The singular operator and the Riesz potential operator in the Lebesgue Spaces with Variable Exponent on the real line

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Abstract

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1 Introduction

We consider the singular integral operator

$$Sf(x) = \frac{1}{\pi}\int\limits_{\mathbb{R}^1} \frac{f(y)\,dy}{y-x}, \qquad x \in \mathbb{R}^1, \tag{1.1}$$

and the Riesz potential operator

$$I^\alpha f(x) = \frac{1}{\gamma_1(\alpha)}\int\limits_{\mathbb{R}^1} \frac{f(y)\,dy}{|y-x|^{1-\alpha}}, \qquad x \in \mathbb{R}^1, \tag{1.2}$$

within the frameworks of weighted spaces $L^{p(\cdot)}(\mathbb{R}^1)$ with variable exponent $p(x)$. We refer, for example, to [9], [12], [10] for the Lebesgue spaces with variable exponent.

The progress in boundedness results for singular (and maximal) operators and for potential type operators in the spaces $L^{p(\cdot)}(\Omega)$, $\Omega \subset \mathbb{R}^1$ is mainly related to the case of bounded domains Ω. For unbounded domains we note an important result in [1] which provides the boundedness of the maximal operator in the space $L^{p(\cdot)}(\mathbb{R}^n)$ under the natural assumptions on $p(x)$:

$$1 < p_0 \leq p(x) \leq P < \infty, \quad x \in \mathbb{R}^n \tag{1.3}$$

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \; x, y \in \mathbb{R}^n \tag{1.4}$$

and there exists $\lim_{x \to \infty} p(x) = p(\infty)$ and

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(1 + |x|)}, \quad x, y \in \mathbb{R}^n. \tag{1.5}$$

The Sobolev theorem for the Riesz potential operator was proved in [11],[6]-[5] for bounded domains and in [2] on the whole space $\mathbb{R}^n$ under the assumption that $p(x) = const$ outside some large ball. In [7] Sobolev theorem was proved for $p(x)$ non necessarily constant at infinity, but with some "extra" weight fixed to infinity and under the assumption that $\min_{x \in \mathbb{R}^n} p(x) = p(\infty)$.

Within the framework of unbounded domains the paper [3] is also relevant, in which, in particular, there was obtained a Hardy-type inequality on the half-axis.

Meanwhile, for unbounded domains the following boundedness problems still remain open:

1) weighted estimates for the Hardy maximal operator;

2) weighted estimates for singular operators;

3) weighted (and non-weighted) $L^{p(\cdot)} \to L^{q(\cdot)}$-estimates for the Riesz potential operator.

We treat problems 2) and 3) in the one-dimensional case $n = 1$ for operators on $\mathbb{R}^1$ showing that in this case their solution may be obtained from the known results for bounded domains.

2 Preliminaries.

We deal with the spaces $L^{p(\cdot)}(\mathbb{R}^1)$ with $p(x)$ treated as function on $\dot{\mathbb{R}}^1$ where $\dot{\mathbb{R}}^1$ is the compactification of $\mathbb{R}^1$ by the unique infinite point. We assume that the function $p(x)$ has the logarithmic smoothness property not only locally as in (1.4), but also at infinity in the sense that the function $p(\frac{1}{x})$ has this property for small x, that is,

$$\left|p\left(\frac{1}{x}\right)-p\left(\frac{1}{y}\right)\right| \leq \frac{A_\infty}{\ln\frac{1}{|x-y|}}, \quad |x-y| \leq \frac{1}{2}, \quad |x| \leq 1, \quad |y| \leq 1. \tag{2.1}$$

From (2.1) it follows that there exists the limit

$$p(\infty) := \lim_{x\to\infty} p(x)$$

By $q(x)$ we denote the conjugate exponent, $\frac{1}{p(x)}+\frac{1}{q(x)}=1$.

The equivalence $f(x) \approx g(x)$ for non-negative functions $f(x)$ and $g(x)$ means that there exist positive constants C_1 and C_2 such that

$$C_1 f(x) \leq g(x) \leq C_2 f(x).$$

Let C be a curve on the complex plane and

$$r(t) = \prod_{k=1}^{m} |t-t_k|^{\mu_k}, \quad t_k \in C, \quad k = 1, 2, \dots m. \tag{2.2}$$

In [6]-[5] the following theorem was proved.

