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NONNEGATIVE WEALTH, ABSENCE OF ARBITRAGE, AND
FEASIBLE CONSUMPTION PLANS

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Abstract

A restriction to nonnegative wealth is sufficient to preclude *all* arbitrage opportunities in financial models that have risk neutral probabilities that are valid for all simple strategies. Imposing nonnegative wealth does not constrain agents from making the choice they would make under the standard integrability condition. This conclusion does not depend on whether the markets are complete.

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1. Introduction

While much of the intuition in option pricing and portfolio choice can be exhibited in discrete time models, continuous time models using Itô calculus have been dominant in these areas of finance.¹ One reason is that it is generally easier to derive a closed-form solution to a differential equation than to a difference equation. Early development of continuous-time finance using Itô calculus tended to be intuitively based and assumed sufficiency of the natural first order conditions (see, for example, Merton [1971] and Black and Scholes [1973]). The intuitive appeal of the results was reassuring, as was consistency with limiting versions of discrete-time results (as in Cox, Ross, and Rubinstein [1979]), but at that time no attempt was made to make sure that the mathematical analysis was rigorously correct.

Harrison and Kreps [1979] set out to give the continuous time analysis a rigorous foundation. They showed that this task is not straightforward, since arbitrage profits can be obtained using seemingly reasonable strategies called doubling strategies (after the strategy of doubling one's bet at roulette). Having continuous trading allows one to do in any finite time interval what would take infinitely many turns at the roulette wheel. Presence of the doubling strategies strikes at the core of the continuous time model, rendering it vacuous. Having arbitrage opportunities precludes having a solution to the optimal investment problem (for strictly monotone preferences) and, of course, invalidates option pricing theory based on the *assumption* that there is no arbitrage opportunity. Harrison and Kreps removed arbitrage possibilities by restricting trading strategies to *simple trading strategies* that allow trade only at finitely many times chosen in advance. This restriction allowed them to use and formalize the risk-neutral pricing approach of Cox and Ross [1976]. Cox and Ross argued that in the absence of arbitrage, one could always reassign the probabilities to give all assets the same expected returns. Harrison and Kreps called this approach the martingale approach because of its relation to martingale theory.

Using simple strategies salvaged some of the pricing results, but logically invalidated the results pertaining to particular strategies for hedging or optimal portfolio choice, except for degenerate cases that happened to lie in the restricted set. For example, in the Black and Scholes [1973] model, the hedging strategy used to duplicate a call option is not simple. As a result, we were in the unhappy position of having a whole set of results that had been rendered logically inconsistent. If we restrict ourselves to using simple strategies, hedging strategies and optimal portfolio choices are not available, while if portfolio choice is unrestricted there is arbitrage.

What we need is a condition that rules out all arbitrage possibilities without also ruling out the rich variety of useful strategies. One approach proposed by Harrison and Pliska [1981] is to impose

¹ Arguably, numerical solutions of binomial models such as those of Cox, Ross, and Rubinstein [1979] are now more prevalent in applications, especially in models built by practitioners. The binomial model provides more intuition over a single period, but the solution is a black box when comparative statics can be performed only numerically.

an integrability condition on the trading strategies. The set of strategies satisfying the restriction is hard to justify economically. The best interpretation so far is that within a class of L^p trading strategies with $p > 1$, these are the limits in L^p of simple trading strategies (see, e.g., Duffie and Huang [1985]).

Harrison and Kreps [1979] suggested a more economically interpretable condition. They conjectured that arbitrage is ruled out if trading strategies are restricted to those having nonnegative wealth at all points in time. Note that it is trivial that any lower bound on wealth rules out the doubling strategy, since wealth *does* go arbitrarily negative with a strictly positive probability given a doubling strategy. What is more subtle is to show that a lower bound on wealth rules out *all* trading strategies that generates a free lunch. Dybvig [1980] confirmed Harrison and Kreps' conjecture in the special fixed-coefficient Black and Scholes [1973] model. Latham [1984] has also explored this possibility more heuristically. Still within the fixed-coefficient Black-Scholes model, Heath and Jarrow [1987] have analyzed margin requirements that are similar to nonnegative wealth constraints, and they show that the margin requirements do not rule out the hedging strategies that duplicate put and call options.

