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NONCOOPERATIVE EXCHANGE WITH A CONTINUUM OF TRADERS

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1. Introduction

The noncooperative "market games" that we will study were originated by M. Shubik in [1]. He constructed a market mechanism for trade using a commodity as a means of payment (a "commodity money"). This led to the view of trade in a market as a noncooperative game in strategic form. For an example involving two types of traders and two commodities, Shubik showed that, if there is enough of commodity money in an appropriately defined sense, the prices and allocations at type-symmetric noncooperative equilibria of the market games converge under replication to the competitive prices and allocations of the market. He conjectured that this result would hold true in general, which was indeed subsequently proved by L. S. Shapley and him in [2]. Several variants of this model were outlined and discussed in [3] and [4]; see also [5]. In this paper, we do not go into the modelling considerations that lie behind these models. These have been amply spelled out in [1], [3], and [4]. Our purpose is to examine a continuum version of the model in [1]. We also look at a variant of a model due to M. Shubik and C. Wilson [6] in which fiat money is used for trade with bankruptcy penalties for traders who go bankrupt. We have called this variant the "Unbounded Credit" Model.

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2. The Non-Atomic Market

Let $\{T, \mathcal{C}, \mu\}$ be a non-atomic measure space of traders, where $T \equiv$ the set of traders, $\mathcal{C} \equiv$ the σ-algebra of coalitions, $\mu \equiv$ a non-atomic measure on $\{T, \mathcal{C}\}$. Trade occurs in m commodities. We will denote by Ω^m the non-negative orthant of the Euclidean space of dimension m. Vectors in Ω^m represent commodity bundles. For any $v \in \Omega^m$, v_j is the j^{th} component of v. 0 denotes the origin of Ω^m, and also the number zero (the meaning is clear from the context). A set $S \in \mathcal{C}$ is called null if $\mu(S) = 0$; otherwise it is called non-null. We will use the term "almost all traders" to mean all traders except for a null set.

The data of the market consists of an initial endowment of the traders, $a : T \to \Omega^m$, and their utility functions $\{u^t : t \in T\}$. a is a measurable function, and $a(t)$ --also written a^t --stands for the endowment of trader t. We assume that $\int_T a_j^t d\mu > 0$ for $j = 1, \ldots, m$. The function $u^t : \Omega^m \to \Omega^1$ describes the preferences of t on commodity bundles. We assume that each u^t is continuous, concave, and strictly increasing in each variable.

An allocation is a measurable function $x : T \to \Omega^m$ such that $\int_T x d\mu = \int_T a d\mu$. It describes a redistribution of the commodities among the traders. A <u>competitive equilibrium</u> (C.E.) of the market is a pair (p, x) where $p \in \Omega^m \setminus \{0\}$ is a price vector, and x is an allocation such that, for almost all t, x^t is optimal on t's budget set

$$B^t(p) = \{y \in \Omega^m : p \cdot y \leq p \cdot a^t\},$$

i.e.,

$$u^t(x^t) = \max\{u^t(y) : y \in B^t(p)\}, \quad \text{and} \quad x^t \in B^t(p) .$$

An allocation x [a price $p' \in \Omega^m \backslash \{0\}$] is called <u>competitive</u> if there is a $p \in \Omega^m \backslash \{0\}$ [an allocation x'] such that (p,x) $[(p',x')]$ is a C.E. Let us also recall the notion of "shadow prices of income"* at a C.E. Suppose (p,x) is a C.E. Then each trader t maximizes $u^t(y)$ subject to $y \in \Omega^m$, $p \cdot y - p \cdot a^t \leq 0$, and the maximum is obtained at $x^t \in B^t(p)$. By the Kuhn-Tucker theorem there exists a number $\lambda^t \geq 0$ such that x^t is also a solution of the following unconstrained problem:

$$\max[u^t(y) + \lambda^t(p \cdot a^t - p \cdot y)] , \quad \text{subject to} \quad y \in \Omega^m .$$

