Intermediate inequality and welfare

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Abstract. We introduce a new centrist or intermediate inequality concept, between the usual relative and absolute notions, which is shown to be a variant of the α-ray invariant inequality measures in Pfingsten and Seidl (1997). We say that distributions $\mathbf{x}$ and $\mathbf{y}$ have the same $(\mathbf{x}, \pi)$-inequality if the total income difference between them is allocated among the individuals as follows: $100\pi\%$ preserving income shares in $\mathbf{x}$, and $100(1-\pi)\%$ in equal absolute amounts. This notion can be made as operational as current standard methods in Shorrocks (1983).

I. Introduction

Most welfare analysis implicitly assume that social or aggregate welfare can be expressed in terms of only two features of the income distribution: the mean, and a notion of vertical inequality. In this context, we are often interested in evaluation methods which require the minimum possible number of value judgments. In particular, we are interested in unambiguous (although incomplete) rankings according to which social welfare increases only if efficiency and distribution both improve.

Dutta and Esteban (1991) show that for this procedure to be justified, among other things we need to specify the type of mean-invariance property we want our inequality indices to satisfy (see also Ebert 1987; Weinhardt 1993). Starting from a given income distribution **x**, two polar cases have been extensively studied so far: a preference for efficiency along rays through **x** from the origin, maintaining constant a relative notion of inequality; and a preference for efficiency along rays through **x** parallel to the line of equality, maintaining constant an absolute notion of inequality. The merit of Shorrocks's (1983) contribution is that he develops operational methods to find out whether one distribution is unambiguously better than another according to all SEFs in wide classes of admissible functions in the relative and the absolute case (for the absolute case, see also Moyes 1987).

In Del Río and Ruiz-Castillo (1996) we have found with this methodology that the 1990 1991 income distribution has less relative inequality but more absolute inequality than the 1980 1981 comparable distribution[1]. The following empirical question cannot be answered with present tools: is the 1990 1991 distribution "barely better" than the 1980 1981 distribution from the relative point of view, and consequently "far away" from it from the absolute one; or is "so much better" from the relative perspective that is "nearly equivalent" to it from the absolute point of view?

To approach this question, we suggest to consider the space of "centrist" or intermediate views on inequality, between the "rightist" (relative) or "leftist" (absolute) cases in Kolm (1976a, b)'s value laden terminology. Informally, in the situation of the example we are interested in knowing how far we can go to the left of the political spectrum within the centrist space, and still claim that the 1990 1991 distribution is less unequal than the 1980 1981 distribution.

To develop this idea we must start by specifying an appropriate notion of intermediate inequality. One possibility is to use Kolm's (1976a, b) suggestion or the single parameter μ-inequality concept proposed by Bossert and Pfingsten (1990). Unfortunately, as pointed out by Pfingsten and Seidl (1997) (or PS for short), both share a serious disadvantage: they approach the rightist position when aggregate income rises, even if the income distribution becomes more unequal according to some inequality measure[2].

Another possibility is to use the ray-invariance concept suggested by PS, which gives rise to a wide class of $\boldsymbol{\alpha}$-invariant inequality measures free from this flaw. In this paper we introduce a new class of inequality measures which is a subset of the $\boldsymbol{\alpha}$-invariant class. We call it $(\mathbf{x}, \pi)$-inequality to stress the dependence on an initial income distribution **x**, as well as on a parameter value π in the unit interval. Like all other notions, it builds upon a monotonicity property conveying the proper division of extra income to leave inequality intact. We say that **x** and **y** have the same $(\mathbf{x}, \pi)$-inequality if the total income difference between the two distributions is allocated among the indi-

[1] Except for Portugal, who has gone through similar political and economic reforms since the mid 1970's, this is a different trend from most OECD countries. For Portugal (see Gouveia and Tavares 1995, Rodrigues 1993), and for the international experience (see, for instance, Atkinson *et al.* 1995, Gottschalk and Smeeding 1997).

[2] For other shortcomings of Kolm's (1976) approach (see Bossert and Pfigsten 1990).

viduals as follows: $\pi 100\%$ preserving income shares in $\mathbf{x}$, and $(1-\pi)100\%$ in equal absolute amounts.

Our reason for defending the new notion is twofold. It has a clear normative interpretation, and it can be made operational in the following way. Given an initial distribution $\mathbf{x}$ and a value of π, we develop empirical methods to test whether any distribution $\mathbf{y}$ has greater social welfare than $\mathbf{x}$ according to all SEFs in a class characterized by the usual assumptions plus a monotonicity property compatible with the $(\mathbf{x}, \pi)$-inequality concept. Suppose now we want to analyze the Spanish situation during the 1980's, an interesting period in which a socialist party occupied power by democratic means for the first time in 40 years. The problem is that we do not have any *a priori* reasons to determine which centrist attitudes, or which range of π values, we should adopt to compare the two distributions. Our strategy is to allow the data to reveal this for us: we estimate the range of π values for which the 1990 1991 distribution is non comparable to the 1980 1981 distribution. In this way, we learn for what type of centrist attitudes there has been a reduction or an increase (to the "right" or the "left" of that range of π values, respectively) in inequality.

The rest of the paper is organized in four sections. Section II presents our notion of intermediate inequality within the larger class of $\boldsymbol{\alpha}$-ray invariant inequality measures proposed by PS. Following up on ideas put forth in Chakravarty (1988), Sect. III describes how our measure can be made operational by using Lorenz comparisons. Section IV concludes. Proofs are included in an Appendix.

II. Ray invariant inequality concepts

II.1. Notation

Let $\mathbf{x} = (x^1, \ldots, x^H) \in R^H_{++}$, $2 \leq H < \infty$, denote an income distribution with $x^1 \leq x^2 \leq \ldots \leq x^H$. Then D denotes the set of all possible ordered income distributions in R^H_{++}, and S the H-dimensional simplex. For any $\mathbf{x} \in D$, let $\mathbf{v}_x = (v^1, \ldots, v^H) \in S$ be the vector of income shares with $v^h = x^h/X$, where $X = \Sigma_h x^h$ is the aggregate income. $\mathbf{1}$ denotes a row vector whose components are all ones, while $\mathbf{e}$ denotes the vector $(1/H)\ \mathbf{1}$ in S. For any two vectors $\mathbf{x}, \mathbf{y} \in D$, let $\mathbf{v_x} L \mathbf{v_y}$ denote weak Lorenz dominance.

Any real valued function I defined on D satisfying continuity, S-convexity and population replication invariance is called an income inequality measure. $I(\,.\,)$ satisfies scale invariance when $I(\mathbf{x}) = I(\lambda\mathbf{x})$ for all $\mathbf{x} \in D$ and for all $\lambda > 0$. $I(\,.\,)$ satisfies translation invariance when $I(\mathbf{x}) = I(\mathbf{x} + \eta\mathbf{1})$ for all $\mathbf{x} \in D$ and for all $\eta \in R$ such that $(\mathbf{x} + \eta\mathbf{1}) \in D$. If an inequality measure satisfies scale or translation invariance it is called a relative or an absolute inequality measure, respectively.

II.2. Centrist inequality attitudes

It appears to be the case that, for technical or other reasons, the vast majority of specialists prefer the relative notion. However, first Dalton (1920) and later

Kolm (1976a, b) observe that many people perceive equiproportional increases in all incomes to increase, and equal incremental increases in all incomes to decrease income inequality. He called such an attitude centrist. The conceptual interest of such views has been enhanced by recent reports on questionnaires which indicate that people are by no means unanimous in their choice between relative, absolute and other intermediate or centrist notions of inequality[3]. As indicated in the conclusions to Ballano and Ruiz-Castillo (1993), if because of the influence of political attitudes to redistribution or other unknown concerns people in large numbers declare to favor absolute or intermediate inequality concepts, then perhaps it is time to change the consensus and use more often other types of inequality measures. This is indeed what Kolm himself, as well as Bossert, Pfingsten and Seidl, for example, recommends.

