One method of solution of an optimum investment portfolio problem for risky assets

Alexander MILNIKOV
Mikheil MAMISTVALOV

Abstract: *The problem for choice of an optimum investment portfolio is considered. The square-law form of risk is presented as two-multiple convolution of covariant tensor of the covariance matrix and contravariant vector of weights. By means of reduction of covariance matrix to the diagonal form, the problem by definition of optimum structure of a portfolio is solved: simple expressions for a minimum of risk and optimum distribution of the weights providing this minimum are received.*

Keywords: tensor, convolution, invariants, risky assets, portfolio, covariance matrix, contravariant vector, optimum structural potentials, relative optimum structural potentials

The primary goal in the theory of a choice of an optimum investment portfolio can be reduced to a multidimensional problem of minimization of a quadratic form with constraints [1, 2]. Here the basic role covariance matrix g_{ij} plays. It reflects a set of relationships among n shares. The vector of weights w, characterizing distribution of invested means, should satisfy equation $\sum w^i = 1$

In this case the mentioned problem of minimization looks as follows.

It is required to minimize the quadratic form representing a square of total risk of the given portfolio.

$$\sigma^2 = g_{ij} w^i w^j \qquad (1)$$

with constraint

$$\sum w_i - 1 = 0 \qquad (2)$$

Thus, the decision of the formulated problem will allow defining such distribution of weights of an investment portfolio, which minimizes risk. The received distribution we refer as optimum structure of a portfolio. We underline, that we do not consider shares with zero risk.

Alexander MILNIKOV *is a professor in the Faculty of Information Tecnologies and Engineering at International Black Sea University, Georgia*

Mikheil MAMISTVALOV *is a senior student in the Finance and Banking Direction at International Black Sea University, Georgia*

It is known, that g_{ij} being a covariance matrix, is positively defined symmetric matrix, therefore its eigen values are always non negative, and eigen vectors orthogonal [3,4]. We will consider it as bivalent covariant metric tensor, then total risk (is more exact, its square) of n shares, written down in the form of (1), is possible to be considered as double convolution of metric tensor g_{ij} with contravariant vector of weights w.

If (1) is a convolution, it is invariant. The last means that total risk does not depend on system of coordinates that allows restating a problem of calculating the diagonal form of tensor g_{ij}. We will define the diagonal form of tensor g_{ij} for what we are calculating its eigen values and corresponding eigen vectors. Thanks to simmetricity and positive definiteness of g_{ij} it's not difficult to do it. We will refer eigen values as μ^2_i (i=1,2, …, n), that is they are represented in the form of squares of positive numbers that is possible thanks to mentioned property of eigen values of positively defined symmetric matrixes. Among eigen values can be multiple ones, that essentially changes nothing. It is easy to define also eigen vectors, corresponding to their eigen values. If we normalize them and write down decomposition of already normalized eigen vectors according to the vectors of current basis, we will receive matrix $p^{i\cdot}_{\cdot j}$ where the top index indicates the number of coordinate of eigen vector (number of a line in matrix P), and bottom - vector's number. The matrix $p^{i\cdot}_{\cdot j}$ can be considered as the operator of transition from old basis to new one. As new basis is orthonormal than the transition matrix (it consists of coordinates of these vectors) appears to be the own orthogonal matrix ($P*P^T = E$ u $|P| = 1$). Coordinates of contravariant vectors are transformed, as it is known, by means of a matrix transposed and inverse to R. However, considering the orthogonality of P, we have $(P^T)^{-1} = P$, that is in our case contarvariant and covariant vectors are transformed by means of the same matrix. Notice that this matrix represents also bivalent, but mixed tensor (once covariant - the bottom index and once contravariant the-top index), that allows to consider P as the operator of n-dimensional space.