The set $\{\lambda^t : t \in T\}$ constitutes the shadow prices at the C.E. (p,x). Note that if (p,x) is a C.E. with shadow prices $\{\lambda^t : t \in T\}$, then for any $K > 0$, (Kp,x) is also a C.E. with shadow prices $\{\frac{1}{K}\lambda^t : t \in T\}$. We will call a C.E. (p,x) <u>normalized</u> if $|p| = 1$, where $| \;|$ denotes the Euclidean norm. Finally define an economy to be <u>competitively bounded</u> if, for almost all $t \in T$,

$$\sup\{\lambda^t : \lambda^t \text{ is a shadow price for } t \text{ at a normalized C.E.}\}$$

is finite. Observe that an economy is competitively bounded if it has a only finite number of (normalized) C.E.'s, for then the sup is taken over a finite set for each t, and is clearly finite. Conditions which guarantee that the number of normalized C.E.'s is finite may be found in [7].

*Henceforth abbreviated "shadow prices."

3. Trade with Unbounded Credit

To recast the market in the form of a noncooperative game we must describe the strategy sets and payoff functions of the traders. As in [3] or [4] we will suppose that there are m trading-posts, one for each commodity. A trader is now permitted to put up commodities for sale in the tradingposts. At the same time he can bid fiat money for purchasing them; this must be borrowed from a bank (which we imagine in the background, even though it has no strategic role in the game). There is no limit on the amount he can bid. However if at the end of the trade he has a deficit in money, i.e., did not receive in trade enough to repay to the bank the amount he borrowed in order to bid, then he is bankrupt and must pay a "bankrupcy penalty," which we will model as a disutility suffered by him depending on the amount of his debt. On the other hand, if a trader ends up with a surplus of fiat money, it has no utility for him.

The formal treatment is as follows. The strategy set of trader t is

$$S^t = \{(b^t, q^t) : b^t \in \Omega^m, q^t \in \Omega^m, q_j^t \leq a_j^t\} .$$

Here b_j^t is the bid of trader t in the j^{th} tradingpost, and q_j^t is the quantity of commodity j that he offers for sale. Given a choice of strategies by the traders, how are the tradingposts cleared? We are beset by a fundamental difficulty in the nonatomic case when we consider the mechanism in [3] or [4]. This is because the mechanism calls for aggregating the bids and offers in each trading-post. These would be $\int_T b_j^t d\mu$ and $\int_T q_j^t d\mu$. But the integrals make sense only if the functions $b : T \to \Omega^m$ and $q : T \to \Omega^m$ are assumed to be measurable. It is not clear how we would justify this assumption heuristically. Why should independent decision-makers behave in a jointly measurable way? We will deal with this problem in Section 5, where a model of noncooperative behavior is suggested which leads to measurable strategies. For the moment let us assume measurability.

Then the price of the j^{th} commodity is obtained as the ratio of bids to offers in the j^{th} trading-post:

$$p_j = \begin{cases} \int_T b_j^t d\mu / \int_T q_j^t d\mu & \text{if} \quad \int_T q_j^t d\mu > 0 \\ 0 & \text{if} \quad \int_T q_j^t d\mu = 0 \,. \end{cases}$$

The commodities and money are disbursed to the traders in proportion to their bids and offers. Letting $x^t \in \Omega^{m+1}$ stand for the vector that t obtains, where x_{m+1}^t is his final holding of money, we have*

$$x_j^t = a_j^t - q_j^t + b_j^t / p_j \quad \text{for} \quad j = 1, \ldots, m$$

$$x_{m+1}^t = -\sum_{j=1}^{m} b_j^t + \sum_{j=1}^{m} p_j q_j^t \,.$$

We must define the utility of $x^t \in \Omega^{m+1}$ to trader t. Let $\underline{x}^t$ be the projection of x^t on Ω^m, i.e., $\underline{x}^t$ denotes the vector of commodities obtained by t. The utility of this is $u^t(\underline{x}^t)$. To this we add a term which represents the bankruptcy penalty. We will take this at the moment to be of the form $\lambda^t \min[0, x_{m+1}^t]$, where $\lambda^t > 0$. (Later we shall see that any "harsher" penalty will also suffice.) Thus the utility of x^t, or the final payoff to trader t, is

$\Pi^t(x^t) = u^t(\underline{x}^t) + \lambda^t \min[0, x_{m+1}^t]$.