As pointed out in PS, a centrist income inequality attitude can be modeled in various ways. For all $\mathbf{x} \in D$, there exists a set of income distributions $E(\mathbf{x})$ such that, first, all $\mathbf{y} \in E(\mathbf{x})$ are perceived to be as equally distributed as $\mathbf{x}$, second, for $\lambda\mathbf{x} > \mathbf{x}$ and $(\mathbf{x} + \eta\mathbf{1}) > \mathbf{x}$ all $y \in E(\mathbf{x})$ are perceived to be more equally distributed than $\lambda\mathbf{x}$ and less equally distributed than $(\mathbf{x} + \eta\mathbf{1})$, and third, for $\mathbf{x} > \lambda\mathbf{x}$ and $\mathbf{x} > (\mathbf{x} + \eta\mathbf{1})$ all $\mathbf{y} \in E(\mathbf{x})$ are perceived to be less equally distributed than $\lambda\mathbf{x}$ and more equally distributed than $(\mathbf{x} + \eta\mathbf{1})$. Given such a centrist inequality attitude, the question arises whether there are E-invariant income inequality measures, i.e., inequality measures $I(\,.\,)$ such that $I(\mathbf{x}) = I(\mathbf{y})$ for all $y \in E(\mathbf{x})$.

As PS indicate, a straightforward case is to assume $E(\mathbf{x})$ to be composed of rays through $\mathbf{x}$[4]. For any $\boldsymbol{\alpha} \in S$, the set $E_\alpha(\mathbf{x})$ of $\boldsymbol{\alpha}$-rays through $\mathbf{x}$ is defined by

$$E_\alpha(\mathbf{x}) = \{\mathbf{y} \in D : \mathbf{y} = \mathbf{x} + \tau\boldsymbol{\alpha}, \tau \in R\}.$$

In accordance with centrist ideas, PS require $\boldsymbol{\alpha}$-rays to be restricted in two ways: first, they Lorenz dominate the original distribution; and, second, they are more unequally distributed than translation invariance would require. Thus, given an income distribution $\mathbf{x} \in D$, define the set $\Omega(\mathbf{x})$ of value judgments (in income share form) which provide a reduction in relative inequality

[3] For example (see Amiel and Cowell 1992, Harrison and Seidl 1994, Seidl and Theilen 1994). In the Spanish case, Ballano and Ruiz Castillo (1993) found that, for the sub sample that showed an acceptable degree of consistency over the questionnaire, only 31 percent supported a relative view of inequality, 24% supported an absolute view, and 27% an intermediate notion (the rest supported other extreme views).

[4] As an alternative, consider the Krtscha (1994) intermediate inequality concept in which, given an initial income distribution $x \in D$, any extra income M should be allo cated among the individuals according to the so called "fair compromise concept": the first extra dollar of income should be distributed so that 50 cents go to the individuals in proportion to the initial income shares, and 50 cents in equal absolute amounts; starting from the new distribution with aggregate income equal to $X + 1$, the second extra dollar of income should be allocated in the same manner, and so on. Notice that, according to this notion, the set of income distributions with the same intermediate inequality as x is no longer a ray but a parabola.

but an increase in absolute inequality relative to $\mathbf{x}$:

$$\Omega(\mathbf{x}) = \{\boldsymbol{\alpha} \in S : \mathbf{e}L\boldsymbol{\alpha}L\mathbf{v}_x\}.$$

In other words, given $\mathbf{x} \in D$ and $\alpha \in \Omega(\mathbf{x})$, every $\mathbf{y} \in E_\alpha(\mathbf{x})$ is derived from $\mathbf{x}$ by superimposing a "more equal" income distribution according to the Lorenz criterion.

To understand in which sense $\mathbf{x}$ and $\boldsymbol{\alpha}$ co-determine the domain of $\boldsymbol{\alpha}$-ray invariant functions, define the set $\Gamma(\boldsymbol{\alpha})$ of income distributions for which $\boldsymbol{\alpha} \in S$ can represent a centrist inequality attitude:

$$\Gamma(\boldsymbol{\alpha}) = \{\mathbf{x} \in D : \boldsymbol{\alpha}L\mathbf{v}_\mathbf{x}\}.$$

Clearly, if $\mathbf{x} \in D$ and $\boldsymbol{\alpha} \in S$ but $\boldsymbol{\alpha} \notin \Omega(\mathbf{x})$ or $\mathbf{x} \notin \Gamma(\boldsymbol{\alpha})$, then the pair $(\mathbf{x}, \boldsymbol{\alpha})$ does not give rise to a centrist inequality relation. Accordingly, a real valued function $F_\alpha : D \to R$ is called $\boldsymbol{\alpha}$-ray invariant in $\Gamma(\boldsymbol{\alpha})$, if and only if for each $\mathbf{x} \in \Gamma(\boldsymbol{\alpha})$,

$$F_\alpha(\mathbf{x}) = F_\alpha(\mathbf{y}) \quad \text{for all } \mathbf{y} \in E_\alpha(\mathbf{x}).$$

Given an $\boldsymbol{\alpha}$-ray invariant function $I_\alpha(\,.\,)$, we say that it is an $\boldsymbol{\alpha}$-ray invariant inequality measure if, in addition, it is continuos, *S*-convex and satisfies the population replication axiom.

In general, $\boldsymbol{\alpha}$-ray invariance requires an inequality measure not to change provided any income change is distributed according to the value judgment represented by the relative pattern $\boldsymbol{\alpha}$. Thus, let $\mathbf{x} = (200, 800)$, so that $\mathbf{v}_\mathbf{x} = (0.2, 0.8)$, and, for example, let $\boldsymbol{\alpha} = (0.4, 0.6)$ so that $\mathbf{e}L\boldsymbol{\alpha}L\mathbf{v}_\mathbf{x}$. Then

$$E_\alpha(\mathbf{x}) = \{\mathbf{y} \in R^2_{++} : \mathbf{y} = (200, 800) + \tau(0.4, 0.6), \tau \in \mathbf{R}\}.$$

Therefore, if we have 100 units of extra income to allocate, to preserve such $\boldsymbol{\alpha}$-ray invariance we must add up the vector $(40, 60)$ to $\mathbf{x}$ to reach $(240, 860)$.

II.3. A new concept of intermediate inequality

In principle, given two distributions $\mathbf{x}, \mathbf{y} \in D$, we could search for τ^* and $\boldsymbol{\alpha}^*$ so that $\mathbf{y}$ is $\boldsymbol{\alpha}^*$-ray invariant inequality equivalent to $\mathbf{x}$, that is, $\mathbf{y} = \mathbf{x} + \tau^*\boldsymbol{\alpha}^*$. In practice, τ^* is given by the total income difference between the two distributions under comparison. In what follows, we assume without loss of generality that $\tau^* \geq 0$. On the other hand, if the two distributions have the same number of individuals, we can always compute $\boldsymbol{\alpha}^* = (\mathbf{u} - \mathbf{t})/\tau^*$[5]. The problem is that, in general, the $\boldsymbol{\alpha}^*$ vector will not have a convenient interpretation. For instance, in the empirical illustration with Spanish data we would have a 24,000-dimensional $\boldsymbol{\alpha}^*$ vector. It would be hard to interpret what is meant by people having more or less demanding inequality views than those represented by such $\boldsymbol{\alpha}^*$ vector.

[5] Otherwise, we can substitute the original distributions by their centiles, for example, and apply the previous expression.

We concentrate our attention on $\boldsymbol{\alpha}$-ray invariant inequality measures which can receive a clear normative interpretation. For that purpose, we start from an initial income distribution $\mathbf{x} \in D$, and a value of $\pi \in [0, 1]$. Then we consider rays through $\mathbf{y} \in D$ constructed so that $\pi 100\%$ of any extra income is allocated to individuals according to income shares in $\mathbf{x}$, and $(1 - \pi)100\%$ in equal absolute amounts. That is, we define

$$P_{(\mathbf{x},\pi)}(\mathbf{y}) = \{\mathbf{z} \in D : \mathbf{z} = \mathbf{y} + \tau(\pi \mathbf{v_x} + (1 - \pi)\mathbf{e}), \tau \in R\}.$$

Clearly, if we let $\boldsymbol{\alpha} = \pi \mathbf{v_x} + (1 - \pi)\mathbf{e}$, then $P_{(\mathbf{x},\pi)}(\mathbf{y}) = E_\alpha(\mathbf{y})$. Correspondingly, we define the subset $\Gamma'(\boldsymbol{\alpha})$ of income distributions for which $\boldsymbol{\alpha}$ can represent a centrist inequality attitude in the following sense:

$$\Gamma'(\boldsymbol{\alpha}) = \{\mathbf{x} \in D : \pi' \mathbf{v_x} + (1 - \pi')\mathbf{e} = \boldsymbol{\alpha} \text{ for some } \pi \in [0, 1]\}.$$

