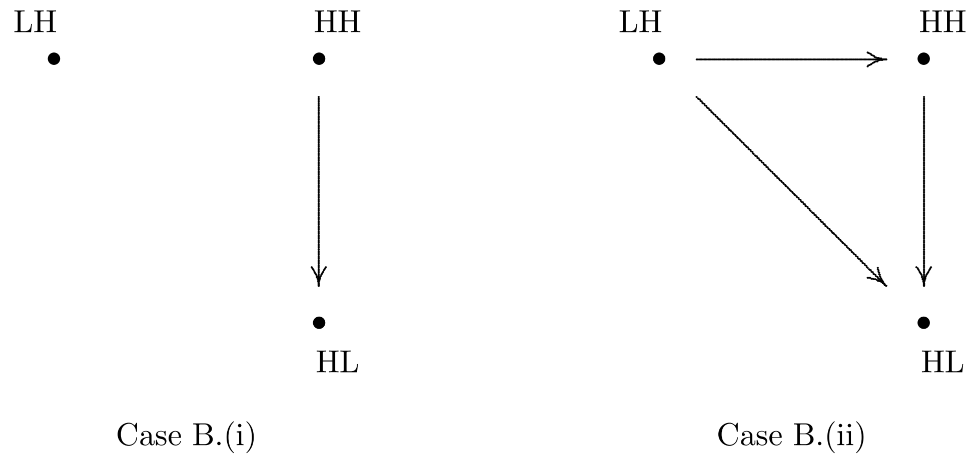

Fig. 4. Patterns of binding constraints – Case B

Under this condition the IC constraint of type HH with respect to type HL, (10), must necessarily be binding and we have the two possible patterns of binding IC constraints as depicted in figure 4.

Case B.(i). Only one IC constraint is binding and this occurs when the following condition holds:

$$\delta^A(\bar{x}_H^A) < \hat{\delta}^B \left(\frac{(1-\beta)\alpha_{HH}}{\alpha_{HL}} \right) \tag{20}$$

(notice that this means that (18) is violated since $\bar{x}_H^A > \bar{x}_L^A$).

Case B.(ii). This case occurs when (20) is violated and we have three IC binding constraints, (10), (16) and (17), as depicted in Figure 4.

Case C.

This case is characterized by the violation of (19). Since it is the mirror image of case B, it will not be treated explicitly.

Proposition 2.

Let $\alpha_{LL} = 0$. In case A the high efficiency type has a positive informational rent, i.e. $R_{HH} > 0$, and first best levels in both activities, i.e.

$x^A_{HH} = \bar{x}^A_H$ and $x^B_{HH} = \bar{x}^B_H$. The low efficiency activities of the mixed types, HL and LH, are distorted downwards, i.e. $x^A_{LH} < \bar{x}^A_L$ and $x^B_{HL} < \bar{x}^B_L$, and both types earn zero rent, thus $R_{HL} = R_{LH} = 0$.

In case B.(i), the LH and HH types have first-best levels in both activities; the level of activity B of the HL type is distorted downward, i.e. $x^B_{HL} < \bar{x}^B_L$. Only type HH has a positive rent while $R_{HL} = R_{LH} = 0$.

In case B.(ii), type LH has first best levels in both activities, $x^A_{LH} = \bar{x}^A_L$ and $x^B_{LH} = \bar{x}^B_H$, while type HH has an upward distortion in activity A, i.e. $x^A_{HH} > \bar{x}^A_H$. The mixed type HL has an upward distortion in activity A and a downward distortion in activity B, i.e. $x^A_{HL} > \bar{x}^A_H$ and $x^B_{HL} < \bar{x}^B_L$. Both, LH and HH have positive rents while $R_{HL} = 0$.

The proof is in Appendix B.

As we did is Section 3 we can consider the situation of perfect simmetry in order to see when the various cases apply. It is not difficult to check that under perfect simmetry (18) and (19) are always met, so that only case A can occur. Then we can conclude that cases B and C can only occur when the model displays strong asimmetries in types as well as activities.

