ROMA TRE
UNIVERSITÀ DEGLI STUDI

COLLANA DEL
DIPARTIMENTO DI ECONOMIA

A SYSTEMATIC APPROACH FOR VALUING EUROPEAN-STYLE INSTALLMENT OPTIONS WITH CONTINUOUS PAYMENT PLAN

Pierangelo Ciurlia

Working Paper n° 115, 2010

I Working Papers del Dipartimento di Economia svolgono la funzione di divulgare tempestivamente, in forma definitiva o provvisoria, i risultati di ricerche scientifiche originali. La loro pubblicazione è soggetta all'approvazione del Comitato Scientifico.

Per ciascuna pubblicazione vengono soddisfatti gli obblighi previsti dall'art. l del D.L.L. 31.8.1945, n. 660 e successive modifiche.

Copie della presente pubblicazione possono essere richieste alla Redazione.

REDAZIONE:
Dipartimento di Economia
Università degli Studi Roma Tre
Via Silvio D'Amico, 77 - 00145 Roma
Tel. 0039-06-57335655 fax 0039-06-57335771
E-mail: dip_eco@uniroma3.it

DIPARTIMENTO DI ECONOMIA

A SYSTEMATIC APPROACH FOR VALUING EUROPEAN-STYLE INSTALLMENT OPTIONS WITH CONTINUOUS PAYMENT PLAN

Pierangelo Ciurlia

A systematic approach for valuing European-style installment options with continuous payment plan

Pierangelo Ciurlia

Department of Economics, University of Rome III
Via Silvio D'Amico 77, 00145 Rome, Italy

Abstract

In this paper we present an integral equation approach for the valuation of European-style installment derivatives when the premium payments, made continuously throughout the contract's life, are assumed to be a function of the asset price and time variables. The contribution of this study is threefold. First, we show that in the Black-Scholes framework the option pricing problem can be formulated as a free boundary problem under very general conditions on payoff structure and installment payment plan. Second, by applying a Fourier transform-based solution technique, we derive a recursive integral equation for the free boundary along with a general integral representation for the option initial premium. Third, within this systematic treatment of the European installment options, we propose a unified and easily applicable method to deal with a broad range of monotonic payoff functions and continuous payment plans depending on the time variable only. Finally, by using the illustrative example of European vanilla installment call options, an explicit valuation formula is obtained for the class of linear time-varying installment payment functions.

Key words: Installment options; free boundary problem; Fourier transform; integral representations.
J.E.L. classification: D81, G13.

1. Introduction

Installment option contracts are exotic derivatives in which the premium (or price) is paid in installments of a payment plan spread over the lifetime of the option, rather than as a lump-sum at time of purchase, and where it can be allowed to lapse the contract at any payment date before maturity. Option holder's right to drop the contract can be viewed as an early exercise feature and leads to free boundary problems similar to that arising for standard American options. An installment option with payments at pre-determined time intervals (usually monthly, quarterly or annually) is referred to as a *discrete-installment* (DI) option, whereas its continuous-time equivalent in which the payment plan is defined by allowing for the installment to be a function of the asset price and time variables is referred to as a *continuous-installment* (CI) option. In this paper we consider the class of European-style CI options, in which the buyer pays a small up-front premium at inception of the trade and then installments of a continuous payment plan to acquire and keep the right, but not the obligation, to buy or sell the underlying asset at the maturity date. However, the holder can choose at any time to terminate installment payments, in which case the option lapses with no further payments on either side.

There are relatively few and quite recent studies on installment options. A complete review of the literature on this topics, with reference to both the type of installment payment plan (discrete or continuous) and style of exercise (American or European), is here carried out and briefly discussed. Davis et al. [8] and Davis et al. [9] derive no-arbitrage bounds for the initial premium of a European DI option and study static versus dynamic hedging strategies within a Black-Scholes framework with stochastic volatility. Applying the concept of compound options, previously considered by Geske [12] and Selby and Hodges [22], they intuitively show that holding an installment option is equivalent to holding the underlying vanilla option plus the right to sell it at any installment date at a price equal

Email address: ciurlia@uniroma3.it (Pierangelo Ciurlia)

to the net present value of all future installment payments. The latter security can be characterized as a Bermuda compound put option written on the vanilla option with time-varying strike price. Ben-Ameur et al. [3] develops a dynamic-programming procedure to price American DI options and investigates, within the geometric Brownian motion framework, some properties of the installment option contract through theoretical and numerical analysis. Extending the concept of compound options, Griebsch et al. [13] derives a closed-form solution to the initial premium of a European DI option in the Black-Scholes model and examines the limiting case of an installment option with continuous payment plan, for which he proves the equivalence to a portfolio consisting of a European vanilla option and an American compound put on that contingent claim with time-dependent strike price. Alobaidi et al. [1] and Alobaidi and Mallier [2] use a partial Laplace transform to solve the free boundary problem arising from the pricing of European CI options. By applying this method, they obtain an integral equation for the position of the free boundary and to deduce through an asymptotic analysis its behavior close to expiry. Ciurlia and Roko [5] and Ciurlia [7] propose three alternative approaches for the valuation of American and European CI options written on assets without dividends or with constant continuous dividend yield. Their analysis can be applied to value derivative securities and investment projects with a continuous payment plan. Ciurlia and Caperdoni [6] extends this pricing framework to the theoretical case of perpetual CI options. The closed-form solution obtained when the underlying asset does not pay dividends allow him to derive some analytical properties of the initial premium and the optimal boundaries. Kimura [17] and Kimura [18] adopt the Laplace transform method to solve the valuation problems of American and European CI options written on dividend-paying assets, obtaining integral representations for the initial premium, the optimal stopping and exercise boundaries and some hedging parameters. Furthermore, Abelian theorems of Laplace transforms allow them to characterize asymptotic properties of the free boundaries at both a time close and an infinite time to expiration. Finally, Yi et al. [25] and Yang and Yi [24] consider parabolic variational inequalities arising from European and American CI call options pricing, and prove the existence and uniqueness of solution to the problems as well as the monotonicity and smoothness properties of the free boundaries.

Among the most actively traded installment options throughout the world currently are the installment warrants on Australian stocks listed on the Australian Stock Exchange and 10-year warrants with 9 annual payments offered by Deutsche Bank. Installment options, which are also frequently traded in foreign exchange markets between banks and corporates, introduce some flexibility in the liquidity management of portfolio strategies in that, instead of paying a lump-sum for a derivative, the option holder will pay the installments as long as the need for being long in the contract is present. Furthermore, the right of terminating the contract by halting the payments with no transaction cost reduces the liquidity risk typically associated with other over-the-counter derivatives. Installment options can be found embedded in other contract, including life insurance contracts, and are also frequently used in financing capital investment projects with some examples given in Dixit and Pindyck [11]. In the field of real options a meaningful model is that due to Majd and Pindyck [20], in which a firm invests in a project continuously and receives no payoff until the project is complete. Although the model of Majd and Pindyck [20] bears many resemblances to a European CI option, it also has some differences, notably that the project can be resumed at a later time without loss of earlier capital outlays, whereas an installment option lapses if the holder halts installment payments.

The aim of this paper is to provide an integral equation approach to pricing European-style installment options with continuous payment plan and general monotonic payoff function. The holder's ability to halt installment payments by dropping the option contract leads to a free boundary separating the region where it is advantageous to hold from that in which early stopping is optimal. In theory, stopping strategy should take place only on this free boundary, which itself is unknown and must be determined along with the up-front price. In order to obtain a general solution to this wide class of free boundary problems we use an appropriate Fourier transform which allows us to derive an integral expression for the initial premium function that involves the optimal stopping boundary. Although the focus is on installment options, in principle our approach is general and applies to more complex derivatives.

The layout of the paper is as follows. Section 2 provides a formal definition of the CI option contract as well as a proposition which shows how to derive within the standard Black-Scholes framework the inhomogeneous partial differential equation (PDE) governing the initial premium function of such an option. Based on this key result, the European CI options pricing problem is formulated as a free boundary problem for a general class of payoff structures and payment plans. Section 3 proceeds to solve this class of pricing problems using the incomplete Fourier transform (IFT) method. By inverting the solution of the transformed pricing problem a general integral expression for the initial premium function is obtained. Section 4 gives a parametric representation of the solution to the European CI options pricing problem which is similar to that first proposed by Kim [16] for standard American options. In §5 we present an application to the valuation of European vanilla CI call options; an explicit pricing formula is obtained for the class of linear time-varying installment payment functions. Concluding remarks are formulated in §6, with a briefly review of the basic properties and theorems of the IFT given in the Appendix.

2. Pricing in the Black-Scholes framework

In this section we briefly review the basic assumptions under the standard Black-Scholes framework and, after giving a formal definition of a CI option contract, we derive the inhomogeneous PDE governing the price to be paid at time of entering into the contract, hereafter called the *initial* (or *up-front*) *premium*, supplementary to a continuous payment plan which is defined as a function of the asset and time variables. Using this key result we consider the free boundary problem for the class of European-style CI options with general monotonic payoff function, to solve which it is necessary to identify the so-called *stopping region* (i.e. the set of asset prices and times at which it is optimal to drop the option contract).

2.1. Black-Scholes PDE for CI options

In the standard Black-Scholes framework it is assumed a financial market model with continuous trading, frictionless, no-arbitrage opportunities, short-selling and consisting of only two assets:

1. a risk-free asset (e.g., a bond) with price process $B = (B_t)_{t\geq 0}$ following a deterministic differential equation
$$dB_t = rB_t dt, \tag{2.1}$$
where $r \geq 0$ is the constant instantaneous risk-free interest rate;
2. a risky asset paying continuous proportional dividends $\delta \geq 0$ and with price process $S = (S_t)_{t\geq 0}$ governed by a geometric Brownian motion
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.2}$$
where $\mu = (r - \delta)$ is the expected rate of return per unit time and $\sigma > 0$ is the instantaneous volatility per unit time. The term dW_t denotes increments of a standard Wiener process $W = (W_t)_{t\geq 0}$ defined on a risk-neutral probability measure $\mathbb{Q}$. If the risky asset is a stock (or equity) δ equals zero, while if it is a foreign currency, δ is replaced by the foreign risk-free interest rate r_f.