Clearly, for any $\mathbf{x} \in \Gamma'(\boldsymbol{\alpha}), \boldsymbol{\alpha} L \mathbf{v_x}$. This means that $\Gamma'(\boldsymbol{\alpha}) \subset \Gamma(\boldsymbol{\alpha})$. Then we say that a real valued function $I_{(\mathbf{x},\pi)} : D \to R$ is a $(\mathbf{x}, \pi)$-inequality measure in $\Gamma'(\boldsymbol{\alpha})$, if and only if it is the restriction to $\Gamma'(\boldsymbol{\alpha})$ of the I_α-ray invariant inequality measure. In this case, of course,

$$I_{(\mathbf{x},\pi)}(\mathbf{y}) = I_{(\mathbf{x},\pi)}(\mathbf{z}) \quad \text{for all } \mathbf{z} \in P_{(\mathbf{x},\pi)}(\mathbf{y}).$$

Alternatively, we have that

$$I_\alpha(\mathbf{y}) = I_\alpha(\mathbf{z}) \quad \text{for all } \mathbf{z} \in E_\alpha(\mathbf{y})^6.$$

In general, the set $\Gamma'(\boldsymbol{\alpha})$ is clearly non-empty[7], so that the $(\mathbf{x}, \pi)$-inequality measures are well defined. This means that they enjoy all the properties discussed by PS for $\boldsymbol{\alpha}$-ray invariant inequality measures.

If we let $\mathbf{x} = (200, 800)$ as before and $\pi = 0.5$, then 50% of all income differences are allocated according to the income shares vector $(1/5, 4/5)$, and 50% in equal absolute amounts according to the proportions $(1/2, 1/2)$. Thus, the $(\mathbf{x}, \pi)$-ray of income distributions through $\mathbf{x}$ is given by

$$P_{(\mathbf{x},\pi)}(\mathbf{x}) = \{\mathbf{y} \in R : \mathbf{y} = \mathbf{x} + \tau(7/20, 13/20), \tau \in R\}.$$

Hence, 100 extra units of income are allocated as $(35, 65)$ to reach the new distribution $(235, 865)$ with the same $(\mathbf{x}, \pi)$-inequality. Informally, we may say that a value of $\pi = 0.9$ reflects a center-right attitude, while a value of $\pi = 0.4$ reflects a center-left perception of inequality. The reason, of course, is that according to the first view inequality is maintained if only 10% of any excess income is distributed according to the more demanding absolute criterion, while the second requires 60% to be allocated that way. On the other hand,

[6] In the 2 dimensional case, all distributions y in $\Gamma(\alpha)$ have the property that α $\pi' v_y + (1 \quad \pi')e$ for some $\pi' \in [0, 1]$. This means that $\Gamma'(\alpha)$ and $\Gamma(\alpha)$ coincide, in which case the (x, π) inequality and the α ray invariant inequality concepts also coincide. In general, of course, the set $\Gamma(\alpha)$ is much richer than $\Gamma'(\alpha)$.

[7] Similarly, the subset $\Omega'(x)$ of $\Omega(x)$, defined by $\Omega'(x) \quad \{\alpha \in S : \alpha \quad \pi' v_x + (1 \quad \pi')e$ for some $\pi' \in [0, 1]\}$, is also non empty.

notice that if $\pi = 1$, $(\mathbf{x}, \pi)$-inequality becomes the relative view, whereas $\pi = 0$ leads to the absolute view.

The dependence of centrist or intermediate inequality measures on an initial situation deserves to be emphasized. Some readers may find this a disadvantage because a certain value judgment is not applicable in all situations. However, we agree with PS when they assert that "... this is indeed an attractive feature ... The meaning of "centrist" need not be decided universally, but can be made contingent on the situations we know and hence can evaluate well". Nevertheless, the way $\boldsymbol{\alpha}$-inequality and (x, π)-inequality depend on the initial situation present some subtle differences worth being discussed.

As we know, $\boldsymbol{\alpha_0} \in S$ and $\mathbf{x} \in D$ can only give rise to a centrist inequality relation if $\mathbf{x} \in \Gamma(\boldsymbol{\alpha_0})$ and $\boldsymbol{\alpha_0} \in \Omega(\mathbf{x})$. Given $\mathbf{y} \in \Gamma(\boldsymbol{\alpha_0})$, if $\mathbf{y} \in E_{\alpha_0}(\mathbf{x})$ then $I_{\alpha_0}(\mathbf{y}) = I_{\alpha_0}(\mathbf{x})$. Otherwise, i.e. if $\mathbf{y} \notin E_{\alpha\mathbf{0}}(\mathbf{x})$, then we can only say that $\mathbf{x}$ and $\mathbf{y}$ do not have the same $\boldsymbol{\alpha}_0$-inequality. In our case, given $\mathbf{x_0} \in D$ and $\pi_0 \in [0, 1]$, $\boldsymbol{\alpha_0} = \pi_0 \mathbf{v}_{\mathbf{x_0}} + (1 - \pi_0)\mathbf{e}$ is determined. Consider now two income distributions $\mathbf{x}, \mathbf{y}\Gamma'(\boldsymbol{\alpha_0})$. Then there exists some $\pi, \pi' \in [0, 1]$ such that $\boldsymbol{\alpha_0} = \pi\mathbf{v}_x + (1 - \pi)\mathbf{e}$ and $\boldsymbol{\alpha_0} = \pi'\mathbf{v}_\mathbf{y} + (1 - \pi')\mathbf{e}$. This means that $(\mathbf{x}, \pi)$-inequality and $(\mathbf{y}, \pi')$-inequality coincides with $(\mathbf{x_0}, \pi_0)$-inequality. The interpretation is clear: the same centrist attitude is captured when, starting from $\mathbf{x}, \pi 100\%$ of the income difference between X and Y is allocated according to $\mathbf{v_x}$ and $(1 - \pi)100\%$ in equal absolute amounts, as when, starting from $\mathbf{y}, \pi' 100\%$ of the income difference is allocated according to $\mathbf{v_y}$ and $(1 - \pi')100\%$ in equal absolute amounts. This can be understood as follows. Suppose first that $\mathbf{y} \in P_{(\mathbf{x}, \pi)}(\mathbf{x})$, so that $\mathbf{x}, \mathbf{y}$ have the same $(\mathbf{x}, \pi)$-inequality. Then, as we show in Proposition 1.i, $\pi' = \pi(X + \tau)/\ (X + \pi\tau)$. Assume without loss of generality that $Y - X > 0$. Then $\mathbf{y}$ has less relative inequality than $\mathbf{x}$ and $\pi' > \pi$. Thus, to get down to $\mathbf{x}$ from $\mathbf{y}$ so as to preserve intermediate inequality, we can follow the pattern $\mathbf{v_y}$ more closely than the pattern $\mathbf{v_x}$ from $\mathbf{x}$; in other words, when we compare income distributions $\mathbf{x}$ and $\mathbf{y}$ from the viewpoint of the latter, the π' which ensures that $I_{(\mathbf{y}, \pi')}(\mathbf{y}) = I_{(\mathbf{y}, \pi')}(\mathbf{x})$ is closer to 1 than π. On the other hand, if $\mathbf{y} \notin P_{(\mathbf{x}, \pi)}(\mathbf{x})$, then we can only state that $\mathbf{x}$ and $\mathbf{y}$ do not have the same $(\mathbf{x}, \pi)$-inequality but, according to Proposition 1. ii, $\pi' \geq \pi$ whenever $\mathbf{y}L\mathbf{x}$.