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*If $p_j = 0$, we define b_j^t / p_j to be 0.

We now have a game in strategic (or normal) form. With the rest of the data fixed, it depends on the choice of $\lambda : T \to \Omega'$, and hence we will denote it by Γ_λ. A non-cooperative equilibrium (N.E.) of this game is a measurable $s_* \in S = \bigtimes_{t \in T} S^t$ such that, for almost all t,

$$\Pi^t(s_*) = \max_{s^t \in S^t} \Pi^t(s_*|s^t) ,$$

where $(s_*|s^t)$ is the same as s_* except that s_*^t is replaced by s^t.

There exists a trivial N.E. of Γ_λ, namely the collection of strategies in which each trader bids and offers nothing. We will focus our attention on <u>active N.E.'s</u>, i.e., those which produce positive prices in each trading-post. An <u>active N.E. allocation</u> of Γ_λ is an allocation produced at an active N.E. of Γ_λ.

We are now ready for our result.

<u>Theorem 1</u>. Active N.E. allocations of Γ_λ are competitive, for every $\lambda > 0$. Moreover, if the economy is competitively bounded, then every competitive allocation is also an active N.E. allocation of $\Gamma_{\hat\lambda}$, where $\hat\lambda^t \geq \sup\{\lambda^t : \lambda^t$ is a shadow price for t at a normalized C.E.$\}$.

<u>Proof</u>. Let (p,x) be the price and allocation at an active N.E. It is directly verified that $\int_T \underline{x} = \int_T a$, and $\int_T x^t_{m+1} d\mu = 0$. Suppose $x^t_{m+1} > 0$ for t in a non-null set S. Then each trader $t \in S$ could improve his payoff by increasing his bids, contradicting that (p,x) occurs at an N.E. Hence $x^t_{m+1} \leq 0$ for a.a.t. This implies that $x^t_{m+1} = 0$ for almost all t. It is now immediate that $(p,\underline{x})$ is a C.E.

Suppose that the economy is competitively bounded, and that x is a competitive allocation. Then there is a normalized p such that (p,x)

is a C.E. Since each u^t is strictly increasing in each vari ble, we must have $p > 0$, and $p \cdot x^t = p \cdot a^t$ for almost all t. Let $\lambda : T \to \Omega'$ be the shadow prices at this C.E. Note that $\lambda^t \leq \hat{\lambda}^t$ for almost all t.

Define $s_*^t = (b^t, q^t)$, for each $t \in T$, as follows:

$$b_j^t = \max[p_j x_j^t - p_j a_j^t, 0]$$

$$q_j^t = \max[a_j^t - x_j^t, 0] .$$

It is obvious that the collection of strategies $\{s_*^t : t \in T\}$ produces the price p and the allocation x. We claim that $\{s_*^t : t \in T\}$ is an N.E. of $\Gamma_{\hat{\lambda}}$.

First observe that since (p,x) is a C.E. with shadow prices λ, we have

$$u^t(x^t) = \max_{y \in \Omega^m} [u^t(y) + \lambda^t(p \cdot a^t - p \cdot y)]$$

for all t. But $\hat{\lambda}^t \geq \lambda^t$ and $p \cdot a^t - p \cdot x^t = 0$, hence

$$u^t(x^t) = \max[u^t(y) + \hat{\lambda}^t(p \cdot a^t - p \cdot y) : y \in \Omega^m, p \cdot a^t - p \cdot y \leq 0]$$