The model studied in the present section exhibits several interesting features expecially in case B. In B.(i) it is quite unusual that the mixed type HL has first-best levels in both activities as well as the most efficient type, HH. Here, as in Section 3, the optimal contract implements pareto-optimal activity levels for two types out of three. Moreover, while HH has a positive rent, the type LH is efficient but does not earn any rent. Thus, in case B.(i) both the mixed types have zero rent.

However, the most striking situation is the one contemplated by case B.(ii). Indeed, it is one of the 'mixed' types, LH, who plays the role of the 'best' type and not the high efficiency agent, as is usual. LH has first-best activity levels while HH exhibits 'upward' distortions in activity A, in the sense that the optimal contract prescribes an activity level above first-best. Here we have a violation of the one-dimensional model rule which says 'no distortion at the top'.

Finally, another peculiar feature is the pattern of distortions imposed on the other mixed type by the optimal contract. Type HL displays upward distortion in the high efficiency activity and downward distortion in the low efficiency activity. The presence of both upward and downward distortions is an all mark of multi-dimensional screening problems as we have seen also in Section 3.

5. The model with only one 'mixed' type

In order to complete the analysis we study a model where one of the mixed types, let us say type LH, is missing, i.e. $\alpha_{LH} = 0$. Therefore, our model consists of the three types LL, HL and HH. The analysis of this case is very similar to the one-dimensional case, indeed, the three types can be completely ordered in terms of overal incremental utility. Following the method of Spence (1980) (see also Brighi D'Amato (1998)) for the analysis of the binding IR and IC constraints, it can be sen that the optimal contract satisfies the conditions $R_{LL} = 0$ and $R_{HL} = \delta^A(x^A_{LL})$ and the pattern of binding constraints is as depicted in figure 5. The principal's optimization problem can be written as follows:

$$\max_{R_{HH}, x^A_{LL}, x^B_{LL}, \dots} L = \alpha_{LL}[w^A_L(x^A_{LL}) + w^B_L(x^B_{LL})] + $$
$$+ \alpha_{HL}[w^A_H(x^A_{HL}) + w^B_L(x^B_{HL}) - (1-\beta)\delta^A(x^A_{LL})] + $$
$$+ \alpha_{HH}[w^A_H(x^A_{HH}) + w^B_H(x^B_{HH}) - (1-\beta)R_{HH}]$$

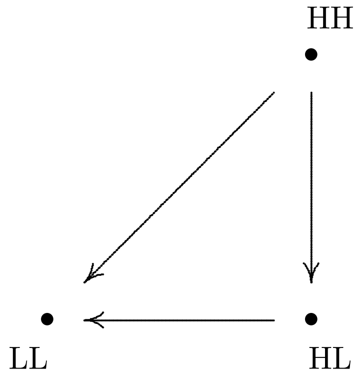

Fig. 5. Pattern of binding constraints

subject to the IC constraints of type HH

$$R_{HH} \geq \delta^A(x^A_{LL}) + \delta^B(x^B_{HL}) \tag{21}$$

$$R_{HH} \geq \delta^A(x^A_{LL}) + \delta^B(x^B_{LL}) \tag{22}$$

Proposition 3.

Let $\alpha_{LH} = 0$. The optimal contract has the following features: The high efficiency type, HH, has a strictly positive rent and first-best levels in both activities, i.e. $x^A_{HH} = \bar{x}^A_H$ and $x^B_{HH} = \bar{x}^B_H$. The 'mixed' type HL has a positive rent, an efficient level in activity A and underproduces in activity B, i.e. $x^A_{HL} = \bar{x}^A_H$, and $x^B_{HL} < \bar{x}^B_L$. The low efficiency type has zero rent and both the activity levels are distorted downward, i.e. $x^A_{LL} < \bar{x}^A_L$ and $x^B_{LL} = x^B_{HL} < \bar{x}^B_L$.

