OPTION IMPLIED TREES WHEN THE PUT-CALL PARITY IS NOT FULFILLED*

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Abstract

Standard methodologies for the derivation of implied trees from option prices are based on the validity of the put-call parity. Muzzioli and Torricelli (2002) propose a methodology which accounts for PCP violations. Based on this latter approach the present paper advances in two main directions. First we propose a different methodology in order to imply the interval of artificial probabilities at each node of the tree. Secondly, we perform an empirical validation of the implied tree obtained, both in the sample and out of sample, by using DAX index options data set covering the period from January 4, 1999 to December 28, 2000. Numerical results are compared with one of the most used standard methodologies, i.e. Derman and Kani's. The results suggest that the estimation proposed, by taking into account the informational content of both call and put prices, highly improves both the in-the-sample fitting and the out-of-sample performance.

Keywords: *Binomial Method, Put-Call Parity, Choquet Pricing, Interval Tree.*

JEL classification: G13, G14.

1. INTRODUCTION

In the literature, various methods for deriving implied trees consistent with the so-called smile effect and the term structure of volatility have been proposed (see among others Derman and Kani (1994), Rubinstein (1994), Barle and Cakici (1995)). However, these methods disregard the issue of a different volatility implied by call and put prices (written on the same underlying and with the same strike price and time to maturity). Empirical evidence is presented, among others, by Chesney, Gibson and Loubergé (1995) and Cavallo and Mammola (2000), about the fact that the volatility implied by call prices is generally lower than the one implied by put prices. When there is

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a substantial difference between the volatility implied by call and put prices, substantial pricing errors may derive from taking into account only one set of options.

Muzzioli and Torricelli (2002) address the issue and obtain the following results. They propose a methodology for the derivation of implied trees that takes into account the information stemming from both call option and put option prices. The method basically extends Derman and Kani's (1994), whereby call (put) prices are also used in the lower (upper) part of the tree thus exploiting the information content of both call and put prices. The tree, called the PC-implied tree, is characterised by interval values for the underlying stock prices and probabilities. In Muzzioli and Torricelli (2002) the authors do not provide any implementation with market data of the methodology proposed and, as a consequence, they do not address the issue of selecting a crisp implied tree among all the possible implied trees.

In Muzzioli and Torricelli (2003), the authors tackle the implementation issue, by providing an example on two dates that shows how the resulting tree captures the different smiles of call and put options. They address two main problems that arise from the implementation issue. First they suggest a way to check for no arbitrage when aggregating the call and put implied trees. Secondly, they provide a methodology in order to select, among all the possible implied trees, a single tree characterised by crisp values for stock prices and probabilities.

The aim of our paper is twofold. First, by dropping assumption A3) of Muzzioli and Torricelli (2002), we propose a different methodology in order to imply the interval of artificial probabilities at each node. This methodology results in a different estimation (w.r.t. Muzzioli and Torricelli (2003)) of the implied tree by means of market data. Secondly, we perform an empirical validation of the implied tree obtained, both in the sample and out of sample, by using the DAX index options data set covering the period from January 4, 1999 to December 28, 2000. Numerical results are compared with Derman and Kani's.

The plan of the paper in the next sections is the following. In Section 2 we briefly recall the Muzzioli and Torricelli (MT) method for the derivation of the put-call (PC) implied tree and we highlight the different methodology proposed in the present article in order to extract the artificial probability interval at each node. In Section 3 we explain the estimation procedure used to select crisp values for the stock prices and the probabilities. In section 4 we present the data set and the methodology used in the implementation. In Sections 5 and 6 we illustrate the in-the-sample and out-of-sample performance respectively, compared to Derman and Kani's (DK). The last section concludes. In the Appendix we show the derivation of the risk neutral probability interval.

2. THE METHODOLOGY FOR THE DERIVATION OF THE PC-IMPLIED TREE

In this section we illustrate, by means of an example, the Muzzioli and Torricelli (2002) method for the derivation of the stock price intervals of the put-call (PC) implied tree, and we highlight the different methodology we propose in order to extract the artificial probability interval at each node. Specifically, the main difference stems from the fact that, while keeping assumptions A1) and A2) of the Muzzioli and Torricelli (2002) method, we drop assumption A3).

Derman and Kani's (1994) method implicitly assumes the validity of the Put-Call-Parity (PCP). In order to apply the same type of methodology to markets characterised by PCP violations and to exploit the whole market information, the MT method basically extends Derman and Kani's by using call prices also in the lower part of the tree and put prices also in the upper part, thus exploiting all the information content of call and put prices. In the following we provide an example that illustrates how the tree is implied (for details, see the original paper).

Essentially, the MT method proposes: a) to construct two implied trees, one using only the information provided by call options and the other using only information provided by put options; b) to aggregate the two trees by taking the implied stock prices as bounds for a price interval.

The initial inputs are: the term structure of risk-free interest rates, the stock price at time zero and the interpolated smile function for call and put prices belonging to a class characterised by a specific date and a specific maturity. In this example we construct a PC-implied tree with n=6 levels, on 6th January 2000, for options with maturity February. The Dax-index value is 6949.47, the interpolated risk-free rate is 3.19%, the smile function for call options is $\sigma_C = 0.38132547 - 0.00001995X$ and the smile function for put options is $\sigma_P = 0.44034707 - 0.00001964X$, where X is the strike price and σ is the volatility.

The same methodology as Derman and Kani's for the derivation of the implied tree is used, the only difference being the use of call options also in the lower part of the tree for the derivation of the call implied tree and the use of put options also in the upper part of the tree for the derivation of the put implied tree. At each node, the no-arbitrage check, the Barle and Cakici (1995) condition is used to exclude arbitrage. The two trees obtained are reported in Figures 2 and 3 respectively.

The MT method suggests the aggregation of the two trees into a single tree characterised by interval values for stock prices, whose bounds are, at each node of the tree, the minimum and the maximum of the stock prices implied by the call and the put options. In this way all the information stemming from call and put prices is summed up in a single tree.

					8073.957
				7490.493	
			7248.876		7249.436
		6995.608		6995.974	
	6750.36		6750.561		6750.764
6494.47		6494.47		6494.47	
	6248.28		6248.094		6247.907
		5992.192		5991.787	
			5747.59		5747.01
				5494.278	
					5254.56

Figure 1. The tree implied by using only call prices.

					8073.957
				7754.148	
			7444.882		7445.777
		7122.997		7123.569	
	6813.899		6814.208		6814.519
6494.47		6494.47		6494.47	
	6190.015		6189.735		6189.452
		5876.976		5876.378	
			5581.05		5580.217
				5278.776	
					4995.976

Figure 2. The tree implied by using only put prices.