Definition 2.1. *A CI option written on the underlying asset S with time of maturity T, payoff function $H(S_t)$, at any time t of exercise, and continuous payment plan which is expressed by the installment payment function $L_t = L(S_t, t)$ is a derivative contract defined by the following clauses*:

- *the holder pays at the time of purchase $t \in [0, T]$ a smaller up-front premium, and then a stream of installments at the rate $L_t = L(S_t, t)$ per unit time throughout the whole life of the option, to acquire and keep the right, but not the obligation, to buy (call option) or sell (put option) one share of the underlying asset at the fixed date T (for European-style exercise) or at any time up to T (for American-style exercise)*;
- *the holder is in no way obligated to continue paying installment premiums until the option expires, in that he has the right to stop making payments, thereby terminating the contract with no claim on either side.*

Note that for a CI option of European type, the holder can choose to stop paying installments only by dropping the contract, while for an American CI option payments can be terminated by either exercising the option or dropping the contract itself. The distinctive feature of a CI option, independently from the type of exercise, is that it gives the holder the possibility to walk away from the contract at any time prior to maturity, which makes it necessary to determine that it is known as the *optimal stopping boundary*, i.e., the trajectory of critical asset prices at which is advantageous to stop installment payments by dropping the contract.

For a contingent claim of the form described by Definition 2.1, we denote the up-front price at time of purchase $t \in [0, T]$ of the claim by $V_t = V(S_t, t; L_t)$, which depends on the current asset price S_t, time t and installment per unit time L_t. Let $V(S_t, t; L_t)$ be a continuously differentiable function in $t < T$ and a.e. twice continuously differentiable in S_t, and let $L(S_t, t)$ be a non-negative real-valued function of bounded variation on an open subset of $\mathbb{R}^2$. Applying

Itô's formula to V_t and combining with eq. (2.2), the initial premium dynamics of the derivative asset is obtained as follows

$$dV_t = \left(\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2}\right)dt + \sigma S_t \frac{\partial V_t}{\partial S} dW_t. \tag{2.3}$$

Proposition 2.1. *Let us assume that the financial market model is specified by eqs.* (2.1-2.2) *and that a contingent claim with price process described by eq.* (2.3) *is to be priced. Then, to avoid risk-free arbitrage opportunities, the initial premium function V_t of any derivative written on S with continuous payment plan L_t must satisfy the following relationship*

$$\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - rV_t = L_t. \tag{2.4}$$

Proof. Let us now form a self-financing portfolio $\pi = (\eta, \theta)$, where η and θ represent positions in the underlying asset and the derivative security, respectively. Denoting by $\Pi_t := \eta S_t + \theta V_t$ the value at time t of the portfolio and using eqs. (2.2-2.3), we obtain the following dynamics for Π_t

$$\begin{aligned} d\Pi_t &= \eta(dS_t + \delta S_t dt) + \theta(dV_t - L_t dt) \\ &= \eta\left[\mu S_t + \delta S_t + \theta\left(\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2}\right)\right]dt + \left(\eta + \theta\frac{\partial V_t}{\partial S}\right)\sigma S_t dW_t. \end{aligned}$$

If we set $\eta = -\Delta$, with $\Delta := \theta\frac{\partial V_t}{\partial S}$, the dW-term in the Π-dynamics of the above equation vanishes completely, leaving us with the equation

$$d\Pi_t = \theta\left(\frac{\partial V_t}{\partial t} - \delta S_t \frac{\partial V_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - L_t\right)dt. \tag{2.5}$$

Using the standard hedging arguments, it is possible to show that the combination of long θ-amount of the derivative security and short Δ-adjusted equivalent amount of the underlying asset produces a locally riskless portfolio, which must grow at the risk-free rate r because of the requirement that the market is free of arbitrage possibilities. Then, by the absence of arbitrage condition, we get that the following relation must hold

$$\begin{aligned} d\Pi_t &= r(\eta S_t + \theta V_t)dt \\ &= r\theta\left(V_t - \frac{\partial V_t}{\partial S} S_t\right)dt. \end{aligned} \tag{2.6}$$

Finally, by comparing eq. (2.5) with eq. (2.6) and rearranging terms, we find that V_t satisfies eq. (2.4). □

Eq. (2.4) is a linear, second-order in S and inhomogeneous PDE of parabolic type, in which the term $L_t \geq 0$ represents the continual input of cash via the installment premium: in a time period dt an amount $L_t dt$ must be paid to keep the derivative alive. If $L_t = 0$, for all $t \geq 0$, we have the usual Black-Scholes PDE for stock options. Independently from the type of exercise, eq. (2.4) is valid only on the continuation region, that is, on the subset of the V-domain where it is advantageous to continue paying installment premiums since the derivative is worth more alive than dead.

2.2. Free boundary problem formulation

We consider a European CI option on an asset whose price process S follows (2.2) with maturity date T, payoff function $H(S_T)$ and installment payment function $L_t = L(S_t, t)$. Let $V_t^E = V^E(S_t, t; L_t)$ be the initial premium function of this option, defined on the domain $\mathcal{D} \equiv \{(S_t, t) \in [0, \infty) \times [0, T]\}$. For each time $t \in [0, T]$, there exists a critical asset price, $A_t = A(t; L_t)$ for the call option and $G_t = G(t; L_t)$ for the put option, at which it is advantageous to terminate installment payments by dropping the option contract. According to these critical asset prices, the initial premium function V_t^E satisfies the following conditions

	For call option	For put option	
$V_t^E = 0,$	if $S_t \in [0, A_t]$	if $S_t \in [G_t, \infty);$	(2.7)
$V_t^E > 0,$	if $S_t \in (A_t, \infty)$	if $S_t \in [0, G_t).$	(2.8)

The optimal stopping (or free) boundary, which is the time path of critical asset prices, i.e., $\{A_t\}_{t\in[0,T]}$ for the call option and $\{G_t\}_{t\in[0,T]}$ for the put option, divide the domain $\mathcal{D}$ into a stopping region $\mathcal{S}$ and a continuation region C, that is

For call option	For put option
$\mathcal{S} = \{(S_t, t) \in [0, A_t] \times [0, T]\}$	$\mathcal{S} = \{(S_t, t) \in [G_t, \infty) \times [0, T]\};$
$C = \{(S_t, t) \in (A_t, \infty) \times [0, T]\}$	$C = \{(S_t, t) \in [0, G_t) \times [0, T]\}.$

To ensure that the fundamental constraint $V_t^E \geq 0$ is satisfied in the domain $\mathcal{D}$, eq. (2.7) impose that, in the stopping region $\mathcal{S}$, the initial premium V_t^E is equal to zero. By contrast, the inequality expressed in eq. (2.8) show that, in the continuation region C, it is advantages to continue paying installment premiums since the option is worth more alive than dead. The initial premium is given by eq. (2.7) if the asset price starts in $\mathcal{S}$, so we assume that the option is alive at the time $t \geq 0$ of entering into the contract, i.e., $S_t \in (A_t, \infty)$ and $S_t \in [0, G_t)$ for call and put options, respectively.

The initial premium function V_t^E of the European CI option satisfies the inhomogeneous Black-Scholes PDE (2.4) in C; that is,

$$\frac{\partial V_t^E}{\partial t} + \mu S \frac{\partial V_t^E}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_t^E}{\partial S^2} - rV_t^E = L_t, \quad \text{on } C. \tag{2.9}$$

Extending the analysis of McKean [21], we determine that V_t^E and the optimal stopping boundary, $\{A_t\}_{t\in[0,T]}$ for the call option and $\{G_t\}_{t\in[0,T]}$ for the put option, jointly solve a free boundary problem consisting of eq. (2.9) subject to the following final and boundary conditions

For call option	For put option		
$V_T^E = H(S_T),$	$V_T^E = H(S_T),$	$0 \leq S_T < \infty;$	(2.10)
$\lim_{S_t \downarrow A_t} V_t^E = 0,$	$\lim_{S_t \uparrow G_t} V_t^E = 0,$	$0 \leq t < T;$	(2.11)
$\lim_{S_t \downarrow A_t} \frac{\partial V_t^E}{\partial S} = 0,$	$\lim_{S_t \uparrow G_t} \frac{\partial V_t^E}{\partial S} = 0,$	$0 \leq t < T.$	(2.12)

The *value matching condition* (2.11) implies that the initial premium function V_t^E is continuous across the free boundary, Furthermore, the *high contact condition* (2.12) implies that also the slope $\partial V_t^E/\partial S$ is continuous. Eqs. (2.11-2.12) are jointly referred to as *smooth fit conditions* and ensure the optimality of the early stopping strategy.

We propose to solve the general pricing problem expressed by eqs. (2.9-2.12) with the integral representation method first introduced in Kim [16] and then developed in Jacka [14], Carr et al. [4] and Jamshidian [15]. Taking the free boundary problem formulation, we use the IFT technique to solve the inhomogeneous Black-Scholes PDE for European-style contingent claims with generic payoff and installment payment functions.

