MATHEMATICS

On the mean summability by Cesaro (C, *a*) and Abel-Poisson methods of trigonometric Fourier series in the weighted Lorentz spaces

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Abstract

In the paper the classical results on the mean convergence and summability of trigonometric Fourier series are extended to the weighted Lorentz spaces.
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Let T be the interval $(-\pi, \pi)$. In the theory of trigonometric Fourier series it is well known (see, [1]) that Cesaro and Abel-Poisson means converges in $L^p(T)$ $(1 \le p \le \infty)$. The problem of mean summability in weighted Lebesgue spaces has been investigated in [2], [3] in the frame of A_p classes.

In the present paper we study the mean summability problems in weighted Lorentz spaces. Let f be 2π-periodic measurable function. Then let w be 2π-periodic nonnegative integrable on T. The last functions are called as weights. For the Borel set e by

$$we = \int_e w(x)dx$$

we denote the Borel measure generated by the function w. For the function f consider its non-increasing rearrangement with respect to measure w:

$$f^*(t) = \sup\{s \ge 0 : w(x \in T : |f(x)| > s) > t\} \tag{1}$$

Then consider the average of f^*:

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(y)\,dy. \tag{2}$$

It is easy to see that $f^*(t) = 0$, when $t > 2\pi$. Let $1 < p$, $s < \infty$. The weighted Lorentz space $L_w^{ps}(T)$ is defined as the set of all measurable functions f, for which

$$\|f\|_{L_w^{ps}} = \left(\int_0^{2\pi} \left(f^{**}(t)\,t^{1/p}\right)^s \frac{dt}{t}\right)^{1/s} < \infty. \tag{3}$$

It is known, that $L_w^{ps}(T)$ is a Banach space (see, example e. g. [4]).

Let $f \in L^1(T)$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{4}$$

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be the Fourier series of function $f \in L^1(T)$. Let

$$\sigma_n^\alpha(x,f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_n^\alpha(t)dt, \qquad \alpha > 0 \tag{5}$$

when

$$K_n^\alpha(t) = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha-1}D_k(t)}{A_n^\alpha},$$

with

$$D_k(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}$$

and

$$A_n^\alpha = \binom{n+\alpha}{n} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

Definition. A weight function w is said to be of class A_p if

$$\sup \frac{1}{|I|}\int_I w(x)dx \left(\frac{1}{|I|}\int_I w^{1-p'}(x)dx\right)^{p-1} < \infty,$$

where the least upper bound is taken over all intervals I, the length of which are not greater than 2π.

Let

$$\tilde{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(x-t)})ctg\frac{t}{2}dt$$

and

$$Mf(x) = \sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|\,dt.$$

In the sequel we need the following two statements.

Proposition 1. Let $1 < p,\ s < \infty$. Then the operator $\tilde{f}$ is bounded in L_w^{ps} if and only if $w \in A_p$.

Proposition 2. Let $1 < p,\ s < \infty$. Then the operator M is bounded in L_w^{ps} if and only if $w \in A_p$.

For the proof of these Propositions see [4] and [5].

Let

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \tag{6}$$

be the conjugate series to the Fourier series (4). Let $S_n(x,f)$ and $\tilde{S}_n(x,f)$ denote the partial sums of (4) and (6) respectively.

Theorem 1. Let $1 < p < \infty$ and $w \in A_p$. Then we have

$$i)\ \lim_{n\to\infty} \|S_n(\cdot,f) - f\|_{L_w^{ps}} = 0$$

and

$$ii) \lim_{n\to\infty} \left\| \widetilde{S}_n(\cdot, f) - f \right\|_{L^{ps}_w} = 0.$$

The proof of Theorem is similar as in classical case for L^p spaces, basing on Proposition 1 (see, [1]).

Theorem 2. Let $1 < p,\ s < \infty$ and let $w \in A_p$. Then

$$\lim_{n\to\infty} \|\sigma_n^\alpha(\cdot,\ f) - f\|_{L^{ps}_w} = 0, \qquad 0 < \alpha \le 1.$$

For $0 \le r < 1$ let

$$U_r(x, f) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\, r^n$$

be the Abel-Poisson means of function f .

Theorem 3. Let $1 < p,\ s < \infty$ and let $w \in A_p$. Then

$$\lim_{r\to 1} \|U_r(\cdot,\ f) - f\|_{L^{ps}_w} = 0.$$

Note that from Proposition 2 and the following estimates (see, [1])

$$\sup_{0<r<1} |U_r(x, f)| \le cMf(x)$$

and for positive functions f

$$c_1 Mf(x) \le \sup_{0<r<1} |U_r(x, f)|$$

we conclude

Proposition 3. Let $1 < p,\ s < \infty$. Then the operator

$$Nf(x) = \sup_{0<r<1} |U_r(x, f)|$$

is bounded in L^{ps}_w if and only if $w \in A_p$.

Proposition 4. Let $1 < p,\ s < \infty$ and let $w \in A_p$. Then the operator

$$f \to \sup_{n\ge 1} |\sigma_n^\alpha(\cdot,\ f)|$$

is bounded in L^{ps}_w.

The last Proposition follows from the known estimate (see [1])

$$\sup_{n\ge 1} |\sigma_n^\alpha(x,\ f)| \le cMf(x)$$

and the Proposition 2.

The proof of Theorem 2. Let us consider the sequence of operator

$$U_n : f \to \sigma_n^\alpha(\cdot,\ f), \qquad n = 1, 2, ...$$

Let us show that each U_n is linear and bounded in L_w^{ps}. The linearity is clear. Applying the estimate

$$K_n^\alpha(t) < 2n$$

we get

$$\|\sigma_n^\alpha(\cdot,\ f)\|_{L_w^{ps}} \leq c_n \left\| \int_{-\pi}^{\pi} |f(t)|\, dt \right\|_{L_w^{ps}} .$$

But using the generalized Holder's inequality for L_w^{ps} (see e.g. [5]) we get

$$\int_T |f(t)|\, dt \leq \|f\|_{L_w^{ps}} \cdot \|l\|_{L_w^{p's'}} \leq c\, \|f\|_{L_w^{ps}} .$$

Thus the operator U_n is bounded for each n. On the other hand by Proposition 4, the sequence of operator norms

$$\|U_n\|_{L_w^{ps} \to L_w^{ps}}$$

is bounded.

The set of continuous functions is dense in L_w^{ps} and the Fourier series of continuous functions on the real line converges uniformly to f. Thus the Cesaro means of continuous functions converges in the norm L_w^{ps}. Applying the Banach-Steinhaus theorem we conclude that we have the (C, α) summability for arbitrary function $f \in L_w^{ps}$.

Theorem is proved.

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