					8073.96
					8073.96
				7754.15	
				7490.49	
			7444.88		7445.78
			7248.88		7249.44
		7123		7123.57	
		6995.61		6995.97	
	6813.9		6814.21		6814.52
	6750.36		6750.56		6750.76
6494.47		6494.47		6494.47	
6494.47		6494.47		6494.47	
	6248.28		6248.09		6247.91
	6190.02		6189.73		6189.45
		5992.19		5991.79	
		5876.98		5876.38	
			5747.59		5747.01
			5581.05		5580.22
				5494.28	
				5278.78	
					5254.56
					4995.98

Figure 3. The P-C implied tree for the stock prices.

When aggregating the two trees, it is necessary to check, at each step, that the newly determined stock prices and probabilities are consistent with the no arbitrage condition, so that the PC-implied tree is arbitrage free. Note that, as the volatility implied by put prices is bigger than the one implied by call prices, the put implied prices are the most external. The resulting tree is reported in Figure 3 and a generalisation of two levels of the same is illustrated in Figure 4.

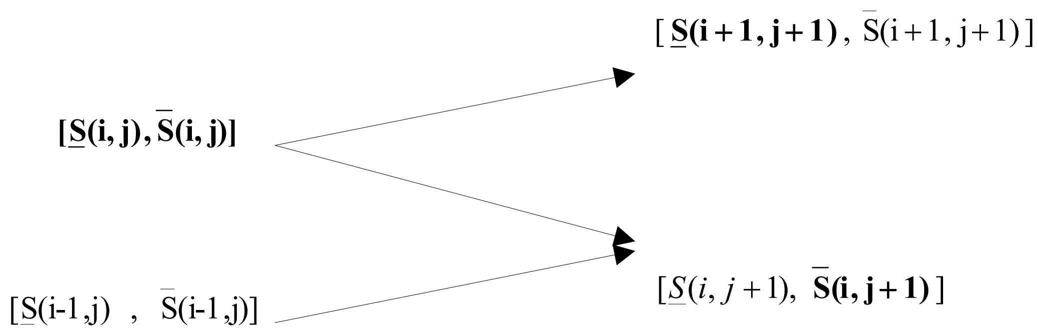

Figure 4. Four nodes of the Pc-implied tree.

Muzzioli and Torricelli (2002), by means of assumption A3), suggest the implication of the interval of artificial probabilities at each node, by taking the minimum and the maximum of the artificial probabilities implied by the call or the put options respectively. In this paper we drop assumption A3) and we derive the artificial probabilities endogenously by using the standard risk neutral valuation formula:

$$p(i+1, j+1) = \frac{S(i, j)e^{r\Delta t} - S(i, j+1)}{S(i+1, j+1) - S(i, j+1)}$$

We are thus in a position to find the largest interval of risk neutral probabilities consistent with the given stock price intervals. Specifically, the set of all the artificial probabilities $[\underline{p}(i+1, j+1), \overline{p}(i+1, j+1)]$ consistent with the stock prices $[\underline{S}(i, j+1), \overline{S}(i, j+1)]$, $[\underline{S}(i+1, j+1), \overline{S}(i+1, j+1)]$ and $[\underline{S}(i, j), \overline{S}(i, j)]$, is given by the following interval:

$$[\underline{p}(i+1, j+1), \overline{p}(i+1, j+1)] = \left[\frac{\underline{S}(i, j)e^{r\Delta t} - \overline{S}(i, j+1)}{\overline{S}(i+1, j+1) - \overline{S}(i, j+1)}, \frac{\overline{S}(i, j)e^{r\Delta t} - \underline{S}(i, j+1)}{\underline{S}(i+1, j+1) - \underline{S}(i, j+1)}\right]. \qquad (1)$$

In the Appendix we show the derivation and the properties of this risk neutral probability interval.

Note that the interval of artificial probabilities proposed in Muzzioli and Torricelli (2002) is contained in this interval.

3. THE ESTIMATION PROCEDURE

In this section we propose a different estimation procedure of the implied tree, consistently with the different methodology used to imply the artificial probability intervals. As stressed in the previous section, the latter differs from Muzzioli and Torricelli (2003) and so does the estimation procedure.

In Muzzioli and Torricelli (2003), the authors propose to identify parameters for both stock prices and probabilities as follows:

$$p_\alpha(i,j) = \alpha \underline{p}(i,j) + (1-\alpha)\overline{p}(i,j)$$

$$S_\alpha(i,j) = \alpha \underline{S}(i,j) + (1-\alpha)\overline{S}(i,j)$$

where $\alpha \in [0,1]$.

and estimate the parameter α, by solving the following non linear optimisation problem:

$$\min_{\alpha} \sum_{i=1}^{m} (f_T(\alpha) - f_M)^2$$

s.t. $\alpha \in [0,1]$ and $S(0,0) = \sum_{j=1}^{n} \lambda_\alpha(j,n) * S_\alpha(j,n)$

where $f_T(\alpha)$ is the theoretical price of the option, f_M is its market price, m is the number of options and the constraint on the stock price is necessary in order to have a risk neutral tree.

In this application we only identify a parameter for the stock prices and we derive the corresponding artificial probabilities (by means of the risk neutral argument).

Two methods are used and compared: Method 1, based on a single parameter, and Method 2, based on two parameters.

Method 1) $S_v(i,j) = \alpha \underline{S}(i,j) + (1-\alpha)\overline{S}(i,j)$, $\quad i = 1,\ldots j+1;\ j = 0,\ldots,n;$. (2)

Method 2) $S_v(i,j) = \begin{cases} \alpha \underline{S}(i,j) + (1-\alpha)\overline{S}(i,j) & i > j \\ \beta \underline{S}(i,j) + (1-\beta)\overline{S}(i,j) & otherwise \end{cases}$ $\quad j = 0,\ldots,n;$ (3)

where $\alpha, \beta \in [0,1]$.

Method 2 better captures the smile effect, since it separates the nodes in the upper part, usually characterised by lower volatility, from the nodes in the lower part of the tree, usually characterised by higher volatility.

At each node (i,j) of the tree, the corresponding risk neutral probabilities are derived as follows:

$$p_v(i,j) = \frac{S_v(i,j-1)e^{r\Delta t} - S_v(i,j)}{S_v(i+1,j) - S_v(i,j)}. \quad (4)$$

The Arrow-Debreu prices $\lambda_v(i,j)$ are computed by forward induction as the sum over all paths leading to node (i, j) of the product of the parameterised risk neutral probabilities of an up move, $p_v(i,j)$, discounted at the risk-free rate at each node in each path.

