The University of Adelaide
School of Economics

Research Paper No. 2010-32
December 2010

Characterizing the Shapley Value in Fixed-Route Travelling Salesman Problems with Appointments

Duygu Yengin

Characterizing The Shapley Value in Fixed-Route Traveling Salesman Problems with Appointments*

Duygu Yengin†

December 11, 2010

Abstract

Starting from her home, a service provider visits several customers, following a predetermined route, and returns home after all customers are visited. The problem is to find a fair allocation of the total cost of this tour among the customers served. A transferable-utility cooperative game can be associated with this cost allocation problem. We introduce a new class of games, which we refer as the *fixed-route traveling salesman games with appointments.* We characterize the Shapley Value in this class using a property which requires that sponsors do not benefit from mergers, or splitting into a set of sponsors.

JEL Classification Codes : C71

Keywords : fixed-route traveling salesman games, routing games, appointment games, the Shapley value, the core, transferable-utility games, merging and splitting proofness, networks, cost allocation.

1 Introduction

Finding the least-costly route that visits a given set of locations and returns to the starting location, the so called "traveling salesman problem (TSP)" is one of the most well-known combinatorial optimization problems in operations research. As first investigated by Fishburn and Pollack (1983), in several TSP problems, the cost of the tour has to be allocated among the locations visited (sponsors). Some examples include distribution planning situations such as delivery of supplies to grocery stores by a manufacturer (see, Engevall et al; 1998), information transmission over a TSP-type of network, a service provider (salesman, repairman, cable guy, parcel delivery guy, private tutor, doctor etc.) visiting his customers, a professor invited by several universities for seminars, and passengers using shuttle buses or car-pooling.

In some of the above examples, the traveling agent may need to follow a route that is not necessarily the least costly one. We study the so called "*fixed-route traveling salesman problems*" where the route is fixed due to outside factors. Here, starting from an original location we call home (e.g., main office, factory, or depot), a set of locations has to be visited following a predetermined route, and after each location is visited exactly once, the tour ends at home.

*The first draft of this paper was written when I was a Ph.D student in the University of Rochester. I am grateful to William Thomson for his guidance and advice. I also thank Anirban Kar for his suggestions.

†School of Economics, the University of Adelaide, Adelaide, SA 5005, Australia; e-mail: duygu.yengin@adelaide.edu.au.

Various factors other than the cost may affect the route. Some of the sponsors may need to be visited before the others due to the urgency of their needs, their higher priority status, or the availability of their free times for a visit. Other examples include a communication network where the flow of information has to follow the specified network structure or a product which has to be processed in several departments in a firm according to the stage of its development (e.g. it can not be sent to the marketing department before quality control).

We assume that each customer is to be visited exactly once but home can be visited more than once, which may be necessary, for instance, when the service provider needs to replenish her supplies, or perform maintenance for the machinery/tools, after visiting a group of customers and before visiting the rest. There may also be customers who come to the main office for the service. Another reason may be that the traveler has appointments to meet with the customers and there is a considerable waiting time between two consecutive appointments. Then, in between those appointments, she would go home (office) and wait there.

Our goal is to find a fair distribution of the total cost generated in a fixed-route TSP among the sponsors. Note that the sponsors only share the total cost of the route. Hence, we implicitly assume that each agent pays the cost of the service provided to her separately.

One approach to solve this cost distribution problem is to define rules that select cost allocations for each fixed-route TSP directly. We follow the second approach which is to associate a *cooperative game with transferable utilities* (TU-game) with each fixed-route TSP and define rules that select allocations for the TU-game. A TU-game is a pair (N, v) where N is a finite set of agents and $v : 2^N \to \mathbb{R}$ is a characteristic function which assigns to each coalition $S \subseteq N$, a value $v(S)$ such that $v(\emptyset) = 0$. In the current context, $v(S)$ represents the cost of the tour in which only the members of S are served by the service provider. Potters et al (1992) formulate a TU-game associated with a fixed-route TSP as follows: for each coalition $S \subset N$, $v(S)$ is defined as the cost of the original route restricted to S, where the salesman visits the members of S in the same order as they were visited in the original route over N, skipping all agents in $N \backslash S$.[1] They refer to these games as *routing games.*

We introduce a new class of games which we refer as the class of *fixed-route traveling salesman games with appointments* (here after, *appointment games*). Consider the case in which each sponsor in N makes an appointment to meet the traveler at a specified time. After all the appointments are made, suppose the members of $S \subseteq N$ decide to hire the traveler without cooperating with the sponsors in $N \backslash S$. That is, the members of S together will pay $v(S)$ to the traveler. This can be thought as if all the sponsors in $N \backslash S$ cancel their appointments. The permissible route over S is the one where the traveler still visits the sponsors in S according to their original appointments. So, the traveler follows the original route, skipping the sponsors who are not in S, and *when she skips a sponsor, she goes home from where she goes to the next unvisited sponsor in* S. The value of a coalition S, is the cost of this permissible route over S.

Our formulation of permissible routes over coalitions makes sense in several TSPs where the service provider makes appointments which satisfy two conditions:

First, when some appointments are cancelled, the remaining ones can not be rescheduled due to the costs of rescheduling, inflexibility of the available or suitable times of other customers, etc. Hence, if some appointments are cancelled, the traveler still has to follow the

[1] Potters et al (1992) also studied TSPs where the route is not fixed. They introduced the *traveling salesman games,* where the value of a coalition is the cost of a least costly tour over the members of that coalition. The salesman is allowed to visit any agent more than once and he is free to visit the agents in a coalition in any order he wishes as long as the cost of the trip is minimized.

initial appointment schedule.

Second, the appointments are sparsely scheduled. This would be the case, for instance, when the service provider has to spend a considerable period of time to complete her service for the sponsor she visits. If an appointment is cancelled, then the traveler has to wait a lengthy period of time till the next appointment. Then, the service provider has two options: either she can wait at (or close by) the previous or next appointee's location till the next appointment starts, or she can go home (main office) and wait there. In many cases, the first option would be too costly and this cost is not reimbursed by the sponsors, hence the traveler would go home. For instance, think of a professor who wants to visit universities in different states at specific dates as a visiting professor. If a university cancels its appointment, there would be a few weeks waiting period till the next appointment. Hence, the professor goes back to her home and waits there until the appointed date for the next university arrives.[2]

Several papers discuss the "core" in traveling salesman games and routing games (see Derks and Kuipers, 1997; Engevall et al., 1998; Kuipers, 1993; Potters et al., 1992; Tamir, 1989). Here, we study another well-known solution, the "Shapley value" (Shapley, 1971). To our knowledge, the results we present here are the only ones so far on the characterizations of the Shapley value in TSPs. Also, we follow an axiomatic approach which differentiates this paper from the other papers analyzing the cost allocation problem in TSPs.

In general, the Shapley value is computationally complex. However, in appointment games, we show that this is not the case. We show that under a mild condition on the costs, the class of appointment games is convex, hence, in this class, the Shapley value is in the core. Moreover, the Shapley value may be an appealing alternative to core since it is always non-empty, single-valued, and is the unique solution satisfying certain desirable properties. The Shapley value has been characterized in general networks by Myerson (1977) and Jackson and Wolinsky (1996), in minimum cost spanning tree games by Kar (2002), and in scheduling and queuing problems by Maniquet (2003), Chun (2006, 2010) and Moulin (2007).

Our characterizations involve several variations of a strategic property called *merging and splitting proofness* which requires that a set of sponsors who follow each other on a route should not gain by merging or a sponsor should not gain by splitting into several sponsors located next to each other. We also analyze the Shapley value in the class of routing games. Our characterizations don't extend to the routing games. Potters et al (1992) specified the conditions which ensure that the core of routing games is non-empty. However, we show that these conditions do not guarantee the convexity of the routing games. Hence, we can not guarantee that Shapley value is in the core whenever it is non-empty.

In Section 2, the model is described. The results for the appointment games are presented in Section 3. Section 4 states the results for the routing games. All proofs are in the Appendix.

2 The Model

2.1 The Economy

Let $N = \{1, ..., n\}$ with $|N| = n \geq 2$ be an ordered list of sponsors and 0 be home. Without loss of generality, we assume that the sponsors are visited in the same order as they appear in N. Let $N^0 \equiv N \cup \{0\}$ and for each $S \subseteq N$, let $S^0 \equiv S \cup \{0\}$. A route $r = (i_1, i_2, ..., i_M)$ is

[2] For more examples and extended introduction, see the working paper version of our paper at http://economics.adelaide.edu.au/research/papers/doc/wp2009-28.pdf.

an ordered list of the agents (sponsors and home) to be visited by a "*traveler*" such that

(i) the route starts from home and ends at home (i.e. $i_1 = i_M = 0$),

(ii) each sponsor is visited exactly once,

(iii) home can be visited more than once,

(iv) after sponsor $i \in N$ is visited, either home or sponsor $i+1$ is to be visited (i.e. the relative order of the sponsors in r respect their order in N).

For each pair $\{i, j\} \subseteq N^0$, i is *connected* to j on a route r (denoted as $i \succ_r j$), if after i, the next agent visited is $j : r = (0, ..., i, j, ..., 0)$.

For each $\{i, j\} \subseteq N^0$, let $c_{i,j} \geq 0$ be the cost of traveling between agents i and j. Let $c_i \equiv c_{0,i} = c_{i,0}$ be the cost of traveling between home and sponsor i. The cost of a route r is $c(r) = \sum_{\{i,j\} \subseteq N^0 \colon i \succ_r j} c_{i,j}$.

Let $\mathbf{c} = \{c_{i,j} : \{i, j\} \subseteq N^0\}$. An *economy* is given by $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$. Let the domain of all economies be $\mathcal{E}$.

A sponsor set $S = \{l, l+1, ..., m-1, m\} \subseteq N$ is a *connected set on* r if and only if $0 \succ_r l \succ_r l+1 \succ_r ... \succ_r m-1 \succ_r m \succ_r 0$. Let $\mathcal{S}_\mathbf{e}$ be the set of all connected sets in economy $\mathbf{e}$.

In order to visualize the problem, we can associate a graph with each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$. The elements of N^0 are called *nodes,* 0 being the *source.* A *link* between nodes i and j (denoted as l_{ij}) is a direct path between them. Let $l_i \equiv l_{0i}$ be the link between home and i. Let $L = \{l_{ij} : \{i, j\} \subseteq N^0\}$ be the set of all links between all agents. A graph g over N^0 is a subset of L. The graph associated with $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ is $g(\mathbf{e}) = \{l_{ij} : \{i, j\} \subseteq N^0$ and $i \succ_r j\}$ where each link l_{ij} in $g(\mathbf{e})$ is associated with weight $c_{i,j}$.

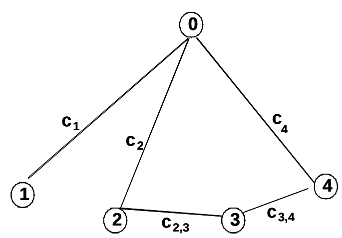

Figure 1

Example 1. Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$ with $r = (i_1, i_2, i_3, ..., i_7) = (0, 1, 0, 2, 3, 4, 0)$. The route r describes a trip where starting from 0 (home), the traveler visits sponsor 1, then goes back home. From home, she visits sponsors 2, 3, and 4, in that order, and returns home and completes the tour.
Here, the connected sets are $S = \{1\}$ and $S' = \{2, 3, 4\}$. Hence, $\mathcal{S}_\mathbf{e} = \{S, S'\}$. The cost of the route is $c(r) = 2c_1 + c_2 + c_{2,3} + c_{3,4} + c_4$. The associated graph $g(\mathbf{e})$ is as in Figure 1.