To appreciate the differences between $\boldsymbol{\alpha_0}$-inequality and $(\mathbf{x_0}, \pi_0)$-inequality from a different perspective, suppose a situation in which $\mathbf{x_1}$ and $\mathbf{x_2}$ are the income distributions of country A in two moments of time, while $\mathbf{y_1}$ and $\mathbf{y_2}$ correspond to the same situation in country B. Given $\boldsymbol{\alpha_0}$, assume that $\mathbf{x_1}$ and $\mathbf{x_2}$, as well as $\mathbf{y_1}$ and $\mathbf{y_2}$, have the same $\boldsymbol{\alpha_0}$-inequality. In our case, given $\mathbf{x_0} \in D$ and $\pi_0 \in [0, 1]$, $\boldsymbol{\alpha_0} = \pi_0 \mathbf{v_{x0}} + (1 - \pi_0)\mathbf{e}$ is determined. Assume that both $\mathbf{x_1}$ and $\mathbf{x_2}$, and $\mathbf{y_1}$ and $\mathbf{y_2}$, have the same $(\mathbf{x_0}, \pi_0)$-inequality. We know that there exist $\pi, \pi' \in [0, 1]$ such that $\boldsymbol{\alpha_0} = \pi\mathbf{v_{x1}} + (1 - \pi)\mathbf{e}$ and $\boldsymbol{\alpha_0} = \pi'\mathbf{v_{y1}} + (1 - \pi')\mathbf{e}$. Suppose, for instance, that $\mathbf{y_1}L\mathbf{x_1}$. Regardless of whether $\mathbf{y_1} \in P_{(\mathbf{x_0}, \pi_0)}(\mathbf{x_1})$ or not, by Proposition 1.ii we know that $\pi' \geq \pi$. Of course, $(\mathbf{x_1}, \pi)$-inequality and $(\mathbf{y_1}, \pi')$-coincide with $(\mathbf{x_0}, \pi_0)$-inequality, but the fact that $\pi' \geq \pi$ reflects the idea that it is different to maintain the same intermediate inequality from $\mathbf{y_1}$ in country B, with less relative inequality, than from $\mathbf{x_1}$ in country A.

Finally, assume that, for some $\pi \in [0, 1]$, in country A the income distributions $\mathbf{x_1}$ and $\mathbf{x_2}$ have the same $(\mathbf{x_1}, \pi)$-inequality while in country $B\mathbf{y_1}$ and $\mathbf{y_2}$ have the same $(\mathbf{y_1}, \pi)$-inequality. Of course, this does not mean that these two inequality concepts capture the same centrist attitude. If we define $\boldsymbol{\alpha}_\mathbf{A} = \pi\mathbf{v}_{\mathbf{x1}} + (1-\pi)\mathbf{e}$ and $\boldsymbol{\alpha}_\mathbf{B} = \pi\mathbf{v}_{\mathbf{y1}} + (1-\pi)\mathbf{e}$, then it is easy to verify that, for example, $\boldsymbol{\alpha}_\mathbf{B} L \boldsymbol{\alpha}_\mathbf{A}$ whenever $\mathbf{y_1} L \mathbf{x_1}$, in which case we can say that $\boldsymbol{\alpha}_\mathbf{B}$ represents a more demanding centrist concept.

Proposition 1. *Let* $\mathbf{x_0} \in D$ *and* $\pi_0 \in [0,1]$, *so that* $\boldsymbol{\alpha}_\mathbf{0} = \pi_0\mathbf{v}_{\mathbf{x0}} + (1-\pi_0)\mathbf{e}$ *is determined. Let* $x \in \Gamma'(\boldsymbol{\alpha}_\mathbf{0})$ *so that* $\boldsymbol{\alpha}_\mathbf{0} = \pi\mathbf{v}_x + (1-\pi)\mathbf{e}$ *for some* $\pi \in [0,1]$.

i) *If* $\mathbf{y} \in P_{(\mathbf{x_0}, \pi_0)}(\mathbf{x}) = P_{(\mathbf{x}, \pi)}(\mathbf{x})$ *so that* $\mathbf{y} = \mathbf{x} + \tau\boldsymbol{\alpha}_\mathbf{0}$ *for some* $\tau \in R$, *then* $\mathbf{y} \in \Gamma'(\boldsymbol{\alpha}_\mathbf{0})$ *and* $\boldsymbol{\alpha}_\mathbf{0} = \pi'\mathbf{v}_y + (1-\pi')\mathbf{e}$ *with* $\pi' = \pi(X+\tau)/(X+\pi\tau)$. *Therefore* $\pi' = \pi$ *if* $\pi = 0$ *or* $\pi = 1$, *and* $\pi' > \pi$ $(\pi' < \pi)$ *as* $\tau > 0$ $(\tau < 0)$.
ii) *If* $\mathbf{y} \notin P_{(\mathbf{x_0}, \pi_0)}(\mathbf{x})$ *but* $\mathbf{y} \in \Gamma'(\boldsymbol{\alpha}_\mathbf{0})$, *then* $\pi' \geq \pi$ $(\pi' \leq \pi)$ *as* $\mathbf{y}L\mathbf{x}(\mathbf{x}L\mathbf{y})$.

(See the proof in the Appendix).

II.4. Social evaluation functions

A Social Evaluation Function (SEF for short) is a real valued function W defined on D, with the interpretation that for each income distribution $\mathbf{x}$, $W(\mathbf{x})$ provides the "social" or, simply, the aggregate welfare from a normative point of view. We need to introduce a social preference for efficiency consistent with the notion of intermediate inequality presented in Sect. II. 3. We first say that a SEF $W: D \to R$ is monotonic along $\boldsymbol{\alpha}$-rays in $\Gamma(\boldsymbol{\alpha})$, if and only if for each $\mathbf{x} \in \Gamma(\boldsymbol{\alpha})$

$$W(\mathbf{x} + \tau\boldsymbol{\alpha}) \geq W(\mathbf{x}) \quad \text{for all scalars } \tau \geq 0.$$

This property of monotonicity along $\boldsymbol{\alpha}$-rays corresponds to a preference for higher incomes keeping $\boldsymbol{\alpha}$-ray invariant inequality constant. Given $\mathbf{x} \in D$ and $\pi \in [0,1]$, so that $\boldsymbol{\alpha} = \pi\mathbf{v}_\mathbf{x} + (1-\pi)\mathbf{e}$, a SEF $W: D \to R$ is called monotonic along $(\mathbf{x}, \pi)$-rays in $\Gamma'(\boldsymbol{\alpha})$, if and only if

$$W(\mathbf{y} + \tau(\pi\mathbf{v}_\mathbf{x} + (1-\pi)\mathbf{e})) \geq W(\mathbf{y}) \quad \text{for all scalars } \tau \geq 0 \quad \text{and all } \mathbf{y} \in \Gamma'(\boldsymbol{\alpha}).$$

This property of monotonicity along $(\mathbf{x}, \pi)$-rays corresponds to a preference for higher incomes keeping $(\mathbf{x}, \pi)$-inequality constant. For any $\mathbf{x} \in D$ and $\pi \in [0,1]$, let $W_{(\mathbf{x}, \pi)}$ be the class of SEF satisfying continuity, population replication invariance, S-concavity and monotonicity along $(\mathbf{x}, \pi)$-rays.

III. Operational methods

Let $m(.)$ denote the income distribution mean. The following theorem, inspired in Chakravarty (1988), summarizes the connection between Lorenz dominance and SEFs in the class $W_{(t,\pi)}$ in the homogeneous case.

Theorem 1. *Let* $\mathbf{t}, \mathbf{u} \in D$. *The following statements are equivalent:*
(1.i) $m(\mathbf{u}) \geq m(\mathbf{t})$, *and*

(1.ii) *there exists some* $\pi^{\#} \in [0, 1]$ *such that, when we define*

$$\mathbf{z} = \mathbf{t} + \tau(\pi^{\#}\mathbf{v_t} + (1 - \pi^{\#})\mathbf{e}) \quad \textit{with } \tau = U - T,$$

we have $\mathbf{v_u} L \mathbf{v_z}$.

(2) $W(\mathbf{u}) \geq W(\mathbf{t})$ *for all* $W \in W_{(\mathbf{t}, \pi^{\#})}$.

Corollary. *Under the conditions of the above Theorem 1,*

$$W(\mathbf{u}) > W(\mathbf{t}) \quad \textit{for all } W \in W_{(\mathbf{t}, \pi)} \quad \textit{with } \pi \in (\pi^{\#}, 1].$$

(See the proofs in the Appendix)

How do we apply these results in practice? Let **t** and **u** be the initial and the final income distributions in a given country after a certain period of time. An empirical situation in which intermediate inequality concepts might prove useful, arises when **u** dominates **t** in the relative Lorenz sense but **t** dominates **u** in the absolute Lorenz sense. Given $\mathbf{x_0} \in D$ and $\pi_0 \in [0, 1]$, suppose that society has centrist views according to which we should judge all income distributions from the point of view of $(\mathbf{x_0}, \pi_0)$-inequality. Assume without loss of generality that $m(\mathbf{u}) \geq m(\mathbf{t})$. If we find that $I_{(\mathbf{x_0}, \pi_0)}(\mathbf{t}) \geq I_{(\mathbf{x_0}, \pi_0)}(\mathbf{u})$, then we can conclude that $W(\mathbf{u}) \geq W(\mathbf{t})$ for all $W \in W_{(\mathbf{x_0}, \pi_0)}$. Otherwise, no intermediate welfare conclusion can be obtained.

