ON SCHACHERMAYER'S EXAMPLE ABOUT THE BANACH–SAKS PROPERTY

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(Received 9 January, 1989)

1. Introduction. A Banach space $(X, \|\,.\,\|)$ is said to have the *Banach–Saks property* (B.S.P.) if, for every bounded sequence (x_n) in X, we can choose a subsequence (x'_n) of (x_n) such that the sequence

$$(y_n) = ((x'_1 + \ldots + x'_n)/n)$$

converges in the X-norm. This property, that a Banach space may enjoy or not, has been extensively studied.

On the other hand, we recall that $L^2([0,1], X)$, which we shall refer to as $L^2(X)$, is the Banach space of the Bochner measurable functions from $[0, 1]$ to X, with the norm

$$\|f\|_2 = \left\{\int \|f(t)\|^2\,dt\right\}^{1/2}.$$

We use **[3]** as our reference for $L^2(X)$ spaces.

It is known that $L^2([0,1])$ has the B.S.P. Nevertheless there are examples (the first ones are due to J. Bourgain and W. Schachermayer) of Banach spaces X which have the B.S.P. but such that $L^2(X)$ does not. The example of Professor W. Schachermayer seems to be the easiest, and has been neatly described in **[1**, p. 152]. Our aim is to present a slight refinement of known results about this space that we will call $(B_1, \|\,.\,\|)$, as in **[1]**. There it is shown that there exists a sequence $(f_n) \subset L^2(X)$ which satisfies:

(a) $\|f_n(t)\| = 1$, $(f_n(t)) \xrightarrow{w} 0$ for every $t \in [0, 1]$ and therefore $(f_n) \xrightarrow{w} 0$;

(b) for each $t \in [0, 1]$, there exists an increasing sequence of integers $(n(k))$—this sequence depending on t—such that for every subsequence $(f'_{n(k)})$ of $(f_{n(k)})$,

$$\frac{1}{m}\left\|\sum_{k \le m} f'_{n(k)}(t)\right\| \to 0 \qquad \text{as } m \to \infty.$$

(c)

$$\lim_{u\to\infty} \inf\left\{\frac{1}{k}\left\|\sum_{i \le k} \varepsilon_i f_{n(i)}\right\|_2 : u \le n(1) < \ldots < n(k),\ \varepsilon_i = \pm 1\right\} = 1,$$

for every $k \in \mathbb{N}$.

(We note that (b) follows from (a) and the fact that B_1 has the B.S.P.) Our improvement is as follows.

(d) For every increasing sequence of integers $(n(k))$

$$\mu\left(\left\{t \in [0,1] : \lim_k \frac{1}{k}\left\|\sum_{i \le k} \varepsilon_i f_{n(i)}(t)\right\| = 1\right\}\right) = 1,$$

where $\varepsilon_i = \pm 1$ and μ is Lebesgue measure.

† Supported in part by CAICYT grant 0338-84. The author wishes to thank Professors F. Bombal, J. Diestel and the referee for their advice.

The result (d) should be compared to previous ones of, for example, [**2**], [**4**] and [**7**]. There, if $(g_n):[0, 1]\to Y$, and for every $t\in[0, 1]$ we can choose a sequence $(n(k))$—depending on t—such that $(g_{n(k)}(t))$ satisfies "something" then we can find a sequence of integers $(m(k))$ such that $(g_{m(k)})$ satisfies "something" a.e.

2. Proof of (d). We use the terminology of [**1**, p. 152]. In order to simplify the notation we shall say that the set $\{e_n : n\in A\}$ is *totally admissible* if $A\subset\mathbb{N}$ is totally admissible.

First, we show that

$$(*)\qquad \mu\left(\left\{t\in[0, 1]: \lim_k \frac{1}{k}\left\|\sum_{i\le k}\varepsilon_i f_i(t)\right\| = 1\right\}\right) = 1.$$

Let $r:\mathbb{N}\to\mathbb{N}$ be any function satisfying:

(a) $r(k)/k\to 0$ as $k\to\infty$,

(b) $\sum_k k(k+1)/2^{r(k)}<\infty$.