The theoretical call and put prices are:

$$C_v = \sum_{j=1}^{n} \lambda_v(j,n) \cdot \max(S_v(j,n) - K,\ 0)$$

$$P_v = \sum_{j=1}^{n} \lambda_v(j,n) \cdot \max(K - S_v(j,n),\ 0)$$

We estimate the parameter vector ν, by solving the following non linear optimisation problem:

$$\min_{v} \sum_{i=1}^{m} (f_T(v) - f_M)^2 \qquad (5)$$

where $f_T(v)$ is the theoretical price of a call (C_v) and a put option (P_v), f_M is its market price, m is the number of options in the class.

4. NUMERICAL EXPERIMENTS

In this section we describe the data set used, and illustrate the implementation of our methodology.

4.1 THE DATA SET

To evaluate the accuracy of the PC-implied tree, with respect to the DK method, in fitting the observed option market prices of options, we provide an application to DAX-index options.

We use the DAX-index options market for two main reasons: first, it is a relatively new European market where short-selling restrictions may induce put-call parity deviations (see e.g. Mittnik S. and Rieken S., 2000); secondly, the nature of the option (European) and of the underlying (dividends reinvested in the index) simplifies the estimation because neither the estimation of the early exercise premium nor the estimation of the dividend payments is required.

DAX-options started trading on the German Options and Futures Exchange (EUREX) in August 1991. They are European options on the DAX-index, which is a capital weighted performance index composed of 30 major German stocks and is adjusted for dividends, stocks splits and changes in capital. Since dividends are assumed to be reinvested into shares, they do not affect the index value. This latter feature, together with the option type (European), avoids the estimation problems of both dividend payments and early exercise premium.

DAX-index options are quoted in index points, carried out one decimal place. The contract value is EUR 5 per DAX index point. The tick size is 0.1 of a point representing a value of EUR 0.50. They are cash settled, payable on the first exchange trading day immediately following the last trading day. The last trading day is the third Friday of the expiry month, if that is an exchange trading day, otherwise the exchange trading day immediately prior to that Friday. The final settlement price is the value of the DAX determined on the basis of the collective prices of the shares contained on the DAX index as reflected in the intra-day trading auction on the electronic system of the Frankfurt Stock Exchange. Expiry months are the three near calendar months within the cycle March, June, September and December as well as the two following months of the cycle June and December.

The data set consists of settlements of DAX-index options, with maturities up to one year, recorded from January 4, 1999 to December 28, 2000. The data set on DAX-index options is kindly provided by Deutsche Börse AG. For the underlying, we used the settlements of the DAX-index in the same time period. As a proxy for the risk-free rate we use the Fibor rates with maturities up to one year: the appropriate yield to maturity is computed by linear interpolation. The data set on DAX-index and risk-free rates is available in Data-Stream.

4.2 THE METHODOLOGY

Options with same expiry month and year are grouped in 2882 classes. In each class the average number of options is 76 and the number of call options is always equal to the number of put options.

The volatility smile is estimated separately for call and put prices in each class, by using market prices to compute the volatility implied by the Black and Scholes formula. The obtained implied volatilities are then interpolated with respect to the strike price by means of a linear[1] function of the following form:

$\sigma(X) = a_0 + a_1 X$

To compute the implied volatilities we use the bisection method in C++. The volatility smile is obtained solving a least square problem implemented in GAMS ver. 20.7, using the solver MINOS 5.4.

As for the Derman and Kani method, the inputs are: the underlying spot price, the risk-free interest rate and the smile. Recall that in this application we assume that the smile is the same at each date until maturity. The DK method is characterised by taking the smile implied by call

[1] The linear function has been chosen after a comparison with other usual polynomial interpolations for two main reasons: first, the shape of the implied volatilities of DAX-index options is much closer to a linear shape; secondly, it avoids over-parameterisation.

options for strikes greater or equal to the value of the underlying and that implied by put options for strikes smaller than the value of the underlying. The DK tree is then derived following the procedure detailed in Derman and Kani (1994), with the only exception of the Barle and Cakici condition which is used to check for no arbitrage violations instead of the one proposed in Derman and Kani (1994).

As for the Muzzioli and Torricelli method, the inputs are: the underlying value, the risk-free interest rate and the two smile functions implied by call and put options. We derive two trees, one using the smile implied by call options and the other using the smile implied by put options. Each tree is derived following the Derman and Kani methodology recalled above. Then we aggregate them in order to have a single implied tree with interval values for the stock prices. At each newly determined node we check the no arbitrage condition by means of the procedure explained in Muzzioli and Torricelli (2003). At each node, we parameterise the stock prices by means of equation (2) for Method 1 and equation (3) for Method 2 and we derive the interval of the artificial probabilities by means of equation (4). Finally, we run the non linear optimisation routine (5) in order to get crisp values for the stock prices and probabilities. The routine is implemented in GAMS ver. 20.7 and solved by MINOS 5.4. through a reduced-gradient algorithm (cf. Wolfe, 1962) combined with a quasi-Newton algorithm that is described in Murtagh and Saunders (1978).

As the choice of the number (odd or even) of the binomial tree levels implies different estimates of the price, both the PC-tree and the DK tree prices are computed as the average between odd and even levels. We assume a binomial tree with 25 and 26 levels.

5. IN-THE-SAMPLE PERFORMANCE

The aim of this section is to evaluate and compare the quality of the three methods (Method 1, Method 2 and DK) in fitting options market prices. To this end, three indicators are selected and computed on each day for each option class and then averaged across the sample: the sum of squared errors (SSE), the sum of squared relative pricing errors (SSRPE) and the index of mispricing (MISP); they are respectively defined as follows:

$$SSE = \frac{1}{m}\sum_{i=1}^{m}(P_i^T - P_i^M)^2$$

$$SSRPE = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{P_i^T - P_i^M}{P_i^M}\right)^2$$

$$MISP=\frac{\sum_{i=1}^{m}\left(\frac{P_i^T-P_i^M}{P_i^M}\right)}{\sum_{i=1}^{m}\left|\frac{P_i^T-P_i^M}{P_i^M}\right|}$$

P_i^T and P_i^M are respectively the theoretical and the market price of option i , $i = 1, \ldots, m$ and m is the number of options in the class.