Moreover, since $\hat{\lambda}^t \min[0, p \cdot a^t - p \cdot y] < \lambda^t(p \cdot a^t - p \cdot y)$ whenever $p \cdot a^t - p \cdot y > 0$, we get

$$u^t(x^t) \geq \max[u^t(y) + \lambda^t(p \cdot a^t - p \cdot y) : y \in \Omega^m, p \cdot a^t - p \cdot y > 0]$$

$$\geq \max[u^t(y) + \hat{\lambda}^t \min[0, p \cdot a^t - p \cdot y] : y \in \Omega^m, p \cdot a^t - p \cdot y > 0] .$$

Combining the two we obtain

$$u^t(x^t) = \max[u^t(y) + \hat{\lambda}^t \min[0,\ p\cdot a^t - p\cdot y] : y \in \Omega^m]$$

for almost all t, which translates easily into:

$$\Pi^t(s_*^t) = \max_{s^t \in S^t} \Pi^t(s_*^t | s^t)$$

for almost all t.

Q.E.D.

Remark 1. Theorem 1 holds if we replace the linear penalties λ by any set of "harsher" penalties. Precisely, given λ, let $\Phi = \{\Phi^t : t \in T\}$ where $\Phi^t : \Omega^m \times R \to R$, $\Phi^t(x_1, \ldots, x_m, x_{m+1}) = u^t(x_1, \ldots, x_m)$ if $x_{m+1} \geq 0$, $\Phi^t(x_1, \ldots, x_m, x_{m+1}) - u^t(x_1, \ldots, x_m) < \lambda^t x_{m+1}$ if $x_{m+1} < 0$. Let Γ_Φ be the game where the payoff to t is $\Phi^t(x^t)$. Theorem 1 clearly holds if we replace Γ_λ by Γ_Φ.

4. Trade Using a Commodity Money

The market mechanism and the rules of the game in our second model are essentially the same as in the previous one, with two differences: a commodity is singled out as money and used for bidding, and every trader is required to put up all of his other commodities for sale. For convenience we will call the distinguished commodity the $(m+1)^{st}$ commodity. The initial endowment of the traders is given by a measurable function $a : T \to \Omega^{m+1}$, and their utility functions are $\{u^t : t \in T\}$, $u^t : \Omega^{m+1} \to \Omega'$. Besides the previous assumptions on u^t, we will require an additional one, which says that the $(m+1)^{st}$ commodity is sufficiently desired by all traders in comparison with the other commodities. Precisely, we require: for any $K > 0$, there is a $B_K > 0$ such that

$$\text{(A)} \qquad \frac{u_j^t(x)}{u_{m+1}^t(x)} < B_K, \quad \text{for all } j = 1, \ldots, m, \text{ and for all } t, \text{ whenever } x_j > K,$$

where "u_j^t" denotes the j^{th} partial derivative of u^t.

Under assumption (A), we will show that when there is a "large" amount of commodity money in the market, distributed in a "non-skewed" manner among the traders, the active N.E. allocations are "nearly" competitive.

First let us define the non-cooperative game. The strategy-set S^t of t consists of bids in the m trading-posts, but he is constrained to bid within a_{m+1}^t:

$$S^t = \{b^t \in \Omega^m : \sum_{j=1}^{m} b_j^t \leq a_{m+1}^t\}.$$

Prices are formed and the markets cleared as before. Let $p \in \Omega^m$ denote the prices in the m trading-posts, and $x^t \in \Omega^{m+1}$ denote the final bundle of t after trade. We have*

$$p_j = \int b_j / \bar{a}_j$$

$$x_j^t = \begin{cases} b_j^t / p_j & \text{if } p_j > 0 \\ 0 & \text{if } p_j = 0 \end{cases}$$

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*We will write "$\int$" for $\int_T d\mu$; and $\bar{a}_j$ for $\int_T a_j d\mu$.

for $j = 1, \ldots, m$; and

$$x_{m+1}^t = a_{m+1}^t - \sum_{j=1}^m b_j^t + \sum_{j=1}^m p_j a_j^t .$$

The payoff to t is the utility of his final bundle, $u^t(x^t)$. N.E.'s and active N.E.'s of this game are defined exactly as before.