(Here $r(n)=[\sqrt{n}]$, where [.] denotes the integer part of any real number, will do.) We want the following equality. If $B_k=\{t\in[0, 1]: \{f_{r(k)}(t), \dots, f_{r(k)+j}(t), \dots, f_k(t)\}$ is a totally admissible set$\}$, then

$$(**)\qquad b(k)=\mu(B_k)=\prod_{j=0}^{k-r(k)}(1-j/2^{r(k)}).$$

To prove (**), we define (for any $j=0, \dots, k-r(k))g_j(t)$ to be the unique element of $\mathbb{N}$ such that

(a) $2^{r(k)}\le g_j(t)<2^{r(k)+1}$,

(b) $t(r(k)+j)\in[g_j(t)/2^{r(k)}, (g_j(t)+1)/2^{r(k)}[$.

Then it is clear that the condition $t\in B_k$ is equivalent to $g_i(t)\ne g_j(t)$ if $i\ne j$. Due to the fact that $\{g_j : j=0, \dots, r(k)\}$ is a set of independent random variables, we obtain (**).

Note now that

$$\left\{t\in[0, 1]: \lim_k \left\|\sum_{i\le k}\varepsilon_i f_i(t)\right\| = 1\right\}\supset\bigcup_n\left(\bigcap_{j>n}B_j\right).$$

In fact, if $t\in\bigcap_{j>n}B_j$, then for $k>n$, we have

$$\frac{1}{k}\left\|\sum_{i\le k}\varepsilon_i f_i(t)\right\|\ge\frac{1}{k}\left\|P_k\left(\sum_{i\le k}\varepsilon_i f_i(t)\right)\right\|,$$

where P_k is the projection on the totally admissible set

$$A_k(t)=\{n(j): f_{r(k)+j}(t)=e_{n(j)}, j=0, \dots, k-r(k)\},$$

and so

$$\frac{1}{k}\left\|\sum_{i\le k}\varepsilon_i f_i(t)\right\|\ge\frac{1}{k}\left(\sum_{r(k)\le i\le k}\|\varepsilon_i f_i(t)\|\right)=(k-r(k)+1)/k.$$

Observing that $\log(1-x) \geq -2x$ if $0 \leq x < \frac{1}{2}$, we have, for k sufficiently large,

$$\log(b(k)) \geq -2 \sum_{j \leq k-r(k)} j/2^{r(k)} \geq -k(k+1)/2^{r(k)}.$$

Now, using the fact that $1-e^{-x} \leq x$ if $x>0$, it is clear that $\sum_k (1-b(k)) < \infty$. We deduce that $\mu\left(\bigcup_n \left(\bigcap_{j>n} B_j\right)\right) = 1$, and so (*) is proved.

Finally, we prove (d). For every increasing sequence of integers $(n(k))$, we let

$$B_k = \{t \in [0,1] : \{f_{n(r(k))}(t), \ldots, f_{n(r(k)+j)}(t), \ldots, f_{n(k)}(t)\} \text{ is a totally admissible set}\}.$$

Obviously the set B_k depends on the sequence of integers $(n(k))$. Then we obtain

$$\mu(B_k) = \prod_{j=0}^{k-r(k)} (1 - j/2^{n(r(k))}) \geq \prod_{j=0}^{k-r(k)} (1 - j/2^{r(k)}).$$

The last inequality holds since $n(i) \geq i$. It only remains for the reader to repeat the analysis of the case $n(k) = k$.

REMARK. The reader should note that the generalization of a property related to the Césaro summation method to other summation methods is straightforward if the convergence that we are studying is the norm convergence of a Banach space (see [**5**] and [**1**, p. 58]), but the convergence a.e. is not of this kind. Nevertheless we can obtain (with the notation of [**5**]) the following result.

(d') For every A u.a.n.r.s.m., and for every sequence of integers $(n(k))$, we have

$$\mu\left(\left\{t \in |0,1| : \liminf_k \left\| \sum_{i<\infty} a_{ki} \varepsilon_i f_{n(i)}(t) \right\| \geq 1\right\}\right) = 1.$$

The proof is similar and we omit it.

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