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NASH EQUILIBRIA OF GAMES WITH MONOTONIC BEST REPLIES

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DIPARTIMENTO DI ECONOMIA

NASH EQUILIBRIA OF GAMES WITH MONOTONIC BEST REPLIES

Filippo L. Calciano

NASH EQUILIBRIA OF GAMES WITH INCREASING BEST REPLIES

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Abstract. We introduce notions of increasingness for the best reply of a game that capture properly the intuitive idea of complementarity among players' strategies. We show, by generalizing the fixpoint theorems of Veinott and Zhou, that the Nash sets of our games with increasing best replies are nonempty complete lattices. Hence we extend the class of games with strategic complementarities.
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1. Introduction

Games whose normal form exhibits an increasing joint best reply are usually referred to, in the economic literature, as games with strategic complementarities (Vives, 2005). The main notion of increasingness for the best reply that is adopted in the literature on games with strategic complementarities is due to Veinott (see Topkis, 1978, and Section 3 of the present paper).

Individual best replies can be made to be Veinott-increasing by means of a theory of monotone comparative statics in which individual payoffs are assumed to be supermodular or quasisupermodular in own strategies, and to satisfy increasing differences or single crossing conditions in the strategies of opponents (See Topkis 1978, Veinott 1992 and Milgrom and Shannon 1994). For this reason, games with strategic complementarities are also known as supermodular games.

In supermodular games, the product of individual strategy sets is assumed to be a complete lattice (see Section 2 for the relevant definitions). For a correspondence defined from a complete lattice into itself, and that is increasing in the sense of Veinott, a fixpoint theorem has been proved by Veinott (1992) and Zhou (1994) showing that the fixpoint set is a nonempty complete lattice. This theorem is an extension to correspondences of the celebrated Tarski's fixpoint theorem (Tarski, 1955). Vives (1990) is also a work in this direction.

These fixpoint results allow to establish, for games with strategic complementarities, that Nash equilibria exist and that the Nash set is a complete lattice. The latter implies in particular that a least and a greatest Nash equilibrium exist; a property that is often used in applications.

Hence, the current theory of games with strategic complementarities is a beautiful merge of monotone comparative statics results and fixpoint results.

However Veinott's increasingness, which is the crucial ingredient in the fixpoint part of the theory, while convenient from a mathematical point of view, does not bear any direct connection with the intuitive idea of complementarity; and this is the main motivation of the present paper.

The intuitive idea of two activities being complements, for example tea and lemon, is that increasing the level of one makes somehow desirable to increase the level of the other. Elaborating on previous work by Calciano (2007), we introduce in this paper notions of increasingness for the joint best reply of a game that are weaker than Veinott's one, and linked more directly with this intuitive idea of complementarity.

We show that games whose joint best reply satisfies our increasingness notions retain the main properties of supermodular games; namely, that a least and a greatest Nash equilibrium point exist and that, furthermore, the Nash set is a nonempty complete lattice. We attain this result by generalizing Veinott's and Zhou's fixpoint theorems. The generalization concerns not only the increasingness notion, but also the space in which the best reply takes values.

To introduce our approach and relate it to the work carried out in Calciano (2007), consider for example a two-player game whose individual strategy spaces are $Z \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$. Let $B : T \to Z$ be the best reply of the corresponding player.

As claimed in Calciano (2007), a good notion of increasingness for B, one that captures our intuition about the meaning of complementarity, is that for all opponent's strategies $x, y \in T$ such that $x \leq y$,

$$\forall a \in B(x), \exists b \in B(y) : a \leq b.$$

Indeed, if after an increase in opponent's strategy from x to y, there is some optimal response $a \in B(x)$ such that every optimal response $b \in B(y)$ is strictly less than a, then we must conclude that the increase in opponent's action has made desirable *to decrease* the action of the player, against our intuitive idea of complementarity.