3. Solving the European CI options pricing problem

3.1. *Transformed pricing problem*

In order to transform the inhomogeneous Black-Scholes PDE (2.9) to an equation with constant coefficients, we set $S_t = e^x$ and $t = T - \tau$. It follows that the transformed function $v_\tau = v(x, \tau)$ is defined by

$$V(S_t, t) = V(e^x, T - \tau) \equiv v(x, \tau), \tag{3.1}$$

on the continuation regione C_v described now by

For call option

$$C_v = \{(x,\tau) \in (\ln a_\tau, \infty) \times [0,T]\}$$

For put option

$$C_v = \{(x,\tau) \in (-\infty, \ln g_\tau) \times [0,T]\},$$

where $a_\tau = a(\tau; l_\tau) \equiv A(T-\tau; L_t)$ and $g_\tau = g(\tau; l_\tau) \equiv G(T-\tau; L_t)$ denote the transformed stopping boundaries of call and put options, respectively. The transformed inhomogeneous Black-Scholes PDE for v_τ is then

$$\frac{\partial v_\tau}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 v_\tau}{\partial x^2} + \rho \frac{\partial v_\tau}{\partial x} - r v_\tau - l_\tau, \quad \text{on } C_v, \tag{3.2}$$

with $\rho \equiv (r - \delta - \frac{1}{2}\sigma^2)$, $l_\tau = l(x,\tau) \equiv L(e^x, T-\tau)$ and associated initial and boundary conditions given by

For call option	For put option		
$v_0 = h(x)$,	$v_0 = h(x)$,	$-\infty < x < \infty$;	(3.3)
$\lim_{x \downarrow \ln a_\tau} v_\tau = 0$,	$\lim_{x \uparrow \ln g_\tau} v_\tau = 0$,	$0 < \tau \leq T$;	(3.4)
$\lim_{x \downarrow \ln a_\tau} \frac{\partial v_\tau}{\partial x} = 0$,	$\lim_{x \uparrow \ln g_\tau} \frac{\partial v_\tau}{\partial x} = 0$,	$0 < \tau \leq T$;	(3.5)

where the payoff function is defined now by $h(x) \equiv H(e^x)$.

In order to be able to apply integral transform methods to solve PDE (3.2) for $v(x,\tau)$, we will consider the domain $\mathcal{D}_v \equiv \{(x,\tau) \in \mathbb{R} \times [0,T]\}$ by expressing eq. (3.2) for call and put options, respectively, as follows

$$\mathcal{H}(x - \ln a_\tau)\left(\frac{\partial v_\tau}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 v_\tau}{\partial x^2} - \rho \frac{\partial v_\tau}{\partial x} + r v_\tau + l_\tau\right) = 0, \tag{3.6}$$

$$\mathcal{H}(\ln g_\tau - x)\left(\frac{\partial v_\tau}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 v_\tau}{\partial x^2} - \rho \frac{\partial v_\tau}{\partial x} + r v_\tau + l_\tau\right) = 0, \tag{3.7}$$

with the initial and boundary conditions that remain unchanged and where $\mathcal{H}(x)$ is the Heaviside step function, defined as

$$\mathcal{H}(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0 \end{cases} \tag{3.8}$$

From the static replication arguments one can establish the following no-arbitrage bounds on the initial premium $v(x,\tau)$ of a European CI option

For call option

$$\lim_{x \to \ln a_\tau^-} \left[W(x,\tau) - \int_0^\tau l(x,\xi)\, e^{-r\xi} d\xi \right] \leq v(x,\tau) \leq \lim_{x \to \infty} e^x, \quad \forall \tau \in (0,T];$$

For put option

$$\lim_{x \to \ln g_\tau^+} \left[W(x,\tau) - \int_0^\tau l(x,\xi)\, e^{-r\xi} d\xi \right] \leq v(x,\tau) \leq \lim_{x \to -\infty} W(x,\tau), \quad \forall \tau \in (0,T];$$

where $W(x,\tau)$ is the expected present value of the payoff at time to maturity τ, while the integral term represents the discounted value of the installment payment stream over the option's lifetime. Note that for a European CI call and put option, with payoff $h(x)$ monotonically increasing and decreasing function of the variable $x \equiv \ln S_t$, the initial premium function $v(x,\tau)$ is unbounded above and bounded, respectively. Although $v(x,\tau)$ is integrable, and so its Fourier transform does exist, the conditions required for Theorem A.3 do not hold because $\lim_{x\to\infty} v(x,\tau) = \infty$ and $\lim_{x\to-\infty} v(x,\tau) < \infty$ for call and put options, respectively. In order to treat this kind of complication we will replace the function $v : \mathcal{D}_v \subset \mathbb{R}^2 \to \mathbb{R}_0^+$ by a real-valued function $y(x,\tau)$ on $\mathcal{D}_v$ satisfying conditions in Theorem A.3.

3.2. Application of the IFT

In this section we will show how the IFT method can be applied to solve the free boundary problem expressed by eqs. (3.3-3.5) and (3.6), in order to obtain an integral representation for initial premium and optimal boundary of the European CI call option. By applying the same results, the solution to the corresponding free boundary problem for the European CI put option, specified by eqs. (3.3-3.5) and (3.7), is achieved.

In order to apply our Fourier transform in a mathematically rigorous way to reduce PDE (3.6) to an ordinary differential equation (ODE) whose solution is readily obtainable, the function $v(x,\tau)$ of a European CI call option is replaced by the still-integrable function $y(x,\tau) := e^{-x} v(x,\tau)$ for which the condition $\lim_{x\to\infty} y(x,\tau) = 0$ does hold. Writing v_τ and its derivatives in term of y_τ and substituting into eq. (3.6), yields

$$\mathcal{H}(x-\ln a_\tau)\left(\frac{\partial y_\tau}{\partial\tau} - \frac{1}{2}\sigma^2\frac{\partial^2 y_\tau}{\partial x^2} - (\rho+\sigma^2)\frac{\partial y_\tau}{\partial x} + \left(r-\rho-\frac{1}{2}\sigma^2\right)y_\tau + e^{-x}\,l_\tau\right) = 0, \tag{3.9}$$

with the associated initial and boundary conditions (3.3-3.5) expressed now by

$$y_0 = e^{-x}h(x), \qquad -\infty < x < \infty; \tag{3.10}$$

$$\lim_{x\downarrow\ln a_\tau} y_\tau = 0, \qquad 0 < \tau \le T; \tag{3.11}$$

$$\lim_{x\downarrow\ln a_\tau}\left(\frac{\partial y_\tau}{\partial x} + y_\tau\right) = 0, \qquad 0 < \tau \le T. \tag{3.12}$$

Now we can use the Fourier transform operator, whose key property of converting differentiation into multiplication by a power of ω allows us to reduce the new PDE with constant coefficients to a far more tractable ODE.

Proposition 3.1. *By applying the definition of IFT to eq.* (3.9)*, we obtain*

$$\mathcal{F}^{a_\tau,\infty}\left\{\frac{\partial y_\tau}{\partial\tau}\right\} = \frac{1}{2}\sigma^2\mathcal{F}^{a_\tau,\infty}\left\{\frac{\partial^2 y_\tau}{\partial x^2}\right\} + \theta\mathcal{F}^{a_\tau,\infty}\left\{\frac{\partial y_\tau}{\partial x}\right\} - \vartheta\mathcal{F}^{a_\tau,\infty}\{y_\tau\} - \mathcal{F}^{a_\tau,\infty}\{u_\tau\}, \tag{3.13}$$

where $\theta \equiv (\rho+\sigma^2)$, $\vartheta \equiv (r-\rho-\sigma^2/2)$ *and* $u_\tau = u(x,\tau) := e^{-x}l(x,\tau)$.

Proof. From Proposition A.2, we see that $\mathcal{F}^{a_\tau,\infty}$ is the IFT operator with respect to x, since it is equivalent to the standard Fourier transformation applied to functions $y(x,\tau)$ and $l(x,\tau)$ in the continuation region C_v. □

In the following proposition we establish three specific properties related to Fourier transforms of derivatives of y_τ that will allow us to convert eq. (3.13) into an ODE for $\mathcal{F}^{a_\tau,\infty}\{y_\tau\}$.

Proposition 3.2. *Let* $\mathcal{F}^{a_\tau,\infty}\{y_\tau\} = \hat{y}(\omega,\tau)$ *be the IFT with respect to* x *of the function* $y_\tau = y(x,\tau)$*, Then, there exist the following three identities for* $\mathcal{F}^{a_\tau,\infty}\{y_\tau\}$

$$\mathcal{F}^{a_\tau,\infty}\left\{\frac{\partial y_\tau}{\partial x}\right\} = i\omega\,\hat{y}(\omega,\tau);$$

$$\mathcal{F}^{a_\tau,\infty}\left\{\frac{\partial^2 y_\tau}{\partial x^2}\right\} = -\,\omega^2\,\hat{y}(\omega,\tau);$$

$$\mathcal{F}^{a_\tau,\infty}\left\{\frac{\partial y_\tau}{\partial\tau}\right\} = \frac{\partial\hat{y}(\omega,\tau)}{\partial\tau}.$$

Proof. From Theorem A.3, we have

$$\mathcal{F}^{a_\tau,\infty}\left\{\frac{\partial y_\tau}{\partial x}\right\} = -\frac{1}{\sqrt{2\pi}} e^{-i\omega \ln a_\tau} \lim_{x \downarrow \ln a_\tau(\tau)} y(x,\tau) + i\omega \hat{y}(\omega,\tau),$$

and by use of the boundary condition (3.11), the first identity is obtained. The proof of the second identity follows easily from the repeated application of Theorem A.3 and using simultaneously the boundary conditions (3.11) and (3.12). Finally, to prove the third identity, we note that

$$\frac{\partial}{\partial \tau}\left(\int_{\ln a_\tau}^{\infty} e^{-i\omega x} y(x,\tau)\, dx\right) = -\frac{\alpha'_\tau}{\alpha_\tau} e^{-i\omega \ln a_\tau} \lim_{x \downarrow \ln \alpha_\tau} y(x,\tau) + \int_{\ln \alpha_\tau}^{\infty} e^{-i\omega x} \frac{\partial y}{\partial \tau}(x,\tau) dx,$$

where $a'_\tau = da(\tau; l_\tau)/d\tau$. Rearranging the above result and then applying the boundary condition (3.11), we get

$$\mathcal{F}^{a_\tau \infty}\left\{\frac{\partial v_\tau}{\partial \tau}\right\} = \frac{\partial}{\partial \tau}\left(\mathcal{F}^{a_\tau,\infty}\{y_\tau\}\right). \qquad \square$$

Note that in deriving the above three identities, we make use of the smooth fit conditions given in eqs. (3.11-3.12), which also assure that the function $y(x,\tau)$ and its first partial derivative $\partial y_\tau/\partial x$ are bounded at the lower bound of the continuation region C_v, where boundary conditions are to be applied since by construction the function $y(x,\tau)$ converges to zero as $x \longrightarrow \infty$.