Theorem 2.1. *Let C be a Lyapunov curve (or a curve of bounded rotation without cusps) and let $p(t)$ be a function defined on C which satisfies conditions (1.3) and (1.4) on C. The operator*

$$S_r f(t) = r(t) \int_{\Gamma} \frac{f(\tau)\, dt}{r(\tau)(\tau - t)} \tag{2.3}$$

is bounded in the space $L^{p(\cdot)}_{\rho}(C)$ if and only if

$$-\frac{1}{p(t_k)} < \mu_k < \frac{1}{q(t_k)}, \qquad k = 1, 2, \dots, m. \tag{2.4}$$

The Sobolev Theorem for the spaces $L^{p(\cdot)}(\Omega)$ in the case of bounded domains in $\mathbb{R}^1$ runs as follows (see [11], Theorem 3.2 and [6]-[5], Theorem B).

Theorem 2.2. *Let $p(x)$ satisfy assumptions (1.3)-(1.4) and the function $\alpha(x) : \Omega \longrightarrow (0, n)$ satisfy the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0 \qquad and \qquad \sup_{x\in\Omega} \alpha(x)p(x) < n. \tag{2.5}$$

Then the potential operator

$$I^{\alpha(\cdot)} f(x) = \int_{\Omega} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\, dy \tag{2.6}$$

is bounded from $L^{p(\cdot)}(\Omega)$ into $L^{p_\alpha(\cdot)}(\Omega)$ with $\frac{1}{p_\alpha(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$.

From the statement of Theorem 2.2 for $n = 1, \Omega = [0, \ell]$, the following result may be easily derived.

Corollary 2.3. *Let C be a Lyapunov curve on the complex plane, $p(t)$ satisfy conditions (1.3)-(1.4) on C and $\alpha(t) : C \longrightarrow (0, 1)$ satisfy the conditions $\inf_{t\in C} \alpha(t) > 0$ and $\sup_{t\in C} \alpha(t)p(t) < 1$. Then the potential type operator*

$$I^{\alpha(\cdot)} f(t) = \int_{C} \frac{f(\tau)}{|\tau - t|^{1-\alpha(t)}}\, d\tau, \tag{2.7}$$

is bounded from the space $L^{p(\cdot)}(C)$ into $L^{p_\alpha(\cdot)}(C)$ with $\frac{1}{p_\alpha(t)} = \frac{1}{p(t)} - \alpha(t)$.

In [13] (see Theorem A and Remark 3.1 in [13]), the following statement was proved, in which it is supposed that the variable order $\alpha(x), x \in \Omega$ satisfies the assumptions

$$\min_{x\in\Omega} \alpha(x) > 0 \qquad \text{and} \qquad \max_{x\in\Omega} \alpha(x) < 1, \tag{2.8}$$

and

$$|\alpha(x) - \alpha(y)| \le \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x-y| \le \frac{1}{2}, \quad x, y \in \Omega. \tag{2.9}$$

Theorem 2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $p(x)$ satisfies conditions (1.3)-(1.4) on $\overline{\Omega}$. Then the weighted potential operator*

$$|x - x_0|^{\beta-\alpha} \int_{\Omega} \frac{f(y)\, dy}{|y - x_0|^{\beta} |x - y|^{n-\alpha(x)}}, \qquad x_0 \in \Omega \tag{2.10}$$

is bounded in the space $L^{p(\cdot)}(\Omega)$ if

$$\alpha(x_0) - \frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}.$$

A similar statement in the case $n = 1$ and $\beta = 0$ was obtained in [3] (see Theorem 2.1 in [3]).