This notion of increasingness, called upper increasingness in Calciano (2007), once imposed on individual best replies is preserved under cartesian products, and hence inherited by the joint best reply of the game. Furthermore, it is much weaker than Veinott-increasingness. Calciano (2007) has shown that upper increasingness yields to the existence of a greatest Nash equilibrium point, while its dual, called lower increasingness, yields to the existence of a least Nash equilibrium point.

Moreover Calciano (2007) has shown that, if the restrictions of the best reply to certain subsets of the strategy space are all lower increasing, then the Nash set is a nonempty complete lattice. However, these subsets could be in principle infinitely many, and no global condition on the best reply was given to assure that the required property would hold for all the restrictions involved.

This paper fix the problem. A new global notion of increasingness is introduced for the best reply which makes, together with other assumptions, the Nash set be a nonempty complete lattice. This new notion is still substantially weaker than Veinott-increasingness.

As it is the case for Veinott's and Zhou's theorems, also our more general fixpoint results can be proved by using Tarski's fixpoint theorem. Accordingly, our results stand in the same relation to Tarski's fixpoint theorem that Kakutani's fixpoint theorem does to Brouwer's.

However, we prove our results without using Tarski's theorem and, in general, without selection arguments. We investigate the relation of our theorems to increasing-selection arguments in the last part of the paper.

The paper is organized as follow. Section 2 contains the necessary background material and terminology. Section 3 introduces our monotonicity notions. Section 4 presents our fixpoint results. Section 5 discusses increasing selections.

2. Background Material

Let X be a nonempty set. A partial order on X is a reflexive, antisymmetric and transitive binary relation $\leq$ on X. The set X together with $\leq$ is called a partially ordered set, or a poset. Its dual, denoted by X^d, is the set X endowed with the relation $\geq$ defined as $x \geq y \Leftrightarrow y \leq x$.

For any nonempty subsets $T \subseteq S \subseteq X$ of a poset X we denote, when they exist, by $\wedge T$ and $\vee T$ respectively the inf and sup of T in X, and by $\wedge_S T$ and $\vee_S T$ respectively the inf and sup of T in S. We denote, when they exists, by $a \wedge b$ and $a \vee b$ respectively the inf and sup in X of the subset $\{a, b\} \subseteq X$. A subset $S \subseteq X$ has a least (greatest) element if $\wedge S \in S$ (if $\vee S \in S$).

A poset X is a join (meet) lattice if every nonempty and finite subset of X has a supremum (an infimum). Dually, X is a join (meet) lattice if and only if X^d is a meet (join) lattice. X is a lattice if it is both a join and meet lattice.

A poset X is a join-complete (meet-complete) lattice if every nonempty subset of X (not necessarily finite) has a supremum (an infimum). X is join-complete (meet-complete) if and only if its dual is meet-complete (join-complete). A join-complete lattice has a greatest element $\mathbf{1}$, and a meet-complete lattice has a least element $\mathbf{0}$.

X is a complete lattice if it is both a join-complete and a meet-complete lattice; or equivalently if it is join (meet) complete and has a least (greatest) element.

A subset $S \subseteq X$ of a poset X is a (complete) lattice whenever it is so in the induced order; that is, whenever for every nonempty (not necessarily) finite subset $T \subseteq S$, $\vee_S T \in S$ and $\wedge_S T \in S$. Subset S is a (subcomplete) sublattice of X whenever for every nonempty (not necessarily) finite subset $T \subseteq S$, $\vee T \in S$ and $\wedge T \in S$.

A subcomplete sublattice of X is a complete lattice, while the converse is false: a subset of X can be a complete lattice without being not even a sublattice of X.

By a correspondence we mean a point-to-set mapping. A correspondence $F : X \to X$ is said to be nonempty if $F(x)$ is nonempty for every $x \in X$. Whenever we write a definition or theorem about correspondences in this paper, we always assume that these are nonempty.

A correspondence $F : X \to X$ is said to have a least (greatest) element if $F(x)$ has a least (greatest) element for every $x \in X$; that is, if for every $x \in X$ there exists some $0_x \in F(x)$ such that $0_x \leq y$ for every $y \in F(x)$.

A selection from F is any function $f : X \to X$ such that $f(x) \in F(x)$ for every $x \in X$.