2.2 Appointment Games

Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $S \subseteq N$. Let the *permissible route over* S (denoted as r_S) be as follows:

Starting from home, the traveler first visits the smallest numbered sponsor in S, let us call this sponsor j_1. Suppose, in the original route r, after visiting sponsor j_1, the traveler visits agent (home or a sponsor) $i \in N^0$ (i.e. $j_1 \succ_r i$). If $i \in S^0$, then in route r_s, the traveler goes to i right after visiting j_1 (i.e. $j_1 \succ_{r_S} j_2 \equiv i$). If $i \notin S$, then it is as if i has cancelled her appointment. In this case, in r_S, after visiting j_1, the traveler goes home and she waits there till it is time to attend the next outstanding appointment with the sponsors in $S\backslash\{j_1\}$. That is, if $j_1 \succ_r i$ and $i \notin S$, then $j_1 \succ_{r_S} 0 \succ_{r_S} l$ where $l = \min\{k : k \in S \text{ and } k > j_1\}$). A similar procedure is followed until all the sponsors in S are visited, then the traveler returns home. Note that each time after the traveler visits home, the next agent she visits is the smallest numbered agent in S that has not been visited so far.

Formally, for some $T \geq |S|$, let $r_S = (0, j_1, j_2, ..., j_T, 0)$ be such that:

(i) for each $t \in \{1, .., T\}$, $j_t \in S^0$, and for each $i \in S$, there is a unique $t \in \{1, .., T\}$ such that $i \equiv j_t$ on r_S,

(ii) $j_1 = \min\limits_{i \in S} i$ and $j_T = \max\limits_{i \in S} i$,

(iii) for each $j_t \in S$ with $t \in \{1, 2, .., T\}$ and each $i \in N$ such that $j_t \succ_r i$, if $i \in S^0$, then $j_t \succ_{r_S} j_{t+1} \equiv i$, otherwise $j_t \succ_{r_S} j_{t+1} \equiv 0$, and

(iv) for each $j_t \equiv 0$ with $t \in \{2, .., T-1\}$, we have $j_t \succ_{r_S} \min\{k : k \in S \text{ and } k > j_{t-1}\}$.

Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$. For each $S \subseteq N$, let $\mathbf{c}_S = \{c_{i,j} \geq 0 : \{i, j\} \subseteq S^0\}$. The economy restricted to S with respect to r_S is $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S \rangle \in \mathcal{E}$.

Let (N, v) be a TU-game where $v : 2^N \rightarrow \mathbb{R}_+$ is a characteristic function such that $v(\emptyset) = 0$. For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, the *fixed-route traveling salesman game with appointments* (in short, *appointment game*) *associated with* $\mathbf{e}$ is $(N, v_{\mathbf{e}})$ where $v_{\mathbf{e}} : 2^N \rightarrow \mathbb{R}_+$ is such that for each $S \subseteq N$, $v_{\mathbf{e}}(S) = c(r_S)$. Let $\mathcal{V}_{\mathcal{E}}$ be the class of appointment games.

Note that $v_{\mathbf{e}}(N) = c(r)$ and for each $S \in \mathcal{S}_{\mathbf{e}}$, $v_{\mathbf{e}}(S) = c(r_S)$. Since $c(r) = \sum\limits_{S \in \mathcal{S}_{\mathbf{e}}} c(r_S)$, $v_{\mathbf{e}}(N) = \sum\limits_{S \in \mathcal{S}_{\mathbf{e}}} v_{\mathbf{e}}(S)$.

Example 2. Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$ where $r = (0, 1, 0, 2, 3, 4, 5, 0, 6, 0, 7, 8, 9, 0)$. Let $S = \{1, 4, 5, 6, 7, 9\}$. Then, $r_S = (0, 1, 0, 4, 5, 0, 6, 0, 7, 0, 9, 0)$. Here, $7 \succ_r 8$ but $8 \notin S$ (i.e. 8 cancelled her appointment). Thus, after visiting 7, the traveler goes home from where she goes to sponsor 9.

The graphs $g(\mathbf{e})$ and $g(\mathbf{e}_S) = \{l_{ij} : \{i, j\} \subseteq S^0 \text{ and } i \succ_{r_S} j\}$ are as in Figures 2a and 2b.

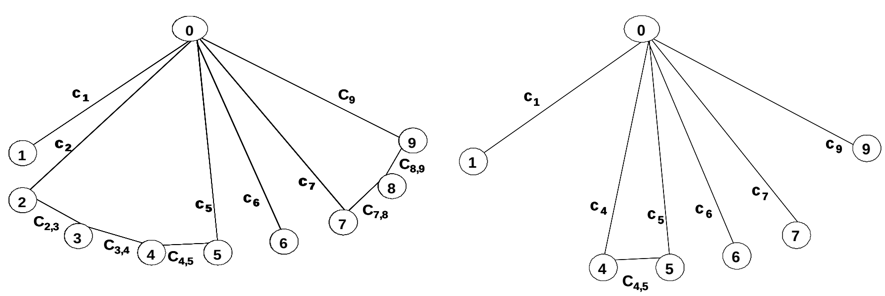

Figure 2a : $g(\mathbf{e})$ *Figure 2b :* $g(\mathbf{e}_S)$

2.3 The Shapley Value

A *solution* F is a mapping that associates with each TU-game (N, v), an allocation vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ where $\sum\limits_{i \in N} x_i = v(N)$. By abuse of notation, instead of $F(N, v)$, let $F(v)$ denote the allocation proposed by F for (N, v).

An example of a solution is the *Shapley value, SV:* for each (N, v) and each $i \in N$,

$$SV_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)].$$

In general, the Shapley value is computationally complex since we need to calculate the marginal contribution of each agent to each possible coalition. But, for appointment games, we can show that the Shapley value takes a simple form.

Proposition 1. *Let* $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$*,* $i \in N$*, and* $S_i \in \mathcal{S}_{\mathbf{e}}$ *be the connected set such that* $i \in S_i$*.*

- *If* $S_i = \{i\}$*, then*

$$SV_i(v_{\mathbf{e}}) = 2c_i.$$

- *If* $S_i \cap \{i-1, i+1\} = j$*, then*

$$SV_i(v_{\mathbf{e}}) = \frac{3c_i + c_{i,j} - c_j}{2}.$$

- *If* $\{i-1, i+1\} \subseteq S_i$*, then*

$$SV_i(v_{\mathbf{e}}) = \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}).$$

In an appointment game, due to the network structure, each agent is essentially cooperating with two sets of agents: the set of agents that come before her on the route, and the set of agents that come after her. Our calculation of the Shapley value in appointment games reflects this fact: for each sponsor $i \in S_i$, her Shapley value in an appointment game is the average of her marginal contribution to the coalition of sponsors that are in the same connected set as her and precede her on the route $(v_{\mathbf{e}}(\{j \in S_i : j \leq i\}) - v_{\mathbf{e}}(\{j \in S_i : j < i\}))$ and her marginal contribution to the coalition of sponsors that are in the same connected set as her and come after her on the route $(v_{\mathbf{e}}(\{j \in S_i : j \geq i\}) - v_{\mathbf{e}}(\{j \in S_i : j > i\}))$. Hence, effectively, for each agent, only her marginal contribution to these two sets of agents matter, even though the original formula of the Shapley value considers her marginal contribution to any subset of agents.

Note that in the appointment games, the Shapley value of a sponsor only depends on the cost of traveling from herself to home, and to the sponsors that are connected to her. A change in the cost of a sponsor to connect home affects only herself and the sponsors who are connected to her. Also, a change in the cost of traveling between two sponsors only affect those sponsors and affect them equally.

3 Characterizations of the Shapley Value in Appointment Games

3.1 The Core and the Shapley Value

In a cost allocation problem, the *core* of a TU-game (N, v) is the set of vectors $\mathbf{x} \in \mathbb{R}^n$ such that for each $S \subseteq N$, $\sum_{i \in S} x_i \leq v(S)$ and $\sum_{i \in N} x_i = v(N)$. If an allocation $\mathbf{x} \in \mathbb{R}^n_+$ is in the core of a game (N, v), then no coalition of sponsors has an incentive to leave the grand coalition N. In general, the core can be empty. Potters et al (1992) state that in the class of routing games, if the route r chosen for the grand coalition is a least-costly tour and triangle inequalities hold for all the agents (i.e. for each triple $\{i,j,k\} \subseteq N^0$, $c_{i,j} + c_{j,k} \geq c_{i,k}$), then the core is non-empty.

In appointment games, a much weaker condition is sufficient for the core to be non-empty. First of all, we do not need that r be a least costly tour for N. Second, we only need that given a route, for each pair of connected sponsors, the sum of their costs of connecting to home is greater than the cost of connecting to each other. Formally, for each r and each pair $\{i,j\} \subseteq N$ such that $i \succ_r j$, $c_i + c_j \geq c_{i,j}$. Let $\mathcal{E}_T$ be the set of economies in which this condition holds. Let $\mathcal{V}_{\mathcal{E}_T}$ be the class of appointment games associated with economies in $\mathcal{E}_T$. Actually, on $\mathcal{V}_{\mathcal{E}_T}$, we achieve more than the non-emptiness of the core. Here, we also have the convexity of the appointment games and hence, by Theorem 7 of Shapley (1971), the Shapley value is an element of the core.

Proposition 2. *On the domain $\mathcal{V}_{\mathcal{E}_T}$, appointment games are convex and the Shapley value is in the core.*

Proposition 2 shows that working on the domain $\mathcal{V}_{\mathcal{E}_T}$ is a sufficient condition for the core to be non-empty. It is easy to see that when there are only two agents, working on domain $\mathcal{V}_{\mathcal{E}_T}$ is also a necessary condition for the non-emptiness of the core: let $\mathbf{x} \in \mathbb{R}^2$ be in the core of $(\{i,j\}, v_{\mathbf{e}})$. The core conditions are: $x_i \leq 2c_i$, $x_j \leq 2c_j$, and $x_i + x_j = v_{\mathbf{e}}(\{i,j\})$. If $i \succ_r j$, that is $r = (0,i,j,0)$, then $v_{\mathbf{e}}(\{i,j\}) = c_i + c_{i,j} + c_j$. By the core conditions, $c_i + c_{i,j} + c_j = x_i + x_j \leq 2c_i + 2c_j$, which implies $c_i + c_j \geq c_{i,j}$. Hence, $(\{i,j\}, v_{\mathbf{e}}) \in \mathcal{V}_{\mathcal{E}_T}$.

In the rest of the paper, unless stated otherwise, the results hold on both of the domains $\mathcal{V}_{\mathcal{E}}$ and $\mathcal{V}_{\mathcal{E}_T}$.

Let us present other axioms that compare the cost shares of sponsors with the values of coalitions in different situations.

Although, the core compares, for each coalition, the sum of the cost shares of the sponsors in the coalition with the value of that coalition, the following two axioms are concerned with only the grand coalition N and singleton coalitions, respectively.

Efficiency: For each (N,v), $\sum_{i \in N} F_i(v) = v(N)$.

Individual Rationality: For each (N,v), $F_i(v) \leq v(\{i\})$.

Note that the Shapley value satisfies *Individual Rationality* only on $\mathcal{V}_{\mathcal{E}_T}$. To see this, let $\mathbf{e} = \langle \{i,j\}, \mathbf{c}, r \rangle$ with $i \succ_r j$ and suppose that $c_i + c_j < c_{i,j}$ (i.e. $\mathbf{e} \notin \mathcal{E}_T$). Then, since $SV_i(v_{\mathbf{e}}) = 1/2(3c_i + c_{i,j} - c_j)$ and $v(\{i\}) = 2c_i$, $SV_i(v_{\mathbf{e}}) > v(\{i\})$.

The following axiom states that in each connected set, the sponsors should together pay the value of that set. Hence, connected sets should not cross-subsidize each other.

Respect of Connected Sets: For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and each connected set $S \in \mathcal{S}_{\mathbf{e}}$,

$$\sum_{i \in S} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S).$$

Note that we define some of the axioms (such as *Efficiency*) for any TU-game where as some (such as *Respect of Connected Sets*) are defined only for those TU-games associated with fixed-route TSPs.

