Aggregate Uncertainty in the Citizen Candidate Model Yields Extremist Parties

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Abstract

We extend the 'citizen candidate' model of party formation to allow for aggregate uncertainty over the distribution of preferences. We discuss and characterize the equilibrium set in this framework and show that two-party equilibria have 'extremist' parties, i.e. the party winning under a left-wing (right-wing) distribution is to the left (right) of the median of that distribution.

1 Introduction

The impact of electoral systems on the number of parties has long been a hotly debated topic in political science. The central hypothesis is the so-called Duverger's Law: A plurality electoral system (also called first-past-the-post) produces a two-party political system. Apart from anecdotal evidence, some empirical studies (see e.g. Wright and Riker [19]) have shown that the first-past-the-post system tends to reduce the number of candidates or parties contesting the election. Comprehensive comparative studies on the impact of electoral systems on the number of parties are contained in Lijphart [10] and Shugart and Taagepera [15].

The most convincing theoretical explanations of the Duverger's Law are produced in models with strategic voting (Palfrey [13], Feddersen, Sened and Wright [7], Feddersen [6] and Fey [8]). The logic is straightforward. A strategic voter should cast her vote to maximize the impact on the final selection of the candidate, so a voter should sort out the two candidates who

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have the highest probability of winning and vote for the one she likes more. In a Nash equilibrium of the voting game only two candidates obtain votes.

The assumption of strategic voting however is difficult to accept in large elections, where the probability that a given vote is pivotal is essentially zero. With sincere voting an argument for the Duverger's law can still be made, but it has to rely on the strategic behavior of *candidates* (or parties) deciding whether or not to contest an election. The role of parties was in fact highlighted by Duverger [4], who claimed that

> The brutal finality of a majority vote on a single ballot forces parties with similar tendencies to regroup their forces at the risk of being overwhelmingly defeated.

When the attention shifts from voter to parties the question of what motivates parties becomes important. One possibility which has been explored is that candidates only care about winning and adopt the platform giving the highest probability of victory (the standard Hotelling model). With a fixed number of parties these models produce 'median voter' kind of results[1]. However Osborne [11] points out that the result is not robust to the possibility of entry. He shows that when there are $n > 2$ potential candidates, for almost all distributions on the political preferences of voters a Nash equilibrium in pure strategies fails to exist. To sum up, models in which candidates only care about winning are unable to produce interesting insights on the long-run political configuration that an electoral system is able to produce.

It is therefore natural to explore models in which candidates have policy preferences, as well as an appetite for power. Osborne and Slivinski [12] (henceforth, OS) and Besley and Coate [1] (henceforth, BC) have proposed the *citizen candidate model* to analyze the entry problem[2]. In the 'citizen candidate model' each voter can decide to become a candidate. Each citizen is endowed with an ideal point over the policy space, and implements the policy corresponding to that ideal point in case of victory. Once a citizen has become a candidate, her political preferences become perfectly known to the players. Thus, citizens can decide whether or not to run, but once they decide to run they cannot choose the political platform. BC assume strategic voting behavior and a finite set of voters, while OS assume sincere voting and a continuum of voters.

[1] Duggan [3] surveys models with a fixed number of parties.

[2] An earlier proponent of such models has been Wittman ([16],[17], [18]). His analysis is restricted to the case of two parties, and it cannot therefore handle the possibility of entry.

Since we are interested in large elections, we will use the OS model as a building block. We introduce two important modifications. First, we want to give to the parties some flexibility in selecting their platforms. In the traditional citizen candidate model the policy implemented by a winning candidate must be identical to her ideal point. When a two-party equilibrium exists, candidates have different political positions and they both receive 50% of the vote, so that each wins with probability $\frac{1}{2}$. This kind of equilibria has the unsatisfactory feature that very slight movements in the political position of a party can insure victory with probability 1. It is hard to believe that existing parties, no matter how ideologically committed, would give up such an opportunity for electoral profit. We therefore explore a model in which parties care about the policy implemented but are also able to perform very slight movements in the platform selected if this is electorally convenient. Formally, if a candidate has ideal point x we assume that the party can choose the platform in the interval $(x-\delta, x+\delta)$, and study equilibria that survive as δ goes to zero. We show that if such a refinement is imposed then the only equilibrium of the OS model is with a single party entering at the median. Thus, there are no equilibria with multiple parties.

Obviously, a model predicting that only one party enters is not very satisfactory. We therefore introduce an additional modification allowing for aggregate uncertainty over the distribution of voters. Uncertainty has been recently introduced in the citizen candidate model by Riviere [14], Eguia [5] and Fey [9]. The common theme in these models is that the exact outcome of an election is subject to uncertainty, for example because there may be computing errors or external factors may prevent some voters from going to the polls. While there is always a candidate positioned at the median of the voters' preferences, an additional candidate close to the median may enter in order to exploit 'mistakes'. The prediction is therefore that equilibria with multiple parties exist, but the parties tend to have similar platforms[3].

We want to explore a different source of uncertainty, namely uncertainty on the political preferences of the electorate. The idea can be described as follows. The actual 'mood' of the electorate is not known at the time a party announces its ideology and hence at that moment a party is never sure how the elctorate would interpret its announced policy. However it is common knowledge that the 'political mood' of the electorate can take

[3] There are other technical differences with our model. Riviere and Fey [9] assume a finite number of ideological positions, rather than a continuum as we do. Riviere [14] and Eguia [5] assume strategic voting.

two forms, a 'left-wing mood' and a 'right-wing mood'. The two 'moods' can be represented by two different distributions on a unidimensional policy space, with the left-wing distribution having a median m^L lower that the median m^R of the right wing distribution; the left-wing mood appears with probability θ and the right-wing mood with probability $(1-\theta)$.

Under some regularity conditions, two-party equilibria exist in this model. In these equilibria the left-wing party wins when the political mood is left-wing and loses otherwise. Usually the winners win decisively, with a sizeable majority. This is in sharp contrast to the standard OS model, where in a two-party equilibrium the two parties get 50% of the vote.

In terms of characterization, the most interesting result is that the two parties are 'extremist'. By this we mean that the left-wing party is to the left of m^L while the right-wing party is to the right of m^R. This appears to be in accordance with the common observation that party activists tend to be ideologically more extreme than party voters. The logic is straightforward. Suppose that a left-wing party is positioned at m^L. A new party can enter at $m^L - \delta$, and for δ small the new party will obtain almost 50% under the left-wing distribution. Since the rest of the vote will be split between the two incumbant parties, the entrant wins the election when the mood is left-wing. Notice that this can be done without changing the outcome under the right-wing distribution, since in that case the right-win party wins anyway. Entry can only be prevented if the two parties are somewhat extremist, so that entry on the wings does not give too high a share of the vote to the entrant. Of course the parties cannot be too extremist, since otherwise entry at the center may become profitable.

There are various papers that obtain parties with divergent platforms. For example, Bernhardt, Duggan and Squintani [2] show that office-seeking parties with private information on the location of the median voter choose different policy points[4]. However, to our knowledge, the rationale for differentiation proposed in this paper is new.

The rest of the paper is organized as follows. In section 2 we discuss the standard citizen candidate model and show that only one-party equilibria survive if parties are allowed to change slightly their platforms. Section 3 introduces aggregate uncertainty on political preferences and discusses two-party equilibria. Equilibria with more than two parties are discussed in section 4, and section 5 contains the conclusions. All proofs are in the

[4]Ex post platform differentiation of course appears when the equilibrium is in mixed strategies, even if the players use identical strategies. In this paper we focus on pure strategy equilibria.

appendix

2 The Citizen Candidate Model

The OS model has a continuum of voters, distributed according to some strictly increasing and continuous c.d.f. F over the real line, which is also the policy space. Each voter has an ideal policy point, corresponding to her index. In particular, a voter with index i obtains a utility $-|x-i|$ when policy x is implemented. Any citizen can become a candidate, paying a fixed cost c, and a winning candidate implements the policy corresponding to her ideal point. Furthermore, the winning agent obtains an 'ego rent' b. Entry decision occurs simultaneously, and whenever a citizen becomes a candidate it is common knowledge that, in case of victory, she will implement her preferred policy. The ideal point of each candidate is perfectly observable, and each citizen votes 'sincerely', that is for the candidate with the closest ideal point.

We are interested in the special case $b = c = 0$. Essentially the idea is that we want to study the long-run equilibrium configuration of political systems, looking in particular at the ideological positions that are likely to emerge. The cost and benefits of entry are important in situations in which a given citizen has to mount anew a campaign to win an election. We are more interested in the emergence of political parties as a long-run phenomenon. In this sense, we can assume that for each ideological position there are many citizens who actually like being politically engaged and therefore have a very small cost of entry. 'Ego rents' and costs are important in determining who exactly will be the party candidates, but not in determining the political positions of the parties. Thus, in the long run the existence of a party will be determined more by the viability of its ideological platform than by any cost of entry or benefit for particular politicians. At any rate, in order to rule out equilibria in which irrelevant entry[5] occurs, we will assume that citizens have a lexicographic preference for not entering so that whenever a citizen is indifferent, she chooses not to contest. Finally, if no party runs the elections, then each citizen earns $-\infty$.