The proof is in Appendix C.

As it was expected, the optimal contract conforms to the standard one-dimensional case so that it obeys the well known rule of 'no distortion at the top, no rent at the bottom'.

6. Summary and conclusions

In this paper we have seen that even in a relatively simple screening model the multidimensionality of agent's private information generates several kinds of optimal contracts. We identify the conditions which discriminate between the various cases and determine the characteristics of the optimal contracts. These conditions have a fairly intuitive economic interpretation and can be easily computed from the primitive data of the model.

The main conclusion of our analysis of models with three types is that when either the most efficient or the least efficient type are missing, the optimal screening contracts may exhibit very unusual and peculiar characteristics. For example, it may well happen that not only the most efficient but also other types of agents have first-best activity levels. Also, we have a case where even the activity levels of the most efficient type, HH, are distorted and this is a striking violation of the single-dimensional optimal contracting rule saying 'no distortion at the top'. As we have seen, these 'anomalies' in optimal contracts are more likely to occur the stronger are the asimmetries in the technology of activities and in the probability of types.

Finally, we notice that typically in continuous type two-dimensional screening models the optimal contracts specify a 'non-participation' region where the least efficient types are excluded from any activity.[4] Accordingly, the model of Section 4, with only the high efficiency type and the two 'mixed' types, is perhaps the one which best approximate the properties of optimal contracts in multidimensional settings and which provides more reliable prescriptions in applications.

[4] This is the main result in Armstrong (1996).

APPENDIX A

The first-order conditions of the optimization problem in Section 3 are

$$\begin{aligned}
w_L^{A\prime}(x_{LL}^A) - \frac{\gamma_2}{\alpha_{LL}}\delta^{A\prime}(x_{LL}^A) &= 0\\
w_L^{B\prime}(x_{LL}^B) - \frac{\gamma_4}{\alpha_{LL}}\delta^{B\prime}(x_{LL}^B) &= 0\\
w_H^{A\prime}(x_{HL}^A) + \frac{\gamma_3}{\alpha_{HL}}\delta^{A\prime}(x_{HL}^A) &= 0\\
w_L^{B\prime}(x_{HL}^B) - \frac{\gamma_3}{\alpha_{HL}}\delta^{B\prime}(x_{HL}^B) &= 0\\
w_L^{A\prime}(x_{LH}^A) - \frac{\gamma_1}{\alpha_{LH}}\delta^{A\prime}(x_{LH}^A) &= 0\\
w_H^{B\prime}(x_{LH}^B) + \frac{\gamma_1}{\alpha_{LH}}\delta^{B\prime}(x_{LH}^B) &= 0
\end{aligned} \qquad (A.1)$$

$$\gamma_2 + \gamma_4 = (1-\beta)(\alpha_{HL} + \alpha_{LH}) \qquad (A.2)$$

$$\gamma_3 + \gamma_4 = \gamma_1 + (1-\beta)\alpha_{LH} \qquad (A.3)$$

where $\gamma_1 \geq 0$ is the Lagrange multiplier of constraint (3), γ_2 of (4), γ_3 of (5) and γ_4 of (5). From (A.2) we know that at least one of the two constraints (4) and (6) must be binding. Moreover, we see that the constraints (3) and (5) can not be simultaneously binding. Indeed, let γ_1, $\gamma_3 > 0$, so that (3) and (5) hold as equalities. Putting them together yields

$$\delta^A(x_{LH}^A) - \delta^A(x_{HL}^A) = \delta^B(x_{LH}^B) - \delta^B(x_{HL}^B) \qquad (A.4)$$

But, from (A.1) it is easy to see that, at a solution, $x_{HL}^A > x_{LH}^A$ and $x_{LH}^B > x_{HL}^B$ therefore (A.4) can not hold. Thus, γ_1, $\gamma_3 > 0$ can not occur.