We wish to study those markets in which there is a large amount of commodity money distributed in a non-skewed manner among the traders. For this purpose we keep all the other data of the market fixed, and vary only the money endowment $a_{m+1} : T \to \Omega'$. a_{m+1} will be called admissible if, for all non-null S ,

$$\frac{\int_S a_{m+1}^t d\mu(t)}{\mu(S) \int_T a_{m+1}^t d\mu(t)} \geq L > 0$$

where L is a fixed constant. This makes precise the idea that a_{m+1} is not too skewed. We shall say that a_{m+1}^t is of level M if

$$\underset{\substack{S \in \mathcal{C} \\ \mu(S)=0}}{\operatorname{Max}} \ \underset{t \in T \setminus S}{\operatorname{Min}} \ a_{m+1}^t \geq M ,$$

which says that all traders--except perhaps those in a null set--have at least the amount M of money.

Let $\hat{\mathcal{E}}_M$ denote the collection of markets* in which the money endowment is admissible and of level M . For any $\mathcal{E}_M \in \hat{\mathcal{E}}_M$, we will represent by $\Gamma(\mathcal{E}_M)$ the noncooperative game corresponding to $\mathcal{E}_M$ that is constructed in the manner described earlier. Our aim is to study prices and allocations at active N.E.'s of $\Gamma(\mathcal{E}_M)$ for large M , and to compare

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* T , $\underline{a}$, and μ are still fixed, of course.

them with competitive prices and allocations. Given a $p \in \Omega^m$ at an active N.E., denote by $\tilde{p}$ the $m+1$ dimensional vector obtained by setting $\tilde{p}_{m+1} = 1$, and $\tilde{p}_j = p_j$ for $j = 1, \ldots, m$. At an active N.E. $\{b^t : t \in T\}$, define a trader t to be <u>interior</u> if $\sum_{j=1}^{m} b_j^t < a_{m+1}^t$.

<u>Lemma 1</u>. Suppose (p,x) are the prices and allocation at an active N.E. of $\Gamma(\mathcal{E}_M)$. If t is interior, then x^t is optimal on $B^t(\tilde{p})$ for t.

<u>Proof</u>. Let $\hat{b}: T \to \Omega^m$ be the N.E., and let $u^t(x^t(b^t))$ be the payoff to t, regarded as a function of his strategy b^t, keeping others' strategies fixed according to $\hat{b}$. Then $\hat{b}^t$ is the solution to

$$\operatorname{Max}\{u^t(x^t(b^t)) : b^t \in \Omega^m, \sum_{j=1}^{m} b_j^t \leq a_{m+1}^t\}.$$

By the Kuhn-Tucker theorem there is a $\sigma^t \geq 0$ such that (a) $\hat{b}^t$ solves

$$\operatorname{Max}\{u^t(x^t(b^t)) + \sigma^t(a_{m+1}^t - \sum_{j=1}^{m} b_j^t) : b^t \in \Omega^m\}$$

and

$$\text{(b)} \quad \sigma^t(a_{m+1}^t - \sum_{j=1}^{m} \hat{b}_j^t) = 0.$$

Since t is interior, $a_{m+1}^t - \sum_{j=1}^{m} \hat{b}_j^t > 0$, hence $\sigma^t = 0$. But then $\hat{b}^t$ solves

$$\operatorname{Max}\{u^t(x^t(b^t)) : b^t \in \Omega^m\}$$

which implies $x^t(\hat{b}^t)$ is optimal on $B^t(\tilde{p})$ for t .

Q.E.D.

At an active N.E. $\hat{b}$ of $\Gamma(\mathcal{E}_M)$, with prices p and allocation x , let $S_M^*(\hat{b})$ denote the set* of traders t who are not optimal on their budget sets $B^t(\tilde{p})$. If $\mu(S_M^*(\hat{b})) = 0$, then (p,x) is a C.E. Thus one can think of $\mu(S_M^*(\hat{b}))$ as a measure of the deviation of (p,x) from competitive prices and allocation. We will show that for large M $\mu(S_M^*(\hat{b}))$ becomes small.