The problem, of course, is that even if we simplify matters by selecting $\mathbf{x_0} = \mathbf{t}$, we do not have any *a priori* reasons to determine which should be the π_0 value. Our strategy is to use Theorem 1 to allow the data to reveal for which π values the income distributions **u** and **t** have the same $(\mathbf{t}, \pi)$-inequality. If we are lucky, there will exist some $\pi \in [0, 1]$ such that $\mathbf{u} = \mathbf{t} + \tau(\pi\mathbf{v}_t + (1 - \pi)\mathbf{e})$ with $\tau = U - T$. Otherwise, we may find a pair of values in the unit interval, π_1 and π_2, with $\pi_1 < \pi_2$, such that

$$I_{(\mathbf{t}, \pi)}(\mathbf{u}) \geq I_{(\mathbf{t}, \pi)}(\mathbf{t}) \quad \text{for all } \pi \in [0, \pi_1],$$

$$I_{(\mathbf{t}, \pi)}(\mathbf{u}) \leq I_{(\mathbf{t}, \pi)}(\mathbf{t}) \quad \text{for all } \pi \in [\pi_2, 1],$$

while for any $\pi \in (\pi_1, \pi_2)$, **u** and **t** are non comparable from the point of view of $(\mathbf{t}, \pi)$-inequality.

A numerical example might be useful at this point. Assume that the data reveals that **t** and **u** are non comparable from the point of view of $(\mathbf{t}, \pi)$-inequality for π's in the interval $(0.4, 0.7)$. Consider the center-right inequality views for which two distributions have the same inequality if, starting from **t**, $(1 - 0.7)100 = 30\%$ or less of any excess income is distributed in absolute terms, and the remaining in relative terms. For all people with such views, in going from **t** to **u** inequality has decreased. For all people with center-left views, for which at least $(1 - 0.4)100 = 60\%$ of excess income should be distributed in absolute terms for intermediate inequality to remain constant, in going from **t** to **u** inequality has increased.

Suppose now that for a different country in the same period, **v** and **z** have non comparable $(\mathbf{v}, \pi)$-inequality for π's in the interval $(0.5, 0.6)$. We can say that, relative to the initial situation **v**, the spectrum of centrist attitudes for which there has been a reduction in inequality is larger. The same can be said of those attitudes for which there has been an increase in inequality. However, the spectrum of inequality views for which inequality cannot be compared has decreased. To appreciate the richness of our approach, notice that with present techniques we can only say that, in both countries, relative inequality decreased while absolute inequality increased. Notice also that to reach our conclusions we do not introduce any new value judgments. What we do is to allow the data to induce a useful partition in the space of centrist attitudes.

Define the absolute and the relative rays through $\mathbf{t}, A(\mathbf{t})$ and $R(\mathbf{t})$, by

$$A(\mathbf{t}) = \{\mathbf{x} \in D : \mathbf{x} = \mathbf{t} + \tau\mathbf{e}, \tau \in R\} = P_{(\mathbf{t},0)}(\mathbf{t}),$$

and

$$R(\mathbf{t}) = \{\mathbf{x} \in D : \mathbf{x} = \mathbf{t} + \tau\mathbf{v_t}, \tau \in R\} = P_{(\mathbf{t},1)}(\mathbf{t}),$$

respectively. Let us call $\mathbf{a}$ and $\mathbf{r}$ the income distributions in $A(\mathbf{t})$ and $R(\mathbf{t})$, respectively, with mean $m(\mathbf{u})$. Since we assume that $\tau = U - T > 0$, we have that $\mathbf{a} = \mathbf{t} + \tau\mathbf{e}$ and $\mathbf{r} = \mathbf{t} + \tau\mathbf{v_t}$. Define the line segment $\{\mathbf{a}, \mathbf{r}\}$ in H-dimensional space by

$$\begin{aligned}\{\mathbf{a}, \mathbf{r}\} &= \{\mathbf{z} \in D : \mathbf{z} = \mathbf{t} + \tau(\pi\mathbf{v_t} + (1-\pi)\mathbf{e}) \quad \text{for some } \pi \in [0,1]\} \\ &= \textstyle\bigcup_{\pi \in [0,1]} P_{(\mathbf{t},\pi)}(\mathbf{t}) \cap \{\mathbf{z} \in D : m(\mathbf{z}) = m(\mathbf{u})\},\end{aligned}$$

This is the set consisting of all income distributions with mean equal to $m(\mathbf{u})$ which can be reached by $(\mathbf{t}, \pi)$-rays through $\mathbf{t}$.

The General Case. Notice that the starting situation can be described by the fact that $\mathbf{v_a} L \mathbf{v_u} L \mathbf{v_r}$. Assume first that the Lorenz dominance relation $\mathbf{v_a} L \mathbf{v_u} L \mathbf{v_r}$ is strict. Then there must exist two values $\pi_1 \in [0,1)$ and $\pi_2 \in [\pi_1, 1]$ which induce the following partition of $\{\mathbf{a}, \mathbf{r}\}$:

$$\{\mathbf{a}, \mathbf{z_1}\} = \{\mathbf{z} \in \{\mathbf{a}, \mathbf{r}\} : \mathbf{z} = \mathbf{t} + \tau(\pi\mathbf{v_t} + (1-\pi)\mathbf{e}), \pi \in [0, \pi_1]\};$$

$$\{\mathbf{z_1}, \mathbf{z_2}\} = \{\mathbf{z} \in \{\mathbf{a}, \mathbf{r}\} : \mathbf{z} = \mathbf{t} + \tau(\pi\mathbf{v_t} + (1-\pi)\mathbf{e}), \pi \in (\pi_1, \pi_2)\};$$

$$\{\mathbf{z}_2, \mathbf{r}\} = \{\mathbf{z} \in \{\mathbf{a}, \mathbf{r}\} : \mathbf{z} = \mathbf{t} + \tau(\pi\mathbf{v_t} + (1-\pi)\mathbf{e}), \pi[\pi_2, 1]\}.$$

The partition has the following property: $\mathbf{v_z} L \mathbf{v_u}$ for all $\mathbf{z} \in \{\mathbf{a}, \mathbf{z_1}\}$; $\mathbf{v}_u L \mathbf{v_z}$ for all $\mathbf{z} \in \{\mathbf{z_2}, \mathbf{r}\}$; and $\mathbf{v_u}$ is either non comparable to $\mathbf{v_z}$ for all $\mathbf{z} \in \{\mathbf{z_1}, \mathbf{z_2}\}$. Since, for instance,

$$\{\mathbf{a}, \mathbf{z_1}\} = \textstyle\bigcup_{\pi \in [0,\pi 1]} P_{(t,\pi)}(\mathbf{t}) \cap \{\mathbf{z} \in D : m(\mathbf{z}) = m(\mathbf{u})\},$$

for every $\mathbf{z} \in \{\mathbf{a}, \mathbf{z_1}\}, I_{(\mathbf{t},\pi)}(\mathbf{z}) = I_{(\mathbf{t},\pi)}(\mathbf{t})$ for some $\pi \in [0, \pi_1]$. Therefore, as we wanted:

$$I_{(t,\pi)}(\mathbf{u}) \geq I_{(\mathbf{t},\pi)}(\mathbf{t}) \quad \text{for all } \pi \in [0, \pi_1].$$

Similarly,

$$I_{(\mathbf{t},\pi)}(\mathbf{u}) \leq I_{(\mathbf{t},\pi)}(\mathbf{t}) \quad \text{for all } \pi \in [\pi_2, 1],$$

while for any $\pi \in (\pi_1, \pi_2)$, $\mathbf{u}$ and $\mathbf{t}$ are non comparable from the point of view of $(\mathbf{t}, \pi)$-inequality.

It would be useful to provide a graphical illustration of the general case. In order not to interrupt the reading of the text, we present a 3-dimensional example in the Appendix.