The SSE is an indicator of the implied tree fit to option prices, it naturally increases with the moneyness of the option. By minimising the SSE, more importance is therefore placed on in- the-money options. In contrast, the SSRPE, being a percentage error, does not suffer from such a drawback. However, for out-of-the-money options (i.e. very low prices), the SSRPE is typically higher than it is for other option classes. The mispricing index ranges from –1 to 1 and indicates, on average, the overpricing (positive MISP) or underpricing (negative MISP) induced by the method. Table 1 reports the results for Method 1, Method 2, and Derman and Kani's (DK).

	SSE	**SSRPE**	**MISP**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DK	1947.77	166.75	-0.61	1618.53	252.45	-0.46	2277.00	81.06	-0.64
Method 1	248.13	11.51	-0.12	250.70	19.73	0.16	245.55	3.29	-0.35
Method 2	230.67	9.11	-0.32	197.40	15.12	-0.01	263.94	3.09	-0.47

Table 1. The results for the whole sample.

DK's SSE is much higher (about nine times) than the one obtained by means of Methods 1 and 2, which suggests that both Methods 1 and 2 obtain a better fit to market prices. The SSRPE is highly in favour of both Methods 1 and 2, particularly for put prices. The MISP indicates that the Derman and Kani method substantially underprices both option classes, while in Methods 1 and 2 the underpricing is lower, especially for call options. Method 1 underprices both classes of options less than Method 2. Overall, Method 2 performs slightly better than Method 1, due to the better pricing of call options, even if it underprices both classes of options more than Method 1.

In order to analyse the different performance of Methods 1 and 2, we report the average parameter estimates in Table 2 (standard errors are in parenthesis).

	Method 1	**Method 2**
alpha	0.65 (0.42)	0.88 (0.25)
beta		0.23 (0.33)

Table 2. The average parameter estimates in Methods 1 and 2.

The overall better fit of Method 2 can thus be explained with the difference in the two parameter estimates and the associated lower standard errors. The two parameter estimates in Method 2 indicate that nodes in the upper part of the tree have a different volatility content than nodes in the lower part, a feature which is only partially captured by Method 1, where the single parameter alpha is roughly a weighted average of the two parameters in Method 2.

Moreover, in Method 2, alpha is bigger than beta, indicating on average a preference for the lower bound of the stock price intervals in the upper part of the tree and a preference for the upper bound of the interval in the lower part of the tree. It follows that on average the volatility implied by Method 2 is less than the one implied by Method 1; as a result, the underpricing of each option class is bigger in Method 2 than in Method 1.

In order to detect which option class is better priced by each method, we divide options according to their time to maturity and their moneyness. In Table 3 options are divided into four classes depending on their time to maturity.

All three methods obtain a better pricing performance for short term options. However, Methods 1 and 2 are much better than Derman and Kani's in relation to the SSE, especially for long term options. Note that the SSE naturally increases from short term to long term options. The SSRPE displays an opposite pattern (it is usually higher for less expensive options) and is better in Methods 1 and 2 except for the two intermediate time to maturity classes. Underpricing is severe with the Derman and Kani method, particularly for puts. Method 2 is slightly better than Method 1 for calls, while the opposite is true for puts.

Overall, Method 2 obtains a better performance than Method 1 in terms of SSE, while it underprices more (overprices less) than Method 1.

DK									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
0-2 month	912.3868	505.2384	-0.5452	844.3079	766.5735	-0.5680	980.4657	243.9033	-0.5438
2-4 month	1430.3877	1.6028	-0.5917	1253.4572	0.0613	-0.4489	1607.3183	3.1444	-0.6134
4-8 month	2500.6870	0.1040	-0.6943	2066.7509	0.0359	-0.4331	2934.6232	0.1722	-0.7500
8-12 month	3435.8224	0.0548	-0.6249	2672.3172	0.0093	-0.3365	4199.3275	0.1003	-0.6990
Method 1									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
0-2 month	229.7189	24.8059	0.0758	248.6046	39.7172	0.0044	210.8332	9.8946	-0.0795
2-4 month	216.8319	14.7690	-0.1321	221.8131	29.4537	0.1075	211.8507	0.0843	-0.3474
4-8 month	273.0739	0.4573	-0.2918	284.0246	0.8802	0.1956	262.1233	0.0343	-0.5435
8-12 month	280.5897	0.0103	-0.2101	247.2621	0.0091	0.3912	313.9173	0.0114	-0.5664
Method 2									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
0-2 month	189.4195	19.0855	-0.1173	174.8312	28.8777	-0.1264	204.0077	9.2934	-0.2423
2-4 month	215.6399	12.4311	-0.3613	172.1900	24.7829	-0.0870	259.0898	0.0792	-0.4311
4-8 month	256.3992	0.4201	-0.4466	234.2797	0.8039	0.0604	278.5188	0.0363	-0.6370
8-12 month	280.4152	0.0107	-0.4356	217.3034	0.0092	0.1880	343.5269	0.0123	-0.6578

Table 3. The three methods and time to maturity.

Table 4 shows how the average parameter estimates vary in Methods 1 and 2, depending on time to maturity. The parameter estimates appear quite stable across classes.

	Method 1 Alpha	**Method2 alpha**	**Method2 beta**
0-2 month	0.76	0.87	0.14
	(0.41)	(0.27)	(0.29)
2-4 month	0.59	0.85	0.18
	(0.44)	(0.28)	(0.30)
4-8 month	0.66	0.88	0.32
	(0.40)	(0.24)	(0.35)
8-12 month	0.53	0.89	0.30
	(0.39)	(0.18)	(0.35)

Table 4. The parameter estimates for Method 1 and Method 2.

In Method 2, alpha is always bigger than beta, indicating that the volatility implied by Method 2 is less than the one implied by Method 1. It follows that the underpricing (overpricing) of each option class is bigger in Method 2 than in Method 1.

In Table 5 we report the performance of the Derman and Kani method, Method 1 and Method 2 in relation to the moneyness of the options. According to the indicator of moneyness $M = S/(Ke^{-rT})$, where S is the underlying value and K the strike price of the option, we divide call and put option prices into the following five classes: DOM (deep-out-of-the-money, call options: $M < 0.9$ and put options: $M > 1.1$), OM (out-of-the-money, call options: $0.9 \leq M < 0.98$ and put options: $1.02 < M \leq 1.1$), AM (at-the-money, call and put options: $0.98 \leq M \leq 1.02$), IM (in-the-money, call options: $1.02 < M \leq 1.1$ and put options $0.9 \leq M < 0.98$), DIM (deep-in-the-money, call options: $M > 1.1$ and put options: $M < 0.9$).