Now we show that both the constraints (4) and (6) must be simultaneously binding , i.e. γ_2, $\gamma_4 > 0$.

If $\gamma_1 = \gamma_3 = 0$ it follows trivially from (A.2) and (A.3) that γ_2, $\gamma_4 > 0$. Let us suppose that $\gamma_3 > 0$ (and hence $\gamma_1 = 0$) and $\gamma_4 = 0$, so that $\gamma_2 > 0$.

Therefore, (4) and (5) hold as equalities and (6) hold as a strict inequality. Substituting and rearranging yields

$$\delta^B(x^B_{HL}) - \delta^B(x^B_{LL}) > \delta^A(x^A_{HL}) - \delta^A(x^A_{LL}) \tag{A.5}$$

From (A.1) and the values of the Lagrange multipliers we easily see that $x^B_{HL} < x^B_{LL}$ and $x^A_{HL} > x^A_{LL}$ and therefore eq. (A.5) can not hold. Thus, if $\gamma_3 > 0$ then $\gamma_4 > 0$, but then, from (A.2) and (A.3), it is easy to see that also $\gamma_2 > 0$.

In a similar way it can be shown that if $\gamma_1 > 0$ then γ_2, $\gamma_4 > 0$. Thus, we can summarize the above analysis as follows:

(i) The 'downward' incentive constraints are always binding, i.e. $\gamma_2 > 0$ and $\gamma_4 > 0$.

(ii) No more than one of the two off-diagonal incentive constraints can be binding.

Therefore, we only have three possible patterns of binding constraints

Case A. γ_1, γ_2, $\gamma_4 > 0$ and $\gamma_3 = 0$.

Case B. γ_3, γ_2, $\gamma_4 > 0$ and $\gamma_1 = 0$.

Case C. γ_2, $\gamma_4 > 0$ and $\gamma_1 = \gamma_3 = 0$.

Let us solve the pricipal problem in Case A. By substituting the equalities (4) and (6) in (3), we obtain

$$\delta^A(x^A_{LH}) - \delta^A(x^A_{LL}) = \delta^B(x^B_{LH}) - \delta^B(x^B_{LL}) \tag{A.6}$$

Now we find necessary and sufficient conditions for the above equation to hold. From (A.2) and (A.3) we write γ_4 and γ_1 in terms of γ_2,

$$\gamma_4 = (1-\beta)(\alpha_{HL} + \alpha_{LH}) - \gamma_2$$

$$\gamma_1 = (1-\beta)\alpha_{HL} - \gamma_2$$

Now, let us substitute γ_1 and γ_4 into (A.1) so that the optimal activity levels only depend on γ_2. Let us define the following functions

$$A(t) = \hat{\delta}^A \left[\frac{(1-\beta)\alpha_{HL} - t}{\alpha_{LH}} \right] - \hat{\delta}^A \left[\frac{t}{\alpha_{LL}} \right]$$
$$B(t) = \hat{\delta}^B \left[-\frac{(1-\beta)\alpha_{HL} - t}{\alpha_{LH}} \right] - \hat{\delta}^B \left[\frac{(1-\beta)(\alpha_{HL} + \alpha_{LH}) - t}{\alpha_{LL}} \right]$$

for $0 \leq t \leq (1-\beta)\alpha_{HL}$. We easily see that, $A(t)$ is increasing and $A(0) < 0$. On the other hand, $B(t)$ is decreasing, $B(0) > 0$ and it is always positive. Therefore, equation (A.6) holds for some $0 < t < (1-\beta)\alpha_{HL}$ if and only if $A[(1-\beta)\alpha_{HL}] > B[(1-\beta)\alpha_{HL}]$. Since for $t = (1-\beta)\alpha_{HL}$ we have $x^A_{LH} = \bar{x}^A_L$ and $x^B_{LH} = \bar{x}^B_H$ we obtain

$$\delta^A(\bar{x}^A_L) - \delta^B(\bar{x}^B_H) > \hat{\delta}^A \left[\frac{(1-\beta)\alpha_{HL}}{\alpha_{LL}} \right] - \hat{\delta}^B \left[\frac{(1-\beta)\alpha_{LH}}{\alpha_{LL}} \right]$$

The proof of Case B is analogous. Finally, the proof of Case C is a trivial consequence. The optimal contracts are computed in each case by solving for the value of the multipliers and using the equations of the IC constraints.