Corollary 2.5. *Let C be a Lyapunov curve on the complex plane and*

$$\rho_\gamma(t) = \prod_{k=1}^{m} |t - t_k|^{\gamma_k} \quad and \quad \rho_\mu(t) = \prod_{k=1}^{m} |t - t_k|^{\mu_k}. \tag{2.11}$$

Suppose that $\alpha(t) : C \to (0,1)$ satisfies assumption (1.2) on C and $0 < \min_{t\in C} \alpha(t)$, $\max_{t\in C} \alpha(t) < 1$ and $p(t)$ satisfies assumptions (1.1)-(1.2). Then the operator

$$\rho_\gamma(t) \int_C \frac{f(t)\, dt}{\rho_\mu(t)|\tau - t|^{1-\alpha(t)}}$$

is bounded in the space $L^{p(\cdot)}(C)$ provided that

$$\gamma_k = \mu_k - \alpha(t_k), \qquad \alpha(t_k) - \frac{1}{p(t_k)} < \mu_k < \frac{1}{q(t_k)}, \qquad k = 1, 2, ...m.$$

We shall also make use of the following result for the weighted maximal operator

$$M^\beta f(x) = |x - x_0|^\beta \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} \frac{|f(y)|}{|y - x_0|^\beta}\, dy, \tag{2.12}$$

where Ω is a bounded open set in $\mathbb{R}^n$ and $x_0 \in \Omega$, proved in [8], [4].

Theorem 2.6. *Let $p(x)$ satisfy conditions (1.3), (1.4). The operator M^β is bounded in $L^{p(\cdot)}(\Omega)$ if and only if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{2.13}$$

Corollary 2.7. *Let C be a smooth curve in the complex plane satisfying the arc-cord condition. The weighted maximal operator on C*

$$M^r f(t) = \sup_{\varepsilon>0} \frac{r(t)}{\varepsilon} \int_{C(t,\varepsilon)} \frac{|f(\tau)|}{r(\tau)} |d\tau|, \tag{2.14}$$

where $C(t,\varepsilon) = \{\tau \in C : |\tau - t| < \varepsilon\}$ and the weight $r(t)$ is defined by (2.2), is bounded in the space $L^{p(\cdot)}(C)$ with $p(t)$ satisfying assumptions (1.3) and (1.4) on C, if and only if conditions (2.4) are fulfilled.

3 Statements of the main results

For the weighted singular operator

$$S_\rho f(x) = \rho(x) \int_{\mathbb{R}^1} \frac{f(y)\, dy}{\rho(y)(y-x)}, \qquad x \in \mathbb{R}^1, \tag{3.1}$$

where $\rho(x) = |x - x_0|^\mu (1 + |x|)^\nu$, we prove the following theorem.

Theorem A. *Let $p(x)$ satisfy assumptions (1.3)-(4.12) and (2.1). The operator S_ρ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^1)$ if and only if*

$$-\frac{1}{p(x_0)} < \mu < \frac{1}{q(x_0)}, \qquad -\frac{1}{p(\infty)} < \mu + \nu < \frac{1}{q(\infty)}. \tag{3.2}$$

We also consider the weighted potential type operator of variable order

$$I^{\alpha(\cdot)}_{\beta,\gamma;\mu,\nu} f(x) = |x - x_0|^\beta (1 + |x|)^\gamma \int_{\mathbb{R}^1} \frac{f(y)\, dy}{|y - x_0|^\mu (1 + |y|)^\nu |x - y|^{1-\alpha(x)}}, \quad x_0 \in \mathbb{R}^1 \tag{3.3}$$

and prove the following statement.

Theorem B. *Let $p(x)$ satisfy assumptions (1.3)-(1.4) and (2.1) and $\alpha(x)$ satisfy assumptions (2.8) and (2.9) on $\Omega = \mathbb{R}^1$ and condition (2.1). Then*

1) the operator $I^{\alpha(\cdot)}_{\beta,\gamma;\mu,\nu}$ *is bounded in the space* $L^{p(\cdot)}(\mathbb{R}^1)$ *if*

$$\beta = \mu - \alpha(x_0), \quad \gamma = \nu - \alpha(\infty), \quad \left(\alpha(\infty) = \lim_{|x|\to\infty} \alpha(x)\right) \tag{3.4}$$

and

$$\alpha(x_0) - \frac{1}{p(x_0)} < \mu < \frac{1}{q(x_0)}, \quad \alpha(\infty) - \frac{1}{p(\infty)} < \nu < \frac{1}{q(\infty)}; \tag{3.5}$$

2) In the case $\min_{x\in\mathbb{R}^1} \alpha(x)p(x) < 1$ *the operator* $I^{\alpha(\cdot)}_{\beta,\gamma;\mu,\nu}$ *with*

$$\beta = \mu = 0 \quad \textit{and} \quad \gamma = 1 - \frac{2}{p(\infty)} - \alpha(\infty), \ \nu = 1 - \frac{2}{p(\infty)} + \alpha(\infty) \tag{3.6}$$

is also bounded from the space $L^{p(\cdot)}(\mathbb{R}^1)$ *into* $L^{p_\alpha(\cdot)}(\mathbb{R}^1)$, *where* $\frac{1}{p_\alpha(x)} = \frac{1}{p(x)} - \alpha(x)$.