The SSE indicates that Methods 1 and 2 perform better than Derman and Kani's in each class of moneyness, particularly for in-the-money calls and out-of-the-money puts. It follows that the worst fit of the DK method is observed in the lower part of the tree, i.e. the one derived by using out-of-the-money puts. The SSRPE of Methods 1 and 2 is always lower than Derman and Kani's, except for IM and DIM puts (but the difference is very low). The MISP indicates that the Derman and Kani method underprices most classes of options: the highest underpricing is observed for DIM and DOM call and put options.

As expected, Method 2 underprices every option class more than Method 1. Method 2 obtains a better pricing performance for IM AM and OM options, the opposite holds for DIM and DOM options. Method 2 is better than Method 1 in the pricing of Call options, while the opposite holds for Put options. As the volatility implied by call options is usually lower than the one implied by put options, Call options were expected to be priced better by the lower volatility tree (Method 2).

DK									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DOM	1922.343	182.0114	-0.70966	177.1344	349.6053	-0.60033	3667.551	14.41749	-0.82091
OM	2274.105	24.21194	-1.25414	528.8513	48.1287	-0.02289	4019.359	0.295178	-0.37643
AM	1812.499	626.631	-0.10748	1440.463	1184.041	-0.01663	2184.536	69.22069	-0.36978
IM	2038.402	0.007021	-0.27001	3081.275	0.010231	-0.19746	995.5284	0.00381	-0.39543
DIM	1785.096	0.001493	-0.38826	2945.494	0.002536	-0.35411	624.6985	0.000449	-0.46122
Method 1									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DOM	179.9587	24.38209	-0.27457	154.5186	47.66418	-0.18519	205.3988	1.099998	-0.51937
OM	321.09	5.387053	0.375748	365.0524	9.058691	0.424837	277.1277	1.715415	0.074633
AM	275.9532	22.42565	0.48038	309.0861	24.05294	0.662258	242.8203	20.79836	0.270609
IM	238.7734	0.00127	0.365786	251.0423	0.001351	0.662797	226.5046	0.001189	0.024515
DIM	260.8418	0.000204	-0.10528	224.1132	0.000184	0.196358	297.5705	0.000225	-0.36111
Method 2									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DOM	187.8214	17.10353	-0.35151	146.9532	33.39706	-0.24941	228.6895	0.810008	-0.60818
OM	281.6242	3.987007	0.513863	271.4583	6.681073	0.282644	291.7902	1.292941	-0.14206
AM	228.3327	21.92721	0.199579	193.673	23.51629	0.448042	262.9924	20.33814	-0.05861
IM	193.5049	0.000924	0.049939	169.1523	0.000822	0.305628	217.8575	0.001026	-0.16525
DIM	265.5256	0.000199	-0.24582	218.9244	0.000168	-0.0448	312.1268	0.00023	-0.39966

Table 5. The three methods and moneyness.

6. OUT-OF-SAMPLE PERFORMANCE

A good in-the-sample fit does not necessarily imply good out-of-sample performance. In particular, the presence of too many parameters may cause overfitting and result in poor out-of-sample performance.

In this section we gauge the quality of the three methods (Method 1, Method 2 and DK) in forecasting the option prices at date t+1. We rely on date t estimated parameters and we compute date t+1 theoretical prices, that are compared with t+1 market prices. The same indicators used to gauge the in-the-sample fit, i.e. the SSE, the SSRPE and the MISP, are computed.

Table 6 shows the out-of-sample performance of the three methods.

	SSE	**SSRPE**	**MISP**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DK	2231.897	118.9831	-0.67734	1869.512	225.1794	-0.46907	2594.282	12.78691	-0.71731
Method 1	369.1174	132.1643	-0.12334	382.2205	167.5055	0.094889	356.0143	96.82318	-0.38221
Method 2	351.5752	54.31403	-0.28665	345.8505	71.26512	-0.04026	357.3	37.36294	-0.54025

Table 6. The results for all the sample.

The results are very similar to the in-the-sample ones. The SSE of the DK method is much higher than the one of both Methods 1 and 2 (about seven times), which suggests that both Methods 1 and 2 obtain a better fit to market prices. In terms of the SSRPE, Method 2 is superior to the other two. The MISP indicates that the Derman and Kani method substantially underprices both option types, more than in Methods 1 and 2.

In order to detect which option class is better priced by each method, we divide options according to their time to maturity and their moneyness. In Table 7 options are divided into four classes depending on their time to maturity.

Methods 1 and 2 price short term options better than other option classes, while Derman and Kani's method obtains mixed evidence. Overall, Methods 1 and 2 are much better than Derman and Kani's, especially for long term options (consistently with in-the-sample performance). The underpricing of the Derman and Kani method is severe, particularly for puts. Method 2 is slightly better than Method 1 for calls, especially for long term ones, while the opposite is true for puts.

In Table 8 we discuss the performance of Derman and Kani's method, Method 1 and Method 2 respectively, in relation to the moneyness of the options.

DK									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
0-2 month	965.6042	33.68494	-0.69642	940.954	35.17427	-0.56826	990.2543	32.19561	-0.77649
2-4 month	1691.243	0.201271	-0.63073	1589.862	0.052324	-0.43734	1792.625	0.350218	-0.72412
4-8 month	2903.526	0.119327	-0.66264	2560.587	0.074496	-0.43226	3246.465	0.164157	-0.76295
8-12 month	4337.149	0.06115	-0.60703	3871.623	0.010899	-0.30816	4802.675	0.111401	-0.7581
Method 1									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
0-2 month	329.1239	391.7411	0.006393	339.1208	493.1993	-0.04593	319.1271	290.2829	-0.17244
2-4 month	332.6885	6.87734	-0.09624	343.828	13.66989	0.062299	321.549	0.084791	-0.35708
4-8 month	399.3219	0.233251	-0.25671	412.7486	0.431242	0.129249	385.8951	0.03526	-0.52808
8-12 month	434.2526	0.009523	-0.20568	453.7423	0.007894	0.306936	414.7629	0.011153	-0.57229
Method 2									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
0-2 month	289.3947	157.1773	-0.16113	294.5701	203.7652	-0.15243	284.2192	110.5894	-0.36957
2-4 month	321.7942	5.574288	-0.27962	312.7215	11.06966	-0.06977	330.867	0.078914	-0.5146
4-8 month	385.8568	0.210148	-0.38599	372.0952	0.382605	0.012706	399.6184	0.03769	-0.6614
8-12 month	441.7773	0.010058	-0.38359	430.9483	0.008013	0.10778	452.6063	0.012103	-0.70304

Table 7. The three methods and time to maturity.