3. Increasing Correspondences

Let X be a poset and $F : X \to X$ be a correspondence. We say that F is *upper (lower) increasing* if for every $x, y \in X$, $x \leq y$ implies that for every $a \in F(x)$ (for every $b \in F(y)$), there is some $b \in F(y)$ (some $a \in F(x)$) such that $a \leq b$.

These properties are dual, in the sense that F is upper (lower) increasing on X if and only if it is lower (upper) increasing on X^d. Antoniadou (2007) uses upper and lower increasingness, which she calls - in terms of the corresponding set relations - the pathwise-lower-than relation, to study the comparative statics of consumer problems. Calciano (2007) uses upper and lower increasingness to study fixpoint problems for correspondences defined on posets and lattices.

Let now X be a lattice. We say that F is *Veinott-increasing*[1] if for every $x, y \in X$, $x \leq y$ implies that for every $a \in F(x)$ and every $b \in F(y)$, $a \vee b \in F(y)$ (called join increasingness), and $a \wedge b \in F(x)$ (called meet increasingness).

The two following properties are introduced in this paper for the first time, to be best of our knowledge[2]. We say that F is *strongly upper (lower) increasing* if for every $x, y \in X$, $x \leq y$ implies that for every $a \in F(x)$ and $b \in F(y)$, there is some $q \in F(y)$ (some $p \in F(x)$) such that $a \leq q \leq a \vee b$ (such that $a \wedge b \leq p \leq b$).

[1]Topkis (1978) ascribes this notion of increasingness to Arthur Veinott.

[2]Calciano (2007, Def. 2), misleadingly enough, uses the same names but to denote different properties.

Notice that we do not require neither $a \vee b$ to belong to $F(y)$, nor $a \wedge b$ to belong to $F(x)$.

Clearly, F is strongly upper (lower) increasing on X if and only if it is strongly lower (upper) increasing on X^d.

If F is a function, all the notions of increasingness introduced so far boil down to the function being increasing, by which we mean that if $x \leq y$, then $F(x) \leq F(y)$. Upper (lower) increasingness is implied by strong upper (lower) increasingness, which in turn is implied by join (meet) increasingness. But the converses do not hold.

4. Fixpoint Theorems

In this section we prove fixpoint theorems for increasing correspondences. Our theorems generalize those of Veinott (1992, Ch. 4, Th. 14) and Zhou (1994, Th. 1). Our main result is Theorem 3, where we establish conditions in terms of the increasingness of the correspondence and of existence of extremal elements that make the correspondence fixpoint set be a nonempty complete lattice.

Similarly to Zhou's theorem, also our fixpoint results can be proved by increasing-selection arguments, hence mainly by Tarski's fixpoint theorem. Accordingly, our results stand in the same relation to Tarski's fixpoint theorem that Kakutani's fixpoint theorem does to Brouwer's.

However, we prove our results without using Tarski's theorem and, in general, without selection arguments. Zhou as well does not use Tarki's theorem, while Veinott uses it directly. We will relate our theorems to increasing-selection arguments in the next section.

Associate to a correspondence $F : X \to X$, where X is a poset, the two sets

$$A := \{x \in X : F(x) \cap [x, +\infty) \neq \emptyset\},$$

$$B := \{x \in X : F(x) \cap (-\infty, x] \neq \emptyset\}.$$

A is the set of elements x of X at which F stays above the diagonal, by which we mean that some element of $F(x)$ is greater than or equal to x. Set B is the dual of A. The fixpoint set of F is exactly the intersection of A and B.

We report here, for completeness and ease of comparison with our results, both Tarski's and Veinott-Zhou's fixpoint theorems[3].