Special cases. If $\mathbf{u} \in \{\mathbf{a}, \mathbf{r}\}$, then $\mathbf{u} = \mathbf{t} + \tau(\pi \mathbf{v_t} + (1 - \pi)\mathbf{e})$ for some $\pi \in [0, 1]$. Similarly, if there is some $z \in \{\mathbf{r}, \mathbf{a}\}$ which is Lorenz equivalent to $\mathbf{v}_u$, then $\pi_2 = \pi_1 = \pi$ with $\mathbf{z} = \mathbf{t} + \tau(\pi \mathbf{v_t} + (1 - \pi)\mathbf{e})$. In both cases $I_{(\mathbf{t}, \pi)}(\mathbf{u}) = I_{(\mathbf{t}, \pi)}(\mathbf{t})$. On the other hand, if $\mathbf{v_a}$ is Lorenz equivalent to v_u, then $\pi_1 = \pi_2 = 0$; but if $\mathbf{v_a}$ is non comparable to $\mathbf{v_u}$, then there exists no $\pi_1 \in [0, 1]$. Similarly, if $\mathbf{v_u}$ is Lorenz equivalent to $\mathbf{v_t}$, then $\pi_1 = \pi_2 = 1$, while if $\mathbf{v_u}$ is non comparable to $\mathbf{v_t}$, then there exists no $\pi_2 \in [0, 1]$.

IV. Concluding remarks

Suppose we want to compare two income distributions **u** and **t** in two different moments of time, and assume that distribution **u** has a greater mean than **t**. If distribution **u** dominates **t** in the absolute Lorenz sense, then we believe there is a consensus that nothing else need to be done. Who would deny that there has been an unambiguous increase in social welfare? Only people who believe that to maintain inequality constant any excess income should be distributed so as to assign greater absolute amounts to the poor than to the rich.

Suppose, however, that distribution **u** dominates distribution **t** in the relative Lorenz sense, but that **t** dominates **u** in the absolute Lorenz sense. The main claim of this paper is that we can improve upon this type of evaluation without bringing in new value judgments. Conditional on a given income distribution **x**, we propose a continuum of inequality notions which can be intuitively ordered from the relative notion to the absolute one in terms of a parameter π which varies in the unit interval. Then we provide operational methods to partition such continuum of inequality notions into subsets with a clear normative interpretation.

For example, in the Spanish case during the 1980's we reach the following result for the total population and an intermediate value of the parameter $\Theta = 0.4$[8]. For a rather small set of center-right perceptions of inequality (according to which inequality remains constant if, say, 13% or less of any excess income is distributed in absolute amounts while the remaining is distributed according to the relative shares in the initial situation), inequality has decreased. For a second set of politically more demanding centrist attitudes (according to which inequality remains constant if approximately 29% or more of any excess income is distributed in absolute amounts), inequality has increased. For the remaining subset of centrist attitudes, inequality in 1990 1991 is equivalent, or statistically indistinguishable, to inequality in 1980 1981. We may take this result as implying that the decrease in inequality in Spain during this period has been "small".

[8] For a discussion of the heterogeneous case and other empirical details (see Del Río and Ruiz Castillo 1997).

Whether social welfare went unambiguously down according to measurement instruments consistent with a relative inequality notion, is a very important piece of knowledge to have. However, in situations like the Spanish one, to know precisely under which set of centrist value judgments inequality has increased, decreased, or remained equivalent, generates some value added worth having. In our opinion, the methodology presented in this paper goes one step in the direction pointed out by Atkinson (1989), when he indicates that we ought to follow procedures and, above all, report empirical estimates, making clear their dependence on the various axioms and value judgments involved.

Finally, what do we have to say if distribution **u** is dominated by **t** in the relative Lorenz sense? Again, we believe it is worth knowing whether distribution **u**'s departure from the relative ray through **t** is "large" or "small". Think for simplicity in the two dimensional case. We know that the income share received by the poor in **u** has decreased. Assume, in addition, that the absolute amount of income received by the poor person in **u** has not decreased relative to **t**. Consider the set of income distributions in which any excess income is assigned to the rich person in **t**. They belong to what we may call the Paretian ray through **t**. Under the above assumptions, the distribution **u** lies somewhere between the Paretian ray and the relative ray through **t**. The question we are interested in can now be rephrased as follows: is the distribution u "very far" apart from the relative ray through **t**, and therefore "close" to the Paretian ray, reflecting a large increase in inequality? Del Río (1996) extends the methods presented in this paper to provide an operative answer to this question.

Appendix

Proposition 1. *Let* $\mathbf{x_0} \in D$ *and* $\pi_0 \in [0,1]$, *so that* $\boldsymbol{\alpha_0} = \pi_0 \mathbf{v_{x_0}} + (1-\pi_0)\mathbf{e}$ *is determined. Let* $\mathbf{x} \in \Gamma'(\boldsymbol{\alpha_0})$ *so that* $\boldsymbol{\alpha_0} = \pi \mathbf{v_x} + (1-\pi)\mathbf{e}$ *for some* $\pi \in [0,1]$.

i) *If* $\mathbf{y} \in P_{(\mathbf{x_0},\pi_0)}(\mathbf{x}) = P_{(\mathbf{x},\pi)}(\mathbf{x})$ *so that* $\mathbf{y} = \mathbf{x} + \tau\boldsymbol{\alpha_0}$ *for some* $\tau \in R$, *then* $\mathbf{y} \in \Gamma'(\boldsymbol{\alpha_0})$ *and* $\boldsymbol{\alpha_0} = \pi' \mathbf{v_y} + (1-\pi')\mathbf{e}$ *with* $\pi' = \pi(X+\tau)/(X+\pi\tau)$. *Therefore* $\pi' = \pi$ *if* $\pi = 0$ *or* $\pi = 1$, *and* $\pi' > \pi (\pi' < \pi)$ *as* $\tau > 0$ $(\tau < 0)$.

ii) *If* $\mathbf{y} \notin P_{(\mathbf{x_0},\pi_0)}(\mathbf{x})$ *but* $\mathbf{y} \in \Gamma'(\boldsymbol{\alpha_0})$, *then* $\pi' \geq \pi$ $(\pi' \leq \pi)$ *as* $\mathbf{y}L\mathbf{x}(\mathbf{x}L\mathbf{y})$.

Proof of Proposition 1: i) We want to prove that, given $\boldsymbol{\alpha_0} = \pi\mathbf{v_x} + (1-\pi)\mathbf{e}$, for any $y \in D$ with $I_{(x,\pi)}(\mathbf{y}) = I_{(x,\pi)}(\mathbf{x})$, there exists a $\pi \in [0,1]$ such that,

$$\mathbf{y} = \mathbf{x} + \tau[\pi'\mathbf{v_y} + (\mathbf{1}-\pi')\mathbf{e}].$$

Taking into account that $\mathbf{y} = \mathbf{x} + \tau\boldsymbol{\alpha_0} = \mathbf{x} + \tau[\pi\mathbf{v_x} + (1-\pi)\mathbf{e}]$, we have

$$\mathbf{v_y} = \frac{X}{X+\tau}\left(\mathbf{1} + \pi\frac{\tau}{X}\right)\mathbf{v_x} + (\mathbf{1}-\pi)\frac{\tau}{X+\tau}\mathbf{e}$$

$$= \frac{X+\pi\tau}{X+\tau}\mathbf{v}_x + \left(1 - \frac{X+\pi\tau}{X+\tau}\right)\mathbf{e} = \lambda v_x + (1-\lambda)\mathbf{e},$$

with $\lambda = (X + \pi\tau)/(X + \tau) = (X/X + \tau)(1 - \pi) + \pi$, which implies that $\lambda \geq \pi$. Rearranging terms and substituting $\mathbf{v_x}$ in $\boldsymbol{\alpha_0} = \pi\mathbf{v_x} + (1 - \pi)\mathbf{e}$ we have

$$\alpha_0 = \pi\frac{\mathbf{v_y}}{\lambda} - \pi\frac{(1-\lambda)}{\lambda}\mathbf{e} + (1-\pi)\mathbf{e} = \left(\frac{\pi}{\lambda}\right)\mathbf{v}_y + \left(1 - \frac{\pi}{\lambda}\right)\mathbf{e}.$$