APPENDIX B

The first-order conditions of the optimization problem in Section 4 are

$$\begin{aligned}
w_H^{A'}(x_{HH}^A) + \frac{\mu_1}{\alpha_{HH}}\delta^{A'}(x_{HH}^A) &= 0 \\
w_H^{B'}(x_{HH}^B) + \frac{\lambda_1}{\alpha_{HH}}\delta^{B'}(x_{HH}^B) &= 0 \\
w_H^{A'}(x_{HL}^A) + \frac{\mu_2}{\alpha_{HL}}\delta^{A'}(x_{HL}^A) &= 0 \\
w_L^{B'}(x_{HL}^B) - \frac{\gamma_1 + \mu_2}{\alpha_{HL}}\delta^{B'}(x_{HL}^B) &= 0 \\
w_L^{A'}(x_{LH}^A) - \frac{\gamma_2 + \lambda_2}{\alpha_{LH}}\delta^{A'}(x_{LH}^A) &= 0 \\
w_H^{B'}(x_{LH}^B) + \frac{\lambda_2}{\alpha_{LH}}\delta^{B'}(x_{LH}^B) &= 0
\end{aligned} \qquad (B.1)$$

$$\gamma_1 + \gamma_2 = \lambda_1 + \mu_1 + (1-\beta)\alpha_{HH} \qquad (B.2)$$

$$\lambda_0 + \lambda_1 + \lambda_2 = \gamma_1 + \mu_2 + (1-\beta)\alpha_{HL} \qquad (B.3)$$

$$\mu_0 + \mu_1 + \mu_2 = \gamma_2 + \lambda_2 + (1-\beta)\alpha_{LH} \qquad (B.4)$$

where γ_1 and γ_2 are the Lagrange multipliers of constraints (10) and (11), λ_0, λ_1 λ_2 are the multipliers of (12), (13) and (14) and μ_0 μ_1, μ_2 are the multipliers of (15), (16) and (17). The complementary slackness conditions are omitted for simplicity.

From (B.2) at least one of the γ multipliers must be positive. Let us consider first the case where γ_1, $\gamma_2 > 0$. Thus, both the IC constraints of type HH are binding and we have

$$R_{HH} = R_{LH} + \delta^A(x_{LH}^A) = R_{HL} + \delta^B(x_{HL}^B) \qquad (B.5)$$

In can be easily checked that the constraints (13), (14), (16) and (17) are satisfied with strict inequality. For instance, let us take the constraint (13). From (B.1) we have $x_{HH}^B > x_{HL}^B$ so that $\delta^B(x_{HH}^B) > \delta^B(x_{HL}^B)$ and $R_{HL} > (R_{HL} + \delta^B(x_{HL}^B)) - \delta^B(x_{HH}^B)$ so that by equality (10) we have that (13) is satisfied with strict inequality. In a similar way it can be seen that also (14), (16) and (17) are strictly satisfied so that we must have $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. From (B.3) and (B.4) $\lambda_0 > 0$, $\mu_0 > 0$ so that $R_{HL} = R_{LH} = 0$ and from (B.5) it follows

$$\delta^A(x^A_{LH}) = \delta^B(x^B_{HL}) \qquad (B.6)$$

In order to find necessary and sufficient conditions for $(B.6)$ to hold let us express γ_1 in terms of γ_2, by using (B.2), i.e.