Proposition 3.3. *The IFT of the inhomogeneous Black-Scholes* PDE (3.13) *with respect to x satisfies the following* ODE

$$\frac{d\hat{y}(\omega,\tau)}{d\tau} + \left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)\hat{y}(\omega,\tau) = -\hat{u}(\omega,\tau), \tag{3.14}$$

with the initial condition $\mathcal{F}^{a_\tau,\infty}\{y(x,0)\} \equiv \hat{y}(\omega,0)$ *calculated from eq.* (3.10).

Proof. Taking eq. (3.13), which is the IFT of eq. (3.9) with respect to x, and using the three identities of Proposition 3.2, gives

$$\frac{\partial\hat{y}(\omega,\tau)}{\partial\tau} + \left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)\hat{y}(\omega,\tau) = -\hat{u}(\omega,\tau).$$

Finally, the initial condition is obtained by definition. $\square$

Instead of solving a PDE for the function $y(x,\tau)$, we are now faced with the simpler task of solving the ODE (3.14) for the function $\hat{y}(\omega,\tau)$. This can then be inverted via the inverse Fourier transform as stated in Proposition ?? to recover the desired function $y(x,\tau)$. Before concluding this section, we obtain the solution to eq. (3.14).

Proposition 3.4. *The solution* $\hat{y}(\omega,\tau)$ *to the* ODE (3.14) *is given by*

$$\hat{y}(\omega,\tau) = \hat{y}(\omega,0)e^{-\left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)\tau} - \int_0^\tau e^{-\left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)(\tau-s)}\hat{u}(\omega,s)ds. \tag{3.15}$$

Proof. Note that the ODE (3.14) is of the form

$$\frac{d\hat{y}}{d\tau}(\omega,\tau) + \varphi(\omega)\hat{y}(\omega,\tau) = -\hat{u}(\omega,s),$$

where

$$\varphi(\omega) := \left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right).$$

Using the integrating factor $e^{\varphi(\omega)\tau}$, the solution to the ODE (3.14) may be expressed as

$$\hat{y}(\omega,\tau)e^{\varphi(\omega)\tau} - \hat{y}(\omega,0) = -\int_0^\tau e^{\varphi(\omega)s}\hat{u}(\omega,s)ds,$$

which is readily reduced to eq. (3.15). □

3.3. Inversion of the IFT

Given the solution $\hat{y}(\omega,\tau)$ to the ODE (3.14), we now proceed to show how it can be recovered the function $y(x,\tau)$, recalling that it is only a suitable transformation of the initial premium function $v(x,\tau)$ of a European CI call option in the (x,τ) plane.

Proposition 3.5. *By applying the inverse Fourier transform to eq.* (3.15) *and the definition of the Heaviside step function, we obtain the following representation for the function* $y(x,\tau)$

$$y(x,\tau) := y_1(x,\tau) - y_2(x,\tau), \qquad \ln a_\tau < x < \infty, \tag{3.16}$$

with

$$y_1(x,\tau) := \mathcal{F}^{-1}\left\{\hat{y}(\omega,0)\, e^{-\left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)\tau}\right\};$$

$$y_2(x,\tau) := \mathcal{F}^{-1}\left\{\int_0^\tau e^{-\left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)(\tau-s)}\hat{u}(\omega,s)\, ds\right\}.$$

As it will be shown in the following proposition, to determine explicit expressions for the functions $y_1(x,\tau)$ and $y_2(x,\tau)$ we use the Convolution Theorem for Fourier transform, which subsequently involves evaluating integrals of the exponential of a quadratic function.

Proposition 3.6. *In the representation of eq.* (3.16) *the functions* $y_1(x,\tau)$ *and* $y_2(x,\tau)$ *can be expressed respectively as*

$$y_1(x,\tau) = \frac{e^{-r\tau-x}}{\sigma\sqrt{2\pi\tau}}\int_{\ln a_{0^+}}^\infty e^{-\frac{(x-u+\rho\tau)^2}{2\sigma^2\tau}}h(u)du; \tag{3.17}$$

$$y_2(x,\tau) = e^{-x}\int_0^\tau\int_{\ln a_s}^\infty \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}}e^{-\frac{(x-u+\rho(\tau-s))^2}{2\sigma^2(\tau-s)}}l(u,s)duds. \tag{3.18}$$

Proof. Let us first consider the definition of $y_1(x,\tau)$

$$y_1(x,\tau) := \mathcal{F}^{-1}\left\{\hat{y}(\omega,0)\, e^{-\left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)\tau}\right\}.$$

This inverse Fourier transform can be evaluated by the Convolution Theorem A.4, i.e., by making use of the following identity

$$\mathcal{F}^{-1}\{F(\omega,\tau_1)G(\omega,\tau_2)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x-\xi,\tau_1)g(\xi,\tau_2)d\xi, \tag{3.19}$$

where $F(\omega,\tau_1)$ and $G(\omega,\tau_2)$ are the Fourier transforms with respect to x of the functions $f(x,\tau_1)$ and $g(x,\tau_2)$, respectively. In order to apply the above result, we first let

$$F(\omega,\tau_1) = e^{-\left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)\tau}.$$

Applying the inverse Fourier transform, we get

$$f(x,\tau_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}\omega^2\tau - i\omega[-(x+\theta\tau)]} e^{-\vartheta\tau} d\omega = \frac{e^{-\vartheta\tau}}{\sigma\sqrt{\tau}} e^{-\frac{(x+\theta\tau)^2}{2\sigma^2\tau}},$$

where the last expression is obtained by setting $\lambda_1 = \frac{\sigma^2}{2}\tau$ and $\lambda_2 = i[-(x+\theta\tau)]$, with $n = 0$, and making use of the following identity

$$\int_{-\infty}^{\infty} e^{-\lambda_1\omega^2 - \lambda_2\omega}\omega^n d\omega = (-1)^n \sqrt{\frac{\pi}{\lambda_1}} \frac{\partial^n}{\partial\lambda_2^n} e^{\frac{\lambda_2^2}{4\lambda_1}}, \tag{3.20}$$

in which λ_1 and λ_2 are complex functions not involving the integration variable ω, with $Re(\lambda_1) \geq 0$ and $n \in \mathbb{N}_0$. Substituting $\rho = (r - \delta - \sigma^2/2)$ into the expression of ϑ and rearranging terms of $(x+\theta\tau)^2$, the function $f(x,\tau_1)$ can be simplified as follows

$$f(x,\tau_1) = \frac{e^{-\left(r-\rho-\sigma^2/2\right)\tau}}{\sigma\sqrt{\tau}} e^{-\frac{\left(x^2+\rho^2\tau^2+2x\rho\tau\right)+\left((\sigma^2+2\rho)\tau^2\sigma^2+2x\sigma^2\tau\right)}{2\sigma^2\tau}} = \frac{e^{-r\tau-x}}{\sigma\sqrt{\tau}} e^{-\frac{(x+\rho\tau)^2}{2\sigma^2\tau}}.$$

Next we let $G(\omega,\tau_2) = \hat{y}(\omega,0)$. Hence, we have

$$\begin{aligned} g(x,\tau_2) &= \mathcal{H}(x - \ln a_{0^+})y(x,0) \\ &= \mathcal{H}(x - \ln a_{0^+})e^{-x}h(x). \end{aligned}$$

Substituting for $f(x-\xi,\tau_1)$ and $g(\xi,\tau_2)$ into eq. (3.19) and using the Heaviside step function on the continuation region C_v, we obtain eq. (3.17).

The function $y_2(x,\tau)$ can be written as

$$\begin{aligned} y_2(x,\tau) &:= \mathcal{F}^{-1}\left\{\int_0^{\tau} e^{-\left(\frac{\sigma^2}{2}\omega^2 + \rho i\omega + r\right)(\tau-s)} \hat{l}(\omega,s)\,ds\right\} \\ &= \int_0^{\tau} \mathcal{F}^{-1}\left\{e^{-\left(\frac{\sigma^2}{2}\omega^2 + \theta i\omega + \vartheta\right)(\tau-s)} \hat{u}(\omega,s)\right\} ds. \end{aligned}$$

Since the above integral can be evaluated using the Convolution Theorem A.4, we let

$$\begin{aligned} &F(\omega,\tau_1) = e^{-\left(\frac{\sigma^2}{2}\omega^2 - \theta i\omega + \vartheta\right)(\tau-s)}; \\ &G(\omega,\tau_2) = \hat{u}(\omega,s). \end{aligned}$$