The purpose of this paper is to provide a general analysis of the nonnegative wealth condition that is not restricted to the fixed-coefficient Black-Scholes example, and to demonstrate the relation between the nonnegative wealth constraint and the integrability condition. We show that the nonnegative wealth constraint precludes arbitrage opportunities quite generally. We also show that for agents whose choice space is the set of consumption plans satisfying the integrability condition, the set of feasible choices is the same up to closure under the two restrictions. None of the results in the paper require completeness of the market, although in one case we give a simple alternative proof under completeness.

The rest of this paper is organized as follows. Section 2 contains our assumptions and the proof that the nonnegative wealth constraint precludes arbitrage. Results are in Section 3. Some concluding remarks are in Section 4.

2. Nonnegative Wealth and Absence of Arbitrage

All uncertainty in the model is generated by an N -dimensional standard Brownian motion, denoted by $w = \{w(t); t \in [0, 1]\}$. Formally, all random variables are defined on the complete probability space (Ω, \mathcal{F}, P) , where each $\omega \in \Omega$ is a complete description of the state of the nature, where \mathcal{F} is a sigma-algebra of distinguishable events, and where P is the common probability belief of agents. Let $\mathbf{F} = \{\mathcal{F}_t; t \in [0, 1]\}$ be the *filtration* generated by w . (A filtration is an increasing family of sub-sigma-fields of \mathcal{F} , and is the formal description of how information arrives over time for an agent who observes w and no other useful information.) We assume that all agents in the economy are endowed with the same information, specified by \mathbf{F} , and therefore the sample paths

of w completely specify distinguishable events ($\mathcal{F}_1 = \mathcal{F}$). Note that since a standard Brownian motion starts from zero almost surely, \mathcal{F}_0 is almost trivial, that is, all events in \mathcal{F}_0 have probability zero or one. All the stochastic processes to appear will be adapted to \mathbf{F} , and unless otherwise noted, all relations involving random variables are to hold almost surely with respect to P for all $t \in [0, 1]$. We denote by $E[\cdot|\mathcal{F}_t]$ the expectation operator under P conditional on \mathcal{F}_t , and by $E[\cdot]$ the unconditional expectation (which is the same as $E[\cdot|\mathcal{F}_0]$).

We consider a frictionless securities market with $M + 1$ long-lived securities, indexed by $j = 0, 1, 2, \dots, M$. Security zero is a locally riskless bond that pays interest at the instantaneous rate $r(t)$. Security zero is not necessarily riskless over longer time periods, because the interest rate $r(t)$ may be random. We assume (w.l.o.g.) that security zero is worth one at time zero and never has any distributions, and therefore its price process is given by $B(t) = \exp\{\int_0^t r(s)ds\}$. For our analysis we will assume that $B(t)$ is bounded below from zero uniformly across states and time, as it will be, for example, if r is never negative.

For $j = 1, 2, \dots, M$, security j can be locally risky (as it will be except in degenerate cases). We want to allow our analysis to be general enough to allow consumption payments and dividends to be lumpy (at a point in time), continuous, or a combination of the two. Also, while we require consumption to be nonnegative, we allow negative dividends (i.e., assessments), to allow financial assets that are not of limited liability. The simplest and most intuitive way to represent a consumption pattern is by a process with nonnegative and nondecreasing sample functions, representing accumulated consumption over time. Similarly, we use processes with bounded variation sample functions to represent accumulated dividends.

Therefore, we will represent the investment opportunities presented by the risky security j using the ex-dividend price at time t , denoted by $S_j(t)$, and cumulative dividends from time 0 time t , denoted by $D_j(t)$. To define properly the integrals characterizing the wealth process, we need to choose carefully the accounting conventions for lumpy dividends and price changes at a point in time. The convention we choose is for the dividend or stock price change at t to be included in the cumulative dividend process or price at t . In technical terms, this says that for each j , both $D_j(t)$ and $S_j(t)$ are assumed to be right continuous, and their left limits $D_j(t-)$ and $S_j(t-)$ are assumed to exist. The lump net dividend paid out at time t by security j is thus $\Delta D_j(t) \equiv D_j(t) - D_j(t-)$, and $\Delta S_j(t) \equiv S_j(t) - S_j(t-)$ is the change in price as the stock goes ex-dividend. To maintain this interpretation for $t = 0$, we follow the convention that $D_j(0-) \equiv 0$. If $D_j(t)$ is absolutely continuous, the dividend rate $D'_j(t)$ exists for almost all t and $D_j(t)$ is its integral. As another special case, if dividends occur only in lumps, then $D_j(t)$ is a random step function. Because it is of no real use to us to have a dividend payment at $t = 0$, we assume, without loss of generality, that $D_j(0-) = D_j(0) = 0$ and that $S_j(0-) = S_j(0)$, $\forall j$.