Observe that a result similar to statement 2) of Theorem B in the case of the half-axis $\mathbb{R}^1_+$ and special values of weight exponents was earlier obtained for fractional integrals in [3], see Theorems 2.3 and 2.4 in [3].

4 Proof of Theorems A and B.

The proof may be obtained via the mapping of $\mathbb{R}^1$ onto the unit circle. For simplicity we take $x_0 = 0$.

Let $\Gamma = \{t \in \mathbb{C} : |t| = 1\}$. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$ we put

$$\frac{x-i}{x+i} = t, \quad \frac{y-i}{y+i} = \tau \in \Gamma \tag{4.1}$$

so that

$$x = i\frac{1+t}{1-t}, \quad y = i\frac{1+\tau}{1-\tau} \in \mathbb{R}^1$$

for $t, \tau \in \Gamma$.

In the sequel we use the notation

$$p^*(t) = p\left(i\frac{1+t}{1-t}\right), \quad \alpha^*(t) = \alpha\left(i\frac{1+t}{1-t}\right), \quad t \in \Gamma.$$

Lemma 4.1. *The function $p(x)$ satisfies both conditions (1.4) and (2.1) on $\mathbb{R}^1$, if and only if the function $p^*(t)$ satisfies the condition*

$$|p^*(t) - p^*(\tau)| \leq \frac{B}{ln\frac{3}{|t-\tau|}} \tag{4.2}$$

for all $t, \tau \in \Gamma$ with some $B > 0$.

Proof. It is easily seen that both conditions (1.4) and (2.1) may be written in a unified way:

$$|p(x) - p(y)| \leq \frac{A}{ln\frac{\sqrt{1+x^2}\sqrt{1+y^2}}{|x-y|}}$$

which is nothing else but (4.2). □

Let $\mathcal{K}(x, y)$ be any kernel and

$$A_\rho f(x) = \rho_1(x) \int\limits_{\mathbb{R}^1} \frac{\mathcal{K}(x, y)}{\rho_2(y)} f(y)\, dy \tag{4.3}$$

where

$$\rho_1(x) = |x|^\beta (1 + |x|)^\gamma, \quad \rho_2(x) = |x|^\mu (1 + |x|)^\nu.$$

Under passage (4.1) we have

$$\rho_1\left(i\frac{1+t}{1-t}\right) \approx |1+t|^\beta |1-t|^{-\beta-\gamma}, \quad \rho_2\left(i\frac{1+t}{1-t}\right) \approx |1+t|^\mu |1-t|^{-\mu-\nu}$$

so that

$$\left|A_\rho f\left(i\frac{1+t}{1-t}\right)\right| \approx \frac{|1+t|^\beta}{|1-t|^{\beta+\gamma}} \left|\int\limits_\Gamma \frac{\mathcal{K}\left(i\frac{1+t}{1-t}, i\frac{1+\tau}{1-\tau}\right)}{|1+\tau|^\mu |1-\tau|^{\frac{2}{q^*(1)}-\mu-\nu}} a(\tau)\psi(\tau)\, d\tau\right| \tag{4.4}$$

where $\frac{1}{q^*(t)} = 1 - \frac{1}{p^*(t)}$, $a(\tau) = \frac{|1-\tau|^2}{(1-\tau)^2} (|1-\tau| + |1+\tau|)^\nu$, and

$$\psi(\tau) = \frac{f\left(i\frac{1+\tau}{1-\tau}\right)}{|1-\tau|^{\frac{2}{p^*(1)}}}.$$