We also consider the following weakening of *Respect of Connected Sets* where for each connected set S, the sum of the cost shares of the sponsors in S sum up to an amount that depends on the value of S. Hence, instead of the cost of visiting all the sponsors in S, the traveler collects an amount from S which is a function of this cost. For instance, the service provider may charge a flat fee to each connected set, regardless of the cost of visiting them or may use markup pricing. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$.

Weak Respect of Connected Sets with respect to ϕ : For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and each connected set $S \in \mathcal{S}_{\mathbf{e}}$,

$$\sum_{i \in S} F_i(v_{\mathbf{e}}) = \phi(v_{\mathbf{e}}(S)).$$

Consider the difference between the value of a coalition consisting of only one sponsor and the cost share of this sponsor in the grand coalition. This difference measures how much a sponsor benefits from cooperating with the other sponsors rather than being alone. The following fairness axiom requires that in a two-sponsor TU-game, the sponsors should equally benefit from cooperation. In a sense, in two-sponsor games, we require the sponsors to have equal bargaining powers when it comes to sharing the benefits from cooperation.

Equal Benefit: For each $(\{i, j\}, v)$,

$$v(\{i\}) - F_i(v) = v(\{j\}) - F_j(v).$$

Hart and Mas-Colell (1989) call a solution F "standard for two-person games" if it satisfies *Efficiency* and *Equal Benefit*. For each $(\{i, j\}, v)$, such a solution divides the surplus $v(\{i, j\}) - v(\{i\}) - v(\{j\})$ equally among the sponsors.[3] Most solutions satisfy this requirement, one of them being the Shapley value.

Remark 1. A solution F satisfies *Efficiency* in two-sponsor TU-games and *Equal Benefit* if and only if for each (N, v) with $n = 2$ and each $i \in N$,

$$F_i(v) = v(\{i\}) + \frac{1}{2}\left[v(N) - v(\{i\}) - v(N \backslash \{i\})\right] = SV_i(v).$$

3.2 Mergers and the Shapley Value

There are several ways in which agents can collude. In the context of TU-games, two approaches can be noticed: either a new game with the same player set evolves when agents make binding agreements (Haller, 1994) or a group of agents merge into one player so that the set of players for the new game is reduced (Lehrer, 1988; Derks and Tijs, 2000, Knudsen and Østerdal, 2005). In some of the papers, only bilateral agreements/amalgamations are studied (e.g. Lehrer, 1988; Haller, 1994) or there is a given partition of the agent set that dictates which coalitions can merge (Derks and Tijs, 2000).

We consider mergers which result in a reduced player set. Also, instead of any group of sponsors, we only allow sponsors who follow each other on a route to merge or a sponsor to split into a set of consecutive sponsors. This requirement is intuitive especially when we think that sponsors can only effectively communicate with their neighbors in the network or when mergers of (or splits into) non-consecutive sponsors are easily detected. Suppose a group of

[3] Hart and Mas-Colell (1989) introduce the concept of "preservation of differences" which can be regarded as a generalization of the "equal division of the surplus" idea for two-person problems.

consecutive sponsors $K = \{k, k+1, k+2, ..., l\}$ for some $\{k, l\} \subseteq N$, forms a coalition and represents itself as a single sponsor $k \in K$ (i.e. K merges into k).[4] Note that K does not have to be a connected set, the traveler may visit home in between visiting any two sponsors in K. However, the traveler does not visit any sponsor outside K in between visiting any two sponsors in K.

In some other models, merging or splitting of agents is studied in a more literal sense of merging or splitting of the characteristics of the agents. For instance, in queuing problems, multiple jobs in the queue can literally merge and become one job. However, we analyze a different strategic situation: when sponsors in K merge, they do not change their physical locations. The traveler still has to visit the locations of these sponsors. However, the sponsors in K now represent themselves as a single entity which may change their bargaining power in the grand coalition. Hence, we are not considering changes in the economy (i.e., the agents do not merge their locations), but we are interested in changes in the cooperation/bargaining structure as a result of some agents' forming alliances between themselves. Therefore, we do not adopt the approach of defining rules and axioms directly for the economy. Instead, we define a TU-game that represents the cooperation structure within the economy and define our axioms for this TU-game.

Think of an electrician who provides maintenance service to a big firm with separate office buildings located closely. These different units can either bill the service provider separately, or they can bill together under the name of the firm which they belong to. Would it make a difference in their total cost share if the units acted separately or as a group?

Before the sponsors in set K merge, each sponsor in K had the authority to bargain/cooperate with other coalitions by herself, so her contribution to other coalitions mattered in the calculation of her cost share. After the merger, agents in K must act together as one entity to cooperate with other sponsors. By requiring K to act as a single entity, we are imposing restrictions on which coalitions can form: previously all subsets of N could interact, after the merger, the number of entities that can bargain within themselves falls to $n - |K| + 1$.

If K merges and acts like a single sponsor $k \in K$, then we assume that as a group, K is willing to pay the traveler up to $v_{\mathbf{e}}(K)$. This is because, the locations of the members in the merging coalition are still the same as before the merger, hence the stand-alone value of coalition K is same before and after the merger. If members of K hire the traveler alone without cooperating with others, they still have to cover the cost of being visited as a group, and this cost is still $v_{\mathbf{e}}(K)$. The resulting TU-game with the restricted cooperation structure after the merger can be defined as follows.

Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $K \subseteq N$ be such that $K = \{k, k+1, k+2, ..., l\}$ for some $1 \leq k < l \leq n$. Let $\widehat{v} : 2^{(N \setminus K) \cup \{k\}} \rightarrow \mathbb{R}_+^{n-|K|+1}$ be such that

- $\widehat{v}(\{k\}) = v_{\mathbf{e}}(K)$,
- for each $S \subseteq N \setminus K$; $\widehat{v}(S) = v_{\mathbf{e}}(S)$, and
- for each $S \subseteq N \setminus K$; $\widehat{v}(S \cup \{k\}) = v_{\mathbf{e}}(S \cup K)$.

We refer $((N \setminus K) \cup \{k\}, \widehat{v})$ as the TU-game obtained from $(N, v_{\mathbf{e}})$ when K merges into a single sponsor k.

The following axiom states that no group of consecutive sponsors can benefit from a merger or no sponsor benefits from splitting into a group of consecutive sponsors.

[4] Note that the choice of k as the representative of K is arbitrary, K can merge into any $i \in K$.

Merging and Splitting Proofness: For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, each $K = \{k, k+1, k+2, ..., l\}$ with $1 \leq k < l \leq n$, and each $((N \backslash K) \cup \{k\}, \widehat{v})$ as described above,

$$F_k(\widehat{v}) = \sum_{i \in K} F_i(v_{\mathbf{e}}).$$

We can strengthen *Merging and Splitting Proofness* by allowing for the possibility that there may be more than one merger at the same time. In this case, for each of the merging groups, the total cost its members pay should remain unchanged. Although, our proofs will also work with this stronger requirement, for our results, we only need a much weaker (but less intuitive) version of this requirement: Suppose the grand coalition is partitioned into two groups of consecutive sponsors and each of these groups merge into a single sponsor. Then, for none of these two groups, their total cost share should change by these mergers. Formally, let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $\{K, K'\} \subseteq 2^N$ be such that $K = \{1, 2, ..., k\}$ and $K' = \{k+1, k+2, ..., n\}$ for some $1 \leq k < n$. Let $\widetilde{v} : 2^{\{k,k'\}} \to \mathbb{R}^2_+$ be such that $\widetilde{v}(\{k\}) = v_{\mathbf{e}}(K)$, $\widetilde{v}(\{k'\}) = v_{\mathbf{e}}(K')$, and $\widetilde{v}(\{k, k'\}) = v_{\mathbf{e}}(N)$. Let $(\{k, k'\}, \widetilde{v})$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ when K merges into a single sponsor k and K' merges into a single sponsor $k' \in K'$.

Merging and Splitting Proofness-2: For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, each $\{K, K'\} \subseteq 2^N$, and each $(\{k, k'\}, \widetilde{v})$ as described above,

$$F_k(\widetilde{v}) = \sum_{i \in K} F_i(v_{\mathbf{e}}) \text{ and } F_{k'}(\widetilde{v}) = \sum_{i \in K'} F_i(v_{\mathbf{e}}).$$

Another variable population property is concerned with departures from the original economy in the following way. Let S be a connected set in $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$. Now, suppose all the sponsors which do not belong to S leave after paying their cost shares. Note that since S is a connected set, the cost of visiting the sponsors in S is same both before and after the departure of sponsors in $N \backslash S$. Hence, fairness may require that whether the sponsors in S cooperate with the grand coalition or not should not affect their cost shares. In other words, the sponsors in S should not be affected when the other sponsors leave the economy. Let $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S \rangle \in \mathcal{E}$ be the reduced economy after the departure of $N \backslash S$.

Consistency over Connected Sets: For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, each $S \in \mathcal{S}_{\mathbf{e}}$, and each $i \in S$,

$$F_i(v_{\mathbf{e}}) = F_i(v_{\mathbf{e}_S}).$$

Between the axioms stated so far, certain (sometimes rather obvious) logical relations hold as the following remark presents.

Remark 2. a) If a solution satisfies *Weak Respect of Connected Sets with respect to* ϕ for some $\phi : \mathbb{R}_+ \to \mathbb{R}$, *Merging and Splitting Proofness-2*, and *Equal Benefit*, then ϕ is an identity function (i.e. the solution satisfies *Respect of Connected Sets*).

b) *Efficiency* and *Merging and Splitting Proofness* together imply *Merging and Splitting Proofness-2.*

c) *Efficiency, Individual Rationality,* and *Merging and Splitting Proofness* together imply *Respect of Connected Sets.*

d) *Efficiency* in two-sponsor TU-games, *Merging and Splitting Proofness-2,* and *Equal Benefit* together imply *Respect of Connected Sets.*

e) *Core* implies *Respect of Connected Sets* which in turn implies *Efficiency* in two-sponsor TU-games.

f) *Efficiency* and *Consistency over Connected Sets* together imply *Respect of Connected Sets.*

By Remark 1, *Efficiency* in two-sponsor TU-games and *Equal Benefit* characterize the Shapley value for two sponsor TU-games. By Remark 2d, we also have *Respect of Connected Sets*. This axiom and *Merging and Splitting Proofness-2* lifts the characterization from two sponsor games to larger economies as stated in our main theorem.

Theorem 1. *The Shapley value is the only solution which satisfies Efficiency in two-sponsor TU-games, Merging and Splitting Proofness-2, and Equal Benefit.*

Several alternative combinations of axioms still characterize the Shapley value in appointment games due to the logical relations stated in Remark 2. For instance, by Remark 2e and Theorem 1, the Shapley value is also the only solution that satisfies *Respect of Connected Sets*, *Merging and Splitting Proofness-2*, and *Equal Benefit*. Moreover, by Remark 2a, we can weaken *Respect of Connected Sets* and still characterize the Shapley value. Also, by Remark 2b and Theorem 1, we have the following: *the Shapley value is the only solution which satisfies Efficiency, Merging and Splitting Proofness, and Equal Benefit*

One may argue that only those sponsors that belong to the same connected set can effectively communicate and hence can merge into a single sponsor. That is, the network structure does not permit sponsors in different connected sets to merge. We can weaken *Merging and Splitting Proofness-2* to take into account this argument by requiring that we can only apply the axiom when the agent set N itself is a connected set. In Theorem 1, if we replace *Merging and Splitting Proofness-2* with its weaker version and add *Consistency over Connected Sets*, we can still characterize the Shapley value (for a formal statement, see Proposition 2 in our working paper version).[5]

Hart and Mas-Colell (1989) characterize the Shapley value in general TU-games using *Efficiency* in two-sponsor TU-games, *Equal Benefit*, and a consistency property which, if adapted to our setting, is stronger than *Consistency over Connected Sets* since it allows the departure of any set of agents (not only the sponsors that are outside a given connected set) after paying their cost shares. Theorem 1 indicates that in appointment games, instead of consistency, we can use *Merging and Splitting Proofness-2* and still obtain the Shapley value.