DK									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DOM	1872.58	60.68011	-0.6994	190.278	80.21822	-0.5391	3554.883	41.14201	-0.8176
OM	2224.094	0.281155	-0.33895	607.766	0.31329	-0.11783	3840.422	0.24902	-0.50886
AM	1844.872	1.494983	-0.29154	1663.136	2.942323	-0.14774	2026.607	0.047644	-0.44633
IM	2189.924	0.007403	-0.28278	3375.471	0.010941	-0.22743	1004.378	0.003866	-0.38233
DIM	1988.69	0.001638	-0.31682	3244.629	0.002744	-0.28341	732.7516	0.000532	-0.38793
Method 1									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DOM	162.9669	18.3653	-0.29407	141.126	35.84651	-0.19327	184.8078	0.884094	-0.55881
OM	296.8426	42.01578	0.281002	349.5409	70.56175	0.374597	244.1443	13.46981	0.058412
AM	346.9351	972.9726	0.380268	395.2703	1164.24	0.547142	298.5998	781.7049	0.170706
IM	481.0347	0.00216	0.237822	484.915	0.002201	0.445035	477.1544	0.002119	0.013654
DIM	534.0856	0.000421	-0.05929	497.3995	0.000417	0.116536	570.7717	0.000425	-0.23939
Method 2									
	SSE	**SSRPE**	**MIS**	**SSE Call**	**SSRPE Call**	**MISP Call**	**SSE Put**	**SSRPE Put**	**MISP Put**
DOM	168.6209	11.10171	-0.36931	132.7025	21.63429	-0.24457	204.5394	0.569121	-0.66367
OM	260.8862	16.44184	0.112091	269.7659	29.35314	0.244414	252.0066	3.530529	-0.21347
AM	302.5933	390.867	0.113091	314.0793	478.0555	0.309581	291.1072	303.6786	-0.13808
IM	433.0772	0.001779	0.051547	423.3172	0.00181	0.241822	442.8373	0.001748	-0.14074
DIM	532.8754	0.000409	-0.13305	487.8545	0.000401	0.013539	577.8962	0.000418	-0.27858

Table 8. The three methods and moneyness.

The SSE indicates that Methods 1 and 2 perform better than Derman and Kani's in each class of moneyness. In terms of the SSE, Method 2 is better than Method 1 for each option class except for DOM options. The SSRPE in Methods 1 and 2 is lower than in Derman and Kani's except for OM and AM options. The MISP indicates that the Derman and Kani method underprices most classes of options: the highest underpricing is observed for OM and DOM call and put options.

As expected, Method 2 underprices every option class more than Method 1. Method 2 obtains a better pricing performance in terms of mispricing index for IM AM and OM options, the opposite holds for DIM and DOM options. Method 2 is better than Method 1 in the pricing of Call options, while the opposite is true for Put options.

In order to see how out-of-sample performance varies if a different number of levels (n) in the tree is used, the results for n equal to 50 and 51 are reported in Table 9 (we compute the average between odd and even levels performance).

	SSE	SSRPE	MISP	SSE Call	SSRPE Call	MISP Call	SSE Put	SSRPE Put	MISP Put
DK	3433,344	10,717408	-0,77354	3179,245	1,988628	-0,61759	3687,442	19,44619	-0,84928
Method 1	359,4763	209,9826	-0,1559	377,7006	274,1591	0,107417	341,252	145,8061	-0,43878
Method 2	343,7804	142,4489	-0,23541	352,8553	185,9835	0,068775	334,7054	98,91425	-0,53067

Table 9. The out of sample performance for n equal to 50 and 51.

Compared to n=25 and 26, the SSE is slightly lower for Methods 1 and 2, while it is higher for Derman and Kani, the SSRPE is lower for all three methods, and the MISP is lower only for Method 2. As for the mispricing index, it points at a worst pricing performance. Therefore, we can say that using more levels slightly improves the pricing performance of Methods 1 and 2, but does not improve the performance of the Derman and Kani method. Differently from the standard Cox, Ross and Rubinstein binomial model with constant volatility that improves the pricing performance as the number of levels tends to infinity, the Derman and Kani implied tree model, which uses different volatilities at each node, does not obtain better results if the number of levels is increased. Due to the presence of more parameters w.r.t. Derman and Kani's, Methods 1 and 2 obtain a slightly better result.

Given that in Methods 1 and 2 we have to estimate parameters α and β, we can now check how out-of-sample performance varies, by means of a different objective function for in-the-sample minimisation. Table 10 shows the results for the following function:

$$\min_{v} \sum_{i=1}^{m} \left(\frac{f_T(v) - f_M}{f_M} \right)^2 \quad .$$

	SSE	SSRPE	MISP	SSE Call	SSRPE Call	MISP Call	SSE Put	SSRPE Put	MISP Put
Method 1	373,9527	131,061	-0,08276	393,6426	165,9019	0,144531	354,2629	96,22009	-0,38126
Method 2	384,5589	54,09393	-0,17903	385,4704	71,13082	0,075156	383,6474	37,05705	-0,46693

Table 10. Out-of-sample performance using a different objective function in the sample.

Compared with the results obtained by using the objective function in equation (5), the SSE is higher in both methods, while the SSRPE is slightly lower, as expected. As a matter of fact, the function that we minimise is the SSRPE multiplied by the number of options in the class. The overall performance of both methods is better than the one obtained by using the objective function in equation (5), but if we analyse each option class we see that in this case the MISP is higher in absolute terms. We can conclude that changing the objective function to be minimised does not substantially change the out-of-sample performance of the two methods.

7. CONCLUSIONS.

In this paper we proposed an alternative way for the estimation of implied trees from observed option prices in markets where the put-call parity is not fulfilled. Implied trees that closely reflect the market price of standard European options are particularly useful in the pricing and hedging of exotic path-dependent options.

We used the same procedure as in Muzzioli and Torricelli (2002) to imply the interval tree, the only difference being the endogenous derivation of the risk neutral probability interval. In order to compare theoretical prices with market prices, we parameterised the stock price at each node of the tree and we minimised the sum of squared errors, in order to extract a single implied tree. Two Methods have been compared with Derman and Kani's in the DAX-index options market: Method 1, based on a single parameter, and Method 2, based on two different parameters.

The results of in-the-sample and out-of-sample performance are very similar. In both cases Method 2 obtains a slightly better performance than Method 1. It is characterised by a lower volatility than Method 1 and, as a consequence, it prices call options and around-the-money (IM, AM and OM) options better. The out-of-sample performance of Methods 1 and 2 does not substantially vary if we employ a different objective function to be minimised in the sample and if we increase the number of levels in the tree.