Applying now the inverse Fourier transform to the functions $F(\omega,\tau_1)$ and $G(\omega,\tau_2)$, we get

$$\begin{aligned} &f(x,\tau_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}\omega^2(\tau-s) - i\omega[-(x+\theta(\tau-s))]} e^{-\vartheta(\tau-s)} d\omega = \frac{e^{-\vartheta(\tau-s)}}{\sigma\sqrt{\tau-s}} e^{-\frac{\left[x+\theta(\tau-s)\right]^2}{2\sigma^2(\tau-s)}}; \\ &g(x,\tau_2) = \mathcal{H}(x - \ln a_s)e^{-x}l(x,s); \end{aligned}$$

where, setting $\lambda_1 = \frac{\sigma^2}{2}(\tau - s)$ and $\lambda_2 = i[-(x+\theta(\tau-s))]$, with $n = 0$, the last expression of $f(x,\tau_1)$ is obtained by identity (3.20). Now, it is quite easy to express function $f(x,\tau_1)$ in the following form

$$f(x,\tau_1) = \frac{e^{-\delta(\tau-s)}}{\sigma\sqrt{\tau-s}} e^{-\frac{\left[x+\rho(\tau-s)\right]^2+\left[(\sigma^2+2\rho)\sigma^2(\tau-s)^2+2x\sigma^2(\tau-s)\right]}{2\sigma^2\tau}} = \frac{e^{-r(\tau-s)-x}}{\sigma\sqrt{\tau}} e^{-\frac{\left[x+\rho(\tau-s)\right]^2}{2\sigma^2\tau}}.$$

Finally, substituting for $f(x-\xi,\tau_1)$ and $g(\xi,\tau_1)$ into eq. (3.19), yields

$$y_2(x,\tau) = \int_0^\tau \left(\int_{-\infty}^{\infty} \frac{e^{-r(\tau-s)-(x-\xi)}}{\sigma\sqrt{2\pi(\tau-s)}} e^{-\frac{[x-\xi+\rho(\tau-s)]^2}{2\sigma^2(\tau-s)}} \mathcal{H}(\xi - \ln a_s) e^{-\xi} l(\xi,s) d\xi \right) ds,$$

which is readily reduced to eq. (3.18) once the Heaviside step function on C_v is used. □

Using the explicit expressions of $y_1(x,\tau)$ and $y_2(x,\tau)$ given by Propositions 3.6 and recovering $v(x,\tau)$ as $e^x y(x,\tau)$, we obtain for the initial premium of a European CI call option with generic payoff and continuous payment plan the following integral representation

$$\begin{aligned} v(x,\tau) &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln a_{0^+}}^{\infty} e^{-\frac{(x-u+\rho\tau)^2}{2\sigma^2\tau}} h(u)du \\ &- \int_0^\tau \int_{\ln a_s}^{\infty} \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}} e^{-\frac{[x-u+\rho(\tau-s)]^2}{2\sigma^2(\tau-s)}} l(u,s)duds, \end{aligned} \tag{3.21}$$

for $\tau \in (0,T]$ and $x \in (\ln a_\tau, \infty)$.

In a similar way, by expressing PDE (3.7) and associated conditions (3.3-3.5) in terms of the newly defined auxiliary function $y(x,\tau) := e^x v(x,\tau)$ we are able to apply the IFT - since the required condition $\lim_{x\to-\infty} y(x,\tau) = 0$ is fulfilled - and then, by making use of identities similar to those in Proposition 3.2, to obtain the solution $\hat{y}(\omega,\tau)$ to the resulting ODE. Hence, taking the inverse Fourier transform of this solution in conjunction with the Convolution Theorem A.4 and then recovering the original function $v(x,\tau)$, we derive for the initial premium of a European CI put option the following integral representation

$$\begin{aligned} v(x,\tau) &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln g_{0^+}} e^{-\frac{(x-u+\rho\tau)^2}{2\sigma^2\tau}} h(u)du \\ &- \int_0^\tau \int_{-\infty}^{\ln g_s} \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}} e^{-\frac{[x-u+\rho(\tau-s)]^2}{2\sigma^2(\tau-s)}} l(u,s)duds, \end{aligned} \tag{3.22}$$

for $\tau \in (0,T]$ and $x \in (-\infty, \ln g_\tau)$.

Eqs. (3.21) and (3.22) express the up-front prices of European CI call and put options in terms of the respective optimal stopping boundary. Although this free boundary remains unknown, we are able to obtain a recursive integral equation that determine it by requiring the expression for $v(x,\tau)$ to satisfy the boundary condition (3.4). Therefore, by using our Heaviside step function, the optimal stopping boundary is found to satisfy the following relationship

For call option	For put option	
$0 = v(\ln a_\tau, \tau)$	$0 = v(\ln g_\tau, \tau).$	(3.23)

Finally, it is important to note that by using the IFT method an analytical expression for $v(x,\tau)$ has been derived without specifying the functional forms of $h(x)$ and $l(x,\tau)$, beyond a few basic properties. A result of comparable generality cannot be easily obtained when using alternative methods (e.g., Kim's (1990) compound option approach), and illustrates the main advantage attainable by using integral transform techniques. Hence, to price a European CI option with monotonic payoff structure and installment payment function of bounded variation, one must first solve integral equation (3.23) using numerical methods to find the optimal stopping boundary, since analytical solution seems impossibile. Once this is found, the function $v(x,\tau)$ can be evaluated via numerical integration.

4. A representation of the solution to the European CI option pricing problem

The Fourier transform method allows us to handle a wide class of payoff structures and continuous payment plans in a general form, as demonstrated by integral representations (3.21) and (3.22) of the initial premium of European call and put options, respectively. Although alternative approaches, such as Kim [16] and Carr et al. [4], are tied rather strongly to the explicit form of the payoff being considered, the results they obtain are much easier to interpret in an economic sense. Thus, to ensure that our Fourier transform technique is seen to be both applicable to a wide class of derivatives and economically meaningful, we will derive a parametric representation of the solution to the European CI option pricing problem by assuming that $L(S_t, t)$ is a function of time t only.

4.1. Kim's representation of the European CI option initial premium

Although it is convenient to define the initial premium in terms of time to maturity $\tau = T-t$, the log-transformation $S_t = e^x$ has little economic interpretation. Furthermore, it is not possible to apply directly the integral representations (3.21) and (3.22) once the functional forms of $H(S_t)$ and $L(S_t, t)$ are specified. For these reasons we will derive a parametric representation of the up-front price in terms of the original variable, S_t, and setting only some basic properties of the functions $H(S_t)$ and $L(S_t, t)$.

Proposition 4.1. *Let $H(S_t)$ be a monotonic increasing function of S_t, for $t \geq 0$, and $L(S_t, t) \equiv L(t)$ be an integral function with respect to t, for $t \geq 0$, and defined for any $S_t \geq 0$. Then, the initial premium function $C(S_t, \tau)$ of a European CI call can be expressed as*

$$\mathcal{H}(S_t - a_\tau)C(S_t, \tau) = c_E^{\mathrm{BS}}(S_t, \tau) - \Lambda^{\mathrm{C}}(S_t, \tau; a(\cdot)), \tag{4.1}$$

with $a_\tau \leq S_t < \infty$, for $\tau \in (0, T]$, and where

$$\begin{aligned} c_E^{\mathrm{BS}}(S_t, \tau) &:= S_t e^{-\delta\tau} \int_{-\infty}^{d_1(S_t, a_{0^+}, \tau)} \frac{h'(u)}{e^u} \frac{e^{-\frac{(z(u))^2}{2}}}{\sqrt{2\pi}} dz(u) \\ &\quad + e^{-r\tau} \int_{-\infty}^{d_2(S_t, a_{0^+}, \tau)} [h(u) - h'(u)] \frac{e^{-\frac{(\zeta(u))^2}{2}}}{\sqrt{2\pi}} d\zeta(u); \end{aligned} \tag{4.2}$$

$$\Lambda^{\mathrm{C}}(S_t, \tau; a(\cdot)) := \int_0^\tau \int_{-\infty}^{d_2(S_t, a_\xi, \tau-\xi)} e^{-r(\tau-\xi)} \frac{e^{-\frac{(\eta(u))^2}{2}}}{\sqrt{2\pi}} l(\xi) d\eta(u) d\xi; \tag{4.3}$$

are the generalized Black-Scholes European call pricing formula and the expected present value of payment stream along the optimal stopping boundary, respectively. Furthermore, the free boundaries a_τ is given by

$$0 = c_E^{\mathrm{BS}}(a_\tau, \tau) - \Lambda^{\mathrm{C}}(a_\tau, \tau; a(\cdot)). \tag{4.4}$$

Proof. Let us consider the first integral term in the right hand side of eq. (3.21). Now by adding and subtracting the term $h'(x) := dh(x)/dx$ to the integrand, yields

$$\begin{aligned} v_1(x, \tau) &:= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln a_{0^+}}^{\infty} e^{-\frac{(x-u+\rho\tau)^2}{2\sigma^2\tau}} h'(u) du + \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln a_{0^+}}^{\infty} e^{-\frac{(x-u+\rho\tau)^2}{2\sigma^2\tau}} [h(u) - h'(u)] du \\ &= I_1(x, \tau) + I_2(x, \tau). \end{aligned}$$

Multiplying and dividing by e^u in the integrand and rearranging terms, the integral $I_1(x, \tau)$ can be written as follows

$$I_1(x, \tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln a_{0^+}}^{\infty} \frac{h'(u)}{e^u} e^{-\frac{\left[u-(x+(\rho+\sigma^2)\tau)\right]^2}{2\sigma^2\tau}} e^{x+\rho\tau+\frac{\sigma^2\tau}{2}} du,$$

and recalling that $\rho = r - \delta - \sigma^2/2$, we obtain for $I_1(x, \tau)$ the following expression

$$I_1(x, \tau) = e^x e^{-\delta\tau} \lim_{c\to\infty} \int_{\ln a_{0^+}}^{c} \frac{h'(u)}{e^u} \frac{e^{-\frac{1}{2}\left[\frac{u-(x+(\rho+\sigma^2)\tau)}{\sigma\sqrt{\tau}}\right]^2}}{\sigma\sqrt{2\pi\tau}} du.$$