In almost all characterizations of the Shapley value, *Efficiency* (or *Respect of Connected Sets*) is used. In Theorem 1, we weakened *Efficiency* so that it is only required to hold in two-sponsor TU-games. To see how far we would move away from the Shapley value when we drop the requirement that the traveler collects the cost of visiting sponsors, let us consider *Weak Respect of Connected Sets with respect to* ϕ for some $\phi : \mathbb{R}_+ \to \mathbb{R}$. Remark 2a states that if $\phi : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary function and not necessarily the identity function, then *Weak Respect of Connected Sets with respect to* ϕ, *Merging and Splitting Proofness-2*, and *Equal Benefit* are incompatible. For instance, if the traveler wants to use markup pricing or a flat fee, she can not use a solution which satisfies *Merging and Splitting Proofness-2* and

[5] Note that the same result can be obtained if we replaced *Weak Merging and Splitting Proofness-2* with the following stronger but more intuitive requirement: there may be more than one merger at the same time but only the sponsors that belong to the same connected set are allowed to merge, then, for each of the merging groups, the total cost its members pay should remain unchanged.

Equal Benefit. The proof of this incompatibility result requires using economies with at least three connected sets. Let $\mathcal{E}^2 = \{\mathbf{e} \in \mathcal{E} : |\mathcal{S}_\mathbf{e}| \leq 2\}$ be the set of economies with at most two connected sets and $\mathcal{V}_{\mathcal{E}^2}$ be the class of appointment games associated with economies in $\mathcal{E}^2$. If F is defined on $\mathcal{V}_{\mathcal{E}^2}$, then F can satisfy the aforementioned three axioms and yet ϕ does not have to be an identity function. The interesting point is that F would still be closely related to the Shapley value: in economies where it is defined, F coincides with the Shapley value for each sponsor $i \in N$ that is connected to both $i-1$ and $i+1$. Hence, weakening *Respect of Connected Sets* only affects the cost shares of the first and the last sponsors to be visited in any connected set. This result shows that, on $\mathcal{V}_{\mathcal{E}^2}$, the traveler can be flexible in her pricing strategy and still not move too far away from the Shapley value in appointment games.

Proposition 3. *Let F be solution defined on $\mathcal{V}_{\mathcal{E}^2}$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$.*

a) *Solution F satisfies Weak Respect of Connected Sets with respect to ϕ, Merging and Splitting Proofness-2, and Equal Benefit if and only if for each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ such that $|\mathcal{S}_\mathbf{e}| = 1$ and each $i \in N$,*

$$\begin{aligned} F_i(v_\mathbf{e}) &= \tfrac{1}{2}(\phi(v_\mathbf{e}(N)) + v_\mathbf{e}(\{i\}) - v_\mathbf{e}(N\backslash\{i\})), && \text{if } i \in \{1, n\}, \\ &= SV_i(v_\mathbf{e}), && \text{if } i \notin \{1, n\}. \end{aligned}$$

b) *Solution F satisfies Weak Respect of Connected Sets with respect to ϕ, Merging and Splitting Proofness-2, Equal Benefit, and Consistency over Connected Sets if and only if for each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ such that $|\mathcal{S}_\mathbf{e}| = 2$, each $i \in N$ and $S_i \in \mathcal{S}_\mathbf{e}$ with $i \in S_i$,*

$$\begin{aligned} F_i(v_\mathbf{e}) &= \phi(v_\mathbf{e}(\{i\})), && \text{if } |\{i-1, i+1\} \cap S_i| = 0, \\ &= \tfrac{1}{2}(\phi(v_\mathbf{e}(\{S_i\})) + v_\mathbf{e}(\{i\}) - v_\mathbf{e}(S_i\backslash\{i\})), && \text{if } |\{i-1, i+1\} \cap S_i| = 1, \\ &= SV_i(v_\mathbf{e}), && \text{if } |\{i-1, i+1\} \cap S_i| = 2. \end{aligned}$$

4 The Shapley Value in Routing Games

Potters et al (1992) introduced the *routing games* to analyze the cost allocation problem in fixed-route TSPs, and they studied the *Core* in these games. Here, we analyze the Shapley value in routing games.

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $S \subset N$. The permissible route over S in a routing game is the one where the traveler follows the original route r, skipping all the sponsors who are absent in S. Let r^*_S be the resulting route over S. The *routing game* associated with $\mathbf{e}$ is $(N, v^*_\mathbf{e})$ where $v^*_\mathbf{e} : 2^N \to \mathbb{R}_+$ is such that for each $S \subseteq N$, $v^*_\mathbf{e}(S) = c(r^*_S)$. The axioms in Section 3 can be stated for routing games just by replacing all $v_\mathbf{e}$ with $v^*_\mathbf{e}$, r_S with r^*_S, etc.

Example 3. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ where $r = (0,1,0,2,3,4,5,0,6,0,7,8,9,0)$. Let $S = \{1,4,5,6,7,9\}$. Then, $r^*_S = (0,1,0,4,5,0,6,0,7,9,0)$ and $v^*_\mathbf{e}(S) = 2c_1 + c_4 + c_{4,5} + c_5 + 2c_6 + c_7 + c_{7,9} + c_9$.

The results we derived in Section 3 do not carry over to the class of routing games. First of all, in the class of routing games, the Shapley value doesn't reduce into a simple formula as it does in the class of appointment games. Moreover, Theorem 1 no longer holds in the class of routing games since the Shapley value violates *Merging and Splitting Proofness-2.* The following example demonstrates this fact and the calculation of the Shapley value for a 3-sponsor economy.

Example 4. Let $N = \{1, 2, 3\}$ and $r = (0, 1, 2, 3, 0)$. For each $S \subseteq N$, let $|S| = s$ and $f(s) = \frac{s!(n-s-1)!}{n!}$. Then, $(N, v^*_{\mathbf{e}})$ is such that

$f(s)$	$S : 1 \notin S$	$v^*_{\mathbf{e}}(S \cup \{1\}) - v^*_{\mathbf{e}}(S)$	$S : 2 \notin S$	$v^*_{\mathbf{e}}(S \cup \{2\}) - v^*_{\mathbf{e}}(S)$	$S : 3 \notin S$	$v^*_{\mathbf{e}}(S \cup \{3\}) - v^*_{\mathbf{e}}(S)$
$2/6$	$\emptyset$	$2c_1$	$\emptyset$	$2c_2$	$\emptyset$	$2c_3$
$1/6$	$\{2\}$	$c_1 + c_{1,2} - c_2$	$\{1\}$	$c_2 + c_{1,2} - c_1$	$\{1\}$	$c_3 + c_{1,3} - c_1$
$1/6$	$\{3\}$	$c_1 + c_{1,3} - c_3$	$\{3\}$	$c_2 + c_{2,3} - c_3$	$\{2\}$	$c_3 + c_{2,3} - c_2$
$2/6$	$\{2, 3\}$	$c_1 + c_{1,2} - c_2$	$\{1, 3\}$	$c_{1,2} + c_{2,3} - c_{1,3}$	$\{1, 2\}$	$c_3 + c_{2,3} - c_2$

Since for each $i \in N$, $SV_i(v^*_{\mathbf{e}}) = \sum_{S \subseteq N \setminus \{i\}} f(s)[v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S)]$, we have

$$\begin{aligned} SV_1(v^*_{\mathbf{e}}) &= \frac{4}{3}c_1 - \frac{1}{2}c_2 - \frac{1}{6}c_3 + \frac{1}{2}c_{1,2} + \frac{1}{6}c_{1,3} \\ SV_2(v^*_{\mathbf{e}}) &= c_2 - \frac{1}{6}c_1 - \frac{1}{6}c_3 + \frac{1}{2}c_{1,2} - \frac{1}{3}c_{1,3} + \frac{1}{2}c_{2,3} \\ SV_3(v^*_{\mathbf{e}}) &= \frac{4}{3}c_3 - \frac{1}{2}c_2 - \frac{1}{6}c_1 + \frac{1}{6}c_{1,3} + \frac{1}{2}c_{2,3}. \end{aligned}$$

Let $c_1 = 30$, $c_2 = 6$, $c_3 = 15$, $c_{1,2} = 25$, $c_{1,3} = 16$, $c_{2,3} = 20$. Note that $\mathbf{e} \in \mathcal{E}_T$. We have $SV_1(v^*_{\mathbf{e}}) = \frac{149}{3}$, $SV_2(v^*_{\mathbf{e}}) = \frac{47}{3}$, and $SV_3(v^*_{\mathbf{e}}) = \frac{74}{3}$.

Let sponsors 1 and 2 merge into a single sponsor denoted by k. Let $((N \setminus \{1,2\}) \cup \{k\}, \widehat{v})$ be the TU-game obtained from $(N, v^*_{\mathbf{e}})$ by this merger. Thus, $\widehat{v}(\{k\}) = v^*_{\mathbf{e}}(\{1,2\}) = c_1 + c_{1,2} + c_2$, $\widehat{v}(\{k,3\}) - \widehat{v}(\{3\}) = c_1 + c_{1,2} + c_{2,3} - c_3$. Then, $SV_k(\widehat{v}) = \frac{1}{2}(2c_1 + c_2 - c_3 + 2c_{1,2} + c_{2,3}) = \frac{121}{2}$. Since, $SV_k(\widehat{v}) \neq SV_1(v^*_{\mathbf{e}}) + SV_2(v^*_{\mathbf{e}})$, SV is not *Merging and Splitting Proof* or *Merging and Splitting Proof-2.*

Potters et al (1992) state that in the class of routing games, if the route r chosen for the grand coalition is a least-costly tour and triangle inequalities hold for all the agents (i.e. for each triple $\{i, j, k\} \subseteq N^0$, $c_{i,j} + c_{j,k} \geq c_{i,k}$), then the core is non-empty. Let $\mathcal{E}^*_T$ be the set of economies in which these conditions hold. Note that to ensure the convexity of appointment games (which also implies the non-emptiness of the core), we only needed the triangle inequalities to hold for those sponsors who are connected rather than all sponsors and we did not need the route r be a least-costly route for the economy. Hence, $\mathcal{E}_T$ is a larger set of economies than $\mathcal{E}^*_T$. Let $\mathcal{V}_{\mathcal{E}^*_T}$ be the class of routing games associated with economies in $\mathcal{E}^*_T$

We know that if a class of TU-games is convex, then the Shapley value is an element of the core. In general, routing games are not convex. Here, we show that even under the conditions Potters et al (1992) specify for the non-emptiness of the core, the routing games are still not convex. Hence, we do not know for sure that Shapley value is in the core whenever it is non-empty. It is an open question to characterize the conditions under which the Shapley value is in the core of routing games.

Proposition 4. *On the domain $\mathcal{V}_{\mathcal{E}^*_T}$, the routing games are not convex.*

5 Appendix

Proof of Proposition 1:

Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, $i \in N$, and $S_i \subseteq N$ be the connected set such that $i \in S_i$. For each $S \subseteq N$, let $|S| = s$ and $f(s) = \frac{s!(n-s-1)!}{n!}$.

- If $S_i = \{i\}$, then since for each $S \subseteq N\backslash\{i\}$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = v_{\mathbf{e}}(\{i\}) = 2c_i$, we have

$$SV_i(v_{\mathbf{e}}) = 2c_i.$$

- If $S_i \cap \{i-1, i+1\} = j$, then since

for each $S \subseteq N\backslash\{i\}$ such that $j \in S$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = c_i + c_{i,j} - c_j$, and

for each $S \subseteq N\backslash\{i\}$ such that $j \notin S$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = 2c_i$, we have

$$\begin{aligned} SV_i(v_{\mathbf{e}}) &= \sum_{S \subseteq N\backslash\{i\}} f(s)\,(v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S)) \\ &= \sum_{S \subseteq N\backslash\{i\}: j \in S} f(s)\,(c_i + c_{i,j} - c_j) + \sum_{S \subseteq N\backslash\{i,j\}} f(s)\,(2c_i) \\ &= (c_i + c_{i,j} - c_j) \sum_{s=1}^{n-1} \binom{n-2}{s-1} f(s) + 2c_i \sum_{s=0}^{n-2} \binom{n-2}{s} f(s) \\ &= (c_i + c_{i,j} - c_j)\frac{1}{2} + (2c_i)\frac{1}{2} \\ &= \frac{3c_i + c_{i,j} - c_j}{2}. \end{aligned}$$

Here, $\binom{n-2}{s-1}$ is the number of $(s-1)$-combinations from the set $N\backslash\{i,j\}$. It gives us the number of subsets of $N\backslash\{i\}$ that contains j and has s number of sponsors: to find such subsets, we need to pick $s-1$ sponsors from the set $N\backslash\{i,j\}$. Similar interpretation applies to $\binom{n-2}{s}$ and all other binomial coefficients from now on.