Since $0 \leq (\pi/\lambda) \leq 1$, it follows that $\mathbf{y} \in \Gamma'(\boldsymbol{\alpha_0})$ and

$$\pi' = \frac{\pi}{\lambda} = \pi\frac{1 + \frac{\tau}{X}}{\mathbf{1} + \pi\frac{\tau}{X}} = \pi\frac{X + \tau}{X + \pi\tau}.$$

ii) Since $\mathbf{x}, \mathbf{y} \in \Gamma'(\boldsymbol{\alpha_0})$, we can write

$$\boldsymbol{\alpha_0} = \pi'\mathbf{v_y} + (1 - \pi')\mathbf{e} = \pi\mathbf{v_x} + (1 - \pi)\mathbf{e}.$$

Assume $\mathbf{y}L\mathbf{x}$. By contradiction, suppose that $\pi' < \pi$. This means that $\pi = \pi' + \varepsilon$ with $\varepsilon > 0$. By substituting π in this expression we obtain

$$\pi' v_y + (1 - \pi')\mathbf{e} = \pi'\mathbf{v}_x + (1 - \pi')\mathbf{e} + (\mathbf{v}_x - \mathbf{e})\varepsilon.$$

This implies that $v_y^h > v_x^h$ for the rich ($v_x^h > (1/H)$) and $v_y^h < v_x^h$ for the poor ($v_x^h < (1/H)$) in the income distribution $\mathbf{x}$. This means that $\mathbf{y}$ can be obtained from $\mathbf{x}$ by transferring income from the poor to the rich, and hence $\mathbf{x}L\mathbf{y}$, a contradiction. Q.E.D.

Theorem 1. *Let* $\mathbf{t}, \mathbf{u} \in D$. *Then the following statements are equivalent*:

(1.i) $m(\mathbf{u}) \geq m(\mathbf{t})$, *and*

(1.ii) *there exists some* $\pi^{\#} \in [0, 1]$ *such that, when we define*

$$\mathbf{z} = \mathbf{t} + \tau(\pi^{\#}\mathbf{v_t} + (1 - \pi^{\#})\mathbf{e}) \quad \text{with } \tau = U - T,$$

we have $\mathbf{v_u}L\mathbf{v_z}$.

(2) $W(\mathbf{u}) \geq W(\mathbf{t})$ *for all* $W \in W_{(\mathbf{t}, \pi^{\#})}$.

Corollary. *Under the conditions of the above Theorem 1*,

$$W(\mathbf{u}) > W(\mathbf{t}) \quad \text{for all } W \in W_{(\mathbf{t}, \pi)} \quad \text{with } \pi \in (\pi^{\#}, 1].$$

Proof of Theorem 1: 1) $\Rightarrow$ 2): As $m(\mathbf{u}) \geq m(\mathbf{t})$, for any SEF $W \in W_{(\mathbf{t}, \pi^{\#})}$ we have:

$$W(\mathbf{z}) = W(\mathbf{t} + (U - T)(\pi^{\#}\mathbf{v_t} + (1 - \pi^{\#})\mathbf{e})) \geq W(\mathbf{t}). \tag{1}$$

Moreover, as $\mathbf{u}$ Lorenz-dominates $\mathbf{z}$ and both distributions have the same mean, $m(\mathbf{u})$, we know that

$$W(\mathbf{u}) \geq W(\mathbf{z}) \tag{2}$$

for any S-concave function, W (see Dasgupta et al. 1973). By combining (1) and (2), we conclude that

$$W(\mathbf{u}) \geq W(\mathbf{t}) \quad \text{for all } W \in W_{(\mathbf{t}, \pi^{\#})}.$$

2) $\Rightarrow$ 1): Let $\mathbf{x} \in D$ and $\mathbf{z}' = \mathbf{x} + (U - X)[\pi^{\#}\mathbf{v_t} + (1 - \pi^{\#})\mathbf{e}]$. Suppose that

$$W(\mathbf{x}) = (m(\mathbf{x}))^n f[z'], \tag{3}$$

where $n \geq 0$, and $f(\,.\,)$ is a continuous, S-concave function satisfying population replication invariance. It can be seen that any function W verifying (3) is monotonic along $(\mathbf{t}, \pi^{\#})$ rays, so that:

$$W(\mathbf{x}) \leq W(\mathbf{x} + \tau'(\pi^{\#} V_{\mathbf{t}} + (1 - \pi^{\#})\mathbf{e})) = \left(m(\mathbf{x}) + \frac{\tau'}{H}\right)^{n} f[\mathbf{z}']$$

for any $\tau' \geq 0$. Notice that continuity, population replication invariance, and S-concavity of f imply that W satisfies the same properties. Therefore, expression (3) ensures that function $W(\cdot)$ satisfies the assumptions of the theorem. Since $W(\mathbf{t}) \leq W(\mathbf{u})$, by choosing $f(\cdot) = 1$ we obtain condition (1.i):

$$W(\mathbf{t}) = (m(\mathbf{t}))^{n} \leq (m(\mathbf{u}))^{n} = W(\mathbf{u}).$$

On the other hand, if $n = 0$ then we get

$$W(\mathbf{t}) = f[\mathbf{z}'] = f[\mathbf{z}] \leq f[\mathbf{u}] = W(\mathbf{u}).$$

Since $m(\mathbf{z}) = m(\mathbf{u}) > 0$, and $f(\cdot)$ is any arbitrary S-concave function, we conclude that $\mathbf{u}L\mathbf{z}$ (see Dasgupta et al. 1973). Q.E.D.

Proof of Corollary: Let $\pi \in (\pi^{\#}, 1]$, so that $\pi^{\#} = \pi - \beta$ for some $\beta > 0$. Then we can write:

$$\pi^{\#}\mathbf{v}_{\mathbf{t}} + (1 - \pi^{\#})\mathbf{e} = \pi\mathbf{v}_{\mathbf{t}} + (1 - \pi)\mathbf{e} - \beta(\mathbf{v}_{\mathbf{t}} - \mathbf{e}).$$

It can be shown that $\pi^{\#}\mathbf{v}_{\mathbf{t}} + (1 - \pi^{\#})\mathbf{e}$ is obtained from $\pi\mathbf{v}_{\mathbf{t}} + (1 - \pi)\mathbf{e}$ by using a sequence of order preserving transformations transferring income from the rich to the poor. Thus, $\pi^{\#}\mathbf{v}_{\mathbf{t}} + (1 - \pi^{\#})\mathbf{e}$ strictly dominates $\pi\mathbf{v}_{\mathbf{t}} + (1 - \pi)\mathbf{e}$ in the Lorenz sense. Using that

$$\mathbf{z}' = \mathbf{t} + \tau[\pi\mathbf{v}_{\mathbf{t}} + (1 - \pi)\mathbf{e}], \quad \tau = U - T,$$

we conclude that $\mathbf{v}_{z}$ strictly dominates $\mathbf{v}_{z'}$ in the Lorenz sense. Therefore, under the assumptions of Theorem 1, the expression

$$W(\mathbf{t}) = W(\mathbf{z}') < W(\mathbf{z}) \leq W(\mathbf{u})$$

must hold for any function $W \in W_{(t,\pi)}$, with $\pi \in (\pi^{\#}, 1]$. Q.E.D.

A graphical illustration of the empirical procedure in the general case

In an economy consisting of three individuals, the two income distributions at two moments in time are $\mathbf{t} = (0.5, 1.0, 1.5)$ and $\mathbf{u} = (8, 9, 13)$. Clearly, $\mathbf{u}$ dominates $\mathbf{t}$ according to the relative Lorenz criterion but the opposite is the case according to the absolute notion.

It is easy to find the vector $\boldsymbol{\alpha}^{*} \in S$, such that $\mathbf{u} = \mathbf{t} + \tau\boldsymbol{\alpha}^{*}$ with $\tau = U - T = 30 - 3 = 27$. It turns out that $\boldsymbol{\alpha}^{*} = (0.27, 0.29, 0.42)$. Of course, $\mathbf{u}$ and $\mathbf{t}$ have the same $\boldsymbol{\alpha}^{*}$-inequality, but we cannot find a clear intuitive interpretation of such an statement. In particular, we cannot say whether this means that inequality has been reduced by a little or by a lot relative to the initial situation $\mathbf{t}$.