Theorem 0 (Tarski 1955, Th. 1): *Let X be a complete lattice and $f : X \to X$ be an increasing function. (a) $\vee A$ and $\wedge B$ are, respectively, the greatest and least fixpoint of f; and (b) the fixpoint set of f is a complete lattice.*

Theorem 1 (Veinott 1992, Ch. 4, Th. 14. Zhou 1994, Th. 1): *Let X be a complete lattice and $F : X \to X$ be a correspondence. If F is Veinott-increasing*

[3]Veinott's fixpoint theorem is indeed cast in a broader context. For a correspondence $F : X \times T \to X$, where X is a complete lattice, T a poset, and F is Veinott-increasing on $X \times T$ and subcomplete-sublattice-valued for each t, Veinott proves that the sets of selections and increasing selections from the associated fixpoint correspondence $E : T \to X$ are nonempty complete lattices with common least and greatest elements. For the special case of $T = \{t\}$, this amounts to the statement of Theorem 1 upon recognizing that, in such a case, the sets of selections and increasing selections from $E : T \to X$ coincide.

and $F(x)$ is a subcomplete sublattice of X for every $x \in X$, then (a) $\vee A$ and $\wedge B$ are, respectively, the greatest and least fixpoint of F; and (b) the fixpoint set of F is a complete lattice.

We prove now, in its general form, a simple result which will be used in proving the theorems which follow. Zhou, in his proof of Theorem 1, prove the result separately for the special cases of A and of arbitrary subsets of the fixpoint set of F, while Veinott (1992, Ch. 4, Th. 11) proves the result for the special case of increasing functions.

LEMMA 1: *Let X be a poset and $F : X \to X$ be a correspondence. (i) If F is upper increasing and has a greatest element, then for every nonempty subset $S \subseteq A$, if $\vee S$ exists then it belongs to A. (ii) If F is lower increasing and has a least element, then for every nonempty subset $T \subseteq B$, if $\wedge T$ exists then it belongs to B.*

PROOF: We prove (i). Point (ii) then follows by duality. Pick any $S \subseteq A$ and any $x \in S$. By the definition of A, there is $y \in F(x)$ such that $x \leq y$. Since $x \leq \vee S$, by upper increasingness of F for such $y \in F(x)$ there exists $z \in F(\vee S)$ such that $y \leq z$. Let $1 \in F(\vee S)$ be the greatest element of $F(\vee S)$. Clearly $x \leq y \leq z \leq 1$. Hence 1 majorizes S and so $\vee S \leq 1$. Thus $\vee S \in A$. Q.E.D.

By Lemma 1, when X is a complete lattice A is a join-subcomplete sublattice of X and, having a least element (the least element of X), it is also a meet-complete lattice. Of course, A it is not necessarily a subcomplete sublattice of X. Dually, B is a meet-subcomplete and join-complete lattice [4].

Theorem 2 below generalizes point (a) of Theorem 1 in two respects. First, it requires the correspondence F to be upper (lower) increasing, instead of Veinott increasing. Second, it requires for any $x \in X$ that $F(x)$ has a greatest (least) element, instead of requiring it to be a subcomplete sublattice of X.

We state Theorem 2 in the context of posets. This is to underscore the fact that, in point (a) of Theorems 0 and 1, the completeness of lattice X plays the only role of making the theorems non-vacuous, but is not a crucial ingredient of the proofs. In particular, assuming that X is a complete lattice guarantees that A and B are nonempty and that $\vee A$ and $\wedge B$ exist[5].

THEOREM 2: *Let X be a poset and $F : X \to X$ be a correspondence. (i) If F is upper increasing and has a greatest element then $\vee A$, whenever it exists, is the greatest fixpoint of F. (ii) If F is lower increasing and has a least element then $\wedge B$, whenever it exists, is the least fixpoint of F.*

PROOF: Point (i). Let $1 \in F(\vee A)$ be the greatest element of $F(\vee A)$. By Lemma 1, $\vee A \in A$ and so there exists $y \in F(\vee A)$ such that $\vee A \leq y$. Thus, for every $x \in A$, $x \leq \vee A \leq y \leq 1$, were $1 \in F(\vee A)$ is the greatest element of $F(\vee A)$. By upper

[4]Notice that the intersection of A and B, that is, the fixpoint set of F, needs not to be a complete lattice (indeed, not even a lattice). Otherwise we would not need Theorem 3 in this paper. Take for example any unordered a, b in X and set $A = \{\mathbf{0}, a, b, a \vee b\}$ and $B = \{\mathbf{1}, a, b, a \wedge b\}$. The intersection $A \cap B$ is not a lattice.