$$\gamma_1 = (1-\beta)\alpha_{HH} - \gamma_2$$

Substitute γ_1 in (B.1) and let γ_2 vary between 0 and $(1-\beta)\alpha_{HH}$. Therefore let us define

$$A(\gamma_2) = \hat{\delta}^A\left(\frac{\gamma_2}{\alpha_{LH}}\right) \qquad \text{and} \qquad B(\gamma_2) = \hat{\delta}^B\left(\frac{(1-\beta)\alpha_{HH} - \gamma_2}{\alpha_{HL}}\right)$$

The function $A(\gamma_2)$ is decreasing and $A(0) = \delta^A(\bar{x}^A_L)$ and $A((1-\beta)\alpha_{HH}) = \delta^A((1-\beta)\alpha_{HH}/\alpha_{LH})$. The function $B(\gamma_2)$ is increasing and $B(0) = \delta^B((1-\beta)\alpha_{HH}/\alpha_{HL})$ and $B((1-\beta)\alpha_{HH}) = \delta^B(\bar{x}^B_L)$.

Therefore, we see that there exists $0 < \gamma_2^* < (1-\beta)\alpha_{HH}$ such that (B.6) holds if and only if

$$\hat{\delta}^A(0) > \hat{\delta}^B\left(\frac{(1-\beta)\alpha_{HH}}{\alpha_{HL}}\right)$$

and

$$\hat{\delta}^A\left(\frac{(1-\beta)\alpha_{HH}}{\alpha_{LH}}\right) < \hat{\delta}^B(0)$$

and these are respectively conditions (18) and (19) in the text.

Let us turn to case B and suppose that $\gamma_1 > 0$ and $\gamma_2 = 0$ so that

$$R_{HH} = R_{HL} + \delta^B(x^B_{HL}) > R_{LH} + \delta^A(x^A_{LH}) \qquad (B.7)$$

From (B.1) we know that $x^B_{HH} > x^B_{HL}$ and $x^B_{LH} > x^B_{HL}$, therefore, it is not difficult to see that at the optimal contract the IC constraints of type HL, i.e. (13) and (14), are satisfied with a strict inequality. Hence $\lambda_1 = \lambda_2 = 0$ so that, by (B.3), $\lambda_0 > 0$. Thus we have $R_{HL} = 0$ and, from (10), $R_{HH} = \delta^B(x^B_{HL})$.

Let us consider the remaining constraints of type LH and the μ multipliers. Consider first the case where $\mu_0 > 0$, so that $R_{LH} = 0$. It can be shown that either (i) both μ_1 and μ_2 are equal to zero or (ii) both are strictly positive. Indeed, let $\mu_1 = 0$, i.e. (16) holds as a strict inequality so that we have $\delta^A(x^A_{HL}) > \delta^B(x^B_{HL})$. From $B.1$ we have $x^A_{HH} \geq x^A_{HL}$ so that also (17) holds a s a strict inequality and $\mu_2 = 0$. Using a similar argument it can be shown that also $\mu_2 = 0$ implies $\mu_1 = 0$. This proves (i). Point (ii) is proved similarly.

Case B.(i) in Section 4 corresponds to the following pattern of multipliers: $\mu_0 > 0$, $\mu_1 = 0$ and $\mu_2 = 0$. Indeed, by (B.1) we know that $x^A_{HH} = x^A_{HL} = \bar{x}^A_H$. The IC constraints (15) and (16) require in this case

$$\delta^B(x^B_{HL}) - \delta^A(x^A_{HL}) < 0 \tag{$*$}$$

Moreover, since $\gamma_1 = (1-\beta)\alpha_{HH}$, by (B.1) we have

$$\delta^B(x^B_{HL}) = \hat{\delta}^B(\gamma_1 = (1-\beta)\alpha_{HH}/\alpha_{HL})$$

so that (*) corresponds to condition ??(20).