We will use vector notation for the price and dividend processes for the risky assets: $S^\top \equiv (S_1, S_2, \dots, S_M)$ and $D^\top \equiv (D_1, D_2, \dots, D_M)$, where $^\top$ denotes the transpose. Also, we will use $|\sigma|^2$ to denote the trace of $\sigma\sigma^\top$, when σ is a matrix or a vector.

Because we will not consider any taxes or other frictions in our model, the change in wealth from holding one share of any security for a period depends only on the sum of income and capital gains. We will assume that this total change in value includes a local drift $\zeta(t)$ and a local standard deviation $\sigma(t)$. In Itô differential notation,

$$dS(t) + dD(t) = \zeta(t)dt + \sigma(t)d\omega(t), \quad (2.1)$$

where $dS(t)$ is the local capital gain and $dD(t)$ is the local income. Note that ζ is an M -dimensional vector random process and σ is an $M \times N$ matrix random process. (In other words, σ embodies the local variances and covariances.) For (2.1) to be well-defined we assume that

$$\int_0^1 |dD(t)|dt < \infty, \quad (2.2)$$

$$\int_0^1 |\zeta(t)|dt < \infty, \quad (2.3)$$

and

$$\int_0^1 |\sigma(t)|^2 dt < \infty. \quad (2.4)$$

(See Liptser and Shirayev [1977, Chapter 4]. Recall our convention that statements such as these are implicitly P-a.s.) For sake of generality, we are intentionally avoiding the common convention of stating changes in asset values in terms of return per dollar invested, thus allowing risk and expected change in value to be nonzero even when the value is zero. Our convention allows us to include contracts having unlimited liability.

The consumption set admits fairly arbitrary nonnegative patterns of consumption over time. We will use a nonnegative, nondecreasing, and right-continuous process $C = \{C(t); t \in [0, 1]\}$ to model cumulative consumption, that is, $C(t)$ is the total consumption over the time interval $[0, t]$. By convention, we will take $C(0-) \equiv 0$, and therefore we can make parallel statements to the statements we made about dividends. For example, the lump of consumption at time t is given by $\Delta C(t) \equiv C(t) - C(t-)$. (However, although the dividend at 0 is assumed to be 0, we will allow consumption at 0, i.e., $C(0)$ is not necessarily zero.) Our convention is consistent with continuous consumption, lumpy consumption, or any combination of the two. For example, if all consumption is at the end, $C(t) = 0$ for $t < 1$ and $C(1)$ is the terminal consumption. In Section 3, we will put more structure on the consumption set, but for now we will leave it at this level of generality. The results we obtain in this section for this consumption set will also be immediately valid for any subset,

by nature of the results. For example, sufficient conditions to preclude arbitrage opportunities are still sufficient when choice set is further constrained.

The objects of choice are portfolio strategies. A portfolio strategy $(\alpha, \theta^\top) = (\alpha, \theta_1, \theta_2, \dots, \theta_M)$ is the vector of share holdings in the $M + 1$ securities. We take α and θ to be *predictable* random processes that are limits of left continuous processes with right limits. This is different from our convention for D and S , because we want $(\alpha(t), \theta(t))$ to be the portfolio position held across the dividend payment at time t . In other words, $\theta(t)$ is the risky portfolio position across the dividend payment $D(t) - D(t-)$ and corresponding stock price jump $S(t) - S(t-)$. This convention is consistent with the convention used by Jacod [1979, ch. III] for defining the values of integrals of the form $\int \theta(t)^\top dS(t)$ where both θ and S can have jumps. Because the portfolio strategies are predictable processes, we will interpret $(\alpha(t), \theta(t))$ to be the portfolio held from an instant before time t to time t . The wealth at t is

$$\begin{aligned} W(t) &= \alpha(t)B(t) + \theta(t)^\top(S(t-) + \Delta S(t) + \Delta D(t)) - \Delta C(t), \\ &= \alpha(t)B(t) + \theta(t)^\top(S(t) + \Delta D(t)) - \Delta C(t), \end{aligned} \quad (2.5)$$

which is the value of the portfolio plus the dividend received at t , less payment for consumption at t . For readers who are content to assume that dividends and consumption are never lumpy, S , D , and C are continuous (and therefore have the same value at $t-$, t , and $t+$), and the changes ΔC , ΔD , and ΔS are always zero. In this case, (2.5) is identical to the budget constraint in Merton (1971).