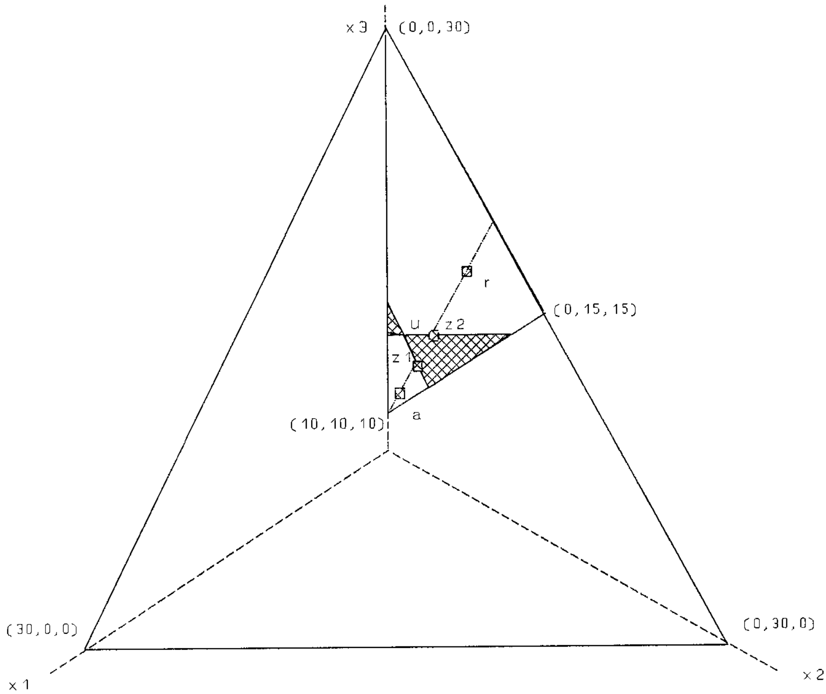

Fig. 1

To understand our approach, it suffices to consider the set of income distributions $\mathbf{z}$ with $m(\mathbf{z}) = m(\mathbf{u}) = 10$ in which the individual rankings in $\mathbf{t}$ are preserved. This is the set in Fig. 1 with vertexes $(0, 0, 30), (10, 10, 10)$ and $(0, 15, 15)$. The subset $\{\mathbf{a}, \mathbf{r}\}$ is the set of income distributions with mean equal to 10 which can be reached by $(\mathbf{t}, \pi)$-rays through $\mathbf{t}$. In particular, the income distribution that results from an equal allocation of the extra 27 income units is $\mathbf{a} = (9.5, 10, 10.5)$, while the income distribution which preserves the income shares in $\mathbf{t}$ is $\mathbf{r} = (5, 10, 15)$.

For any $\mathbf{z} \in \{\mathbf{a}, \mathbf{r}\}$, there exists some $\pi \in [0, 1]$ such that $\mathbf{z} = \mathbf{t} + 27(\pi v_{\mathbf{t}} + (1 - \pi)\mathbf{e})$. That is, every $\mathbf{z} \in \{\mathbf{a}, \mathbf{r}\}$ has been obtained from $\mathbf{t}$ by a meaningful economic procedure: allocating $(1 - \pi)100\%$ of the extra 27 income units in equal absolute amounts among the three individuals, and the remaining $\pi 100\%$ so as to maintain the income shares in $\mathbf{t}$.

In the example, $\mathbf{u} \notin \{\mathbf{a}, \mathbf{r}\}$. However, the values $\pi_1 = 0.33$ and $\pi_2 = 0.56$ with the corresponding income distributions $\mathbf{z}_1 = (8, 10, 12)$ and $\mathbf{z}_2 = (7, 10, 13)$, induce a partition of $\{\mathbf{a}, \mathbf{r}\}$ with the property that $\mathbf{z}L\mathbf{u}$ for all $\mathbf{z} \in \{\mathbf{a}, \mathbf{z}_1\}$, $\mathbf{u}L\mathbf{z}$ for all $\mathbf{z} \in \{\mathbf{z}_2, 1\}$, and $\mathbf{u}$ is Lorenz non comparable with z for all $z \in \{z_1, z_2\}$. The dark zone in Fig. 1 represents income distributions non comparable with income distribution $\mathbf{u}$. Therefore, we conclude that in going from $\mathbf{t}$ to $\mathbf{u}$ income inequality has decreased for centrist attitudes according to which 44% or less of all extra income should be allocated equally among all individuals, while it has increased for those according to which that percentage should be at least equal to 67%. For the remaining attitudes, **t** and **u** are non comparable from the point of view of $(\mathbf{t}, \pi)$-inequality. One may say informally that the data have revealed that income inequality has been reduced by a considerable amount. Therefore, this cardinalization exercise has been carried out without the help of any new value judgments.

References

Amiel Y, Cowell FA (1992) Measurement of Income Inequality. Experimental Test by Questionnaire. Public Econ 47: 3 26

Atkinson AB (1989) Measuring Inequality and Differing Value Judgments. ESRC Program on Taxation, Incentives and the Distribution of Income, Discussion Paper, 129

Atkinson AB, Rainwater L, Smeeding T (1995), Income Distribution in OECD Countries: the Evidence from the Luxembourg Income Study (LIS). OECD, Paris

Ballano C, Ruiz Castillo J (1993) Searching by Questionnaire for the Meaning of Income Inequality. Rev Españ Econ 10: 233 259

Bosser W, Pfingsten A (1990) Intermediate Inequality: Concepts, Indices and Welfare Implications. Math Soci Sci 19: 117 134

Chakravarty S (1998) On Quasi Orderings of Income Profiles. University of Pader born, Methods of Operations Research, 60, XIII Symposium on Operations Re search

Dalton H (1920) The Measurement of Inequality of Income. Econ J 30: 348 361

Dasgupta P, Sen A, Starret D (1973) Notes on the Measurement of Inequality. J Econ Theory 6: 180 187

Del Río C (1996) Desigualdad y pobreza en España, de 1980 81 a 1990 91. Unpub lished Philos. D. Dissertation, Universidad Carlos III de Madrid

Del Río C, Ruiz Castillo J (1996) Ordenaciones de bienestar e inferencia estadística. El caso de las EPF de 1980 81 y 1990 91. In: La desigualdad de recursos. Segundo Simposio sobre la distribución de la renta y la riqueza. Fundación Argentaria, Colección Igualdad, Volumen VI, 9 44

Del Río C, Ruiz Castillo J (1997) Intermediate Inequality and Welfare. The Case of Spain, 1980 81 to 1990 91. Universidad Carlos III de Madrid, Working Paper 97 38, Economic Series 16

Dutta B, Esteban JM (1991) Social Welfare and Equality. Soc Choice Welfare 50: 49 68

Ebert U (1987) Size and Distribution of Incomes as Determinants of Social Welfare. J Econ Theory 41: 23 33

Harrison E, Seidl C (1994) Acceptance of Distributional Axioms: Experimental Find ings. In: Eichorn W (ed) Models and Measurement of Welfare and Inequality.

Gottschalk P, Smeeding T (1997) Crossnational Comparisons of Earnings and Income Inequality. J Econ Lit 35: 633 686

Gouveia M, Tavares J (1995) The Distribution of Household Income and Expenditure in Portugal: 1980 and 1990. Rev Income Wealth 41(1): 1 17

Kolm SC (1976) Unequal Inequalities I, Journal of Economic Theory, 12: 416 442, and "Unequal Inequalities II". J Econ Theory 13: 82 111

Krtscha M (1994) A New Compromise Measure of Inequality. In: Eichorn W (ed) Models and Measurement of Welfare and Inequality.

Moyes P (1987) A New Concept of Lorenz Domination. Econ Lett 23: 203 207

Pfingsten A, Seidl C (1997) Ray Invariant Inequality Measures. In: Zandvakili S, Slotje D (eds) Research on Taxation and Inequality pp 107 129. JAI Press Greenwich
Rodrigues CF (1993) Measurement and Decomposition of Inequality in Portugal, 1980/81 1990/91. Department of Applied Economics, University of Cambridge, E.S.R.C. Discussion Paper, MU 9302
Seidl C, Theilen B (1994) Stochastic Independence of Distributional Attitudes and Social Status. A Comparison of German and Polish Data. Eur J Polit Econ 10: 295 310
Shorrocks A (1983) Ranking Income Distributions. Economica 50: 3 17
Weinhardt C (1993) The Central Role of Efficiency in Inequality and Welfare Mea surement Theory. In: Diewert, Spremann, Stheling (eds) Mathematical Modelling in Economics