- If $\{i-1, i+1\} \subseteq S_i$, then since

for each $S \subseteq N\backslash\{i\}$ such that $\{i-1, i+1\} \subseteq S$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$,

for each $S \subseteq N\backslash\{i\}$ such that $S \cap \{i-1, i+1\} = \{j\}$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = c_i + c_{i,j} - c_j$, and

for each $S \subseteq N\backslash\{i\}$ such that $S \cap \{i-1, i+1\} = \emptyset$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = 2c_i$, we have

$$\begin{aligned} SV_i(v_{\mathbf{e}}) = &\sum_{S \subseteq N\backslash\{i\}: \{i-1,i+1\} \subseteq S} f(s)\,(c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}) \\ &+ \sum_{S \subseteq N\backslash\{i\}: \{i-1,i+1\} \cap S = \{i-1\}} f(s)\,(c_i + c_{i-1,i} - c_{i-1}) \\ &+ \sum_{S \subseteq N\backslash\{i\}: \{i-1,i+1\} \cap S = \{i+1\}} f(s)\,(c_i + c_{i,i+1} - c_{i+1}) + \sum_{S \subseteq N\backslash\{i\}: \{i-1,i+1\} \cap S = \emptyset} f(s)\,(2c_i) \\ &= (c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}) \sum_{s=2}^{n-1} \binom{n-3}{s-2} f(s) + (c_i + c_{i-1,i} - c_{i-1}) \sum_{s=1}^{n-2} \binom{n-3}{s-1} f(s) \\ &+ (c_i + c_{i,i+1} - c_{i+1}) \sum_{s=1}^{n-2} \binom{n-3}{s-1} f(s) + 2c_i \sum_{s=0}^{n-3} \binom{n-3}{s} f(s) \\ &= (c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1})\,\tfrac{1}{3} + (c_i + c_{i-1,i} - c_{i-1})\,\tfrac{1}{6} + (c_i + c_{i,i+1} - c_{i+1})\,\tfrac{1}{6} + (2c_i)\tfrac{1}{3} \end{aligned}$$

$= \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}).$ □

Proof of Proposition 2: Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}_T$ and $i \in N$. Let $K = \{j \in N\backslash\{i\} :$ either $i \succ_r j$ or $j \succ_r i\}$.[6] Note that on $\mathcal{E}_T$, for each $j \in K$,

$$c_i + c_j \geq c_{i,j}. \tag{1}$$

We need to show that for each $S \subseteq T \subseteq N\backslash\{i\}$,

$$v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) \geq v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T). \tag{2}$$

There are 6 possible cases. We will show that in each case, (2) holds.

1. $K \cap S = \emptyset$. Then, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = 2c_i$.
a) $K \cap T = \emptyset$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = 2c_i$. Hence, (2) holds.
b) $K \cap T = \{j\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{j,i} + c_i - c_j$. Hence, by (1), (2) holds.
c) $K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, by (1), (2) holds.

2. $K \cap S = \{j\}$. Then, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = c_{j,i} + c_i - c_j$.
a) $K \cap T = \{j\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{j,i} + c_i - c_j$. Hence, (2) holds.
b) $K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, by (1), (2) holds.

3. $K \cap S = K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, (2) holds. □

Proof of Remark 1: Let F satisfy *Efficiency* in two-sponsor TU-games and *Equal Benefit*. Let (N, v) be such that $n = 2$. Then, for each $i \in N$ and $j = N\backslash\{i\}$, by *Equal Benefit*, (I) $F_i(v) - F_j(v) = v(\{i\}) - v(\{j\})$, and by *Efficiency*, (II) $v(N) = F_i(v) + F_j(v)$. By (I) and (II), for each $i \in N$, $F_i(v) = \frac{1}{2}[v(N) + v(\{i\}) - v(\{j\})] = SV_i(v)$. □

Proof of Remark 2:

a) Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ and F satisfy the first 3 axioms listed in Remark 2a. We need to show that for each $a \in \mathbb{R}_+$, $\phi(a) = a$. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ be such that $|\mathcal{S}_{\mathbf{e}}| \geq 3$ and there is $S \in \mathcal{S}_{\mathbf{e}}$ with $S = \{l, l+1, ..., m\}$ for some $1 < l \leq m < n$ and $v_{\mathbf{e}}(S) = a$. Let $K_1 = \{i \in N : i < l\}$ and $K_2 = \{i \in N : i > m\}$.

Let K_1 and S merge into a single sponsor denoted by $k_1 \in K_1$ and K_2 merge into a single sponsor denoted by n. Let $(\{k_1, n\}, v^1)$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by these mergers. Thus, $v^1(\{k_1\}) = v_{\mathbf{e}}(K_1 \cup S) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S)$, $v^1(\{n\}) = v_{\mathbf{e}}(K_2)$, and $v^1(\{k_1, n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

Since $(\{k_1, n\}, v^1)$ is a two-sponsor TU-game, by *Equal Benefit,*

$$F_{k_1}(v^1) - F_n(v^1) = v^1(\{k_1\}) - v^1(\{n\}) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_2). \tag{3}$$

[6] If $0 \succ_r i \succ_r 0$, then $K = \emptyset$. If $i-1 \succ_r i \succ_r 0$, then $K = \{i-1\}$. If $0 \succ_r i \succ_r i+1$, then $K = \{i+1\}$. If $i-1 \succ_r i \succ_r i+1$, then $K = \{i-1, i+1\}$.

By *Merging and Splitting Proofness-2,* $F_{k_1}(v^1) = \sum_{i\in K_1} F_i(v_{\mathbf{e}}) + \sum_{i\in S} F_i(v_{\mathbf{e}})$ and $F_n(v^1) = \sum_{i\in K_2} F_i(v_{\mathbf{e}})$. These equalities and (3) together imply

$$\sum_{i\in K_1} F_i(v_{\mathbf{e}}) + \sum_{i\in S} F_i(v_{\mathbf{e}}) - \sum_{i\in K_2} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_2). \tag{4}$$

Note that K_1 is a union of connected sets and so is K_2. Let $\Phi(K_1) = \sum_{S'\subseteq K_1 : S'\in\mathcal{S}_{\mathbf{e}}} \phi(v_{\mathbf{e}}(S'))$ and $\Phi(K_2) = \sum_{S'\subseteq K_2 : S'\in\mathcal{S}_{\mathbf{e}}} \phi(v_{\mathbf{e}}(S'))$. By *Weak Respect of Connected Sets* and (4),

$$\phi(v_{\mathbf{e}}(S)) = \Phi(K_2) - \Phi(K_1) + v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_2). \tag{5}$$

Now, let K_2 and S merge into a single sponsor denoted by $k_2 \in K_2$ and K_1 merge into a single sponsor denoted by 1. Let $(\{1, k_2\}, v^2)$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by these mergers. Again, by *Equal Benefit,* (I) $F_{k_2}(v^2) - F_1(v^2) = v_{\mathbf{e}}(K_2) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_1)$. By *Merging and Splitting Proofness-2,* (II) $F_{k_2}(v^2) = \sum_{i\in K_2} F_i(v_{\mathbf{e}}) + \sum_{i\in S} F_i(v_{\mathbf{e}})$ and $F_1(v^2) = \sum_{i\in K_1} F_i(v_{\mathbf{e}})$. By (I), (II), and *Weak Respect of Connected Sets,*

$$\phi(v_{\mathbf{e}}(S)) = \Phi(K_1) - \Phi(K_2) + v_{\mathbf{e}}(K_2) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_1). \tag{6}$$

By (5) and (6),

$$\Phi(K_2) - v_{\mathbf{e}}(K_2) = \Phi(K_1) - v_{\mathbf{e}}(K_1). \tag{7}$$

Since $v_{\mathbf{e}}(S) = a$, substituting (7) into (5), $\phi(a) = a$. Since we can repeat this procedure for any $a \in \mathbb{R}_+$, ϕ is the identity function. Therefore, F satisfies *Respect of Connected Sets.*

b) Let F satisfy the first two axioms in Remark 1b. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $\{K, K'\} \subseteq 2^N$ be such that $K = \{1, 2, ..., k\}$ and $K' = \{k+1, k+2, ..., n\}$ for some $1 \le k < n$. Let $(\{k\} \cup K', \widehat{v})$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ when K merges into a single sponsor k. By *Efficiency,* $F_k(\widehat{v}) = \widehat{v}(\{k\} \cup K') - \sum_{i\in K'} F_i(\widehat{v})$ and $\sum_{i\in K} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(N) - \sum_{i\in K'} F_i(v_{\mathbf{e}})$. Since $\widehat{v}(\{k\} \cup K') = v_{\mathbf{e}}(N)$ and by *Merging and Splitting Proofness,* $F_k(\widehat{v}) = \sum_{i\in K} F_i(v_{\mathbf{e}})$, we have (I) $\sum_{i\in K'} F_i(\widehat{v}) = \sum_{i\in K'} F_i(v_{\mathbf{e}})$.

Now, let $(\{k, k'\}, \widetilde{v})$ be the TU-game obtained from $(\{k\} \cup K', \widehat{v})$ when K' merges into a single sponsor $k' \in K'$. By *Merging and Splitting Proofness,* $F_{k'}(\widetilde{v}) = \sum_{i\in K'} F_i(\widehat{v})$. This equality and (I) together imply (II) $F_{k'}(\widetilde{v}) = \sum_{i\in K'} F_i(v_{\mathbf{e}})$. Also, by *Efficiency,* $F_{k'}(\widetilde{v}) = \widetilde{v}(\{k, k'\}) - F_k(\widetilde{v})$ and $\sum_{i\in K'} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(N) - \sum_{i\in K} F_i(v_{\mathbf{e}})$. Since $\widetilde{v}(\{k, k'\}) = v_{\mathbf{e}}(N)$, by (II), $F_k(\widetilde{v}) = \sum_{i\in K} F_i(v_{\mathbf{e}})$. This equality and (II) together imply that F satisfies *Merging and Splitting Proofness-2.*

c) Let F satisfy the first 3 axioms listed in Remark 2c. Suppose, by contradiction, that F does not satisfy *Respect of Connected Sets.* Then, there are $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}_T$ and $\{S', S''\} \subseteq \mathcal{S}_{\mathbf{e}}$ such that $\sum_{i\in S''} F_i(v_{\mathbf{e}}) < v_{\mathbf{e}}(S'')$ and (I) $\sum_{i\in S'} F_i(v_{\mathbf{e}}) > v_{\mathbf{e}}(S')$. Such S' and S'' exist since by *Efficiency,* $\sum_{S\in\mathcal{S}_{\mathbf{e}}} \left(\sum_{i\in S} F_i(v_{\mathbf{e}})\right) = v_{\mathbf{e}}(N) = \sum_{S\in\mathcal{S}_{\mathbf{e}}} v_{\mathbf{e}}(S)$.