Overall, Methods 1 and 2, by capturing the different information carried by call and put prices, provide a better fit of the implied tree to market option prices than Derman and Kani's. This better pricing performance can be attributed to a better fit of the lower part of the implied tree and a better pricing of long term and far-from-the-money options.

REFERENCES

Barle S., N. Cakici, 1995, Growing a smiling tree, *Risk*, 8, 76-81.

Black F., Scholes M., 1973, "The pricing of options and corporate liabilities" *Journal of Political Economy*, 81, 637-654.

Brooke A., Kendrick D., Meeraus A., Raman R., 1998, "GAMS. A User's Guide", *GAMS Development Corporation*.

Buckley J.J., Eslami E., Feuring T., 2002, "*Fuzzy mathematics in economics and engineering*", Phisica Verlag.

Cavallo L., Mammola P., 2000, "Empirical tests of efficiency of the Italian index options market" *Journal of Empirical Finance,* 7, 173-193.

Chesney M., Gibson R., Loubergé H., 1995, "Arbitrage trading and index option pricing at SOFFEX: an empirical study using daily and intradaily data" *Finanzmarkt und Portfolio Management,* 1, 35-60.

Cox J., Ross S. and Rubinstein M., 1979, "Option pricing, a simplified approach" *Journal of Financial Economics*, 7, 229-263.

Derman E., Kani I., 1994, "Riding on a smile" *Risk*, 7, (2), 32-39.

Dumas B., Fleming J. and Whaley R.E., 1998, "Implied volatility functions: empirical tests", *Journal of Finance*, 8 (6), 2059-2106.

Evnine J., Rudd A., 1985, "Index options, the early evidence" *Journal of Finance*, 40, 743-756.

Gould J.P., Galai D., 1974, "Transaction costs and the relationship between put and call prices" *Journal of Financial Economics,* 1,105-129.

Kamara A., Miller T.W. Jr., 1995, "Daily and intradaily tests of European put call parity", *Journal of Financial and Quantitative Analysis,* 30, (4), 519-539.

Klemkosky R.C., Resnick B.G., 1979, "Put-call parity and market efficiency" *The Journal of Finance,* 34, (5), 1141-1155.

Mittnik S., Rieken S., 2000, "Put – call parity and the informational efficiency of the German DAX index options market" *International Review of Financial Analysis,* 9, 259-279.

Murtagh B.A., Saunders M.A., 1978, "Large-scale linearly constrained optimization" *Mathematical Programming*, 14, 41-72.

Murtagh B.A., Saunders M.A., Murray W., Gill P.E., Raman R., Kalvelagen E., 2002, "GAMS/MINOS: A Solver for Large-scale Nonlinear Optimization Problems", *GAMS Development Corporation*.

Muzzioli S., Torricelli C., 2002, "Implied trees in illiquid markets, a Choquet pricing approach" *International Journal of Intelligent Systems,* 17, (6), 577-594.

Muzzioli S., Torricelli C., 2003, "The pricing of options on an interval binomial tree, an application to the DAX-index option market" *European Journal of Operations Research,* accepted.

Rubinstein, 1994, Implied binomial trees, *Journal of Finance* 49, 771-818.

Stoll H.R., 1969, "The relationship between put and call options prices" *Journal of Finance* 24, 801-824.

Wolfe P., 1962, *The reduced-gradient method*, unpublished manuscript, RAND Corporation.

Appendix. The derivation of the risk-neutral probability interval.

Let us briefly recall the assumptions of Muzzioli and Torricelli (2002). Let $r \geq 0$ be the continuously compounded risk free rate; X be the real line, representing a set of monetary values and let F be the set of all intervals of X, i.e. the set of closed bounded sets of real numbers:

A1) $[\underline{S}(i,j), \overline{S}(i,j)] \in F$ is the stock price at time j in state i, where $\underline{S}(i,j) \leq \overline{S}(i,j)$; as a special case, $S(0,0) \in X$ is the spot price at time zero and it is crisp (i.e. a collapsed interval). The bounds of the stock price interval at each node are determined by both call and put options.

A2) No arbitrage opportunities are allowed:

$\overline{S}(i, j+1) < \underline{S}(i, j) \cdot e^{r\Delta t} \leq \overline{S}(i, j) \cdot e^{r\Delta t} < \underline{S}(i+1, j+1)$,

(see the corresponding values, in bold, in Figure 5). As we have interval values for the stock prices, this condition guarantees that if the interval collapses to a crisp value, the "standard" no arbitrage condition is fulfilled (i.e. $S(i, j+1) < S(i, j) \cdot (1+r) < S(i+1, j+1)$, where $S(i, j+1), S(i, j), S(i+1, j+1) \in X$).

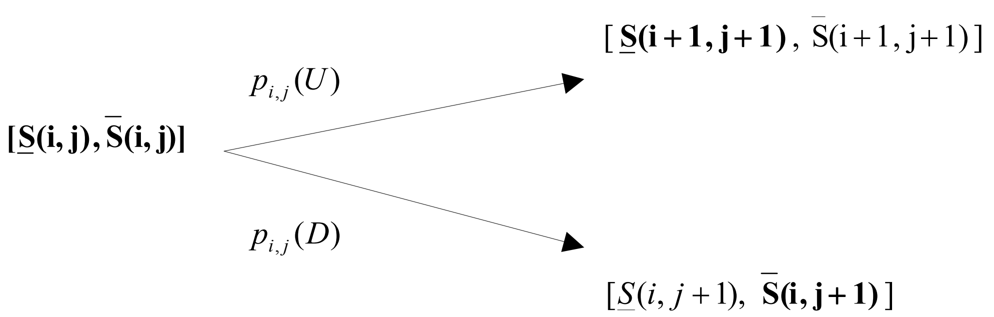

Figure 5. Three nodes of the tree with the corresponding risk neutral probabilities.

Under assumptions A1) and A2) we show that there exists a risk neutral probability measure s.t. any contingent claim C can be valued as:

$\widetilde{C}(j) = e^{-r\Delta t} E_p(\widetilde{C}(j+1))$.

Let us examine the one period binomial sub tree of length $\Delta t = T/n$ represented in Figure 5. Two are the securities traded in the market: the money market account and the stock. The money market account is worth *1* at time *j* and $e^{r\Delta t}$ at time *j+1*, *j=0,...,n*. The stock price is $\widetilde{S}(i,j) = [\underline{S}(i,j), \overline{S}(i,j)]$

at time j, and at time j+1 is $\widetilde{S}(i+1,j+1)=[\underline{S}(i+1,j+1),\overline{S}(i+1,j+1)]$ if it moves up, $\widetilde{S}(i,j+1)=[\underline{S}(i,j+1),\overline{S}(i,j+1)]$ if it moves down.