Following OS's notation, let F be the distribution of ideal points and m its median. Without loss of generality we will normalize the total mass of voters to unity. Notice that if there are two parties, one located at $m-\varepsilon$ and the other at $m+\varepsilon$, and a third party enters at $s \in (m-\epsilon, m+\epsilon)$ then

[5] By this we mean the entry by parties whose presence does not change the outcome of the election.

the party located at $m-\varepsilon$ obtains a share $F\left[\frac{1}{2}(m-\varepsilon+s)\right]$ of the vote and the party located at $m+\varepsilon$ obtains a share $1-F\left[\frac{1}{2}(m+\varepsilon+s)\right]$. We define $s(\varepsilon, F)$ as the position by an entrant that minimizes the highest share of the vote obtained by one of the existing parties. Clearly, this occurs when the share of the vote of the two parties is equal, that is $s(\varepsilon, F)$ is defined as:

$$F\left[\frac{1}{2}(m-\epsilon+s(\epsilon, F))\right]=1-F\left[\frac{1}{2}(m+\epsilon+s(\epsilon, F))\right]$$

We collect here some results of the OS model that we are particularly interested in.

Proposition 1 *Assume $b=c=0$ and citizens have lexicographic preference for not entering. Then:*

1. *There is always an equilibrium in which a single candidate enters. In all such equilibria, this candidate is the median voter.*
2. *An equilibrium in which two parties win with positive probabilities exists only if $s(\epsilon, F)=m$ for some $\epsilon>0$.*

The basic intuition can be explained as follows. For the one-party equilibrium, it is clear that if the voter with $i=m$ enters no other candidate can enter successfully. Thus, having a single party positioned at m is always an equilibrium. No other equilibrium with a single party can exist because it would be profitable for the voter located at m to enter.

Regarding two-party equilibria, first notice that there is no two-party equilibrium with both parties on one side of the median, as in this case the median voter could enter and win the election outright. Thus, all two-party equilibria have one party on the right and one party on the left, say x_L and x_R. Both parties must win with positive probability, since otherwise the loser would exit, which implies $\frac{x_L+x_R}{2}=m$. In other words, we can find $\varepsilon>0$ such that $x_L=m-\varepsilon$ and $x_R=m+\varepsilon$. No voter positioned to the right of x_R or to the left of x_L will want to enter, since this gives victory for sure to the more distant party. Entry in the interval $(m-\varepsilon, m+\varepsilon)$ may occur for two reasons. The first is the obvious one: when ε is large, so that the two parties are 'extremists' then entry at the center ensures victory. This puts an upper bound on ε. The second reason, highlighted by OS, is subtler. A citizen may decide to create a party even if she knows it will lose for sure, because it will subtract most of the votes to the party she dislike most. If, for example, $s(\varepsilon, F)>m$ a citizen in $(m, s(\varepsilon, F))$ can enter and subtract more votes from the left-wing party than to the right-wing party. A similar reasoning applies if $s(\varepsilon, F) < m$. Thus, to avoid entry it is necessary that $s(\varepsilon, F) = m$.

2.1 Robust Equilibria in the Citizen Candidate Model

The existence of two-party equilibria seems to be a natural and desirable property of a game theoretic model of the first-past-the-post system. When $b = c = 0$ the OS model produces this kind of equilibria only for distributions symmetric around the median[6]. The equilibria however have the disturbing feature that *any candidate can obtain victory for sure with a minimal change in position.* Slightly moving to the right ensures victory with probability 1 (rather than 0.5) for x_L; similarly x_R can win with probability 1 moving slightly to the left.

While it is true that parties or candidates usually have a reputation and are perceived by the electorate to be located at some point in the policy space, we think that the extreme assumption that a candidate can do nothing to change such perceptions is unrealistic and obscures some important elements of the strategic situation faced by potential candidates in an election. It is not hard to think of real life mechanisms that may be used to 'marginally correct' the perception of the electorate. For example, a presidential candidate who is thought to be too far to the left can choose a more moderate candidate for vice-president. Similarly, a party which is thought to be too extreme may decide to choose more moderate candidates or present a more moderate manifesto. We want to explore a model in which parties have a reputation for ideological preferences but are able to marginally correct the perception that the electorate has of the party.

Formally, assume that a candidate with ideal point x can enter at any point $x' \in (x - \delta, x + \delta)$, where δ is small. Let $X(\delta) = \left\{x_1^\delta, x_2^\delta, \ldots, x_{k_\delta}^\delta\right\}$ be a political equilibrium for a given δ. We are interested in $\lim_{\delta \longrightarrow 0} X(\delta)$, i.e. we are interested in equilibria that are robust to the possibility of small changes in the party position. Looking at the equilibrium set for $\delta \longrightarrow 0$ means that the assumption that parties are able to modify marginally their platform is used essentially as an equilibrium selection device; in our analysis the parties end up entering exactly at the preferred policy point and the

[6] Osborne and Slivinski show that when $c > 0$ a necessary condition for the existence of a two-party equilibrium is $c \geq |s(\varepsilon, F) - m|$. Thus, even if we allow the cost to be positive it remains true that the distribution cannot be 'too asymmetric' if we want to ensure that a two-party equilibrium exists.

possibility of choosing a policy different from the preferred one is never exploited.

The following proposition provides a full characterization of equilibria of such equilibria.

Proposition 2 *Consider the citizen candidate model with $b = c = 0$ and lexicographic preference for not entering, and assume that a candidate positioned at x can choose the platform in the interval $(x - \delta, x + \delta)$. There is a unique pure strategy Nash equilibrium that survives for each $\delta > 0$. In the equilibrium only the citizen located at the median enters.*

The proposition implies that this version of citizen-candidate model is highly unsatisfactory when we allow candidates some possibility of choosing their platform. In the next section we show that by introducing uncertainty on the aggregate distribution of preferences we get more interesting and realistic predictions.

3 Aggregate Uncertainty

In this section we consider a model in which candidates decide to enter in the political competition and decide their platform *before* some aggregate uncertainty about the state of political opinion is resolved. This is meant to capture the idea that building a party and making its platform known is a lengthy process, so that parties have to make their decisions when the exact shape of political opinion is not known. It could also be the case that having exact information regarding political opinions of an electorate is not achievable.

We model aggregate uncertainty by assuming that there are two possible distributions of political opinion F^L and F^R, where F^L denotes the 'left-wing' distribution and F^R the 'right-wing' distribution. Both functions are strictly increasing and continuous and we denote with m^i the median of distribution F^i and assume $m^L < m^R$. The actual distribution is F^L with probability $\theta \in (0, 1)$ and F^R with probability $1 - \theta$. This framework is common knowledge and decisions about entry have to be taken before the realization of the true state of nature. The preferences of citizens are as in the previous section. In particular, we assume that the utility of each citizen does not depend on the state of nature but only on her ideal point, the policy chosen and the decision to run.

A *political configuration* is a collection $X = \{x_1, \ldots, x_h\}$ of parties running for election, with $x_1 < x_2 < \ldots < x_h$. Given a political configuration and a distribution of political opinion F^q, $q \in \{L, R\}$, the share of the vote going to party x_i is:

$$v^q(x_i) = F^q\left(\frac{x_i + x_{i+1}}{2}\right) - F^q\left(\frac{x_i + x_{i-1}}{2}\right)$$

where we use the convention $F^q\left(\frac{x_h + x_{h+1}}{2}\right) = 1$ and $F^q\left(\frac{x_1 + x_0}{2}\right) = 0$. For a given distribution F^q the set of winners is defined as

$$W(F^q, X) = \{x_i \in X \,|\forall x_j \neq x_i \; v^q(x_i) \geq v^q(x_j)\}$$

Let $n_q(X)$ be the cardinality of $W(F^q, X)$ when the political configuration is X. At last, for a given political configuration X, for each $x_i \in X$ we build the the following notations and events:

$$X^{-x_i} \equiv \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_h\},$$

$$X(x_i') \equiv X^{-x_i} \cup x_i',$$

$$X^{+x_j} \equiv X \cup x_j$$

X^{-x_i} corresponds to the political configuration that would result should x_i decide to drop out of the electoral race. $X(x_i')$ is the political configuration that would result should x_i decide to change political platform from x_i to x_i'. Finally, X^{+x_j} is the political configuration which results when a new party with platform x_j enters. Given a political configuration X, the expected utility of a citizen with ideal point x is:

$$U(x, X) = -\sum_{q \in \{L,R\}} \theta^q \left(\frac{1}{n_q(X)} \sum_{x_j \in W(F^q, X)} |x_j - x|\right)$$

This leads us to introduce the following equilibrium notion.

Definition 1 *A **political equilibrium** is a political configuration X that satisfies the following properties:*

No Entry *For each $x \notin X$, $U(x, X) \geq U(x, X^{+x})$.*

No Exit *For each $x_i \in X$, $U(x_i, X) > U(x_i, X^{-x_i})$.*

Robustness *For each $x_i \in X$, there exists $\delta > 0$ such that $U(x_i, X) \geq U(x_i, X(x_i'))$ for each $x_i' \in (x_i - \delta, x_i + \delta)$.*

In words, a political equilibrium is a situation in which no citizen wants to create a new party and each existing party prefers to maintain the current platform rather than either disbanding or changing its position slightly. Notice that we use a strict inequality for the No Exit condition, thus incorporating the requirement that a candidate prefers to stay out of the race when the expected utility of running is equal to the expected utility of not running.