To ensure that the wealth process is well-defined, we will assume that

$$\int_0^1 |\alpha(t)B(t)r(t) + \theta(t)^\top \zeta(t)| dt < \infty \quad (2.6)$$

and

$$\int_0^1 |\theta(t)^\top \sigma(t)|^2 dt < \infty. \quad (2.7)$$

(We will refer to these conditions after we write down the wealth process in (2.8).) Let H denote the space of trading strategies satisfying these two conditions. By the Minkowski Inequality (see Royden (1968, p.114)), H is a linear space. Note that (2.2), (2.3), and (2.4) imply that H contains all the simple strategies² considered by Harrison and Kreps [1979].

Now we want to define the budget constraint. To avoid cumbersome notation, we again sacrifice complete generality by assuming that there are no intermediate cash inflows. In discrete time, we would start with some given amount of wealth at time zero. The change of wealth from the end of

²A strategy is said to be simple if it is bounded and changes its values at a finite number of nonstochastic time points. Formally, (α, θ) is a simple trading strategy if there exist time points $0 = t_0 < t_1 < \dots < t_N = 1$ and bounded random variables x_n, y_{jn} , $n = 0, \dots, N - 1$ and $j = 1, \dots, M$, such that x_n and y_{jn} are measurable with respect to \mathcal{F}_{t_n} and $\alpha(t) = x_n$ and $\theta_j(t) = y_{jn}$ if $t \in (t_n, t_{n+1}]$.

one period to the end of the next period is equal to net gains plus net income, less any consumption. Also, to preclude borrowing without repayment, terminal wealth cannot be negative.

The continuous time budget constraint is given by analogous conditions. We start with a given initial wealth level $W_0 > 0$. Recall from (2.5) that the wealth at time t , denoted by $W(t)$, is the value of the portfolio plus the value of the dividends at t , minus the consumption at t . For all $t > 0$, the change in wealth over the interval dt must equal to the change in value of investments inclusive of income, less consumption withdrawal:

$$dW(t) = \alpha(t)B(t)r(t)dt + \theta(t)^\top(\zeta(t)dt + \sigma(t)dw(t)) - dC(t) \quad t > 0. \quad (2.8)$$

Because $C(0-) = 0$, the integral form of (2.8) is

$$W(t) = W(0-) + \int_0^t \alpha(s)B(s)r(s)ds + \int_0^t \theta(s)^\top(\zeta(s)ds + \sigma(s)dw(s)) - C(t). \quad (2.9)$$

Relation (2.8) is a natural budget constraint and is often referred to as the *self-financing* constraint. Besides (2.8), an agent is constrained by initial wealth, and nonnegative final wealth (or else the agent could borrow without repayment):

$$W(0-) = W_0, \quad (2.10)$$

$$W(1) \geq 0. \quad (2.11)$$

We will say that the consumption plan $C = \{C(t); t \in [0, 1]\}$ is *financed* by the portfolio strategy (α, θ) with initial wealth W_0 if $(\alpha, \theta) \in H$, and (2.8), (2.10), and (2.11) are satisfied when we define $W(t)$ by (2.5). Note that (2.8) (or equivalently (2.9)) is well-defined as a consequence of our conditions (2.6) and (2.7) defining H . Therefore, every consumption plan financed by a feasible trading strategy has a finite cumulative consumption, that is, $C(1) < \infty$.

At this point we give a definition for an arbitrage opportunity. An arbitrage opportunity is a consumption plan C which is nonzero and nonnegative and financed by some $(\alpha, \theta) \in H$ given an initial wealth $W_0 = 0$. More formally, an arbitrage opportunity is a consumption plan C financed by $(\alpha, \theta) \in H$ such that $W_0 = 0$ and total consumption $C(1) > 0$ with a strictly positive probability.

Kreps [1981] and Huang [1985] showed that a sufficient condition for absence of arbitrage in the L^2 limit of consumption patterns given by simple strategies is the existence of an equivalent probability reassignment under which all assets are priced by expected present value, where present value is computed using the rolled-over spot rate. (Two probability measures are said to be equivalent to each other if they have the same sets of probability zero.) This probability reassignment is referred to by Harrison and Kreps as an *equivalent martingale measure*, which is another term for the artificial *risk neutral probabilities* of Cox and Ross [1976]. We will always assume such a probability reassignment exists.