Setting $z(u) := \frac{x-u+(\rho+\sigma^2)\tau}{\sigma\sqrt{\tau}}$ and by defining $d_1(x,y,\tau) := \frac{\ln(x/y)+(\rho+\sigma^2)\tau}{\sigma\sqrt{\tau}}$, yields

$$I_1(x,\tau) = e^x e^{-\delta\tau} \int_{-\infty}^{d_1(e^x, a_{0^+}, \tau)} \frac{h'(u)}{e^u} \frac{e^{-\frac{(z(u))^2}{2}}}{\sqrt{2\pi}} dz(u).$$

Similarly, the integral $I_2(x,\tau)$ can be written as

$$I_2(x,\tau) = e^{-r\tau} \lim_{c\to\infty} \int_{\ln a_{0^+}}^{c} [h(u) - h'(u)] \frac{e^{-\frac{1}{2}\left[\frac{x-u+\rho\tau}{\sigma\sqrt{\tau}}\right]^2}}{\sigma\sqrt{2\pi\tau}} du.$$

Setting $\zeta(u) := \frac{x-u+\rho\tau}{\sigma\sqrt{\tau}}$ and by defining $d_2(x,y,\tau) := \frac{\ln(x/y)+\rho\tau}{\sigma\sqrt{\tau}}$, we obtain

$$I_2(x,\tau) = e^{-r\tau} \int_{-\infty}^{d_2(e^x, a_{0^+}, \tau)} [h(u) - h'(u)] \frac{e^{-\frac{(\zeta(u))^2}{2}}}{\sqrt{2\pi}} d\zeta(u).$$

For the second integral term in the right hand side of eq. (3.21), by splitting up the interval of integration at a point $c \in (\ln a_s, \infty)$, as c tends to infinity we get

$$v_2(x,\tau) := \int_0^{\tau} \left[\lim_{c\to\infty} \int_{\ln a_s}^{c} \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}} e^{-\frac{1}{2}\left[\frac{x-u+\rho(\tau-s)}{\sigma\sqrt{\tau-s}}\right]^2} l(s) du \right] ds.$$

Performing the change of variable $\eta(u) := \frac{x-u+\rho(\tau-s)}{\sigma\sqrt{\tau-s}}$ and using the above definition of $d_2(\cdot,\cdot,\cdot)$, yields

$$v_2(x,\tau) = \int_0^{\tau} \int_{-\infty}^{d_2(e^x, a_s, \tau-s)} e^{-r(\tau-s)} \frac{e^{-\frac{(\eta(u))^2}{2}}}{\sqrt{2\pi}} l(s) d\eta(u) ds.$$

Substituting the new expressions of $v_1(x,\tau)$ and $v_2(x,\tau)$ into eq. (3.21), rearranging terms and reverting back to the original space-variable via $S_t = e^x$, we obtain eq. (4.1) in conjunction with the expressions (4.2) and (4.3) defining $c_E^{BS}(S_t,\tau)$ and $\Lambda^c(S_t,\tau;a(\cdot))$, respectively. Finally, because early stopping decision is optimal when $S_t = a_\tau$, by applying the value-matching condition (3.23) we get eq. (4.4). □

Proceeding in a way similar to that described above, we obtain a parametric representation of the solution (3.22) to the pricing problem of European CI put options. Thus, if $H(S_t)$ is a monotonic decreasing function of S_t, for $t \geq 0$, and $L(S_t,t) \equiv L(t)$ is an integrable function of t for any $S_t \geq 0$, then the initial premium function $P(S_t,\tau)$ of a European CI put can be expressed as

$$\mathcal{H}(g_\tau - S_t)P(S_t,\tau) = p_E^{BS}(S_t,\tau) - \Lambda^P(S_t,\tau;g(\cdot)), \tag{4.5}$$

with $-\infty < S_t \leq g_\tau$, for $\tau \in (0,T]$, and where

$$\begin{aligned} p_E^{BS}(S_t,\tau) &:= S_t e^{-\delta\tau} \int_{-\infty}^{-d_1(S_t, g_{0^+}, \tau)} \frac{h'(u)}{e^u} \frac{e^{-\frac{(z(u))^2}{2}}}{\sqrt{2\pi}} dz(u) \\ &\quad - e^{-r\tau} \int_{-\infty}^{-d_2(S_t, g_{0^+}, \tau)} [h(u) - h'(u)] \frac{e^{-\frac{(\zeta(u))^2}{2}}}{\sqrt{2\pi}} d\zeta(u); \end{aligned} \tag{4.6}$$

$$\Lambda^P(S_t,\tau;g(\cdot)) := \int_0^{\tau} \int_{-\infty}^{-d_2(S_t, g_\xi, \tau)} e^{-r(\tau-\xi)} \frac{e^{-\frac{(\eta(u))^2}{2}}}{\sqrt{2\pi}} l(\xi) d\eta(u) d\xi; \tag{4.7}$$

are the generalized Black-Scholes European put pricing formula and the expected present value of payment stream along the optimal stopping boundary, respectively. Furthermore, if we evaluate $P(S_t,\tau)$ at $S_t = g_\tau$, we obtain the following recursive integral equation for the free boundaries g_τ

$$0 = p_E^{BS}(g_\tau,\tau) - \Lambda^P(g_\tau,\tau;g(\cdot)). \tag{4.8}$$

Since the solution forms (4.1-4.3) and (4.5-4.7) were first found for the case of standard American options by an approach due to Kim [16], we henceforth refer to these as Kim's representations for the initial premium of a European CI call and put option, respectively.

5. An example: European vanilla CI call options with linear installment payment function

Now we want to consider an interesting class of examples which illustrates the flexibility and generality of the results in this paper. The following proposition gives an explicit formula for the up-front price of European CI call options with well-known payoff specification and installment payment function of linear form.

Proposition 5.1. *Let $H(S_T)$ and $L(t)$ be given respectively by*

$$H(S_T) := (S_T - K)^+;$$
$$L(t) := \frac{q_T - q_t}{T}t + q_t, \qquad 0 \le t \le T;$$

where $(x)^+ = \max(x,0)$ and $K \ge 0$ is the exercise (or strike) price of the option, and where q_t and q_T are fixed positive constants with either $0 \le q_t \le q_T$ or $0 \le q_T \le q_t$. Then, the initial premium function $C(S_t,\tau)$ of a European CI call in eq. (4.1) can be expressed as

$$\begin{aligned} C(S_t,\tau) &= S_t e^{-\delta\tau}N(d_1(S_t,K,\tau)) - Ke^{-r\tau}N(d_2(S_t,K,\tau)) - q_T\int_0^\tau e^{-r(\tau-\xi)}N(d_2(S_t,a_\xi,\tau-\xi))d\xi \\ &+ \frac{q_T-q_0}{T}\int_0^\tau e^{-r(\tau-\xi)}\xi N(d_2(S_t,a_\xi,\tau-\xi))d\xi, \end{aligned} \tag{5.1}$$

with $a_\tau \le S_t < \infty$ and where $N(\cdot)$ is the standard normal cumulative distribution function given by

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}v^2}dv.$$

The recursive integral equation (4.4) for the optimal stopping boundary a_τ can be written as

$$\begin{aligned} 0 &= a_\tau e^{-\delta\tau}N(d_1(a_\tau,K,\tau)) - Ke^{-r\tau}N(d_2(a_\tau,K,\tau)) - q_T\int_0^\tau e^{-r(\tau-\xi)}N(d_2(a_\tau,a_\xi,\tau-\xi))d\xi \\ &+ \frac{q_T-q_0}{T}\int_0^\tau e^{-r(\tau-\xi)}\xi N(d_2(a_\tau,a_\xi,\tau-\xi))d\xi. \end{aligned} \tag{5.2}$$

Proof. If now we set $H(S_T) = \max(S_T - K, 0)$, this implies that $h(x) = \max(e^x - K, 0)$ and then $h'(x) = \max(e^x, 0)$. By substituting for $h(u)$ and its first derivative into eq. (4.2), simplifying and using the definition of standard normal cumulative distribution function, we can express $c_E^{BS}(S_t,\tau)$ as

$$c_E^{BS}(S_t,\tau) = S_t e^{-\delta\tau}N(d_1(S_t,a_{0^+},\tau)) - Ke^{-r\tau}N(d_1(S_t,a_{0^+},\tau)),$$

where $a_{0^+} := \lim_{\tau\to 0^+} a_\tau$, i.e., the optimal stopping boundary at the maturity date T, is equal to the strike price K as proven in Ciurlia [7] and Kimura [18]. Furthermore, setting $l(\xi) = \frac{q_T-q_t}{T}(T-\xi) + q_t$ and substituting this expression for $l(\xi)$ into eq. (4.3), we can simplify $\Lambda^C(S_t,\tau;a(\cdot))$ as follows

$$\Lambda^C(S_t,\tau;a(\cdot)) = \int_0^\tau e^{-r(\tau-\xi)}\left(q_T - \frac{q_T-q_0}{T}\xi\right)N(d_2(S_t,a_\xi,\tau-\xi))d\xi.$$

Substituting for $c_E^{BS}(S_t, \tau)$ and $\Lambda^C(S_t, \tau; a(\cdot))$ into eq. (4.1) and applying the definition of the Heaviside step function, yields the explicit expression (5.1) for the initial premium function $C(S_t, \tau)$. Finally, if we evaluate $C(S_t, \tau)$ at $S_t = a_\tau$, we obtain the integral equation (5.2) for the free boundary a_τ. □

Remark. While the above proposition is fully sufficient to dealing with European vanilla CI call options when the payment plan is a linear function of time, we note for completeness that we can give the following abstract characterization of a wide class of contingent claims. European-style vanilla installment derivatives with continuous, positive and linear strictly increasing or decreasing installment payment function $L(t)$ on time interval $[0, T]$ are given by setting $0 < q_t < q_T$ or $0 < q_T < q_t$, respectively. It is worth noting that the special case $q_t = q_T \equiv q$ corresponds to the class of installment derivatives with constant payment stream, previously treated in the literature (see, for instance, Ciurlia [7] and Kimura [18]), whereas the Black-Scholes option pricing model is recovered in the limiting case as $q_t \to 0^+ \wedge q_T \to 0^+$.