3.1 Characterization of Political Equilibria

To prove existence of a political equilibrium we will have to add regularity conditions on the environment. Before dealing with existence, however, we can obtain results about the characteristics that a political equilibrium should have. Our first proposition provides some general results.

Proposition 3 *If a configuration $X = \{x_1, \ldots, x_h\}$ is a political equilibrium then:*

1. *there are at least two parties, i.e. $h \geq 2$, and all parties are located at different positions;*

2. *x_1 wins with positive probability under one distribution for which x_h loses with probability 1 while x_h wins with positive probaility under the other distribution for which x_1 loses with probability 1.*

The proposition is intuitive, but it shows how different political equilibria are when aggregate uncertainty is introduce in the citizen-candidate model. One party equilibria are ruled out, since at least one of the medians will not be occupied thus making entry profitable. Extreme parties must win because otherwise they are better off not entering, letting their votes to go to the closest party.

The next proposition characterizes two-party equilibria and shows that the parties must be 'extremists'.

Proposition 4 *Suppose that a two-party political equilibrium $X = \{x_L, x_R\}$ exists. Then:*

1. *x_L wins with probability 1 when the distribution is F^L and x_R wins with probability 1 when the distribution is F^R.*

2. *If both parties get a strictly positive share of the vote when losing then $x_L < m^L$ and $x_R > m^R$.*

The proposition shows that two-party equilibria in the citizen-candidate model with aggregate uncertainty have some natural properties. *First*, the two parties both win with positive probability. Differently from the OS model however this is not because they both get 50% of the vote. Rather, what happens is that the left-wing party wins with a clear majority when the 'political mood' is leftist and loses decisively otherwise. *Second*, the parties tend to be somewhat extremist. When the left-wing mood prevails then the policy implemented will be to the left of m^L, i.e. to the left of the 'leftist median voter', while the policy implemented under F^R will be to the right of m^R. This centrifugal tendency is in stark contrast with models without aggregate uncertainty or with aggregate uncertainty on the vote count. Proposition 2 shows that, absent aggregate uncertainty, the citizen-candidate model gives the same outcome as the classic Downsian model, i.e. the implementation of the policy preferred by the median voter. Similarly, when uncertainty is on the exact vote count as in Riviere [14], Eguia [5] and Fey [9], the parties tend to concentrate around the expected median. Aggregate uncertainty on the actual political preferences instead a stronger form of divergence. Not only the policy implemented will move when the median voter moves, but in fact it will shift further to the right or to the left than the position of the median voter.

3.2 Existence of Two-party Equilibria

When does a two party equilibrium exist? Call x_L the left-wing party and x_R the right-wing party, so that $x_L < x_R$. The key condition for $\{x_L, x_R\}$ to be a political equilibrium is that no other citizen can gain from entry. Entry may occur either at the wings or at the center.

It is clear that the most dangerous entry at the wings is the one just to the left of x_L or just to the right of x^R. If a new party is created at $x_L - \varepsilon$ it may have the chance of winning the election when the distribution is F^L, while not changing the outcome when the distribution is F^R (since in that case x_R would win anyway). Thus, we have to make sure that $x_L - \varepsilon$ does not win when the distribution is F^L for each $\varepsilon > 0$. This leads to the following necessary condition for a pair $\{x_L, x_R\}$ to be a two-party equilibrium:

$$F^L(x_L) \leq \max\left\{1 - F^L\left(\frac{x_L + x_R}{2}\right), F^L\left(\frac{x_L + x_R}{2}\right) - F^L(x_L)\right\} \quad (1)$$

where $F^L(x_L)$ is the share of the vote of a party entering right to the left of x_L, $F^L\left(\frac{x_L+x_R}{2}\right) - F^L(x_L)$ is the share of x_L vote after entry and $1 - F^L\left(\frac{x_L+x_R}{2}\right)$ is the share of x_R vote (which is not affected by entry).

A similar reasoning on the right side of the political spectrum yields the necessary condition:

$$1 - F^R(x_R) \leq \max\left\{F^R\left(\frac{x_L + x_R}{2}\right), F^R(x_R) - F^R\left(\frac{x_L + x_R}{2}\right)\right\} \quad (2)$$

Consider now potential entry by citizens located at $y \in (x_L, x_R)$. A third party may enter either to win outright or to defeat the less favored between x_L and x_R.

A new party entering at $y \in (x_L, x_R)$ will collect a share of the vote equal to $F^q\left(\frac{y+x_R}{2}\right) - F^q\left(\frac{y+x_L}{2}\right)$ when the distribution is F^q. The share of x_L is $F^q\left(\frac{y+x_L}{2}\right)$ and the share of x_R is $1 - F^q\left(\frac{y+x_R}{2}\right)$. Let

$$v^q(y; x_L, x_R) = F^q\left(\frac{y + x_R}{2}\right) - F^q\left(\frac{y + x_L}{2}\right) - \max\left\{1 - F^q\left(\frac{y + x_R}{2}\right), F^q\left(\frac{y + x_L}{2}\right)\right\}$$

and define

$$\Phi^L(x_L, x_R) = \max_{y \in \left[x_L, \frac{x_L + x_R}{2}\right]} v^L(y; x_L, x_R).$$

If $\Phi^L(x_L, x_R) > 0$ then a citizen entering at some $y \in \left(x_L, \frac{x_L+x_R}{2}\right)$ can win the election when the distribution is F^L without changing unfavorably the outcome[7] when the distribution is F^R. Thus, a necessary condition to avoid entry at the center is

$$\Phi^L(x_L, x_R) \leq 0 \quad (3)$$

Consider next a citizen at $x \in \left(\frac{x_L+x_R}{2}, x_R\right)$ and define

$$\Phi^R(x_L, x_R) = \max_{y \in \left[\frac{x_L + x_R}{2}, x_R\right]} v^R(y; x_L, x_R).$$

If $\Phi^R(x_L, x_R) > 0$ then a citizen entering at some $y \in \left(\frac{x_L+x_R}{2}, x_R\right)$ can win the election when the distribution is F^R without changing unfavorably the outcome when the distribution is F^L. Thus, a necessary condition for no entry to be profitable is

$$\Phi^R(x_L, x_R) \leq 0 \quad (4)$$

The 4 inequalities 1, 2, 3, 4 define a region of possible pairs (x_L, x_R) which can be two-party equilibria.

We now make the regularity assumption that ensures the existence of two-party equilibria.

[7] Without entry party x_R wins under F^R. With entry either x_R keeps winning or the winner is in the set $\{x_L, y\}$. Since x_L is closer to y than x_R the outcome can only change favorably for y.

Assumption 1 (No large peaks) *The distributions F^L and F^R can be represented by strictly positive densities f^L and f^R. The densities have the same support and for each $i \in \{L, R\}$ we have $f^q(x) < 2f^q(y)$ for each pair x, y in the support.*

Essentially, assumption 1 requires that there are no strong concentration of voters at some points of the ideological spectrum. Large concentrations of voters may jeopardize the existence of a two party equilibrium because an entrant positioned at a large peak can capture a large share of the vote under at least one distribution. On the other hand, if the preferences are sufficiently dispersed then it becomes possible to position the two parties in such a way that no new party can profitably enter.

The proof of existence will be constructive, i.e. we will show how to compute a two-party equilibrium when Assumption 1 is satisfied.

Define

$$\Psi^L(x_1, x_2) = \max\left\{F^L\left(\frac{x_1 + x_2}{2}\right) - F^L(x_1), 1 - F^L\left(\frac{x_1 + x_2}{2}\right)\right\} - F^L(x_1) \tag{5}$$

$$\Psi^R(x_1, x_2) = \max\left\{F(x_2) - F^R\left(\frac{x_1 + x_2}{2}\right), F^R\left(\frac{x_1 + x_2}{2}\right)\right\} - (1 - F(x_2)). \tag{6}$$

We can now prove the following result.

Proposition 5 *Suppose that Assumption 1 holds. Then there exists a pair (x_1^*, x_2^*) such that*

$$\Psi^L(x_1^*, x_2^*) = \Psi^R(x_1^*, x_2^*) = 0$$

and $X = \{x_1^*, x_2^*\}$ is a political equilibrium.

The Proposition offers a sufficient condition for the existence of a two-party equilibrium. We now present two examples. The first shows that a two party equilibrium may fail to exist. The second shows that a two party equilibrium may exist even if assumption 1 is violated, thus showing that the assumption is not necessary.