[5]Theorem 2 has been already proved in Calciano (2007, Th. 1), but without recurring explicitly to Lemma 2.

increasingness of F, from $\vee A \leq 1$ it follows that there is some $y \in F(1)$ such that $1 \leq y$. Thus $1 \in A$, and so $1 \leq \vee A$. Hence $1 = \vee A \in F(\vee A)$. Thus $\vee A$ is a fixpoint of F. It is the greatest fixpoint since the fixpoint set of F is a subset of A.

Point (ii). Consider the dual poset X^d of X. Since F is lower increasing on X, it is upper increasing on X^d, and since $F(x)$ has a least element for every $x \in X$, it has a greatest element for every $x \in X^d$. Furthermore, $\wedge B$ in X coincides with $\vee A$ in X^d. Hence, by applying to X^d the result obtained in (i) for X, we have done. Q.E.D.

Theorem 3 below generalizes point (b) of Theorem 1 in a similar way as Theorem 2 generalized point (a) of Theorem 1. Namely, Theorem 3 assumes a form of increasingness for F weaker than Veinott-increasingness, and requires existence of certain extremal elements for F instead of requiring it to be subcomplete-sublattice-valued.

Some new notation is needed. Let X be a poset with a greatest element $\mathbf{1}$ and $F : X \to X$ be a correspondence. For a fixed $h \in X$, define the correspondence $F_h : [h, \mathbf{1}] \to [h, \mathbf{1}]$ as

$$F_h(x) = F(x) \cap [h, \mathbf{1}].$$

Notice that this correspondence may well be empty.

Theorem 3: *Let X be a complete lattice and $F : X \to X$ be a correspondence. If F has a greatest element, is upper increasing, is strongly lower increasing, and if for every $h \in A$ the correspondence F_h has a least element whenever nonempty, then the fixpoint set of F is a nonempty complete lattice.*

Remark. With the assumptions of Veinott-Zhou's theorem F would be subcomplete-sublattice-valued and so F_h, whenever nonempty, would be subcomplete-sublattice-valued as well, and hence would have both a least and a greatest element.

Proof of Theorem 3: Call E the fixpoint set of F. Since X is complete, then B is nonempty and $\wedge B$ exists. Observe that the correspondence $F_{\mathbf{0}}$ coincides with F, which is assumed to be nonempty. So F has a least element and, being strongly lower increasing, it is lower increasing. Thus by point (ii) of Theorem 1, E is nonempty and $\wedge B$ is its least element.

We now show that E is a join-complete lattice. Hence, having a least element, it is a complete lattice. Pick any nonempty subset $T \subseteq E$. We want to show that $\vee_E T$ exists, which is equivalent to show that the set $E \cap [\vee T, \mathbf{1}]$ has a least element. To prove the latter, consider the correspondence $F_{\vee T}$. Observe that the fixpoint set of $F_{\vee T}$ is exactly $E \cap [\vee T, \mathbf{1}]$. We show that $F_{\vee T}$ has a least fixpoint.

Claim 1. The correspondence $F_{\vee T}$ is nonempty. Indeed, we prove a little more, namely that for every $h \in A$, the correspondence F_h is nonempty. Then, because by Lemma 1 we know that $\vee T \in A$, Claim 1 is proved. Indeed, pick any $h \in A$ and any $x \in [h, \mathbf{1}]$. Since $h \in A$ there is some $y \in F(h)$ such that $h \leq y$, and since $h \leq x$ and F is upper increasing, for such $y \in F(h)$ there is some $z \in F(x)$ such that $y \leq z$. Hence $z \in F(x) \cap [h, \mathbf{1}]$. □