6. Conclusions

In this paper we have examined the valuation of European continuous-installment options when the payment plan is a function of the asset price and time variables. By expressing this class of option pricing problems as a free boundary problem under very general conditions, and then applying the incomplete Fourier transform technique, we were able to provide optimal stopping strategies and valuation formulas based only on knowing a few basic properties of payoff and installment payment functions. This high degree of flexibility and robustness substantially simplifies the pricing of these derivatives because it reduces the determination of the initial premium and free boundary to the identification of some key parameters. In the case of a constant installment option the pricing formula of Ciurlia [7] and Kimura [18] is easily derived once a specific form of the installment payment function has been identified.

Although our focus is on options, our approach is general and applies to more complex derivatives. For instance, many of our results, which leads to a unified treatment of the pricing of European installment options with general monotone payoff functions, can be extend to cover convex or concave payoff structures. Examples include not only the well known option trading strategies, such as strangles and condors, but also a class of installment derivatives whose initial premium satisfies the inhomogeneous partial differential equation, though the final and boundary conditions for these claims may be changed. When the underlying asset price is modeled by a more general diffusion process (e.g. jump-diffusion models), this approach is amenable to further extensions and can be used to construct an analytic treatment of a range of valuation and hedging problems.

Finally, the most challenging issue concerns the class of high-dimensional problems that arise, for example, in pricing options on multiple assets and in pricing options under additional risk factors such as stochastic volatility, interest rates and exchange rates. While it is possible to extend our approach to handle low-dimensional problems involving direct numerical integration, it does not resolve the general multidimensional case. For future research in the field, other numerical procedures such as Monte Carlo simulation may be more appropriate to deal with high-dimensional pricing problems.

A. The incomplete Fourier transform: Basic properties and theorems

Here we present a collection of results concerning the application of the standard Fourier transform to the class of absolutely integrable functions on a finite or semi-infinite interval. Finally, the convolution theorem for Fourier transforms is formulated.

Definition A.1 (Fourier transform and its inverse). *The Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$ is denoted by $\mathcal{F}\{f(x)\} = F(\omega)$ and defined by the integral*

$$\mathcal{F}\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx,$$

where $\mathcal{F}$ is called the Fourier transform operator or the Fourier transformation.

The inverse Fourier transform, denoted by $\mathcal{F}^{-1}\{F(\omega)\} = f(x)$, is defined by

$$\mathcal{F}^{-1}\{F(\omega)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} F(\omega)d\omega,$$

where $\mathcal{F}^{-1}$ is called the inverse Fourier transform operator.

Theorem A.1. *Let $\mathcal{F}\{f(x)\} = F(\omega)$ be the Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$. Then, $\mathcal{F}\{f(x)\}$ satisfies the following properties*

(*a*) (*Shifting*) $\mathcal{F}\{f(x-a)\} = e^{-i\omega a}\mathcal{F}\{f(x)\}$;

(*b*) (*Scaling*) $\mathcal{F}\{f(ax)\} = \frac{1}{|a|}\mathcal{F}\{f(x)\}$;

(*c*) (*Conjugate*) $\mathcal{F}\left\{\overline{f(-x)}\right\} = \overline{\mathcal{F}\{f(x)\}}$;

(*d*) (*Translation*) $\mathcal{F}\{e^{iax}f(x)\} = F(\omega - a)$;

(*e*) (*Linearity*) $\mathcal{F}\{a_1 f_1(x) + a_2 f_2(x)\} = a_1\mathcal{F}\{f_1(x)\} + a_2\mathcal{F}\{f_2(x)\}, \quad \forall a_1, a_2 \in \mathbb{C}$;

(*f*) (*Duality*) $\mathcal{F}\{F(x)\} = f(-\omega)$.

Proof. To prove (*a*), we have, by definition,

$$\begin{aligned}
\mathcal{F}\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x-a)dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(\xi+a)} f(\xi)d\xi \qquad (x - a = \xi) \\
&= e^{-i\omega a}\mathcal{F}\{f(x)\}.
\end{aligned}$$

The proofs of results (*b*)-(*e*) follow easily from the definition of the Fourier transform. We give a proof of the duality property (*f*). We have, by definition,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} F(\omega)d\omega = \mathcal{F}^{-1}\{F(\omega)\}.$$

Interchanging x and ω, and replacing ω by $-\omega$, we obtain

$$f(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} F(x)dx = \mathcal{F}\{F(x)\}. \qquad \square$$

Proposition A.1. *Let $f : (a,b) \subset \mathbb{R} \to \mathbb{R}$ be a function defined on a finite real segment (a,b). If the x-domain is extended to all real numbers by expressing $f(x)$ as*

$$[\mathcal{H}(b-x) - \mathcal{H}(a-x)]f(x),$$

with $\mathcal{H}(x)$ the Heaviside step function given by

$$\mathcal{H}(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0 \end{cases}$$

then the incomplete Fourier transform (IFT), denoted by $\mathcal{F}^{a,b}\{f(x)\} = \hat{F}(\omega)$, is defined as

$$\mathcal{F}^{a,b}\{f(x)\} = \hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\omega x} f(x) dx. \tag{A.1}$$

Proof. We have, by definition,

$$\mathcal{F}\Big\{[\mathcal{H}(b-x) - \mathcal{H}(a-x)]f(x)\Big\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x}[\mathcal{H}(b-x) - \mathcal{H}(a-x)]f(x)dx,$$

and applying the linearity of the integral operator, yields

$$\begin{aligned} \mathcal{F}^{a,b}\{f(x)\} &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{b} e^{-i\omega x} f(x)\, dx - \int_{-\infty}^{a} e^{-i\omega x} f(x) dx\right) \\ &= \frac{1}{\sqrt{2\pi}} \lim_{v \to -\infty} \left(\int_a^v e^{-i\omega x} f(x)\, dx + \int_v^b e^{-i\omega x} f(x) dx\right), \end{aligned}$$

which is expression (A.1). □

Proposition A.2. *The IFT of a function $f : (-\infty, b) \subset \mathbb{R} \to \mathbb{R}$, denoted as $\mathcal{F}^{-\infty,b}\{f(x)\} = \hat{F}(\omega)$, is given by*

$$\mathcal{F}^{-\infty,b}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^b e^{-i\omega x} f(x) dx. \tag{A.2}$$

The IFT of a function $f : (a, \infty) \subset \mathbb{R} \to \mathbb{R}$, denoted as $\mathcal{F}^{a,\infty}\{f(x)\} = \hat{F}(\omega)$, is given by

$$\mathcal{F}^{a,\infty}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-i\omega x} f(x) dx. \tag{A.3}$$

Proof. Taking the limit of eq. (A.1) as either $a \to -\infty$ or $b \to \infty$, we obtain eq. (A.2) and eq. (A.3), respectively. □

Proposition A.3. *The inverse Fourier transform of a function $f : (a,b) \subset \mathbb{R} \to \mathbb{R}$, denoted by $\mathcal{F}^{-1}\{\hat{F}(\omega)\} = f(x)$, is given by*

$$\mathcal{F}^{-1}\{\hat{F}(\omega)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_a^b e^{-i\omega x} f(x)\, dx\right) e^{i\omega x} d\omega, \quad a < x < b. \tag{A.4}$$

Proof. Let $g(x) = [\mathcal{H}(b-x) - \mathcal{H}(a-x)]f(x)$ be a function defined for all $x \in \mathbb{R}$. Then, applying the inverse Fourier transform operator to $g(x)$, yields

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\xi} g(\xi)\, d\xi\right) e^{i\omega x} d\omega, \qquad -\infty < x < \infty.$$

According to the definition of the Heaviside step function, the right- and left-hand sides of the above expression can be written respectively as

$$\begin{aligned}
\text{RHS} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega\xi}[\mathcal{H}(b-\xi)-\mathcal{H}(a-\xi)]f(\xi)\,d\xi\right)e^{i\omega x}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{\upsilon\to-\infty}\left(\int_{a}^{\upsilon} e^{-i\omega\xi}f(\xi)\,d\xi+\int_{\upsilon}^{b} e^{-i\omega\xi}f(\xi)\,d\xi\right)\right]e^{i\omega x}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{a}^{b} e^{-i\omega\xi}f(\xi)\,d\xi\right)e^{i\omega x}d\omega,
\end{aligned}$$

$$\text{LHS} = (\mathcal{H}(b-x)-\mathcal{H}(a-x))f(x) = \begin{cases} f(x), & a<x<b \\ \dfrac{f(x)}{2}, & x=a \vee x=b \\ 0, & x<a \vee x>b \end{cases}$$

Putting together, we have

$$[\mathcal{H}(b-x)-\mathcal{H}(a-x)]f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{b}^{a} e^{-i\omega\xi}f(\xi)\,d\xi\right)e^{i\omega x}d\omega, \qquad -\infty<x<\infty,$$

or alternatively

$$f(x) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left(\int_{b}^{a} e^{-i\omega\xi}f(\xi)\,d\xi\right)e^{i\omega x}d\omega, & a<x<b \\ \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}\left(\int_{b}^{a} e^{-i\omega\xi}f(\xi)\,d\xi\right)e^{i\omega x}d\omega, & x=a \vee x=b \end{cases} \quad \square$$

Proposition A.4. *The inverse Fourier transform of a function* $f\,:\,(-\infty,b)\subset\mathbb{R}\to\mathbb{R}$, *denoted by* $\mathcal{F}^{-1}\{\hat{F}(\omega)\}=f(x)$, *is given by*

$$\mathcal{F}^{-1}\{\hat{F}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{b} e^{-i\omega x}f(x)\,dx\right)e^{i\omega x}d\omega, \quad -\infty<x<b. \tag{A.5}$$