Example: Non-existence of a two party equilibrium. Consider a sequence of distributions $\{F_n^L\}_{n=10}^{+\infty}$ represented by the densities:

$$f_n^L(x) = \begin{cases} 2 & \text{if} \quad x \in \left[0, \frac{1}{8}\right) \\ \frac{2}{3} & \text{if} \quad x \in \left[\frac{1}{8}, \frac{1}{2}\right) \\ \frac{1 - 2\frac{1}{n}}{2\left(\frac{1}{n} + \frac{3}{10}\right)} & \text{if} \quad x \in \left[\frac{1}{2}, \frac{4}{5} + \frac{1}{n}\right) \\ \frac{\frac{1}{n}}{\frac{1}{5} - \frac{1}{n}} & \text{if} \quad x \in \left[\frac{4}{5} + \frac{1}{n}, 1\right] \end{cases}$$

with median $m_n^L = \frac{1}{2}$ for each n and a sequence of distributions $\left\{F_n^R\right\}_{n=10}^{+\infty}$ represented by the densities

$$f_n^R(x) = \begin{cases} \frac{1}{n-1} & \text{if} \quad x \in \left[0, \frac{n-1}{n}\right) \\ n-1 & \text{if} \quad x \in \left[\frac{n-1}{n}, 1\right]. \end{cases}$$

We will show that for $\frac{1}{n}$ small enough there is no two party equilibrium, since at any candidate pair there is profitable a profitable entry.

Assume that for each n there is a two party equilibrium $\{x_L^n, x_R^n\}$. Start observing than for each n it must be $x_R^n > 1 - \frac{1}{n}$, since otherwise the citizen located at $1 - \frac{1}{n}$ could enter and win the election under F_n^R, leaving at the same time the outcome unchanged under F_n^L. This in turn implies $\lim_{n \longrightarrow +\infty} x_R^n = 1$.

Also, since the distribution are represented by strictly positive densities, $x_L^n < \frac{1}{2} = m_\varepsilon^L$ for each n. We observe that for n large enough any entry $y \leq \frac{5}{8}$ does not change the outcome under F_n^R. Thus, if entry changes favorably the outcome under F_n^L it will occur. We consider two cases.

<u>Case 1.</u> For each n' there is $n > n'$ such that $x_L^n < \frac{3}{8}$. We will show that if this is the case there is profitable entry at $y = \frac{9}{16}$.

Take a converging subsequence $\{x_L^n\}$ such that $x_L^n < \frac{3}{8}$ for each element of the subsequence. First notice that since both x_L^n and x_R^n are less than 1 we have $\max\left\{\frac{\frac{9}{16}+x_L^n}{2}, \frac{\frac{9}{16}+x_R^n}{2}\right\} \leq \frac{\frac{9}{16}+1}{2} < \frac{4}{5}$. Also, since $x_L^n \geq 0$, we have $\frac{\frac{9}{16}+x_L^n}{2} \geq \frac{9}{32} > \frac{1}{8}$. Thus, entry at $y = \frac{9}{16}$ yields the following shares of the vote when the distribution is F_n^L:

$$v_n^L(x_L^n) = \frac{1}{4} + \left(\frac{x_L^n + \frac{9}{16}}{2} - \frac{1}{8}\right)\frac{2}{3} = \frac{1}{3}x_L^n + \frac{17}{48}$$

$$v_n^L(y) = \left(\frac{1}{2} - \frac{x_L^n + \frac{9}{16}}{2}\right)\frac{2}{3} + \left(\frac{x_R^n + \frac{9}{16}}{2} - \frac{1}{2}\right)\frac{1 - 2\frac{1}{n}}{2\left(\frac{1}{n} + \frac{3}{10}\right)}$$

$$v_n^L(x_R^n) = \left(\frac{4}{5} + \frac{1}{n} - \frac{x_R^n + \frac{9}{16}}{2}\right)\frac{1 - 2\frac{1}{n}}{2\left(\frac{1}{n} + \frac{3}{10}\right)} + \frac{1}{n}.$$

Let $x_L^* = \lim_{n \longrightarrow +\infty} x_L^n$, and observe that it must be $x_L^* \leq \frac{3}{8}$. Also observe that $\lim_{n \longrightarrow +\infty} x_R^n = 1$. The condition $\lim_{n \longrightarrow +\infty} v_n^L(y) > \lim_{n \longrightarrow +\infty} v_n^L(x_L^n)$ is therefore

$$\left(\frac{1 + \frac{9}{16}}{2} - \frac{1}{2}\right)\frac{5}{3} > \frac{2}{3}x_L^* + \frac{10}{48}. \tag{7}$$

which is satisfied since $x_L^* \leq \frac{3}{8}$.

The condition $\lim_{n\longrightarrow+\infty} v_n^L(y) > \lim_{n\longrightarrow+\infty} v_n^L(x_L^n)$ is

$$\frac{7}{48} + \left(\frac{1+\frac{9}{16}}{2} - \frac{1}{2}\right)\frac{5}{3} - \left(\frac{4}{5} - \frac{1+\frac{9}{16}}{2}\right)\frac{5}{3} > \frac{1}{3}x_L^*$$

which again is satisfied since $x_L^* \leq \frac{3}{8}$.

We conclude that in this case there is always a profitable entry at $y = \frac{9}{16}$ for n large enough, contradicting that $\{x_L^n, x_R^n\}$ is a two-party equilibrium for each n.

<u>Case 2.</u> There is n' such that for each $n > n'$ we have $x_L^n \geq \frac{3}{8}$. Consider again a converging subsequence and let $x_L^* = \lim_{n\longrightarrow+\infty} x_L^n \geq \frac{3}{8}$. In this case consider entry right at the left of x_L^n, i.e. at $y = x_L^n - 2\delta_n$, with δ_n such that $\lim_{n\longrightarrow+\infty} \delta^n = 0$. The shares of the vote are

$$v_n^L(y) = \frac{1}{4} + \frac{2}{3}\left(x_L^n - \delta_n - \frac{1}{8}\right) = \frac{2}{3}x_L^n - \frac{2}{3}\delta_n + \frac{1}{6}$$

$$v_n^L(x_L^n) = \frac{2}{3}\left(\frac{1}{2} - (x_L^n - \delta_n)\right) + \frac{1-2\frac{1}{n}}{2\left(\frac{1}{n}+\frac{3}{10}\right)}\left(\frac{x_R^n + x_L^n - 2\delta_n}{2} - \frac{1}{2}\right)$$

$$v_n^L(x_R^n) = \frac{1-2\frac{1}{n}}{2\left(\frac{1}{n}+\frac{3}{10}\right)}\left(\frac{4}{5} + \frac{1}{n} - \frac{x_R^n + x_L^n}{2} - \delta_n\right) + \frac{1}{n}$$

The condition $\lim_{n\longrightarrow+\infty} v_n^L(y) > \lim_{n\longrightarrow+\infty} v_n^L(x_L^n)$ is

$$\frac{2}{3}x_L^* + \frac{1}{6} > \frac{2}{3}\left(\frac{1}{2} - x_L^*\right) + \frac{5}{3}\left(\frac{1+x_L^*}{2} - \frac{1}{2}\right)$$

which is satisfied since $x_L^* \geq \frac{3}{8}$.

The condition $\lim_{n\longrightarrow+\infty} v_n^L(y) > \lim_{n\longrightarrow+\infty} v_n^L(x_R^n)$ is

$$\frac{2}{3}x_L^* + \frac{1}{6} > \frac{5}{3}\left(\frac{4}{5} - \frac{1+x_L^*}{2}\right)$$

which again is satisfied since $x_L^* \geq \frac{3}{8}$.

We conclude that for n large enough there cannot be a two-party equilibrium.

The example illustrates the reason why a two-party equilibrium may fail to exist. Fior n large the distribution F_n^R is very concentrated, imposing severe constraints on the location of x_R^n (essentially, it has to be very close to 1). On the other hand, the distribution F_n^L has 'peaks' in the interval $\left[0, \frac{1}{8}\right]$ and $\left[\frac{1}{2}, \frac{4}{5}\right]$. This implies that if x_L^n is 'large' (close to $\frac{1}{2}$) an entrant on the left will be able to capture enough votes and win, while if x_L^n is 'low' an entrant in the center can win.

Example: The 'no large peaks' assumption is not necessary. Let F^L be represented by the density function

$$f^L(x) = \begin{cases} 1 & \text{if} \quad x < \frac{1}{3} \\ 3 & \text{if} \quad \frac{1}{3} \leq x \leq \frac{5}{12} \\ \frac{1}{3} & \text{if} \quad \frac{5}{12} < x \leq \frac{2}{3} \\ 1 & \text{if} \quad x > \frac{2}{3} \end{cases}$$

with median $m^L = \frac{7}{18}$, and F^R by

$$f^R(x) = \begin{cases} 1 & \text{if} \quad x < \frac{1}{3} \\ \frac{1}{3} & \text{if} \quad \frac{1}{3} \leq x \leq \frac{7}{12} \\ 3 & \text{if} \quad \frac{7}{12} < x \leq \frac{2}{3} \\ 1 & \text{if} \quad x > \frac{2}{3} \end{cases}$$

with median $m^R = \frac{11}{18}$. We want to show that there is a two-party equilibrium $\{x_L, x_R\}$ with $x_L = \frac{1}{3}$ and $x_R = \frac{2}{3}$.