Claim 2. $F_{\vee T}$ is lower increasing. Again, we prove a little more, namely that for every $h \in X$ such that F_h is nonempty, F_h is lower increasing. Pick indeed any $x, y \in [h, \mathbf{1}]$ such that $x \leq y$. Pick any $b \in F_h(y)$ and any $a \in F_h(x)$. If $a \leq b$ we are done. If a is unordered with b, then by strong lower increasingness of F there is some $p \in F(x)$ such that $a \wedge b \leq p \leq b$. Since h minorizes the set $\{a, b\}$, then $h \leq a \wedge b$, and so p is in $F_h(x)$. If $b < a$, by strong lower increasingness of F there is some $p \in F(x)$ such that $b = a \wedge b \leq p \leq b$, and since $h \leq a \wedge b$, then b is in $F_h(x)$. Hence F_h is lower increasing. □

By Claim 1, the set $F_{\vee T}(\mathbf{1}) = F(\mathbf{1}) \cap [\vee T, \mathbf{1}]$ is nonempty, and since $F(\mathbf{1}) \cap [\vee T, \mathbf{1}] \equiv F_{\vee T}(\mathbf{1}) \cap [\vee T, \mathbf{1}]$, then $\mathbf{1}$ belongs to the set

$$B_T := \{x \in [\vee T, \mathbf{1}] : F_{\vee T}(x) \cap [\vee T, x] \neq \emptyset\},$$

which is then nonempty. Furthermore $\wedge B_T$ exists by completeness of X, and it clearly belongs to $[\vee T, \mathbf{1}]$ because $\vee T$ minorizes B_T. Finally, the correspondence $F_{\vee T}$ has a least element by assumption and hence satisfies all the assumptions of point (ii) of Theorem 1. As a result, for the least element $0 \in F_{\vee T}(\wedge B_T)$, $0 = \wedge B_T \in F_{\vee T}(\wedge B_T)$. Hence $\wedge B_T$ is the least fixpoint of $F_{\vee T}$. Since T was arbitrary, E is a join-complete lattice and the theorem is proved. Q.E.D.

5. Increasing Selections

We have proved our Theorems 2 and 3 without recurring to increasing-selection arguments, hence without using Tarski's fixpoint theorem. It is natural, then, to pose the following question: there would be any added-value in using increasing-selection arguments instead, or in other words, are we assuming too much in order to prove Theorems 2 and 3 without recurring to Tarski's result?

We address these questions with respect to the increasingness assumptions for the correspondence F that we have done in Theorems 2 and 3. We consider these assumptions for F 'minimal' if they are equivalent to the increasingness of some selection f from F, where f - upon being increasing - allows to prove the theorem by applying Tarski's fixpoint result.

In Theorem 4 below we show that, keeping fixed all the other assumptions of Theorem 2, assuming that F is lower increasing is equivalent as assuming that a certain selection from F is increasing, this selection having the same least fixpoint as F (an analogous result holds for the greatest fixpoint of F).

In Theorem 5 we show that, keeping fixed all the other assumptions of Theorem 3, assuming that F is strongly lower increasing implies that a certain selection from F is increasing, this selection having the same fixpoint set of F.

Unfortunately, in Theorem 5 the equivalence does not hold. Hence the notion of increasingness for F is minimal in Theorem 2, but it is not in Theorem 3.

5.1. **Extremal Fixpoints.** If one is interested only in the existence of extremal fixed point, then the proof of Theorem 2 suggests how to pick selections from F that have the same extremal fixpoints as F. In particular, if F has a least (greatest) element and a least (greatest) fixpoint, then by Tarski's theorem the least (greatest) element, whenever it is an increasing function, has the same least (greatest) fixpoint as F, and so represents a proper selection in this context.

The next theorem shows that, assuming F lower (upper) increasing, is equivalent as assuming that the least (greatest) element is an increasing selection from F, and hence it is the weakest form of increasingness for correspondences that implies this property for the least (greatest) element.

In this sense, we cannot relax the assumption of lower (upper) increasingness in Theorem 2 to show existence of extremal fixed points. We are using, given the other assumptions of Theorem 2, the weakest form of increasingness for F in order to get the desired result.