It is easily seen that

$$\int\limits_{\mathbb{R}^1} |f(x)|^{p(x)}\, dx = 2\int\limits_{\Gamma} \frac{\left|f\left(i\frac{1+t}{1-t}\right)\right|^{p^*(t)}}{|1-t|^2}|dt| \approx \int\limits_{\Gamma} |\psi(t)|^{p^*(t)}\,|dt| \tag{4.5}$$

where we have used the fact that

$$|1-t|^{\frac{2}{p^*(t)}} \approx |1-t|^{\frac{2}{p^*(1)}} \tag{4.6}$$

in view of condition (4.2). Therefore,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^1)} \approx \|\psi\|_{L^{p^*(\cdot)}(\Gamma)}. \tag{4.7}$$

Note that $a(t)$ is a function bounded from below and above: $|a(\tau)| = (|1-\tau|+|1+\tau|)^\nu$, so that

$$2^\nu \le |a(\tau)| \le (2\sqrt{2})^\nu, \qquad \text{if} \qquad \nu \ge 0$$

and

$$(2\sqrt{2})^\nu \le |a(\tau)| \le 2^\nu, \qquad \text{if} \qquad \nu \le 0$$

which is easily checked for $\tau = e^{i\theta}$.

A). The case of the singular operator. We choose $\mathcal{K}(x,y) = \frac{1}{x-y}$ and $\beta = \mu$ and $\gamma = \nu$ in (4.3) so that the operator A_ρ turns to be the singular operator S_ρ defined in (3.3). Then

$$\mathcal{K}\left(i\frac{1+t}{1-t}, i\frac{1+\tau}{1-\tau}\right) = 2i\frac{(1-t)(1-\tau)}{\tau - t}$$

and relation (4.4) takes the form

$$\left|S_\rho f\left(i\frac{1+t}{1-t}\right)\right| \approx \frac{|1+t|^\mu}{|1-t|^{\mu+\nu-1}}\left|\int\limits_{\Gamma} \frac{|1-\tau|^{\mu+\nu+1-\frac{2}{q^*(1)}}\widetilde{f}(\tau)\,d\tau}{|1+\tau|^\mu}\right| \tag{4.8}$$

where

$$\widetilde{f}(\tau) = a(\tau)\psi(\tau) = a(\tau)\,\frac{f\left(i\frac{1+\tau}{1-\tau}\right)}{|1-\tau|^{\frac{2}{p^*(1)}}} \tag{4.9}$$

and

$$\int_{\mathbb{R}^1} |f(x)|^{p(x)}\, dx \approx \int_{\Gamma} |\widetilde{f}(t)|^{p^*(t)}\, |dt|. \tag{4.10}$$

From (4.8) according to (4.10)-(4.9) we have

$$\int_{\mathbb{R}^1} |S_\rho f(x)|^{p(x)}\, dx \approx \int_{\Gamma} \left| |1-t|^{-\frac{2}{p^*(1)}} (S_\rho f)\left(i\frac{1+t}{1-t}\right)\right|^{p^*(t)} |dt|$$

$$\approx \int_{\Gamma} \left| r(t) \int_{\Gamma} \frac{\widetilde{f}(\tau)\, d\tau}{r(\tau)(\tau - t)}\right|^{p^*(t)} |dt| \tag{4.11}$$

where

$$r(t) = |1+t|^{\mu}|1-t|^{\nu_1}, \qquad \nu_1 = 1 - \mu - \nu - \frac{2}{p^*(1)}.$$

Taking Lemma 4.1 into account, we apply Theorem 2.1 and conclude that the operator S_ρ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^1)$ if and only if

$$-\frac{1}{p^*(-1)} < \mu < \frac{1}{q^*(-1)} \quad \text{and} \quad -\frac{1}{p^*(1)} < \nu_1 < \frac{1}{q^*(1)}. \tag{4.12}$$

Since $p^*(-1) = p(0)$ and $p^*(1) = p(\infty)$, it is easy to check that conditions (4.12) coincide with assumptions (3.2), which proves Theorem A.