Observe first that the left-wing party wins when the distribution is f^L, and the right-wing party wins when the distribution is f^R, so they both want to enter. To check that no new entry is profitable, consider first entrants at $y < \frac{1}{3}$. When the distribution is F^R the right-wing party still wins, so entry has no effect. When the distribution is F^L the entrant obtains strictly less than $F^L\left(\frac{1}{3}\right) = \frac{1}{3}$. Since the right-wing party still collects $\frac{1}{3}$ of the vote, the entrant loses for sure and either does not change the original outcome or it causes the right wing party to win. We conclude that entry is not profitable. An identical reasoning shows that entry at $y > \frac{2}{3}$ is not profitable. We are left with entry in the interval $y \in \left(\frac{1}{3}, \frac{2}{3}\right)$. First observe that no entrant can win an election under both distributions, since in both cases the left-wing party and the right-wing party collect at least $\frac{1}{3}$ of the vote. Thus the issue is whether an entrant can change the result of the election in a way which is favorable. It is enough to observe that an entrant at $\frac{1}{2}$ does not change the outcome. Since an entrant at $y > \frac{1}{2}$ subtracts fewer votes to the left wing party, it cannot change favorably the outcome, and similarly for $y < \frac{1}{2}$. The reasoning for f^R is symmetric, so no entry occurs.

4 Equilibria with More than Two Parties

While equilibria with two parties seem to be quite natural in this model, there may be other equilibria. In particular we may both have equilibria in which more than two parties win and equilibria in which only two parties win but more than two parties enter. In this case the entry of 'losing' parties is justified by the fact that entry still changes the set of winners in a way which is favorable to the losing entrant. However, it is possible to characterize quite tightly the number of winners in any equilibrium. In fact, it turns out that either there is a single winner for each distribution or there are exactly two winners for each distribution. We now make the statement precise.

For a given political configuration X let $n^q = |W(F^q, X)|$ the cardinality of the set of winners under distribution F^q, $q \in \{L, R\}$. When no ambiguity arises we will write $W^q = W(F^q, X)$.

Proposition 6 *If $X = \{x_1, \ldots, x_k\}$ is a political equilibrium then,*

1. *either $n^L = n^R = 1$ or $n^L = n^R = 2$. No other combinations are possible in equilibrium and*
2. *Equilibria in which $n^L = n^R = 2$ have $W^L = \{x_1, x_{k-1}\}$ and $W^R = \{x_2, x_k\}$.*

Essentially, in order to accept tieing under F^L party x_1 must be unwilling to move slightly to the right. This happens only of x_2 ties under F^R, so that a movement to the right by x_1 causes x_2 to lose under F^R.

Equilibria with $n^L = n^R = 1$ tend to be quite robust, in the sense that by moving slightly the positions of x_L and x_R or by changing slightly the distributions F^L and F^R the equilibrium is preserved. This is not true for equilibria with $n^R = n^L = 2$, which tend instead to be quite fragile. In fact $n^R = n^L = 2$ requires that parties tie; in order to avoid profitable deviations a very careful balance has to be struck between the expected gain and the expected loss in case of any deviation. The next proposition illustrates this point for the case of a three-party equilibrium.

Proposition 7 *In a three-party equilibrium $\{x_L, x_M, x_R\}$ in which all three parties win with strictly positive probability we must have $x_M = \theta x_L + (1-\theta) x_R$ with $W^L = \{x_L, x_M\}$ and $W^R = \{x_M, x_R\}$.*

To understand the point observe that x_M can win under F^L by moving slightly to the left, thus gaining an expected value $\frac{\theta}{2}(x_M - x_L)$, at the cost of letting x_R win with probability 1, thus suffering an expected loss of $\frac{(1-\theta)}{2}(x_R - x_L)$. Thus

$$\theta (x_M - x_L) \leq (1-\theta)(x_R - x_L)$$

must be true in equilibrium. An identical reasoning with regard to a slight movement to the right yields the reverse inequality, thus leading to the conclusion that in equilibrium

$$(1-\theta)(x_R - x_M) = \theta (x_M - x_L) . \tag{8}$$

From Proposition 6 we also know that x_L and x_M must tie under F^L, and x_M and x_R must tie under under F^R. This adds two equalities that have to be satisfied. Together with (8) we have a system of three equations in three unknowns. Generically the system will have a finite number of solutions. If a solution is indeed an equilibrium (i.e. if it can be shown that there is no incentive to enter) then the positions of the parties are exactly determined, and slight perturbations are not equilibria.

We now provide two examples of three-party equilibria, one in which all parties win with positive probability and another in which the centrist party never wins.

Example. A three-party equilibrium with $n^R = n^L = 2$. Assume $\theta = \frac{1}{2}$. The distribution F^L has density

$$f^L(x) = \begin{cases} \frac{5}{4} & \text{if} \quad 0 \leq x \leq \frac{4}{5} \\ 0 & \text{if} \quad \frac{4}{5} < x \leq 1, \end{cases}$$

while distribution F^R has density

$$f^R(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < \frac{1}{5} \\ \frac{5}{4} & \text{if} \quad \frac{1}{5} \leq x \leq 1. \end{cases}$$

We now show that the three party political configuration

$$X = \left\{ x_L = \frac{1}{6}, x_M = \frac{1}{2}, x_R = \frac{5}{6} \right\}$$

is a political equilibrium where all the three parties win with positive probability.

Notice that

$$v^L(x_L) = v^L(x_L) = \frac{5}{12}, \qquad v^L(x_R) = \frac{1}{6}$$

$$v^R(x_L) = \frac{1}{6}, \qquad v^R(x_M) = v^R(x_R) = \frac{5}{12}$$

so that x_L, x_M win under F^L and x_M, x_R win under F^R. Since $x_M = \frac{1}{2}x_R + \frac{1}{2}x_L$ the existing parties do not want to move. Entry at the left of x_L is unprofitable since it cause x_L to lose under F^L and does not change the outcome under F^R. A similar argument proves that entry to the right of x_R is not profitable.

Now consider entry on the interval (x_L, x_M). Observe that any entrant here receives a vote of $\frac{1}{6}$ under F^L which is always less that the maximum of what x_L or x_M receives under F^L. Moreover, such an entry does not affect the share of x_R under any distribution. Hence, such an entry never wins. Notice that any such entry closer to x_L either does not affect the outcome or defeats x_L (the ideologically closer party) under F^L and ensures victory for x_R (the ideologically farthest party) under F^R. So there cannot be any entry in the interval (x_L, x_M) which is closer to x_L. Now let the entry be at the center of the interval (x_L, x_M). Here the outcome remains unaffected under F^L while under F^R, x_R (the ideologically farthest party) wins with probability 1. Hence such an entry is not profitable. Now consider an entry closer to x_M. In this case, x_L wins for sure under F^L and x_R wins for sure under F^R. Since x_M is the closest party for this entrant, such entry is never beneficial. By similar arguments, there is no profitable entry in the interval (x_M, x_R).

Example. A three-party equilibrium with $n^R = n^L = 1$. Let the total mass of voters be $\frac{39}{10}$. The distribution F^L is given by the density

$$f^L = \begin{cases} 11 & \text{if} \quad 0 \leq x < \frac{1}{10} \\ 2 & \text{if} \quad \frac{1}{10} \leq x \leq \frac{5}{10} \\ 10 & \text{if} \quad \frac{5}{10} < x \leq \frac{7}{10} \\ 0 & \text{if} \quad 1 \geq x > \frac{7}{10}, \end{cases}$$

while F^R is given by the density

$$f^R = \begin{cases} \frac{24}{7} & \text{if} \quad 0 \leq x < \frac{7}{10} \\ 5 & \text{if} \quad 1 \geq x \geq \frac{7}{10}. \end{cases}$$

We want to show that the political configuration

$$X = \left\{ x_L = \frac{2}{10}, x_M = \frac{4}{10}, x_R = \frac{8}{10} \right\}$$

is a political equilibrium where x_M never wins.

First notice the voter indifferent between x_L and x_M is at $\frac{3}{10}$, the voter indifferent between x_M and x_R is at $\frac{6}{10}$ and the voter indifferent between x_L and x_R is at $\frac{5}{10}$ (this is relevant if x_M withdraws).

Under F^L the shares of the vote are as follows:

$$v^L(x_L) = \frac{15}{10} \qquad v^L(x_M) = \frac{14}{10} \qquad v^L(x_R) = \frac{10}{10}$$

so that x_L wins. Under F^R the shares of the vote are as follows:

$$v^R(x_L) = \frac{72}{70} \qquad v^R(x_M) = \frac{72}{70} \qquad v^R(x_R) = \frac{129}{70}$$

so that x_R wins. Since there are no ties, no slight movement is profitable. Furthermore, it is easy to check that if x_M exits then x_R wins under both distributions. Since x_M is closer to x_L, it strictly prefers to stay. The only thing left to do is to check that no party wants to enter.

Consider first the wings. A party entering right at the left of $\frac{2}{10}$ collects a share of vote equal to $\frac{13}{10}$ under F^L, making x_M the winner. Under F^R the outcome is unchanged. Thus the only change is unfavorable, and entry on the left of x_L is not profitable.

A party entering right to the right of x_R obtains $\frac{70}{70}$ of the vote when the distribution is F^R, so that in this case the x_L and x_M win with probability $\frac{1}{2}$, while the outcome does not change under F^L. Again, the only changes are unfavorable and entry is not profitable.