Now, let S' merge into a single sponsor denoted by $s' \in S'$. Let $((N \backslash S') \cup \{s'\}, \widehat{v})$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by this merger. Thus, (II) $\widehat{v}(\{s'\}) = v_{\mathbf{e}}(S')$. By *Merging and Splitting Proofness*, (III) $F_{s'}(\widehat{v}) = \sum_{i \in S'} F_i(v_{\mathbf{e}})$. By *Individual Rationality*, (IV) $F_{s'}(\widehat{v}) \leq \widehat{v}(\{s'\})$. By (II), (III), and (IV), $\sum_{i \in S'} F_i(v_{\mathbf{e}}) \leq v_{\mathbf{e}}(S')$ which contradicts (I).

d) Let F satisfy the first 3 axioms listed in Remark 2d. Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $\mathcal{S}_{\mathbf{e}} = \{S_1, S_2, ..., S_T\}$ for some $T \leq n$. The proof is by induction.

$*$*Base Step:* Let S_1 merge into a single sponsor denoted by 1 and $N \backslash S_1$ merge into a single sponsor denoted by n. Let $(\{1, n\}, v^1)$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by these mergers. Thus, $v^1(\{1\}) = v_{\mathbf{e}}(S_1)$, $v^1(\{n\}) = v_{\mathbf{e}}(N \backslash S_1)$, and $v^1(\{1, n\}) = v_{\mathbf{e}}(N)$. Note that since S_1 is a connected set, $v_{\mathbf{e}}(N) = v_{\mathbf{e}}(S_1) + v_{\mathbf{e}}(N \backslash S_1)$. These equalities and Remark 1 together imply

$$\begin{aligned} F_1(v^1) &= \tfrac{1}{2}\left[v^1(\{1, n\}) + v^1(\{1\}) - v^1(\{n\})\right], \\ &= v_{\mathbf{e}}(S_1). \end{aligned} \tag{8}$$

By *Merging and Splitting Proofness-2*, $F_1(v^1) = \sum_{i \in S_1} F_i(v_{\mathbf{e}})$. This equality and (8) together imply $\sum_{i \in S_1} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S_1)$.

$*$*Induction Step:* Let $k < T$. Assume that for each $t < k$, $\sum_{i \in S_t} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S_t)$. We will prove that $\sum_{i \in S_k} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S_k)$.

Let $\{S_1, S_2, ..., S_k\}$ merge into a single sponsor denoted by k, and $\{S_{k+1}, .., S_T\}$ merge into a single sponsor denoted by n. Let $(\{k, n\}, v^k)$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by these mergers. Thus, $v^k(\{k\}) = v_{\mathbf{e}}(\bigcup_{t=1}^{k} S_t) = \sum_{t=1}^{k} v_{\mathbf{e}}(S_t)$, $v^k(\{n\}) = v_{\mathbf{e}}(\bigcup_{t=k+1}^{T} S_t) = \sum_{t=k+1}^{T} v_{\mathbf{e}}(S_t)$, and $v^k(\{k, n\}) = v_{\mathbf{e}}(N) = \sum_{t=1}^{k} v_{\mathbf{e}}(S_t) + \sum_{t=k+1}^{T} v_{\mathbf{e}}(S_t)$.

These equalities and Remark 1 together imply

$$F_k(v^k) \quad = \quad v_{\mathbf{e}}(S_k) + \sum_{t=1}^{k-1} v_{\mathbf{e}}(S_t). \tag{9}$$

By *Merging and Splitting Proofness-2*, $F_k(v^k) = \sum_{i \in S_k} F_i(v_{\mathbf{e}}) + \sum_{t=1}^{k-1} \sum_{i \in S_t} F_i(v_{\mathbf{e}})$. This equality, (9), and the induction hypothesis together imply $\sum_{i \in S_k} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S_k)$.

$*$*Conclusion Step:* By the Base and the Induction steps, for each $t < T$, $\sum_{i \in S_t} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S_t)$. Now, consider $(\{T-1, n\}, v^{T-1})$ obtained from $(N, v_{\mathbf{e}})$ when S_T merges into a single sponsor denoted by n and $N \backslash S_T$ merge into a single sponsor denoted by $T-1$. Similar to the argument in the Base step, we have $\sum_{i \in S_T} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S_T)$. Therefore, F satisfies *Respect of Connected Sets*.

e) For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, let $F(v_{\mathbf{e}})$ be in the *Core* of $(N, v_{\mathbf{e}})$. Then, $\sum_{S\in\mathcal{S}_{\mathbf{e}}} \sum_{i\in S} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(N)$ and for each $S \in \mathcal{S}_{\mathbf{e}}$, $\sum_{i\in S} F_i(v_{\mathbf{e}}) \leq v_{\mathbf{e}}(S)$. Since $v_{\mathbf{e}}(N) = \sum_{S\in\mathcal{S}_{\mathbf{e}}} v_{\mathbf{e}}(S)$, for each $S \in \mathcal{S}_{\mathbf{e}}$, $\sum_{i\in S} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S)$ and F satisfies *Respect of Connected Sets.*

Now, let F satisfy *Respect of Connected Sets.* Let $(\{i, j\}, v)$ be a two-sponsor TU-game. Let $\mathbf{e} = \langle\{i, j\}, \mathbf{c}, r\rangle$ be such that $c_i = v(\{i\})/2$, $c_j = v(\{j\})/2$, $c_i + c_{i,j} + c_j = v(\{i, j\})$, and $r = (0, i, j, 0)$. By *Respect of Connected Sets,* (I) $F_i(v_{\mathbf{e}}) + F_j(v_{\mathbf{e}}) = v_{\mathbf{e}}(\{i, j\})$. Since for each $S \subseteq \{i, j\}$, $v_{\mathbf{e}}(S) = v(S)$, we have $(N, v) \equiv (N, v_{\mathbf{e}})$. This equivalency and (I) together imply that F satisfies *Efficiency* in two-sponsor TU-games.

f) Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $S \in \mathcal{S}_{\mathbf{e}}$. Consider $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S\rangle \in \mathcal{E}$. By *Efficiency,* (I) $\sum_{i\in S} F_i(v_{\mathbf{e}_S}) = v_{\mathbf{e}_S}(S)$. By *Consistency over Connected Sets,* for each $i \in S$, (II) $F_i(v_{\mathbf{e}}) = F_i(v_{\mathbf{e}_S})$. Note that by definition, $v_{\mathbf{e}}(S) = c(r_S) = v_{\mathbf{e}_S}(S)$. Hence, by (I) and (II), $\sum_{i\in S} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S)$. That is, F satisfies *Respect of Connected Sets.* ■

Proof of Theorem 1:

It is easy to see that the Shapley value satisfies *Efficiency* and *Equal Benefit.* Next, we show it satisfies *Merging and Splitting Proofness.* Then, by Remark 2b, it also satisfies *Merging and Splitting Proofness-2.*

Merging and Splitting Proofness:

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $K \subseteq N$ be such that $K = \{k, k+1, k+2, ..., l\}$ for some $1 \leq k < l \leq n$. Let $((N\backslash K) \cup \{k\}, \widehat{v})$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ when K merges into k.

Note that K may involve some connected sets. For some $1 \leq M \leq |K|$, let $P_K = \{K_1, K_2, ..., K_M\}$ be the partitioning of K such that

$*$ for each $m \in \{1, M-1\}$, each $i \in K_m$, and each $j \in K_{m+1}$, we have $i < j$,
$*$ for each $m \in \{1, M\}$, $K_m \subseteq S$ for some $S \in \mathcal{S}_{\mathbf{e}}$, and
$*$ for each $m \notin \{1, M\}$, $K_m \in \mathcal{S}_{\mathbf{e}}$.

For example, if $r = (0, 1, 2, 3, 4, 0, 5, 6, 0, 7, 0, 8, 9, 0)$ and $K = \{3, 4, .., 8\}$, then $P_K = \{\{3, 4\}, \{5, 6\}, \{7\}, \{8\}\}$.

Note that there are $n - |K| + 1$ agents in the game $((N\backslash K) \cup \{k\}, \widehat{v})$. For each $S \subseteq N$, let $|S| = s$ and $g(s) = \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$. For each $1 \leq m \leq M$, let $K_m = \{k_m, k_m + 1, ..., l_m\}$. The following four cases are possible.

1) $P_K \subseteq \mathcal{S}_{\mathbf{e}}$: That is, for each $1 \leq m \leq M$, K_m is a connected set. Then, for each $S \subseteq N\backslash K$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = \widehat{v}(\{k\}) = v_{\mathbf{e}}(K) = \sum_{m=1}^{M} v_{\mathbf{e}}(K_m) = \sum_{m=1}^{M} (c_{k_m} + \sum_{t=k_m}^{l_m-1} c_{t,t+1} + c_{l_m}). \quad (10)$$

Hence, $SV_k(\widehat{v}) = \widehat{v}(\{k\})$. By *Respect of Connected Sets*, for each $K_m \in P_K$, $\sum_{i\in K_m} SV_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(K_m)$. These equalities and (10) together imply that $SV_k(\widehat{v}) = \sum_{m=1}^{M} \sum_{i\in K_m} SV_i(v_{\mathbf{e}}) = \sum_{i\in K} SV_i(v_{\mathbf{e}})$.

2) $P_K \backslash \mathcal{S}_{\mathbf{e}} = \{K_1\}$ and either $M \geq 2$ or $l = n$: That is, except for K_1, each $K_m \in P_K$ is a connected set. Then, for each $S \subseteq N \backslash K$ such that $k_1 - 1 \notin S$, (10) holds. For each $S \subseteq N \backslash K$ such that $k_1 - 1 \in S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1}) - c_{k_1-1}. \tag{11}$$

Then,

$$SV_k(\widehat{v}) = \sum_{S \subseteq N\backslash K: k_1-1 \in S} g(s)(\widehat{v}(S \cup \{k\}) - \widehat{v}(S)) + \sum_{S \subseteq N\backslash K: k_1-1 \notin S} g(s)(\widehat{v}(S \cup \{k\}) - \widehat{v}(S))$$

$$= \sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s)[c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1}) - c_{k_1-1}] + \sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) v_{\mathbf{e}}(K).$$

Note that $\sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s) = \sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) = \frac{1}{2}$.[7] Hence,

$$SV_k(\widehat{v}) = v_{\mathbf{e}}(K) + \tfrac{1}{2}(c_{k_1-1,k_1} - c_{k_1} - c_{k_1-1})$$

$$= \tfrac{1}{2}(2c_{l_1} + 2\sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{k_1-1,k_1} + c_{k_1} - c_{k_1-1}) + \sum_{m=2}^{M} v_{\mathbf{e}}(K_m)$$

$$= \sum_{i \in K_1} SV_i(v_{\mathbf{e}}) + \sum_{m=2}^{M} \sum_{i \in K_m} SV_i(v_{\mathbf{e}})$$

$$= \sum_{i \in K} SV_i(v_{\mathbf{e}}).$$

3) $P_K \backslash \mathcal{S}_{\mathbf{e}} = \{K_M\}$ and either $M \geq 2$ or $k = 1$: That is, except for K_M, each $K_m \in P_K$ is a connected set. Then, for each $S \subseteq N \backslash K$ such that $l_M + 1 \notin S$, (10) holds. For each $S \subseteq N \backslash K$ such that $l_M + 1 \in S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = (v_{\mathbf{e}}(K) - c_{l_M}) + c_{l_M,l_M+1} - c_{l_M+1}. \tag{12}$$

Then,

$$SV_k(\widehat{v}) = \sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s)[(v_{\mathbf{e}}(K) - c_{l_M}) + c_{l_M,l_M+1} - c_{l_M+1}] + \sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) v_{\mathbf{e}}(K)$$

$$= v_{\mathbf{e}}(K) + \tfrac{1}{2}(c_{l_M,l_M+1} - c_{l_M} - c_{l_M+1})$$

$$= \tfrac{1}{2}(2c_{k_M} + 2\sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} + c_{l_M} - c_{l_M+1}) + \sum_{m=1}^{M-1} v_{\mathbf{e}}(K_m)$$

$$= \sum_{i \in K_M} SV_i(v_{\mathbf{e}}) + \sum_{m=1}^{M-1} \sum_{i \in K_m} SV_i(v_{\mathbf{e}})$$

$$= \sum_{i \in K} SV_i(v_{\mathbf{e}}).$$

4) $P_K \backslash \mathcal{S}_{\mathbf{e}} = \{K_1, K_M\}$: That is, except for K_1 and K_M, each $K_m \in P_K$ is a connected set. Note that this case covers the possibility that $K = K_1 = K_M$ and $K \notin \mathcal{S}_{\mathbf{e}}$.