Let $p_{i,j}(U)$ be the artificial probability measure of an up move from node (i,j) to node $(i+1,j+1)$ and $p_{i,j}(D)$ be the artificial probability measure of a down move from node (i,j) to node $(i,j+1)$.

We derive the set of risk neutral probability measures by means of the risk neutral argument, i.e. by solving the following system:

$$\begin{cases} e^{-r\Delta t}\left(p_{i,j}(U)e^{r\Delta t}+p_{i,j}(D)e^{r\Delta t}\right)=1 \\ e^{-r\Delta t}\left(p_{i,j}(U)[\underline{S}(i+1,j+1),\overline{S}(i+1,j+1)]+p_{i,j}(D)[\underline{S}(i,j+1),\overline{S}(i,j+1)]\right)=[\underline{S}(i,j),\overline{S}(i,j)] \end{cases} \tag{6}$$

where the first and the second equation represents the risk neutral valuation of the money market account and of the stock respectively.

By simplifying system (6) we get:

$$\begin{cases} p_{i,j}(U)+p_{i,j}(D)=1 \\ p_{i,j}(U)[\underline{S}(i+1,j+1),\overline{S}(i+1,j+1)]+p_{i,j}(D)[\underline{S}(i,j+1),\overline{S}(i,j+1)]=[\underline{S}(i,j),\overline{S}(i,j)]e^{r\Delta t} \end{cases} \tag{7}$$

In order to solve this system we resort to interval analysis and we use the solution method proposed in Buckley et al. (2002). Note that by the no arbitrage assumption A2) the determinant of system (7) is never zero.

Buckley et al. (2002) propose to solve the corresponding crisp system:

$$\begin{cases} p_{i,j}(U)+p_{i,j}(D)=1 \\ p_{i,j}(U)S(i+1,j+1)+p_{i,j}(D)S(i,j+1)=S(i,j)e^{r\Delta t} \end{cases}$$

by using Cramer's rule to solve for each unknown:

$$\begin{cases} p_{j,j}(U)=\dfrac{S(i,j)e^{r\Delta t}-S(i,j+1)}{S(i+1,j+1)-S(i,j+1)} \\ p_{j,j}(D)=\dfrac{S(i+1,j+1)-S(i,j)e^{r\Delta t}}{S(i+1,j+1)-S(i,j+1)} \end{cases}$$

and investigate the interval solution, $[\underline{p}_{i,j}(K),\overline{p}_{i,j}(K)]$ for $K=\{U,D\}$ as follows:

$$\underline{p}_{i,j}(K)=\min\left\{ p_{i,j}(K) \;\middle|\; \begin{array}{l} S(i,j)\in[\underline{S}(i,j),\overline{S}(i,j)], S(i+1,j+1)\in[\underline{S}(i+1,j+1),\overline{S}(i+1,j+1)], \\ S(i,j+1)\in[\underline{S}(i,j+1),\overline{S}(i,j+1)] \end{array} \right\}$$

$$\overline{p}_{i,j}(K)=\max\left\{ p_{i,j}(K) \;\middle|\; \begin{array}{l} S(i,j)\in[\underline{S}(i,j),\overline{S}(i,j)], S(i+1,j+1)\in[\underline{S}(i+1,j+1),\overline{S}(i+1,j+1)], \\ S(i,j+1)\in[\underline{S}(i,j+1),\overline{S}(i,j+1)] \end{array} \right\}$$

By noting that: $\frac{\partial p_{i,j}(U)}{\partial S(i,j)} > 0$, $\frac{\partial p_{i,j}(U)}{\partial S(i+1,j+1)} < 0$, $\frac{\partial p_{i,j}(U)}{\partial S(i,j+1)} < 0$, $\frac{\partial p_{i,j}(D)}{\partial S(i,j)} < 0$, $\frac{\partial p_{i,j}(U)}{\partial S(i+1,j+1)} > 0$, $\frac{\partial p_{i,j}(U)}{\partial S(i,j+1)} > 0$ we get:

$$\underline{p}_{i,j}(U) = \frac{\underline{S}(i,j)e^{r\Delta t} - \overline{S}(i,j+1)}{\overline{S}(i+1,j+1) - \overline{S}(i,j+1)}$$

$$\overline{p}_{i,j}(U) = \frac{\overline{S}(i,j)e^{r\Delta t} - \underline{S}(i,j+1)}{\underline{S}(i+1,j+1) - \underline{S}(i,j+1)}$$

$$\underline{p}_{i,j}(D) = \frac{\underline{S}(i+1,j+1) - \overline{S}(i,j)e^{r\Delta t}}{\underline{S}(i+1,j+1) - \underline{S}(i,j+1)}$$

$$\overline{p}_{i,j}(D) = \frac{\overline{S}(i+1,j+1) - \underline{S}(i,j)e^{r\Delta t}}{\overline{S}(i+1,j+1) - \overline{S}(i,j+1)}$$

Let $\Omega = \{U, D\}$ be the sample space and $\mathrm{M} = \{\varnothing, U, D, \Omega\}$ be an algebra of subsets of Ω, it is easy to check that $\underline{p}_{i,j}(K)$ and $\overline{p}_{i,j}(K)$ are upper and lower probabilities, since they satisfy the following properties $\forall A, B \in \mathrm{M}$:

1) $0 \le \underline{p}(A) \le 1$, $0 \le \overline{p}(A) \le 1$,

2) $\underline{p}(\varnothing) = 0$, $\overline{p}(\varnothing) = 0,$ $\underline{p}(\Omega) = 1$, $\overline{p}(\Omega) = 1$

3) if $A \subseteq B$ then $\underline{p}(A) \le \underline{p}(B)$ $\overline{p}(A) \le \overline{p}(B)$

4) $\underline{p}(A) + \overline{p}(A^C) = 1$

5) $\underline{p}(A \cup B) \ge \underline{p}(A) + \underline{p}(B), \quad A \cap B = \varnothing$

6) $\overline{p}(A \cup B) \le \overline{p}(A) + \overline{p}(B), \quad A \cap B = \varnothing$

5’) $\underline{p}(A \cup B) + \underline{p}(A \cap B) \ge \underline{p}(A) + \underline{p}(B)$

6’) $\overline{p}(A \cup B) + \overline{p}(A \cap B) \le \overline{p}(A) + \overline{p}(B)$.