Let us turn to case B.(ii) which is characterized by the following pattern of multipliers: $\mu_0 = 0$, $\mu_1 > 0$ and $\mu_2 > 0$. Since $\mu_0 = 0$ we have $R_{LH} > 0$. From (B.4) at least one of the μ's must be strictly positive. Let us suppose that $\mu_2 > 0$, hence (16) holds as an equality and $R_{LH} = \delta^B(x^B_{HL}) - \delta^A(x^A_{HL}) > 0$. Moreover, using (17) we obtain $\delta^A(x^A_{HH}) \geq \delta^A(x^A_{HL})$, that by (B.1) requires $\mu_1 > 0$. Similarly it can be shown that $\mu_1 > 0$ implies $\mu_2 > 0$. Finally, the pattern of multipliers implies $\delta^A(x^A_{HH}) = \delta^A(x^A_{HL})$ so that we have $x^A_{HH} = x^A_{HL}$.

APPENDIX C

The first-order conditions of the optimization problem in Section 5 are

$$
\begin{aligned}
w_L^{A\prime}(x_{LL}^A) - \frac{(\gamma_1 + \gamma_2 + (1-\beta)\alpha_{HL})}{\alpha_{LL}}\delta^{A\prime}(x_{LL}^A) &= 0 \\
w_L^{B\prime}(x_{LL}^B) - \frac{\gamma_2}{\alpha_{LL}}\delta^{B\prime}(x_{LL}^B) &= 0 \\
w_H^{A\prime}(x_{HL}^A) &= 0 \\
w_L^{B\prime}(x_{HL}^B) - \frac{\gamma_1}{\alpha_{HL}}\delta^{B\prime}(x_{HL}^B) &= 0 \\
w_H^{A\prime}(x_{HH}^A) &= 0 \\
w_H^{B\prime}(x_{HH}^B) &= 0
\end{aligned}
\qquad (C.1)
$$

$$
\gamma_1 + \gamma_2 = (1-\beta)\alpha_{HH} \qquad (C.2)
$$

where γ_1 and γ_2 are the Lagrange multiplier of IC constraints of type HH. From C.1 and C.2 it is easily seen that both the multipliers must be strictly positive so that, by (21) and (22), we have $x_{HL}^B = x_{LL}^B$ and we obtain

$$
\begin{aligned}
\gamma_1 &= (1-\beta)\frac{\alpha_{HL}\alpha_{HH}}{\alpha_{LL} + \alpha_{HL}} \\
\gamma_2 &= (1-\beta)\frac{\alpha_{LL}\alpha_{HH}}{\alpha_{LL} + \alpha_{HL}}
\end{aligned}
$$

From the values of the multipliers it is easy to compute all the feature of the optimal contract.

References

Armstrong M. (1996). Multiproduct nonlinear pricing. Econometrica, 64, 51-75.

Armstrong M. and J-C. Rochet (1999). Multi-Dimensional Screening: A User's Guide. European Economic Review, 43, 959-979.

Baron, D. and R. Myerson (1982). Regulating a Monopolist with Unknown Costs. Econometrica, 50, 911-930.

Brighi, L. and M. D'Amato (2002). Two-dimensional screening: A case of monopoly regulation. Research in Economics, 56, 251-264.

Dana, J. D. (1993). The Organization and Scope of Agents: Regulating Multiproduct Industries. Journal of Economic Theory, 59, 288-310.

Rochet, J-C. and P. Choné (1998). Ironing, sweeping and multidimensional screening. Econometrica, 66, 783-826.

Rochet, J-C. and L. A. Stole (2003). The Economics of Multidimensional Screening, in Dewatripont, M. L.P. Hansen and S. Turnowsky (eds.) Advances in Economics and Econometrics: Theory and Applications. Cambridge, CUP.

Spence, A. M. (1980). Multi-product Quantity-dependent Prices and Profitability Constraints. Review of Economic Studies, 47, 821-842.