Consider now entry at the center. First consider the interval $\left[\frac{2}{10}, \frac{4}{10}\right]$. No party entering in this region can change the outcome at F^R. Also, no party entering in this region can win under F^L (the total mass of voters in the interval is $\frac{4}{10}$, less that $v^L(x_R)$). Thus, entry can be profitable only if it changes the winner under F^L in a favorable way, i.e. making sure that x_M wins. Entering at $y \in \left[\frac{2}{10}, \frac{4}{10}\right]$ yields the following shares of the vote:

$$v^L(x_L) = y + \frac{11}{10} \qquad v^L(y) = \frac{1}{5} \qquad v^L(x_M) = \frac{8}{5} - y \qquad v^L(x_R) = 1.$$

For x_M to win it must be $y + \frac{11}{10} < \frac{8}{5} - y$, or $y < 0.25$. But citizens with ideal point $y \in (0.2, 0.25)$ prefer x_L to x_M.

Next consider the interval $\left[\frac{4}{10}, \frac{8}{10}\right]$. First observe that any such entry will not change the outcome under F^L. When the distribution is F^R it is impossible to make sure that x_M wins, since x_L gets strictly more votes if $y \in \left(\frac{4}{10}, \frac{8}{10}\right)$ wins. Thus the only possible changes that y may induce are either victory by x_L or victory by y. The highest share of the vote is obtained through entry just at the left of x_R, and it is $\frac{59}{70}$. Thus, outright victory is impossible.

Consider now citizens with $y \in (0.4, 0.5)$. Those citizens prefer x_L to x_R and would be happy to enter if this would shift victory to x_L under F^R. Clearly $x_R = \frac{5}{10}$ is the citizen with the best chance to give victory to x_L. If $y = 0.5$ enters the distribution of votes is

$$v^R(x_L) = \frac{72}{70} \qquad v^R(x_M) = \frac{36}{70} \qquad v^R(y) = \frac{36}{70} \qquad v^R(x_R) = \frac{117}{70}$$

so x_R still wins. We conclude that no entry is profitable and X is a political equilibrium.

5 Conclusions

We have shown that the introduction of aggregate uncertainty in the citizen candidate model has important consequences in terms of characterization of the equilibrium. In our model parties decide to enter before the distribution of voters' preferences is determined. Political opinion may be left-wing or right-wing, with certain probabilities. The model has some interesting predictions. Two-party equilibria in such a model have some intriguing characterstics. First, there are no ties: parties win when the political mood is favorable, and when they do they win decisively. Second, parties tend to be 'extremist': the right wing party is to the right of the median right-wing voter and the left-wing party is to the left of the left-wing voter.

Appendix

Proof of Proposition 1. Point 1 follows from Proposition 1 in OS for the case $b = c = 0$. Point 2 is a corollary of Proposition 2 in OS for the case $b = c = 0$. ■

Proof of Proposition 2. If the citizen located at $i = m$ enters then no citizen can enter and change the outcome. Thus, this is a Nash equilibrium for each $\delta > 0$.

To prove that this is the unique Nash equilibrium, first observe that no equilibrium with a single candidate not at the median can exist, since the median citizen could profitably enter. Suppose that at $\delta > 0$ there is an equilibrium with $n > 1$ parties, and let $x_1^\delta, x_2^\delta, \ldots, x_n^\delta$ be the positions occupied by the entering parties. Consider the party with platform x_1^δ. It cannot be the case that the party wins with probability 1, since otherwise the other parties would be better off not entering. On the other hand x_1^δ party must win with positive probability, since otherwise not entering would be a profitable deviation. But if the party wins with strictly positive probability at x_1^δ then it can win with probability 1 by moving to any point $x_1^\delta + \varepsilon$ such that $x_1^\delta + \varepsilon < x_2^\delta$ and $\varepsilon > 0$. For ε sufficiently small the deviation is profitable, contradicting the fact that the proposed configuration is an equilibrium. ■

Proof of Proposition 3. To prove point 1 observe that if a single party enters at location x then there is at least one distribution F^q such that $x \neq m^q$. If the voter with ideal point m^q enters then she surely wins the election when the distribution is F^q, which is strictly better than getting x with probability 1. We conclude that m^q would enter, contradicting the fact that in equilibrium there is only one entrant.

To prove point 2, suppose that x_1 never wins. By exiting either the outcome is unchanged or party x_2 is more likely to win. Thus, exiting is a profitable deviation, a contradiction. This implies that x_1 is a winner with strictly positive probability. The argument for x_h is similar. We now show that if the distribution is F^R, then x_1 loses with probability 1. So suppose not. Then x_1 wins with positive probability on F^L and F^R which is impossible since then x_1 will re-position itself arbitrarily small and win with probability 1 under both distributions, a contradiction. ■

Proof of Proposition 4. Proposition 3 implies that in any two-party equilibrium it must be the case that one party wins with probability 1 when the distribution is F^L and the other wins with probability one when the distribution is F^R. Suppose x_L wins under F^R and x_R wins under F^L. Then $\frac{x_L+x_R}{2} > m^R$ and $\frac{x_L+x_R}{2} < m^L$, contradicting $m^R > m^L$. We conclude that in any two-party equilibrium the 'natural' outcome occurs: the left-wing party wins with probability 1 under the left-wing distribution and loses with probability 1 under the right-wing distribution.

To prove point 2, suppose $x_L \geq m^L$. Consider a citizen located at $y_\varepsilon = m^L - \varepsilon$. By entering the citizen obtains a share of the vote $F^L\left(\frac{y_\varepsilon+x_L}{2}\right)$ when the distribution is F^L. Since, by assumption, x_R obtains a strictly positive share of the vote when the distribution is F^L and F^L is strictly increasing and continuous we can find ε small enough such that

$$F^L\left(\frac{y_\varepsilon+x_L}{2}\right) > \max\left\{F^L\left(\frac{x_L+x_R}{2}\right) - F^L\left(\frac{y_\varepsilon+x_L}{2}\right), 1 - F^L\left(\frac{x_L+x_R}{2}\right)\right\},$$

where $F^L\left(\frac{x_L+x_R}{2}\right) - F^L\left(\frac{y_\varepsilon+x_L}{2}\right)$ is the share of x_L vote under distribution F^L and $1 - F^L\left(\frac{x_L+x_R}{2}\right)$ is the share of the x_R vote under distribution F^L. Thus, for ε small enough, y_ε can enter and win with probability 1 under F^L. Furthermore entry does not change the outcome when the distribution is F^R, since the share of x_R vote does not change and it remains greater than 50%. Thus, $x_L \geq m^L$ cannot be part of an equilibrium because it would cause entry on the left. Similarly, $x_2 \leq m^R$ cannot be part of an equilibrium because it would cause entry on the right. ■

Proof of Proposition 5. The proof will be in two steps. In the first step we show that there is a pair (x_1^*, x_2^*) such that $\Psi^L(x_1^*, x_2^*) = \Psi^R(x_1^*, x_2^*) = 0$, $x_1^* < m_L$, $x_2^* > m^R$ and $m^L < \frac{x_1^*+x_2^*}{2} < m^R$. This implies that x_1^* wins under F^L and x_2^* wins under F^R, so that neither of the two parties wants to exit or move.

In the second step we show that when there are two parties positioned at the pair (x_1^*, x_2^*) with the above described properties there is no profitable entry.

<u>Step 1.</u> Fix $x_2 \geq m^R$ and consider the equation in x_1 given by

$$\Psi^L(x_1, x_2) = 0.$$

The function $\Psi^L(\cdot, x_2)$ is continuous and differentiable almost everywhere in x_1. Furthermore, there exists x_1 such that $\Psi^L(x_1, x_2^*) > 0$, $\Psi^L\left(m^L, x_2\right) < 0$ and $\frac{d\Psi^L}{dx_1} < 1$ wherever the derivative is defined. We can therefore conclude that a solution exists and it is unique. Let $h(x_2)$ be the solution and observe that h is continuous and $h(x_2) < m^L$ for each x_2.

Next consider the equation

$$\Psi^R(h(x_2), x_2) = 0.$$

The function is continuous in x_2 since it's the composition of continuous functions. Furthermore $\Psi^R(h(m^R), m^R) < 0$ and there exsits x_2 sich that $\Psi^R(h(x_2), x_2) > 0$. Thus, there is a point x_2^* such that $\Psi^R(h(x_2^*), x_2^*) = 0$. Let now $x_1^* = h(x_2^*)$. By construction we have $\Psi^L(x_1^*, x_2^*) = \Psi^R(x_1^*, x_2^*) = 0$, $x_1^* < m_L$, $x_2^* > m^R$.

To complete Step 1 we have to show $m^L < \frac{x_1^*+x_2^*}{2} < m^R$. Suppose $\frac{x_1^*+x_2^*}{2} \leq m^L$. Then $1 - F^L\left(\frac{x_1^*+x_2^*}{2}\right) \geq \frac{1}{2} > F^L(x_1^*)$, since $x_1^* < m^L$. But this implies $\Psi^L(x_1^*, x_2^*) > 0$, a contradiction; thus $m^L < \frac{x_1^*+x_2^*}{2}$. A similar argument shows that $\frac{x_1^*+x_2^*}{2} < m^R$.

Step 2. Consider a pair (x_1^*, x_2^*) satisfying the properties described in Step 1. We now show that no new party can be profitably formed.