THEOREM 4: *Let X be a poset and $F : X \to X$ be a correspondence with a least (greatest) element. F is lower (upper) increasing if and only if the least (greatest) element of F is an increasing selection from F.*

PROOF: We prove the theorem for the least element. The case for the greatest element follows then by duality. For z in X, let 0_z denote the least element of $F(z)$. Pick x, y in X with $x \leq y$. Assume that the least element of F is an increasing selection. Hence, for any b in $F(y)$, $0_x \leq 0_y \leq b$, hence F is lower increasing. In the other direction, if F is lower increasing, given 0_y there is some a in $F(x)$ such that $0_x \leq a \leq 0_y$. Q.E.D.

5.2. **Structure of the Fixpoint Set.** Theorem 4 can be used also in the context of Theorem 3; that is, when one is indeed interested in the whole order-structure of the fixpoint set. In fact, the proof of Theorem 3 makes evident that what matters, in order for the (nonempty) fixpoint set of F to be a complete lattice, is that for each subset T of the fixpoint set of F, the correspondence $F_{\vee T}$ be lower increasing. In the presence of Theorem 4, this is equivalent to assuming that, for every T, the least element of $F_{\vee T}$ is an increasing selection. We have indeed introduced strong lower increasingness to provide for a global condition on F which guarantees that each correspondence $F_{\vee T}$ is lower increasing.

However, there is a certain flaw in the selection-argument in Theorem 4: that we need an ad-hoc selection for each fixpoint of F. And this would be even more the case in the context of Theorem 3.

We introduce now a selection from F that works for every fixpoint of F; that is, which has exactly the same fixpoint set of F. We show that this selection is increasing if F is strongly lower increasing. The vice-versa however is not true, that is, our selection can be increasing without F being strongly lower increasing. In such sense, there may be some room to relax the assumption of strong lower increasingness for F in Theorem 3.

Consider the set A and the function $l : A \to X$ defined as

$$l(x) = min\{F(x) \cap [x, \mathbf{1}]\}.$$

Under the assumptions of Theorem 3, in particular that F has a greatest element and is upper increasing and that, for each $h \in A$, the correspondence F_h has a least element, the function l is indeed well-defined (see Claim 1 of Theorem 3), in the sense that the min does exist. It is furthermore evident that the fixpoint set of l is exactly that of F. Function l is a selection from the restriction of F to the set A. Notice that when we study the fixpoint set of F, the behavior of F outside set A is indeed irrelevant.

In order to apply Tarski theorem to l, we would need A to be a complete lattice, l to map A into itself, and l to be increasing. All these properties are guaranteed by the assumptions of Theorem 3.

In particular, if X is a complete lattice and F is upper increasing and has a greatest element, Lemma 1 guarantees that A is a complete lattice. We address the other two conditions below. Add to the assumptions stated to prove the next results those, aforementioned, that make selection l well-defined.

LEMMA 2: *If F is upper increasing, then l maps A into itself.*

PROOF: Pick any $h \in A$. We want to show that the least element $l(h) := 0_h$ is in A. By the definition of l, we know that $0_h \in F(h)$ and $h \leq 0_h$. But then, since F is upper increasing, for such $0_h \in F(h)$ there exists some $x \in F(0_h)$ with $0_h \leq x$, which proves the claim. Q.E.D.

THEOREM 5: *If F is strongly lower increasing, then l is increasing.*

PROOF: Pick any $x, y \in A$ with $x \leq y$. Set $l(x) := 0_x$ and $l(y) = 0_y$. Pick any $a \in F(x) \cap [x, \mathbf{1}]$ and consider 0_y. By strong lower increasingness of F, there is some $p \in F(x)$ such that $0_y \wedge a \leq p \leq 0_y$. But $x \leq a$ and $y \leq 0_y$, and so the inequality $x \leq y$ implies that x minorizes $\{a, 0_y\}$. Hence $x \leq 0_y \wedge a$, which implies that $p \in F(x) \cap [x, \mathbf{1}]$. Thus $0_x \leq p$ and hence $0_x \leq 0_y$. Q.E.D.

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