Then, for each $S \subseteq N \backslash K$ such that $\{k_1 - 1, l_M + 1\} \cap S = \emptyset$, (10) holds. For each $S \subseteq N \backslash K$ such that $k_1 - 1 \in S$ and $l_M + 1 \notin S$, (11) holds. For each $S \subseteq N \backslash K$ such that $l_M + 1 \in S$ and $k_1 - 1 \notin S$, (12) holds. For each $S \subseteq N \backslash K$ such that $\{k_1 - 1, l_M + 1\} \subseteq S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1} - c_{l_M}) + c_{l_M,l_M+1} - c_{k_1-1} - c_{l_M+1}.$$

[7] For the calculation of these values, see Appendix 7 in the working version of our paper.

Then,

$SV_k(\widehat{v}) = \sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} g(s) v_{\mathbf{e}}(K) + \sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s)[c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1}) - c_{k_1-1}] + \sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s)[(v_{\mathbf{e}}(K) - c_{l_M}) + c_{l_M,l_M+1} - c_{l_M+1}] + \sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} g(s)[c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1} - c_{l_M}) + c_{l_M,l_M+1} - c_{k_1-1} - c_{l_M+1}]$

Note that $\sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} g(s) = \sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} g(s) = \frac{1}{3}$ and $\sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s) = \frac{1}{6}$.

Hence,

$= v_{\mathbf{e}}(K) + \frac{1}{2}(c_{k_1-1,k_1} - c_{k_1-1} - c_{k_1} + c_{l_M,l_M+1} - c_{l_M} - c_{l_M+1})$

$= \frac{1}{2}(2c_{l_1} + 2\sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{k_1-1,k_1} + c_{k_1} - c_{k_1-1}) + \sum_{m=2}^{M-1} v_{\mathbf{e}}(K_m) + \frac{1}{2}(2c_{k_M} + 2\sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} + c_{l_M} - c_{l_M+1})$

$= \sum_{i \in K_1} SV_i(v_{\mathbf{e}}) + \sum_{m=2}^{M-1} \sum_{i \in K_m} SV_i(v_{\mathbf{e}}) + \sum_{i \in K_M} SV_i(v_{\mathbf{e}})$

$= \sum_{i \in K} SV_i(v_{\mathbf{e}})$.

In all the possible cases, we showed that $SV_k(\widehat{v}) = \sum_{i \in K} SV_i(v_{\mathbf{e}})$. Therefore, the Shapley value satisfies *Merging and Splitting Proofness*.

Now, we show that the Shapley value is the only solution that satisfies the axioms listed in Theorem 1. (For the independence of axioms, see the appendix in our working paper version). Let F satisfy those axioms and $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$. We will show that for each $S \in \mathcal{S}_{\mathbf{e}}$ and each $i \in S$, $F_i(v_{\mathbf{e}}) = SV_i(v_{\mathbf{e}})$.

If $n = 2$, by Remark 1, $F = SV$. Let $n > 2$. By Remark 2d, F satisfies *Respect of Connected Sets*. Hence, for each $\{i\} \in \mathcal{S}_{\mathbf{e}}$, $F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(\{i\}) = SV_i(v_{\mathbf{e}})$.

Now, let $S \in \mathcal{S}_{\mathbf{e}}$ be such that $|S| \geq 2$ and $S = \{l, l+1, ..., m\}$ for some $\{l, m\} \subseteq N$. Let $K_1 = \{i \in N : i < l\}$ and $K_2 = \{i \in N : i > m\}$.[8]

The proof is by induction.

$*$*Base Step:* Let K_1 and $\{l\}$ merge into a single sponsor denoted by l. Let K_2 and $\{l+1, l+2, ..., m\}$ merge into a single sponsor denoted by n. Let $(\{l, n\}, v^l)$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by these mergers. Thus,

$v^l(\{l\}) = v_{\mathbf{e}}(\{1, ..., l\}) = v_{\mathbf{e}}(K_1) + 2c_l$,

$v^l(\{n\}) = v_{\mathbf{e}}(\{l+1, ..., n\}) = c_{l+1} + \sum_{t=l+1}^{m-1} c_{t,t+1} + c_m + v_{\mathbf{e}}(K_2) = (v_{\mathbf{e}}(S) - c_l - c_{l,l+1} + c_{l+1}) + v_{\mathbf{e}}(K_2)$, and

$v^l(\{l, n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

[8] If $l = 1$, then $K_1 = \emptyset$ and if $m = n$, then $K_2 = \emptyset$. Note that K_1 is a union of connected sets and so is K_2.

These equalities and Remark 1 together imply

$$\begin{array}{rcl} F_l(v^l) & = & \frac{1}{2}\left[v^l(\{l,n\}) + v^l(\{l\}) - v^l(\{n\})\right], \\ & = & v_{\mathbf{e}}(K_1) + \frac{1}{2}(3c_l + c_{l,l+1} - c_{l+1}). \end{array} \tag{13}$$

By *Respect of Connected Sets,*

$$\sum_{i\in K_1} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(K_1), \ \sum_{i\in S} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(S), \text{ and } \sum_{i\in K_2} F_i(v_{\mathbf{e}}) = v_{\mathbf{e}}(K_2). \tag{14}$$

By *Merging and Splitting Proofness-2*,

$$F_l(v^l) = \sum_{i\in K_1} F_i(v_{\mathbf{e}}) + F_l(v_{\mathbf{e}}). \tag{15}$$

By equalities (13), (14), and (15),

$$F_l(v_{\mathbf{e}}) = \frac{1}{2}(3c_l + c_{l,l+1} - c_{l+1}) = SV_l(v_{\mathbf{e}}). \tag{16}$$

$*$*Induction Step:* Let $l < k \leq m$. Assume that, for each $l < i < k$, $F_i(v_{\mathbf{e}}) = SV_i(v_{\mathbf{e}})$. We will prove that $F_k(v_{\mathbf{e}}) = SV_k(v_{\mathbf{e}})$.

Let K_1 and $S_1 = \{l, l+1, .., k\}$ merge into a single sponsor denoted by k. Let K_2 and $S \backslash S_1$ merge into a single sponsor denoted by n. Let $(\{k,n\}, v^k)$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by these mergers.

If $k < m$, then

$v^k(\{k\}) = v_{\mathbf{e}}(\{1,...,k\}) = v_{\mathbf{e}}(K_1) + c_l + \sum\limits_{t=l}^{k-1} c_{t,t+1} + c_k,$

$v^k(\{n\}) = v_{\mathbf{e}}(\{k+1,...,n\}) = (v_{\mathbf{e}}(S) - c_l - \sum\limits_{t=l}^{k} c_{t,t+1} + c_{k+1}) + v_{\mathbf{e}}(K_2)$, and

$v^k(\{k,n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2).$

These equalities and Remark 1 together imply

$$\begin{array}{rcl} F_k(v^k) & = & \frac{1}{2}\left[v^k(\{k,n\}) + v^k(\{k\}) - v^k(\{n\})\right], \\ & = & v_{\mathbf{e}}(K_1) + \frac{1}{2}(2c_l + 2\sum\limits_{t=l}^{k-1} c_{t,t+1} + c_k + c_{k,k+1} - c_{k+1}). \end{array} \tag{17}$$

By *Merging and Splitting Proofness-2*,

$$F_k(v^k) = \sum_{i\in K_1} F_i(v_{\mathbf{e}}) + \sum_{i=l}^{k-1} F_i(v_{\mathbf{e}}) + F_k(v_{\mathbf{e}}). \tag{18}$$

Note that $\sum\limits_{i=l}^{k-1} SV_i(v_{\mathbf{e}}) = \frac{1}{2}(2c_l + 2\sum\limits_{t=l}^{k-2} c_{t,t+1} + c_{k-1} + c_{k-1,k} - c_k)$. Hence, by the induction hypothesis and equalities (14), (17), and (18),

$$F_k(v_{\mathbf{e}}) = (2c_k + c_{k-1,k} + c_{k,k+1} - c_{k-1} - c_{k+1})/2 = SV_k(v_{\mathbf{e}}).$$

If $k = m$, then
$v^m(\{m\}) = v_{\mathbf{e}}(\{1,...,m\}) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S)$,
$v^m(\{n\}) = v_{\mathbf{e}}(\{m+1,...,n\}) = v_{\mathbf{e}}(K_2)$, and
$v^m(\{m,n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

These equalities and Remark 1 together imply

$$\begin{aligned} F_m(v^m) &= \tfrac{1}{2}\left[v^m(\{m,n\}) + v^m(\{m\}) - v^m(\{n\})\right], \\ &= v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S). \end{aligned} \tag{19}$$

By *Merging and Splitting Proofness-2*,

$$F_m(v^m) = \sum_{i\in K_1} F_i(v_{\mathbf{e}}) + \sum_{i=l}^{m-1} F_i(v_{\mathbf{e}}) + F_m(v_{\mathbf{e}}). \tag{20}$$

Hence, by the induction hypothesis and equalities (14), (19), and (20),

$$F_m(v_{\mathbf{e}}) = (3c_m + c_{m-1,m} - c_{m-1})/2 = SV_m(v_{\mathbf{e}}).$$

This concludes the induction step.

$*$*Conclusion Step:* By the Base and the Induction steps, for each $l \le k \le m$, we have $F_k(v_{\mathbf{e}}) = SV_k(v_{\mathbf{e}})$.

By repeating the induction proof for each $S \in \mathcal{S}_{\mathbf{e}}$, we obtain that for each $i \in S$, $F_i(v_{\mathbf{e}}) = SV_i(v_{\mathbf{e}})$. This completes the proof. ∎

Proof of Proposition 3:

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$. Let F be defined on $\mathcal{V}_{\mathcal{E}^2}$.

a) Let F satisfy *Weak Respect of Connected Sets with respect to* ϕ, *Merging and Splitting Proofness-2,* and *Equal Benefit.* Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}^2$ be such that $|\mathcal{S}_{\mathbf{e}}| = 1$.

First, suppose that $i \in \{1, n\}$. Let $N\backslash\{i\}$ merge into a single sponsor denoted by $j \in N\backslash\{i\}$. Let $(\{i,j\}, v^i)$ be the TU-game obtained from $(N, v_{\mathbf{e}})$ by this merger. Thus, $v^i(\{i\}) = v_{\mathbf{e}}(\{i\})$, $v^i(\{j\}) = v_{\mathbf{e}}(N\backslash\{i\})$, and $v^i(\{i,j\}) = v_{\mathbf{e}}(N)$.

Since $(\{i,j\}, v^i)$ is a two-sponsor TU-game, by *Equal Benefit*, (I) $F_i(v^i) - F_j(v^i) = v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(N\backslash\{i\})$.

By *Merging and Splitting Proofness-2*, (II) $F_i(v^i) = F_i(v_{\mathbf{e}})$ and $F_j(v^i) = \sum\limits_{l\in N\backslash\{i\}} F_l(v_{\mathbf{e}})$.

By (I) and (II), we have (III) $F_i(v_{\mathbf{e}}) - \sum\limits_{l\in N\backslash\{i\}} F_l(v_{\mathbf{e}}) = v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(N\backslash\{i\})$.

By *Weak Respect of Connected Sets,* (IV) $F_i(v_{\mathbf{e}}) + \sum\limits_{l\in N\backslash\{i\}} F_l(v_{\mathbf{e}}) = \phi(v_{\mathbf{e}}(N))$. By (III) and (IV),

$$F_i(v_{\mathbf{e}}) = \frac{1}{2}\left[\phi(v_{\mathbf{e}}(N)) + v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(N\backslash\{i\})\right]. \tag{21}$$

Now, we will show that for each $i \in \{2,..,n-1\}$, $F_i(v_{\mathbf{e}}) = SV_i(v_{\mathbf{e}})$. The proof is by induction.