The inverse Fourier transform of a function $f\,:\,(a,\infty)\subset\mathbb{R}\to\mathbb{R}$, *denoted by* $\mathcal{F}^{-1}\{\hat{F}(\omega)\}=f(x)$, *is given by*

$$\mathcal{F}^{-1}\{\hat{F}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{a}^{\infty} e^{-i\omega x}f(x)\,dx\right)e^{i\omega x}d\omega, \quad a<x<\infty. \tag{A.6}$$

Proof. Taking the limit of eq. (A.4) as either $a\to-\infty$ or $b\to\infty$, we obtain eq. (A.5) and eq. (A.6), respectively. $\square$

Theorem A.2. *Let* $f\,:\,[a,b]\subset\mathbb{R}\to\mathbb{R}$ *be continuously differentiable for all* $x\in(a,b)$ *and such that* $f(x)$ *is bounded as* $x\to a^{+}$ *and* $x\to b^{-}$. *Then, the following equality holds*

$$\mathcal{F}^{a,b}\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega b}f(b)-e^{-i\omega a}f(a)\right]+(i\omega)\hat{F}(\omega). \tag{A.7}$$

Proof. We have, by definition,

$$\mathcal{F}^{a,b}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\omega x} f'(x) dx.$$

Integrating by parts, the right-hand side can be rewritten as

$$\text{RHS} = \left[\frac{e^{-i\omega x} f(x)}{\sqrt{2\pi}} \right]_a^b + \frac{i\omega}{\sqrt{2\pi}} \int_a^b e^{-i\omega x} f(x) dx,$$

which is expression (A.7). $\square$

Theorem A.3. *If $f : (-\infty, b] \subset \mathbb{R} \to \mathbb{R}$ is bounded, continuously differentiable and $f(x) \to 0$ as $x \to -\infty$, then*

$$\mathcal{F}^{-\infty,b}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} e^{-i\omega b} f(b) + (i\omega)\hat{F}(\omega). \tag{A.8}$$

If $f : [a, \infty) \subset \mathbb{R} \to \mathbb{R}$ is bounded, continuously differentiable and $f(x) \to 0$ as $x \to \infty$, then

$$\mathcal{F}^{a,\infty}\{f'(x)\} = -\frac{1}{\sqrt{2\pi}} e^{-i\omega a} f(a) + (i\omega)\hat{F}(\omega). \tag{A.9}$$

Proof. Taking the limit of eq. (A.7) as either $a \to -\infty$ or $b \to \infty$, we obtain eq. (A.8) and eq. (A.9), respectively. $\square$

A repeated application of Theorems A.2 and A.3 to higher derivatives gives the following two corollaries.

Corollary A.1. *If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuously n-times differentiable for all $x \in (a, b)$ and such that $f^{(k)}(x)$ is bounded as $x \to a^+$ and $x \to b^-$, for $k = 0, 1, 2, \ldots, n-1$, then*

$$\begin{aligned}\mathcal{F}^{a,b}\{f^{(n)}(x)\} &= \frac{1}{\sqrt{2\pi}} \Bigg[e^{-i\omega b} \sum_{j=1}^{n} (i\omega)^{j-1} f^{(n-j)}(b) \\ &\quad - e^{-i\omega a} \sum_{j=1}^{n} (i\omega)^{j-1} f^{(n-j)}(a) \Bigg] + (i\omega)^n \hat{F}(\omega).\end{aligned} \tag{A.10}$$

Corollary A.2. *If $f : (-\infty, b] \subset \mathbb{R} \to \mathbb{R}$ is bounded, continuously differentiable and $f^{(k)}(x) \to 0$ as $x \to -\infty$, for $k = 0, 1, 2, \ldots, n-1$, then*

$$\mathcal{F}^{-\infty,b}\{f^{(n)}(x)\} = \frac{1}{\sqrt{2\pi}} e^{-i\omega b} \sum_{j=1}^{n} (i\omega)^{j-1} f^{(n-j)}(b) + (i\omega)^n \hat{F}(\omega), \tag{A.11}$$

If $f : [a, \infty) \subset \mathbb{R} \to \mathbb{R}$ is bounded, continuously differentiable and $f^{(k)}(x) \to 0$ as $x \to \infty$, for $k = 0, 1, 2, \ldots, n-1$, then

$$\mathcal{F}^{a,\infty}\{f^{(n)}(x)\} = -\frac{1}{\sqrt{2\pi}} e^{-i\omega a} \sum_{j=1}^{n} (i\omega)^{j-1} f^{(n-j)}(a) + (i\omega)^n \hat{F}(\omega). \tag{A.12}$$

Definition A.2 (Convolution). *The convolution of two integrable functions $f, g : \mathbb{R} \to \mathbb{R}$, denoted by $f(x) * g(x)$, is defined by*

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi.$$

Theorem A.4 (Convolution Theorem). *Let $\mathcal{F}\{f(x)\} = F(\omega)$ and $\mathcal{F}\{g(x)\} = G(\omega)$ be the Fourier transforms of functions $f(x)$ and $g(x)$, respectively. Then, for the convolution $f(x) * g(x)$ the following equalities hold*

$$\mathcal{F}\{f(x) * g(x)\} = F(\omega)G(\omega),$$

or

$$f(x) * g(x) = \mathcal{F}^{-1}\{F\omega)G(\omega)\},$$

or, equivalently,

$$\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi = \int_{-\infty}^{\infty} e^{i\omega x}F(\omega)G(\omega)d\omega.$$

Proof. Using the definition of the standard Fourier transform, we have

$$\begin{aligned}
\mathcal{F}\{f(x) * g(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}dx \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-i\omega(x-\xi)}f(x-\xi)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-i\omega\eta}f(\eta)d\eta \qquad (x-\xi=\eta) \\
&= G(\omega)F(\omega).
\end{aligned}$$

This completes the proof. □

It can be readily verified that the convolution has the following algebraic properties

(a) (Commutative) $f * g = g * f$;

(b) (Associative) $(f * g) * h = f * (g * h)$;

(c) (Distributive) $f * (g + h) = f * g + f * h$;

(d) (Identity) $f * \sqrt{2\pi}\delta = f = \sqrt{2\pi}\delta * f$.

References

[1] Alobaidi, G., Mallier, R., Deakin, A.S.: Laplace transforms and installment options. *Mathematical Models and Methods in Applied Sciences* **14** (2004) 1167-1189.

[2] Alobaidi, G., Mallier, R.: Installment options close to expiry. *Journal of Applied Mathematics and Stochastic Analysis* (2006) 1-9.

[3] Ben-Ameur, H., Breton, M., François, P.: A dynamic programming approach to price installment options. *European Journal of Operational Research* **169** (2006) 667-676.

[4] Carr, P., Jarrow, R., Myneni, R.: Alternative characterizations of American put options. *Mathematical Finance* **2** (1992) 87-106.

[5] Ciurlia, P., Roko, I.: Valuation of American continuous-installment options. *Computational Economics* **25** (2005) 143-165.

[6] Ciurlia, P., Caperdoni, C.: A note on the pricing problem of perpetual continuous-installment options. *Mathematical Methods in Economics and Finance* **4** (2009) 11-26.

[7] Ciurlia, P.: On the evaluation of European continuous-installment options. *Department of Economics Workin paper*, No 113, University of Rome III (2010).

[8] Davis, M., Schachermayer, W., Tompkins, R.: Pricing, no-arbitrage bounds and robust hedging of instalment options. *Quantitative Finance* **1** (2001) 597-610.

[9] Davis, M., Schachermayer, W., Tompkins, R.: Installment options and static hedging. *Journal of Risk Finance* **3** (2002) 46-52.

[10] Dettman, J.W.: Applied complex variables. *Dover, New York* (1965).

[11] Dixit, A.R., Pindyck, R.S.: Investment under uncertainty. Princeton University Press, New Jersey (1994).

[12] Geske, R.: The valuation of corporate liabilities as compound options. *Journal of Financial and Quantitative Analysis* **12** (1977) 541-552.

[13] Griebsch, S., Kühn, C. Wystup, U.: Instalment options: A closed-form solution and the limiting case. In: Sarychev, A., Shiryaev, A., Guerra, M., Grossinho, M.d.R. (eds.): *Mathematical Control Theory and Finance*. Springer-Verlag, Berlin (2008) 211-229.

[14] Jacka, S.D.: Optimal stopping and the American put. *Mathematical Finance* **1** (1991) 1-14.
[15] Jamshidian, F.: An analysis of American options. *Review of Futures Markets* **11** (1992) 72-80.
[16] Kim, I.J.: The analytical valuation of American options. *Review of Financial Studies* **3** (1990) 547-572.
[17] Kimura, T.: Valuing continuous-installment options. *Discussion Paper Series A*, No 2007-184, Hokkaido University (2007).
[18] Kimura, T.: Valuing continuous-installment options. *European Journal of Operational Research* **201** (2010) 222-230.
[19] Kolodner, I.I.: Free boundary problem for the heat equation with applications to problems of change of phase. *Communications on Pure and Applied Mathematics* **9** (1956) 1-31.
[20] Majd, S., Pindyck, R.S.: Time to build, option value, and investment decisions. *Journal of Financial Economics* **18** (1987) 7-27.
[21] McKean, H.P.: Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review* **6** (1965) 32-39.
[22] Selby, M.J.P., Hodges, S.D.: On the evaluation of compound options. *Management Science* **33** (1987) 347-355.
[23] Van Moerbeke, P.: On optimal stopping and free boundary problems. *Arch. Rational Mech. Anal.* **60** (1976) 101-148.
[24] Yang, Z., Yi, F.: A variational inequality arising from American installment call options pricing. *Journal of Mathematical Analysis and Applications* **357** (2009) 54-68.
[25] Yi, F., Yang, Z., Wang X.: A variational inequality arising from European installment call options pricing. *SIAM Journal on Mathematical Analysis* **40** (2008) 306-326.