B). The case of the potential operator. Under the choice $\mathcal{K}(x,y) = |x-y|^{\alpha(x)-1}$ we have

$$\mathcal{K}\left(i\frac{1+t}{1-t}, i\frac{1+\tau}{1-\tau}\right) \approx \frac{|1-t|^{1-\alpha^*(1)}|1-\tau|^{1-\alpha^*(1)}}{|\tau - t|^{1-\alpha^*(t)}}$$

and then from (4.4) we have

$$\left|\left(I^{\alpha(\cdot)}_{\beta,\gamma;\mu,\nu} f\right)\left(i\frac{1+t}{1-t}\right)\right| \approx \frac{|1+t|^{\beta}}{|1-t|^{\beta+\gamma+\alpha^*(1)-1}} \left| \int_{\Gamma} \frac{|1-\tau|^{\mu+\nu+1-\frac{2}{q^*(1)}-\alpha^*(1)}\psi(\tau)\, d\tau}{|1+\tau|^{\mu}|\tau - t|^{1-\alpha^*(t)}} \right|. \tag{4.13}$$

1) T h e $L^{p(\cdot)} \longrightarrow L^{p(\cdot)}$ - e s t i m a t e.

From (4.13) according to (4.5) and (4.6) we have

$$\int_{\mathbb{R}^1} \left| \left(I^{\alpha(\cdot)}_{\beta,\gamma;\mu,\nu} f \right)(x) \right|^{p(x)} dx \approx \int_{\Gamma} \left| |1+t|^{\beta} |1-t|^{\gamma_1} \int_{\Gamma} \frac{\psi(\tau)\, d\tau}{|1+\tau|^{\mu} |1-\tau|^{\gamma_2} |\tau - t|^{1-\alpha^*(t)}} \right|^{p^*(t)} |dt|,$$

where

$$\gamma_1 = 1 - \frac{2}{p^*(1)} - \beta - \gamma - \alpha^*(1), \quad \gamma_2 = 1 - \frac{2}{p^*(1)} - \mu - \nu + \alpha^*(1).$$

By Corollary 2.5, we arrive at the boundedness statement under the conditions

$$\beta = \mu - \alpha^*(-1), \qquad \gamma = \nu - \alpha^*(1)$$

and

$$\alpha^*(-1) - \frac{1}{p^*(-1)} < \mu < \frac{1}{q^*(-1)}, \quad \alpha^*(1) - \frac{1}{p^*(1)} < \mu + \nu < \frac{1}{q^*(1)},$$

which coincide with conditions (3.4)-(3.5) of Theorem B. Part 1) of Theorem B is proved.

2) T h e $L^{p(\cdot)} \longrightarrow L^{p_\alpha(\cdot)}$ - e s t i m a t e.

Similarly, from (4.13) according to (4.5) and (4.6) in the case (3.6) we have

$$\int_{\mathbb{R}^1} \left| \left(I^{\alpha(\cdot)}_{0,\gamma;0,\nu} f \right)(x) \right|^{p_\alpha(x)} dx$$

$$\approx \int_{\Gamma} \left| |1-t|^{-\left[\gamma+\alpha^*(1)-1+\frac{2}{p^*(1)}\right]} \int_{\Gamma} \frac{|1-\tau|^{1+\nu-\alpha^*(1)-\frac{2}{q^*(1)}}}{|\tau - t|^{1-\alpha^*(t)}} \psi(\tau)\, d\tau \right|^{p^*_\alpha(t)} |dt|$$

$$\approx \int_{\Gamma} \left| \int_{\Gamma} \frac{\psi(\tau)\, d\tau}{|\tau - t|^{1-\alpha^*(t)}} \right|^{p^*_\alpha(t)} |dt|$$

since

$$\gamma + \alpha^*(1) - 1 + \frac{2}{p^*(1)} = 0 \qquad \text{and} \qquad 1 + \nu - \alpha^*(1) - \frac{2}{q^*(1)} = 0$$

by (3.6).

Making use of Corollary 2.3, we arrive at statement 2) of Theorem B.

5 On the boundedness of the maximal operator on $\mathbb{R}^1$.

Let

$$M^\rho f(x) = \sup_{h>0} M_h^\rho f(x), \qquad \text{where} \qquad M_h^\rho f(x) = \frac{\rho(x)}{2h} \int_{x-h}^{x+h} \frac{|f(y)|}{\rho(y)}\, dy \quad (5.1)$$

where $\rho(x) = |x - x_0|^\mu (1 + |x|)^\nu$.