In the new basis the matrix of tensor g will have a diagonal form:

$$g_{ij}\, p^{i\cdot}_{\cdot l}\, p^{j\cdot}_{\cdot m} = g'_{lm} = \begin{cases} \mu_l^2 & l=m \\ 0 & l\neq m \end{cases} \qquad (3)$$

Proceeding from the aforesaid, the coordinates u^i of contarvariant vector of weights w in new basis

$$u^i = p^{i\cdot}_{\cdot j} w^j \tag{4}$$

It's obvious that we also have inverse transformation

$$w^j = p^{\cdot j}_{i\cdot} u^i \tag{5}$$

Thus we have received the transformed initial problem of optimization:

to minimize

$$\sigma^2 = \mu_i^2 (u^i)^2 \qquad (6),$$

at restriction

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p^{\cdot j}_{i\cdot} u^i - 1 = 0 \qquad (7),$$

which after a regrouping will become

$$\sum_{i=1}^{n} u^i \sum_{j=1}^{n} p^{\cdot j}_{i\cdot} - 1 = 0 \qquad (8).$$

It is easy to write down Lagrange function of a problem (5) - (6)

$$\phi(u,\lambda) = \mu_i^2 (u^i)^2 - \lambda(\sum_{i=1}^{n} u^i \sum_{j=1}^{n} p^{\cdot j}_{i\cdot} - 1 = 0)$$

Whence we have

$$\frac{\partial \phi(u,\lambda)}{\partial u^i} = 2\mu_i^2 u^i - \lambda \sum_{j=1}^{n} p^{\cdot j}_{i\cdot} = 0 \qquad (8^1)$$

and

$$\frac{\partial \phi(u,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} u^i \sum_{j=1}^{n} p^{\cdot j}_{k\cdot} - 1 = 0 \qquad (8^2).$$

From (8^1)

$$u^i = \frac{\lambda \sum_{j=1}^{n} p^{\cdot j}_{i\cdot}}{2\mu_i^2} \tag{9}$$

And substituting in (8^2) it is received

$$\lambda = \frac{1}{\sum_{i=1}^{n} \frac{(\sum_{j=1}^{n} p_{i\cdot}^{\cdot j})^2}{2\mu_i^2}} \qquad (10)$$

Let's enter notations

$$\varepsilon_i = \sum_{j=1}^{n} p_{i\cdot}^{\cdot j} \qquad (11^1)$$

and

$$\varepsilon = \sum_{i=1}^{n} \frac{\varepsilon_i^2}{\mu_i^2} \qquad (11^2)$$

Substituting (10) in (9) and using (11^1) and (11^2), we have for optimum weights

$$u_{op}^{i} = \frac{\sum_{j=1}^{n} p_{i\cdot}^{\cdot j}}{\varepsilon\mu_i^2} = \frac{1}{\varepsilon}\frac{\varepsilon_i}{\mu_i^2} = \frac{\pi^i}{\varepsilon} \qquad (12)$$

where $\pi^i = \frac{\varepsilon_i}{\mu_i^2}$

Now it is possible to calculate a minimum of risk

$$\sigma_{\min}^2 = \mu_i^2 (u_{op}^i)^2 = \frac{1}{\varepsilon} \qquad (13)$$

Using transformation (5), it is possible to write down optimum structure for initial variables

$$w_{op}^{i} = p_{i\cdot}^{\cdot j} \frac{\pi^i}{\varepsilon} \qquad (14)$$

Considering, that in similar notations the first the top index changes, it is visible that constant ε_i is the sum of elements of i-th column of transposed matrix P, or the sum of elements of i-th line of matrix P, that is the sum of co-ordinates of i-th eigen vector of covariance matrixes g.

As these constants completely define optimum structure of an investment portfolio it is possible to name them optimum structural potentials of a portfolio, constants π^i- relative optimum structural potentials (they are expressed in terms of variances μ^2_i), and constant ε which is inverse to the minimum risk, - the maximum risk.

References

1. Elton E. J. and Martin J.G., (1991), "Modern Portfolio Theory and Investment Analysis", New York: John Wiley

2. Gordon J. Alexander and Jack Clark Francis, (1986), "Portfolio Analysis", New York: Prentice Hall

3. Kostrikin A.I., Manin JU.I., (1980), "Linear algebra and geometry", M: Moscow State University Publishing house

4. Postnikov M.M., (1982), "Lektsii po geometrii. Linear algebra. A semestre II", M: Nauka