Consider first entry at the left of x_1^*, i.e. entry at $y = x_1^* - \delta$ for some $\delta > 0$. In this case the shares of the vote are as follows

$$v^L(x_1^* - \delta) = F^L\left(x_1^* - \frac{1}{2}\delta\right), \qquad v^L(x_1^*) = F^L\left(\frac{x_1^*+x_2^*}{2}\right) - F^L\left(x_1^* - \frac{1}{2}\delta\right)$$

$$v^L(x_2^*) = 1 - F^L\left(\frac{x_1^*+x_2^*}{2}\right).$$

Since

$$F^L\left(x_1^* - \frac{1}{2}\delta\right) < F^L(x_1^*) =$$

$$\max\left\{F^L\left(\frac{x_1^*+x_2^*}{2}\right) - F^L(x_1^*), 1 - F^L\left(\frac{x_1^*+x_2^*}{2}\right)\right\} \leq \max\left\{v^L(x_1^*), v^L(x_2^*)\right\}$$

we conclude that $x_1^* - \delta$ is defeated when the distribution is F^L. When the distribution is F^R the winner remains x_2^*, so the entry is not profitable. A similar reasoning establishes that no entry at the right of x_2^* is profitable.

What remains is to check that no entry in the interval (x_1^*, x_2^*) is profitable. We first show that no citizen $y \in (x_1^*, x_2^*)$ can enter and win. Under distribution F^q, such an entry yields voting shares

$$v^q(x_1^* | y) = F^L\left(\frac{x_1^*+y}{2}\right), \qquad v^q(y | y) = F^q\left(\frac{y+x_2^*}{2}\right) - F^q\left(\frac{x_1^*+y}{2}\right)$$

$$v^q\left(x_2^*\middle|\, y\right)=1-F^L\left(\frac{y+x_2^*}{2}\right).$$

Consider F^L first, and define the function

$$g\left(y\right)=v^L\left(x_1^*\middle|\, y\right)-v^L\left(y\middle|\, y\right).$$

Since $F^L\left(x_1^*\right)=\max\left\{F^i\left(\frac{x_1^*+x_2^*}{2}\right)-F^i\left(x_1^*\right),1-F^L\left(\frac{x_1^*+x_2^*}{2}\right)\right\}$ we have $g\left(x_1^*\right)\geq 0$. Furthermore

$$g'\left(y\right)=f^L\left(\frac{x_1^*+y}{2}\right)-\frac{1}{2}f^L\left(\frac{y+x_2^*}{2}\right)>0,$$

where the inequality follows from assumption 1. Thus, for each $y\in\left(x_1^*,x_2^*\right)$ and entrant loses against x_1^* at F^L. Furthermore, an entrant cannot change the outcome either. Define

$$z\left(y\right)=v^L\left(x_1^*\middle|\, y\right)-v^L\left(x_2^*\middle|\, y\right).$$

Again, by construction we have $z\left(x_1^*\right)\geq 0$ and

$$z'\left(y\right)=\frac{1}{2}f^L\left(\frac{x_1^*+y}{2}\right)+\frac{1}{2}f^L\left(\frac{y+x_2^*}{2}\right)>0.$$

Thus, any entry $y\in\left(x_1^*,x_2^*\right)$ does not change the outcome under F^L. A similar reasoning establishes that the outcome does not change under F^R either. This completes the proof. ■

Proof of Proposition 6. Suppose first that $\min\left\{n^L,n^R\right\}=1$, and without loss of generality assume $n^R=1$. Proposition 3 part 2 implies that the unique winner under F^R is x_k and that under F^L party x_1 wins with positive probability. But in equilibrium it must be that x_1 is the only winner at F^L, since otherwise it could ensure victory with probability 1 by moving slightly to the right, without affecting the outcome under F^R. We conclude that $\min\left\{n^L,n^R\right\}=1$ implies $n^L=n^R=1$.

Thus, suppose that $\min\left\{n^L,n^R\right\}>1$, i.e. $n^L\geq 2$ and $n^R\geq 2$. Again, by proposition 3 we know that $x_1\in W^L$ and $x_k\in W^R$. Next observe that $n^L\geq 2$ must imply that $x_2\in W^R$. If not, x_1 could move slightly to the right and win with probability 1 (rather than $\frac{1}{2}$) under F^L without affecting the outcome at F^R. A similar argument establishes that $x_{k-1}\in W^L$. We consider now two cases.

Case 1: $x_2\in W^L\cap W^R$. For this to be an equilibrium it must be the case that x_1 does not want to move to the right. If x_1 did move to the right, then the set of winners at F^L would be just x_1, while the set of winners at F^R would be $W^R \backslash \{x_2\}$. Therefore the relevant inequality is

$$-\frac{\theta}{n^L}\left(\sum_{x_q \in W^L \backslash \{x_1\}} |x_q - x_1|\right) - \frac{1-\theta}{n^R}\left(\sum_{x_q \in W^R} |x_q - x_1|\right) \geq$$

$$-\frac{1-\theta}{n^R - 1}\left(\sum_{x_q \in W^R \backslash \{x_2\}} |x_q - x_1|\right) \tag{9}$$

which can be written as

$$(1-\theta)\left(\sum_{x_q \in W^R \backslash \{x_2\}} |x_q - x_1|\right) \geq$$

$$\theta \frac{n^R - 1}{n^L}\left(\sum_{x_q \in W^L \backslash \{x_1\}} |x_q - x_1|\right) + (1-\theta)\frac{\left(n^R - 1\right)}{n^R}\left(\sum_{x_q \in W^R} |x_q - x_1|\right).$$

Since $|x_q - x_1| \geq |x_2 - x_1|$ for each $x_q \in W^R$ this in turn implies

$$(1-\theta)\left(\sum_{x_q \in W^R \backslash \{x_2\}} |x_q - x_1|\right) \geq$$

$$\theta \frac{n^R - 1}{n^L}\left(\sum_{x_q \in W^L \backslash \{x_1\}} |x_q - x_1|\right) + (1-\theta)\left(n^R - 1\right)|x_2 - x_1|. \tag{10}$$

Consider now a slight movement to the left by x_1. This gives victory with probability 1 to x_2 under both distributions. The condition for the deviation not to be profitable can be written as

$$-\frac{\theta}{n^L}\left(\sum_{x_q \in W^L \backslash \{x_1\}} |x_q - x_1|\right) -$$

$$-\frac{1-\theta}{n^R}\left(\sum_{x_q \in W^R \backslash \{x_2\}} |x_q - x_1|\right) - \frac{1-\theta}{n^R}|x_2 - x_1| \geq -|x_2 - x_1|$$

which in turn becomes

$$\left(n^R-(1-\theta)\right)|x_2-x_1|-\theta\frac{n^R}{n^L}\left(\sum_{x_q\in W^L\backslash\{x_1\}}|x_q-x_1|\right)\geq$$

$$(1-\theta)\left(\sum_{x_q\in W^R\backslash\{x_2\}}|x_q-x_1|\right). \tag{11}$$

Combining (10) and (11) we obtain the following a necessary condition for X to be an equilibrium:

$$\left(n^R-(1-\theta)\right)|x_2-x_1|\geq$$

$$\theta\left(\frac{2n^R-1}{n^L}\right)\left(\sum_{x_q\in W^L\backslash\{x_1\}}|x_q-x_1|\right)+(1-\theta)\frac{\left(n^R-1\right)}{n^R}\left(\sum_{x_q\in W^R}|x_q-x_1|\right)$$

which in turn, since $|x_q-x_1|\geq|x_2-x_1|$ for each $x_q\in W^R$ and each $x_q\in W^L\backslash x_1$, implies

$$\left(n^R-(1-\theta)\right)|x_2-x_1|\geq$$

$$\theta\left(\frac{2n^R-1}{n^L}\right)\left(n^L-1\right)|x_2-x_1|+(1-\theta)\left(n^R-1\right)|x_2-x_1|$$

and, after simplifications.

$$2n^R+n^L\geq n^Rn^L+1$$

If $n^R\geq2$ the inequality can be written as

$$n^L\leq\frac{2n^R-1}{n^R-1}$$

Thus, $n^L\leq2$ if $n^R\geq2$. We conclude that if $\min\left\{n^L,n^R\right\}\geq2$ then in fact $n^L=\min\left\{n^L,n^R\right\}=2$. However, remember that $x_{k-1}\in W^L$. Since $\{x_1,x_2\}\subset W^L$ and $n^L=2$ we conclude $x_{k-1}=x_2$ and $k=3$. Thus, if $x_2\in W^L\cap W^R$ then X has exactly three members and $W^L=\{x_1,x_2\}$, $W^R=\{x_2,x_3\}$.