H y p o t h e s i s. Under conditions (1.1)-(1.2) and (2.1)

$$\|M^\rho f(x)\|_{p(\cdot)} \le C\|f\|_{p(\cdot)} \quad (5.2)$$

with $C > 0$ not depending on f, if

$$-\frac{1}{p(x_0)} < \mu < \frac{1}{q(x_0)}, \qquad -\frac{1}{p(\infty)} < \mu + \nu < \frac{1}{q(\infty)}. \quad (5.3)$$

We are able to prove this statement in a weaker form, namely for the maximal operator defined by (5.1), but with $\sup\limits_{0<h<1} M_h^\rho f(x)$ instead of $\sup\limits_{h>0} M_h^\rho f(x)$. Namely, the following theorem is valid.

Theorem C. *Under assumptions (1.3)-(1.4) and (2.1)*

$$\left\| \sup_{0<h<1} M_h^\rho f(x) \right\|_{p(\cdot)} \le C\|f\|_{p(\cdot)} \quad (5.4)$$

if conditions (5.3) are satisfied.

Proof. As in the proof of Theorem A, we map $\mathbb{R}^1$ onto the unit circle Γ, but this time this procedure with respect to the maximal operator is more complicated. Under change of variables (4.1) we have

$$\left(M_h^\rho f\right)(x)\Big|_{x=i\frac{1+t}{1-t}} \approx \frac{\rho^*(t)}{2h} \int_{\varphi_1(h)}^{\varphi_2(h)} \frac{\left|f\left(i\frac{1+\tau}{1-\tau}\right)\right|}{\rho^*(\tau)} \frac{|d\tau|}{|1-\tau|^2} \quad (5.5)$$

where

$$\rho^*(t) = |1 + t|^\mu |1 - t|^{-\mu-\nu}$$

and

$$\varphi_1(h) = \varphi_1(h,t) = \arg \frac{x-h-i}{x-h+i}, \qquad \varphi_2(h) = \varphi_2(h,t) = arg\frac{x+h-i}{x+h+i}.$$

Since $\arg(x-i) = \pi - arctg\frac{1}{x}$ for $x<0$ and $\arg(x-i) = -arctg\frac{1}{x}$ for $x>0$, we have

$$\varphi := \arg \frac{x-i}{x+i} = 2\arg(x-i) = -2arctg\frac{1}{x}$$

and

$$\varphi_1(h) = -2arctg\frac{1}{x-h}, \qquad \varphi_2(h) = -2arctg\frac{1}{x+h}$$

In (5.5) we integrate along the arc $\varphi_1(h) \le \theta \le \varphi_2(h)$, $\theta = \arg\tau$, which is not symmetric with respect to its "center" $t = e^{i\varphi}$, $\varphi = -2arctg\frac{1}{x}$:

$$\varphi_2(h) - \varphi \neq \varphi - \varphi_1(h)$$

(but $\lim_{h\to 0}\frac{\varphi_2-\varphi}{\varphi-\varphi_1} = 1$). Indeed, we have

$$\varphi_2(h) - \varphi = 2\left[arctg\frac{1}{x} - arctg\frac{1}{x+h}\right], \quad \varphi - \varphi_1(h) = 2\left[arctg\frac{1}{x-h} - arctg\frac{1}{x}\right]$$

so that

$$\varphi_2(h) - \varphi = 2\ arctg\frac{h}{x^2+hx+1} = 2\ arctg\frac{h}{\left(x+\frac{h}{2}\right)^2 + 1 - \frac{h^2}{4}},$$

$$\varphi - \varphi_1(h) = 2\ arctg\frac{h}{x^2-hx+1} = 2\ arctg\frac{h}{\left(x-\frac{h}{2}\right)^2 + 1 - \frac{h^2}{4}}.$$

Therefore, for $h \le 2$ we have

$$\varphi_2(h) - \varphi \le \varphi - \varphi_1(h), \qquad \text{if} \quad x > 0,$$

$$\varphi_2(h) - \varphi \ge \varphi - \varphi_1(h), \qquad \text{if} \quad x < 0.$$