<u>Theorem 2</u>. Let $\hat{b}$ be any active N.E. of $\Gamma(\mathcal{E}_M)$. Then $\mu(S_M^*(\hat{b})) < R/M$, where R is a positive constant independent of M and $\hat{b}$.

<u>Proof</u>. We will keep $\hat{b}$ fixed, and write S_M^* for $S_M^*(\hat{b})$. Let S_M = the set* of traders t who are not interior at $\hat{b}$. Note that, by Lemma 1, $S_M^* \subset S_M$. For any $R \in \mathcal{C}$, denote $\int_R a_{m+1}^t d\mu$ by $\alpha(R)$. Since the traders in S_M bid all their money, there is some $j \in \{1, \ldots, m\}$ such that $p_j \geq \alpha(S_M)/m\bar{a}_j$. On the other hand, $p_j \leq \alpha(R)/\bar{a}_j$. Thus the total amount of commodity j purchased by S_M is at least $\alpha(S_M)\bar{a}_j/m\alpha(R)$. Hence there exists a non-null subset S_M' of S_M such that each $t \in S_M'$ purchases at least $\alpha(S_M)\bar{a}_j/\mu(S_M)m\alpha(R)$ of j ; otherwise the net amount of j purchased by S_M is less than $\alpha(S_M)\bar{a}_j\mu(S_M)/\mu(S_M)m\alpha(R) = \alpha(S_M)\bar{a}_j/m\alpha(R)$, a contradiction. Since a_{m+1} is admissible, each $t \in S_M'$ purchases at least $L\bar{a}_j/m$ of j . Put $K = L\bar{a}_j/m$, and let B_K be as in the assumption (A) on the utility functions. Consider any $t \in S_M'$, and suppose t were to decrease his bid $\hat{b}_j^t$ on j by a small amount $\Delta > 0$. Then the change in his utility

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*This set is clearly measurable.

function is

$$\Delta u^t \sim u^t_{m+1}(x^t)\Delta - u^t_j(x^t)(\Delta/p_j) .$$

Since $\hat{b}$ is an N.E., we must have $\Delta u^t \leq 0$ for all $t \in S'_M$. This implies

$$p_j \leq \frac{u^t_j(x^t)}{u^t_{m+1}(x^t)}$$

for $t \in S'_M$. But $x^t_j \geq K$ for $t \in S'_M$, therefore by (A) ,

$$p_j \leq B_K .$$

Since a^t_{m+1} is of level M , $\alpha(S_M) \geq M\mu(S_M)$. So $p_j \geq \alpha(S_M)/m\bar{a}_j \geq$ $\geq M\mu(S_M)/m\bar{a}_j$. We then have

$$\frac{M\mu(S_M)}{m\bar{a}_j} \leq B_K ,$$

i.e.,

$$\mu(S_M) \leq \frac{B_K m\bar{a}_j}{M} .$$

Note that m , $\bar{a}_j$, K and B_K are all independent of M and $\hat{b}$.

Q.E.D.

5. Noncooperative Games in 'Coalitional Strategic' Form

Let $Z : T \rightarrow \Omega^k$ be a mapping of t into subsets of Ω^k, where $Z(t)$ is the strategy set of t. We will assume that (a) Z is measurable, and (b) the graph of Z is bounded from above (component-wise) by some integrable function $b : T \rightarrow \Omega^k$.

Imagine that each player t has an "intended strategy" $g(t) \in Z(t)$. The map $g : T \rightarrow \bigcup_{t \in T} Z(t)$ may not be measurable. How are we to obtain a measurable choice $f : T \rightarrow \bigcup_{t \in T} Z(t)$, $f(t) \in Z(t)$, starting from g?