Case 2: $x_2\notin W^L$. In this case a move to the right by x_1 gives victory to x_1 under F^L and causes x_2 to lose under F^R, so that the relevant inequality is still (9). A slight movement to the left by x_1 gives victory with probability

1 to x_2 under F^R and eliminates x_1 from the set of winners under F^L. Thus the relevant inequality is

$$-\frac{\theta}{n^L}\left(\sum_{x_q \in W^L \setminus \{x_1\}} |x_q - x_1|\right) - \frac{1-\theta}{n^R}\left(\sum_{x_q \in W^R \setminus \{x_2\}} |x_q - x_1|\right) - \frac{1-\theta}{n^R}|x_2 - x_1| \geq$$

$$-\frac{\theta}{n^L - 1}\left(\sum_{x_q \in W^L \setminus \{x_1\}} |x_q - x_1|\right) - (1-\theta)|x_2 - x_1|$$

which can be written as

$$\theta \frac{n^R}{n^L(n^L-1)}\left(\sum_{x_q \in W^L \setminus \{x_1\}} |x_q - x_1|\right) + (1-\theta)\left(n^R - 1\right)|x_2 - x_1| \geq$$

$$(1-\theta)\left(\sum_{x_q \in W^R \setminus \{x_2\}} |x_q - x_1|\right) \tag{12}$$

Combining (10) and (12) we obtain the following condition

$$\theta \frac{n^R}{n^L(n^L-1)}\left(\sum_{x_q \in W^L \setminus \{x_1\}} |x_q - x_1|\right) + (1-\theta)\left(n^R - 1\right)|x_2 - x_1| \geq$$

$$\theta \frac{n^R - 1}{n^L}\left(\sum_{x_q \in W^L \setminus \{x_1\}} |x_q - x_1|\right) + (1-\theta)\left(n^R - 1\right)|x_2 - x_1|$$

which can be written as

$$2n^R + n^L \geq n^R n^L + 1$$

Which is the same as before. Thus, in this case as well we have $n^L = 2$. In particular, $W^L = \{x_1, x_{k-1}\}$.

Now suppose $n^R > 2$. We know that both x_2 and x_k belong to W^R. Since x_1 cannot win under F^R there must be at least 4 parties, that is $k \geq 4$.

Subcase 1.a $x_{k-1} \notin W^R$. Suppose x_k moves slightly to the right. Then it drops out of the winning set under F^R and it gives victory to x_{k-1} with probability 1 under F^L. The relevant inequality can be written as

$$\frac{(1-\theta)}{n^R}\left(\sum_{x_q \in W^R \setminus \{x_k\}} |x_q - x_k|\right) \geq$$

$$\theta\frac{\left(n^R-1\right)}{n^L}\left(\sum_{x_q\in W^L}|x_q-x_k|\right)-\theta\left(n^R-1\right)|x_{k-1}-x_k|. \tag{13}$$

Moving to the left by x_k implies victory to x_k with probability 1 under F^R and x_{k-1} dropping out of W^L. In this case the relevant inequality can be written as

$$\frac{\theta}{n^L-1}\left(\sum_{x_q\in W^L\backslash\{x_{k-1}\}}|x_q-x_k|\right)-\frac{\theta}{n^L}\left(\sum_{x_q\in W^L}|x_q-x_k|\right)\geq$$

$$\frac{1-\theta}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right).$$

A necessary condition is therefore

$$\frac{\theta}{n^L-1}\left(\sum_{x_q\in W^L\backslash\{x_{k-1}\}}|x_q-x_k|\right)\geq$$

$$\theta\frac{n^R}{n^L}\left(\sum_{x_q\in W^L}|x_q-x_k|\right)-\theta\left(n^R-1\right)|x_{k-1}-x_k|.$$

Since $n^L=2$ and $W^L=\{x_1,x_{k-1}\}$, we have

$$|x_{k-1}-x_k|\geq|x_1-x_k|$$

which is impossible.

Subcase 1.b $x_{k-1}\in W^L\cap W^R$. Suppose x_k moves slightly to the right. Then it gives victory with probability 1 to x_{k-1} under both distributions. The relevant inequality is

$$-\frac{\theta}{n^L}\left(\sum_{x_q\in W^L}|x_q-x_k|\right)-\frac{1-\theta}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right)\geq-|x_{k-1}-x_k|.$$

Using $n^L=2$ and $\sum_{x_q\in W^L}|x_q-x_k|=|x_1-x_k|+|x_{k-1}-x_k|$ we obtain

$$\left(1-\frac{\theta}{2}\right)|x_{k-1}-x_k|-\frac{1-\theta}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right)\geq\frac{\theta}{2}|x_1-x_k| \tag{14}$$

Moving to the left by x_k implies victory for x_k with probability 1 under F^R and x_{k-1} dropping out of W^L. Thus the relevant inequality is

$$-\frac{\theta}{n^L}\left(\sum_{x_q\in W^L}|x_q-x_k|\right)-\frac{1-\theta}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right)\geq$$

$$-\frac{\theta}{n^L-1}\left(\sum_{x_q\in W^L\backslash\{x_{k-1}\}}|x_q-x_k|\right)$$

which again can be written as

$$\frac{\theta}{2}|x_1-x_k|\geq\frac{\theta}{2}|x_{k-1}-x_k|+\frac{1-\theta}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right). \tag{15}$$

Combining (14) and (15) we obtain that a necessary condition for equilibrium is

$$\left(1-\frac{\theta}{2}\right)|x_{k-1}-x_k|-\frac{1-\theta}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right)\geq$$

$$\frac{\theta}{2}|x_{k-1}-x_k|+\frac{1-\theta}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right).$$

which, after manipulations, yields

$$|x_{k-1}-x_k|\geq\frac{2}{n^R}\left(\sum_{x_q\in W^R\backslash\{x_k\}}|x_q-x_k|\right).$$

Since $|x_q-x_k|\geq|x_{k-1}-x_k|$ for each $x_q\in W^R\backslash x_k$ this in turn implies

$$|x_{k-1}-x_k|\geq\frac{2}{n^R}\left(n^R-1\right)|x_{k-1}-x_k|$$

which implies $n^R\leq 2$. ■

Proof of Proposition 7. From Proposition 6 we have $W^L=\{x_L,x_M\}$ and $W^R=\{x_M,x_R\}$. This in turn implies that the following equalities have to be satisfied:

$$F^L\left(\frac{x_L+x_M}{2}\right)=F^L\left(\frac{x_M+x_R}{2}\right)-F^L\left(\frac{x_L+x_M}{2}\right) \tag{16}$$

$$1 - F^R\left(\frac{x_M + x_R}{2}\right) = F^R\left(\frac{x_M + x_R}{2}\right) - F^R\left(\frac{x_L + x_M}{2}\right). \tag{17}$$

Furthermore, if this is indeed an equilibrium it must be the case that no agent is willing to move. Consider x_M first. In the postulated equilibrium the expected utility of x_M is

$$-\frac{\theta}{2}\left|x_M - x_L\right| - \frac{1-\theta}{2}\left|x_M - x_R\right|.$$

Moving slightly to the right ensures victory with probability one under F^R and defeat under F^L. Therefore, the following inequality has to be satisfied:

$$\theta\left|x_M - x_L\right| \geq (1-\theta)\left|x_M - x_R\right| \tag{18}$$

A slight movement to the left ensures victory with probability 1 under F^L and defeat under F^R. Thus, the following equality must hold:

$$(1-\theta)\left|x_R - x_M\right| \geq \theta\left|x_M - x_L\right| \tag{19}$$

Combining (19) and (18) we obtain

$$(1-\theta)\left|x_R - x_M\right| = \theta\left|x_M - x_L\right| \tag{20}$$

Consider now x_L. In the postulated equilibrium the expected utility of x_L is

$$-\frac{\theta}{2}\left|x_L - x_M\right| - \frac{1-\theta}{2}\left|x_L - x_M\right| - \frac{1-\theta}{2}\left|x_L - x_R\right|$$

Moving slightly to the right gives victory with probability one under F^L and gives victory with probability 1 to x_R under F^R. Therefore the following inequality has to be satisfied:

$$(1-\theta)\left|x_L - x_R\right| \geq \theta\left|x_L - x_M\right|. \tag{21}$$

A movement to the left gives victory with probability 1 to x_M under both distributions. Thus in equilibrium it must be

$$\left|x_L - x_M\right| \geq (1-\theta)\left|x_L - x_R\right|. \tag{22}$$

Inequalities (21) and (22) imply

$$\left|x_L - x_M\right| = (1-\theta)\left|x_L - x_R\right|. \tag{23}$$

At last, consider x_R. The expected utility of x_R under the proposed equilibrium is

$$-\frac{\theta}{2}\left|x_R - x_L\right| - \frac{\theta}{2}\left|x_R - x_M\right| - \frac{1-\theta}{2}\left|x_R - x_M\right|.$$

Moving slightly to the right yields victory with probability 1 under both distributions to x_M, so it must be the case that

$$\left|x_R - x_M\right| \geq \theta\left|x_R - x_L\right|,$$

while moving slightly to the left ensures victory under F^R and gives victory under F^L to x_L. Thus we must have

$$\theta\left|x_R - x_L\right| \geq \left|x_R - x_M\right|$$

thus yielding

$$\theta\left|x_R - x_L\right| = \left|x_R - x_M\right|. \tag{24}$$

Summing up, if an equilibrium with all three parties winning exists the positions x_L, x_C and x_R must solve the three equations (20), (23) and (24). The three equations are actually the same, that is

$$(1-\theta)\,x_R + \theta x_L = x_M \tag{25}$$

An equilibrium exists if the system of equations (16), (17) and (25) has a solution, and if at the solution there is no incentive to enter at the center (there cannot be incentive to enter at the wings). ∎

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