Let

$$\delta = \delta(h,x) = \max\{\varphi_2(h) - \varphi, \varphi - \varphi_1(h)\} = 2\ arctg\frac{h}{x^2 - h|x| + 1}. \qquad (5.6)$$

We introduce the function

$$\psi(t) = \frac{f\left(i\frac{1+t}{1-t}\right)}{|1-t|^{\frac{2}{p^*(1)}}} \in L^{p^*(\cdot)}(\Gamma)$$

where $p^*(t) = p\left(i\frac{1+t}{1-t}\right)$ is the same as in the proof of Theorems A and B. From (5.5) we have

$$\int_{\mathbb{R}^1} |(M_h^\rho f)(x)|^{p(x)}\ dx \leq \int_\Gamma \left| \sup_{0<h<1} \frac{\rho_2(t)}{2h} \int_{\varphi-\delta}^{\varphi+\delta} \frac{|\psi(t)|}{\rho_1(\tau)}\, |d\tau| \right|^{p^*(t)} |dt|, \qquad (5.7)$$

where

$$\rho_1(t) = \rho^*(t)|1-t|^{\frac{2}{q^*(1)}} = |1+t|^\mu |1-t|^{-\mu-\nu+\frac{2}{q^*(1)}}$$

and

$$\rho_2(t) = \rho_1(t)\ \cdot |1-t|^{-2} = |1+t|^\mu |1-t|^{-\mu-\nu-\frac{2}{p^*(1)}}.$$

It remains to pass from h to δ in (5.7).

From (5.6) we observe that

$$\frac{1}{h} = \frac{1+|x|tg\ \frac{\delta}{2}}{(1+|x|^2)tg\ \frac{\delta}{2}}. \qquad (5.8)$$

According to (5.6) we also see that

$$|x|tg\ \frac{\delta}{2} = \frac{|x|h}{x^2-h|x|+1} \leq 1 \qquad \text{for} \qquad 0<h\leq 1.$$

Therefore,

$$\frac{1}{h} \leq \frac{2}{(1+|x|^2)tg\ \frac{\delta}{2}} \leq \frac{c\ |t-1|^2}{tg\frac{\delta}{2}} \leq c\frac{|t-1|^2}{\delta}. \qquad (5.9)$$

Since $h \leq 1$, we have $|y-x| \leq 1$ so that

$$\frac{1}{2}(1+|x|) \leq (1+|y|) \leq 2(1+|x|)$$

from which it follows that

$$c_1|1-t| \leq |1-\tau| \leq c_2|1-t|.$$

Then from (5.9) we have

$$\frac{1}{h} \leq c\frac{|t-1|\cdot|\tau-1|}{\delta}.$$

Consequently, from (5.7)

$$\int\limits_{\mathbb{R}^1} |(M_h^\rho f)(x)|^{p(x)}\ dx \le c \int\limits_{\Gamma} \left| \sup_{0<h<1} \frac{r(t)}{2\delta} \int\limits_{\varphi-\delta}^{\varphi+\delta} \frac{|\psi(t)|}{r(\tau)}\, |d\tau| \right|^{p^*(t)} |dt|, \qquad (5.10)$$

where $r(t) = |t+1|^\mu |t-1|^{-\mu-\nu+1-\frac{2}{p^*(1)}}$ and

$$\delta = \delta(h,t) = 2\ arctg \frac{h}{x^2 - h|x| + 1}.$$

Then from (5.10)

$$\int\limits_{\mathbb{R}^1} |(M_h^\rho f)(x)|^{p(x)}\ dx \le \int\limits_{\Gamma} \left| \sup_{0<\delta<\pi} \frac{r(t)}{2\delta} \int\limits_{\varphi-\delta}^{\varphi+\delta} \frac{|\psi(t)|}{r(\tau)}\, |d\tau| \right|^{p^*(t)} |dt|. \qquad (5.11)$$

By Corollary 2.7, we arrive at the boundedness statement if the conditions

$$-\frac{1}{p^*(-1)} < \mu < \frac{1}{q^*(-1)}, \qquad -\frac{1}{p^*(1)} < -\mu - \nu + \frac{2}{q^*(1)} < \frac{1}{q^*(-1)}$$

are satisfied, which coincide with assumptions (5.3).

□

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