Introduce the notion of a coalition's strategy. For any S in $\mathcal{C}$, this is a measurable function φ^S from S to $\bigcup_{t \in T} Z(t)$, such that $\varphi^S(t) \in Z(t)$ for $t \in S$. We can represent it equivalently by a vector measure Φ^S on T, with carrier S, given by: $\Phi^S(R) = \int_{S \cap R} \varphi^S(t) d\mu$, for any $R \in \mathcal{C}$.

By a <u>selection</u> we will mean a choice of strategies, in the above sense, by every coalition. We will suppose that there is some sort of decision process which arrives at a selection starting from the intended strategies g. The exact description of this process is not needed, though we give an example of one at the end of this section: all we assume is that the process gives rise to a selection.

Given a selection, we will be able to determine an outcome (and payoffs) for the game for any partition of T into a finite collection of disjoint coalitions $S_1, \ldots, S_k$. Different partitions will in general give different outcomes. The "noncooperative" idea will be embodied in our intention to look only at very fine partitions--passing to the limit in a way that permits the "mesh": $\max[\mu(S_i) : i = 1, \ldots, k]$ of the partition go to zero.

We may imagine that the referee can only "hear" measurable sets. The players in effect band together in order to transmit their moves to the referee, but they do not band together in order to play the game cooperatively in the usual sense.

It is natural now to ask under what circumstances a selection will lead to a measurable function b, and hence to a particular "most noncooperative" outcome of the game.

Given a selection $\Phi = \{\phi^S : S \in \mathcal{C}\}$, and a R in $\mathcal{C}$, define $\xi^R : \mathcal{C} \to \mathbb{R}^k$ by $\xi_i^R(S) = \phi_i^R(T)$. If ξ_i^R is of bounded deviation* for all R in $\mathcal{C}$, and all $i = 1, \ldots, k$, then we shall say that Φ is admissible. It turns out that if Φ is admissible we can ensure a measurable noncooperative outcome "at the limit."

Let $\hat{\mathcal{P}}$ be the collection of (finite) partitions of T. If h is any real-valued function defined on $\hat{\mathcal{P}}$, we define the directed limit

$$\overrightarrow{\lim_{\mathcal{P} \in \hat{\mathcal{P}}}}\, h(\mathcal{P})$$

to be the number λ such that, for every $\epsilon > 0$, there is $\mathcal{P}_\epsilon \in \hat{\mathcal{P}}$ such that

$$|h(\mathcal{P}) - \lambda| < \epsilon$$

for every $\mathcal{P} \in \hat{\mathcal{P}}$ that is a refinement of $\mathcal{P}_\epsilon$. Since any two partitions have a common refinement, the directed limit, if it exists, must be unique. (See e.g. [8], p. 26.)

The noncooperative behavior of the selection Φ at the limit can be now described at the set function $\eta : \mathcal{C} \to \mathbb{R}^k$, where

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*A set function is of bounded deviation if it is the difference of two superadditive functions.

$$\eta_i(R) = \lim_{\mathcal{P}\in\mathcal{P}} \{ \sum_{j=1}^{p} \Phi_i^{S_j}(R) : \{S_1, \ldots, S_p\} = \mathcal{P} \} ,$$

By Theorem 10 in [9], this limit exists for all $R \in \mathcal{C}$ and all i, provided only that Φ is admissible.

It is obvious that η is finitely additive. We assert that it is in fact countably additive. First recall that since b bounds the graph of Z from above

$$\Phi(R) = \int_{S\cap R} \varphi_i^S(t)d\mu \leq \int_{S\cap R} b_i(t)d\mu$$

for all S and R in $\mathcal{C}$, and all $i = 1, \ldots, k$. Let β represent the vector measure generated by the indefinite integral of b. First observe that, for any $R \in \mathcal{C}$,

$$\sum_{j=1}^{\beta} \phi_i^{S_j}(R) \leq \sum_{j=1}^{\beta} \beta_i(S_j \cap R)$$

$$= \beta_i(R)$$

where $(S_1, \ldots, S_p)$ is any partition of T. This immediately implies $\eta_i(R) \leq \beta_i(R)$ for any R and any i. Now suppose $R = \bigcup_{j=1}^{\infty} R_j$, where the R_j's are disjoint. For any k,

$$\eta_i(R) - \sum_{j=1}^{k} \eta_i(R_j) = \eta_i(\bigcup_{k+1}^{\infty} R_j)$$

since η_i is finitely additive.