$*$*Base Step:* Let $\{1,2\}$ merge into a single sponsor denoted by 2 and $N\backslash\{1,2\}$ merge into a single sponsor denoted by n. Let $(\{2,n\},v^2)$ be the TU-game obtained from $(N,v_{\mathbf{e}})$ by these mergers. Similarly, $(\{2,n\},v^2)$ is a two-sponsor TU-game. Hence, by *Equal Benefit*,

$$F_2(v^2)-F_n(v^2)=v_{\mathbf{e}}(\{1,2\})-v_{\mathbf{e}}(N\backslash\{1,2\}). \tag{22}$$

By *Merging and Splitting Proofness-2*, (I) $F_2(v^2)=\sum_{i\in\{1,2\}}F_i(v_{\mathbf{e}})$ and $F_n(v^2)=\sum_{i\in N\backslash\{1,2\}}F_i(v_{\mathbf{e}})$.

By *Weak Respect of Connected Sets,* (II) $\sum_{i\in\{1,2\}}F_i(v_{\mathbf{e}})+\sum_{i\in N\backslash\{1,2\}}F_i(v_{\mathbf{e}})=\phi(v_{\mathbf{e}}(N))$. By (I), (II), and (22)

$$\sum_{i\in\{1,2\}}F_i(v_{\mathbf{e}})=\frac{1}{2}\left[\phi(v_{\mathbf{e}}(N))+v_{\mathbf{e}}(\{1,2\})-v_{\mathbf{e}}(N\backslash\{1,2\})\right].$$

This equality and (21) together imply

$$F_2(v_{\mathbf{e}})=\frac{1}{2}\left[v_{\mathbf{e}}(\{1,2\})-v_{\mathbf{e}}(N\backslash\{1,2\})-v_{\mathbf{e}}(\{1\})+v_{\mathbf{e}}(N\backslash\{1\})\right].$$

Note that $v_{\mathbf{e}}(\{1,2\})-v_{\mathbf{e}}(\{1\})=c_2+c_{1,2}-c_1$ and $v_{\mathbf{e}}(N\backslash\{1\})-v_{\mathbf{e}}(N\backslash\{1,2\})=c_2+c_{2,3}-c_3$. Hence, $F_2(v_{\mathbf{e}})=\frac{1}{2}\left[2c_2+c_{1,2}+c_{2,3}-c_1-c_3\right]=SV_2(v_{\mathbf{e}})$.

$*$*Induction Step:* Let $1<k<n$. Assume that for each $1<j<k$, $F_j(v_{\mathbf{e}})=SV_j(v_{\mathbf{e}})$. That is, for each $j<k$, $\sum_{i\in\{1,..,j\}}F_i(v_{\mathbf{e}})=\frac{1}{2}\left[\phi(v_{\mathbf{e}}(N))+v_{\mathbf{e}}(\{1,...,j\})-v_{\mathbf{e}}(N\backslash\{1,...,j\})\right].$

Let $\{1,..,k\}$ merge into k and $\{k+1,..,n\}$ merge into n. Let $(\{k,n\},v^k)$ be the TU-game obtained from $(N,v_{\mathbf{e}})$ by these mergers. Since $(\{k,n\},v^k)$ is a two-sponsor TU-game, by *Equal Benefit*, (I) $F_k(v^k)-F_n(v^k)=v_{\mathbf{e}}(\{1,..,k\})-v_{\mathbf{e}}(N\backslash\{1,..,k\})$.

By *Merging and Splitting Proofness-2*, (II) $F_k(v^k)=\sum_{i\in\{1,..,k\}}F_i(v_{\mathbf{e}})$ and $F_n(v^k)=\sum_{i\in N\backslash\{1,..,k\}}F_i(v_{\mathbf{e}})$.

By *Weak Respect of Connected Sets,* (III) $\sum_{i\in\{1,..,k\}}F_i(v_{\mathbf{e}})+\sum_{i\in N\backslash\{1,..,k\}}F_i(v_{\mathbf{e}})=\phi(v_{\mathbf{e}}(N))$. By (I), (II), and (III),

$$\sum_{i\in\{1,..,k\}}F_i(v_{\mathbf{e}})=\frac{1}{2}\left[\phi(v_{\mathbf{e}}(N))+v_{\mathbf{e}}(\{1,..,k\})-v_{\mathbf{e}}(N\backslash\{1,...,k\})\right].$$

This equality and the induction hypothesis together imply

$$F_k(v_{\mathbf{e}})=\frac{1}{2}\left[v_{\mathbf{e}}(\{1,..,k\})-v_{\mathbf{e}}(N\backslash\{1,...,k\})-v_{\mathbf{e}}(\{1,..,k-1\})+v_{\mathbf{e}}(N\backslash\{1,...,k-1\})\right].$$

Since $v_{\mathbf{e}}(\{1,..,k\})-v_{\mathbf{e}}(\{1,..,k-1\})=c_k+c_{k-1,k}-c_{k-1}$ and $v_{\mathbf{e}}(N\backslash\{1,...,k-1\})-v_{\mathbf{e}}(N\backslash\{1,...,k\})=c_k+c_{k,k+1}-c_{k+1}$, we have

$$F_k(v_{\mathbf{e}})==\frac{1}{2}\left[2c_k+c_{k-1,k}+c_{k,k+1}-c_{k-1}-c_{k+1}\right]=SV_k(v_{\mathbf{e}}).$$

$*$*Conclusion Step:* By the Base and the Induction steps, for each $i \in \{2,..,n-1\}$, $F_i(v_\mathbf{e}){=}SV_i(v_\mathbf{e})$.

b) Let F satisfy *Weak Respect of Connected Sets with respect to* ϕ, *Merging and Splitting Proofness-2, Equal Benefit,* and *Consistency over Connected Sets.* Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}^2$ be such that $|\mathcal{S}_\mathbf{e}| = 2$, $i \in N$, and $S_i \in \mathcal{S}_\mathbf{e}$ with $i \in S_i$.

If $|\{i-1, i+1\} \cap S_i| = 0$, that is $S_i = \{i\}$, then by *Weak Respect of Connected Sets*, $F_i(v_\mathbf{e}) = \phi(v_\mathbf{e}(\{i\}))$.

Now, suppose that $|\{i-1, i+1\} \cap S_i| \geq 1$. Let $S_i = \{l, l+1, ..., m\}$ for some $\{l, m\} \subseteq N$. Consider $\mathbf{e}_{S_i} = \langle S_i, \mathbf{c}_{S_i}, r_{S_i}\rangle \in \mathcal{E}^2$.

If $|\{i-1, i+1\} \cap S_i| = 1$, that is $i \in \{l, m\}$, then by part (a), $F_i(v_{\mathbf{e}_{S_i}}){=}\frac{1}{2}\left[\phi(v_\mathbf{e}(S_i)){+}v_\mathbf{e}(\{i\}) - v_\mathbf{e}(S_i \backslash \{i\})\right]$ and by *Consistency over Connected Sets* $F_i(v_{\mathbf{e}_{S_i}}){=}F_i(v_\mathbf{e})$.

If $|\{i-1, i+1\} \cap S_i| = 2$, that is $i \in \{l+1, .., m-1\}$, then by part (a), $F_i(v_{\mathbf{e}_{S_i}}){=}SV_i(v_{\mathbf{e}_{S_i}})$ and by *Consistency over Connected Sets* $F_i(v_{\mathbf{e}_{S_i}}){=}F_i(v_\mathbf{e})$ and $SV_i(v_{\mathbf{e}_{S_i}}){=}SV_i(v_\mathbf{e})$. This concludes the proof. ■

Proof of Proposition 4:
Let $\mathbf{e} = \langle \{1,2,3,4\}, \mathbf{c}, r\rangle \in \mathcal{E}_T^*$ be such that $r^* = (0,1,2,3,4,0)$ and $c_1 = c_{1,2} = c_{3,4} = c_4 = 5$, $c_2 = c_3 = c_{2,3} = 3$, $c_{1,4} = 10$, and $c_{1,3} = c_{2,4} = 6$. Note that triangle inequalities hold among all agents and r is a least costly route for $\mathbf{e}$. Let $i = 2$, $S = \{1,4\}$, and $T = \{1,3,4\}$. Since, $v_\mathbf{e}^*(S \cup \{i\}) - v_\mathbf{e}^*(S) = 1$ and $v_\mathbf{e}^*(T \cup \{i\}) - v_\mathbf{e}^*(T) = 2$, and $1 < 2$, $(N, v_\mathbf{e}^*)$ is not convex.

In general, let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}_T^*$ be such that there is $S_i \in \mathcal{S}_\mathbf{e}$ with $|S_i| \geq 4$ and $\{i-1, i, i+1, i+2\} \subseteq S_i$ for some $i \in N$ where $c_{i,i+2} + c_{i-1,i+1} < c_{i-1,i+2} + c_{i,i+1}$. Note that this inequality is compatible with a cost vector satisfying triangle inequalities (as in the example in the previous paragraph). Let $S = \{i-1, i+2\}$ and $T = \{i-1, i+1, i+2\}$. Then, $v_\mathbf{e}^*(S \cup \{i\}) - v_\mathbf{e}^*(S) < v_\mathbf{e}^*(T \cup \{i\}) - v_\mathbf{e}^*(T)$ and $(N, v_\mathbf{e}^*)$ is not convex. ■

6 References

1. Chun Y. (2006). “A pessimistic approach to the queuing problem". Mathematical Social Sciences 51(2) 171–181.

2. Chun Y. (2010). “Consistency and Monotonicity in Sequencing Problems". International Journal of Game Theory (in press).

3. Derks J, and Kuipers J (1997) “On the Core of Routing Games". International Journal of Game Theory 26, 193-205

4. Derks J, and Tijs S (2000) “On Merge Properties of the Shapley Value". International Game Theory Review 2, 249-257

5. Engevall S, Gothe-Lundgren M, and Varbrand P (1998) “The Traveling Salesman Game: An Application of Cost Allocation in a Gas and Oil Company". Annals of Operations Research 82, 453-472

6. Fishburn PC and Pollak HO (1983) “Fixed-route Cost Allocation". The American Mathematical Monthly 90, 366-378

7. Haller H (1994) “Collusion Properties of Values". International Journal of Game Theory 23, 261-281

8. Hart S, and Mas-Colell A (1989) “Potential, Value, and Consistency”. Econometrica 57, 589-614

9. Jackson MO and Wolinsky A (1996) “A Strategic Model of Social and Economic Network". Journal of Economic Theory 71, 44-74

10. Kar A (2002) “Axiomatization of the Shapley Value on Minimum Cost Spanning Tree Games". Games and Economic Behavior 38, 265-277

11. Knudsen PH, and Østerdal LP (2005) “Merging and Splitting in Cooperative Games: Some (Im-)Possibility Results. Discussion Papers, Department of Economics, University of Copenhagen, 05-19

12. Kuipers J (1993) “A Note on the 5-person Traveling Salesman Game". Methods and Models of Operations Research 38, 131-139

13. Lehrer E (1988) “An Axiomatization of the Banzhaf Value". International Journal of Game Theory 17, 89-99

14. Maniquet F. (2003). “A Characterization of the Shapley Value in Queueing Problems". Journal of Economic Theory 109(1) 90–103.

15. Moulin H (2007) “On Scheduling Fees to Prevent Merging, Splitting, and Transferring of Jobs". Mathematics of Operations Research 32, 266-283

16. Myerson R (1977) “Graphs and Cooperation in Games". Mathematics of Operations Research 2, 225-229

17. Potters JAM, Curiel IJ, and Tijs SH (1992) “Traveling Salesman Games". Mathematical Programming 53, 199-211

18. Shapley LS (1971) “Core of Convex Games". International Journal of Game Theory 1, 11-26

19. Tamir A (1989) “On the Core of a Traveling Salesman Cost Allocation Game". Operations Research Letters 8, 31-34