Now β_i is a finite measure, hence $\beta_i(R) = \sum_{j=1}^{\infty} \beta_i(R_j)$, i.e., $\lim_{k\to\infty} \sum_{j=k+1}^{\infty} \beta_i(R_j) = 0$. But

$$\eta_i(\bigcup_{j=k+1}^{\infty} R_j) \leq \beta_i(\bigcup_{j=k+1}^{\infty} R_j)$$

$$= \sum_{j=k+1}^{\infty} \beta_i(R_j) \ .$$

It follows that $\lim_{k\to\infty} \eta_i(\bigcup_{k+1}^{\infty} R_j) = 0$, and thus $\eta_i(R) = \sum_{j=1}^{\infty} \eta_i(R_j)$, showing that η is countably additive.

Note that for $i = 1, \ldots, k$, η_i is absolutely continuous with respect to β_i , and each β_i is absolutely continuous with respect to μ . Therefore each η_i is absolutely continuous with respect to μ . Let $f : R \to \mathbb{R}^k$ be the Radon-Nikodymn of η with respect to μ . We then have $\eta = \int f d\mu$. This f represents the measurable choice of strategies of the players derived from the intended non-measurable choice g .

We now give an example of a process leading from a set of "intended" strategies $g(t)$, $t \in T$, to an admissible selection. In this method, each coalition announces that its members all choose the same strategy, given by the least bid in each commodity that any member of the coalition "intends." (This entails the assumption that the sets $Z(t)$ are comprehensive in Ω^k .) Thus, we have, for any $R \in \mathcal{C}$,

$$\Phi^S(R) = [\inf_{t \in S \cap R} g(t)]\mu(S \cap R) \ ,$$

where the infinum is compnnentwise, and where it is most natural <u>not</u> to exclude sets of measure 0 in taking the infinum, as g is in general not a measurable function. It is easily verified that

$$\Phi^S(R) + \Phi^T(R) \geqq \Phi^{S \cup T}(R)$$

for all $R, S, T \in \mathcal{C}$ with $S \cap T = \emptyset$. In other words, the functions

ξ^R defined above are subadditive and hence of finite deviation, making the selection Φ admissible as claimed.

To see how this might work, let $k = 1$ and let there be a non-measurable set $A \subset T$ of players who "intend" to play $g(t) = a$, while the complementary set $B = T \setminus A$ "intends" to play $g(t) = b \neq a$. Denote the inner measures of these sets by $\underline{\mu}(A)$, $\underline{\mu}(B)$. Then we can measure sets $A_1, B_1 \in \mathcal{C}$ such that $A_1 \subset A$, $B_1 \subset B$, $\mu(A_1) = \underline{\mu}(A)$, $\mu(B_1) = \underline{\mu}(B)$. Let $C_1 = T \setminus (A_1 \cup B_1)$. Any measurable $R \subset C_1$ having positive measure must contain members of both A and B, so for such R we have

$$\Phi^R(S) = \min(a,b)\mu(R \cap S) .$$

From this it follows without difficulty that η is given by

$$\eta(S) = a\mu(S \cap A_1) + b\mu(S \cap B_1) + \min(a,b)\mu(S \cap C_1) .$$

In this example, a and b could be made (measurable) functions of t and a similar argument would go through. Also, in the method itself, we could use "sup" in place of "inf" with the aid of some additional conditions on the $Z(t)$. In that case, the ξ^R would be superadditive rather than subadditive, but still of finite deviation.

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