Assumption 2.1. *There exists a probability measure Q equivalent to P such that $(\forall t < s)$*

$$\frac{S(t)}{B(t)} = E^* \left[\int_t^s \frac{1}{B(\tau)} dD(\tau) + \frac{S(s)}{B(s)} \middle| \mathcal{F}_t \right]. \quad (2.12)$$

Equivalence of P and Q implies that the Radon–Nikodym derivative of Q with respect to P , which we will call η , is strictly positive. Because Q is a probability,

$$\begin{aligned} 1 &= \int_{\Omega} dQ(\omega) = \int_{\Omega} d\eta(\omega) dP(\omega) \\ &= E[\eta], \end{aligned}$$

i.e., the expectation of η is finite and is equal to 1. We can therefore define a process

$$\eta(t) \equiv E[\eta | \mathcal{F}_t].$$

This process is a martingale under P with $\eta(0) = 1$.

Remark 2.1. Since P and Q have the same probability zero sets, statements that are true almost surely with respect to P must be true almost surely with respect to Q , and vice versa. Therefore, our convention that all statements are almost surely- P means equivalently that all statements are almost surely- Q .

We will first record some implications of Assumption 2.1.

Proposition 2.1. *There exists an N -dimensional standard Brownian motion w^* under Q such that*

$$\frac{S(t)}{B(t)} + \int_0^t \frac{1}{B(s)} dD(s) = S(0) + \int_0^t \frac{\sigma(s)}{B(s)} dw^*(s). \quad (2.13)$$

(In other words, if we take the riskless bond as numeraire, we have that each stock price plus cumulative dividends is a driftless Itô integral under Q .) Furthermore, $dw^*(t) = dw(t) - \kappa(t)dt$, where κ consistently gives the local price of taking on the risk in dw , that is,

$$\zeta(t) - r(t)S(t) = -\sigma(t)\kappa(t) \quad \nu - a.e., \quad (2.14)$$

where ν is the product measure of P and the Lebesgue measure on $[0, 1]$.

Proof. Because $\{\eta(t); t \in [0, 1]\}$ is a martingale adapted to the Brownian fields \mathbf{F} , there exists an N -dimensional process $\{\rho(t); t \in [0, 1]\}$ with

$$\int_0^1 |\rho(t)|^2 dt < \infty$$

such that

$$\eta(t) = \eta(0) + \int_0^t \rho(s)^\top dw(s) \quad t \in [0, 1] P - a.s.;$$

see Clark [1970]. Since $\eta(t)$ is strictly positive, Itô's lemma implies that

$$d \ln \eta(t) = \kappa(t)^\top dw(t) - \frac{1}{2} |\kappa(t)|^2 dt,$$

where

$$\kappa(t) \equiv \frac{\rho(t)}{\eta(t)}.$$

Equivalently,

$$\eta(t) = \exp \left\{ \int_0^t \kappa(s)^\top dw(s) - \frac{1}{2} \int_0^t |\kappa(s)|^2 ds \right\}.$$

Girsanov's Theorem then implies that

$$w^*(t) \equiv w(t) - \int_0^t \kappa(s) ds$$

is an N -dimensional standard Brownian motion under Q .

Next, Itô's lemma and the definition of w^* imply that

$$\frac{S(t)}{B(t)} + \int_0^t \frac{1}{B(s)} dD(s) = S(0) + \int_0^t \frac{1}{B(s)} (\zeta(s) - r(s)S(s) + \sigma(s)\kappa(s)) ds + \int_0^t \frac{\sigma(s)}{B(s)} dw^*(s). \quad (2.15)$$

By definition of Q in (2.12), the left side of (2.15) is a martingale under Q , and the integrals on the right side are well-defined by our regularity conditions and the equivalence of P and Q . But an Itô integral is a martingale only if it has zero drift (Lipster and Shirayev [1977]),³ and therefore

$$\zeta(t) - r(t)S(t) + \sigma(t)\kappa(t) = 0, \quad \nu - a.e.,$$

and consequently

$$\frac{S(t)}{B(t)} + \int_0^t \frac{1}{B(s)} dD(s) = S(0) + \int_0^t \frac{\sigma(s)}{B(s)} dw^*(s).$$

■

Our second result shows how expectations under the new measure E^* price the consumption plans that can be purchased when there is a nonnegative wealth constraint. Before we proceed, however, two technical lemmas are included for completeness. The first is a specialization to our context of the result that a local martingale that is bounded below is a supermartingale.

³The converse is false: an Itô integral with zero drift is a local martingale, but may not be a martingale.

Lemma 2.1. *Let the process $y(t)$ be such that*

$$dy(t) = \phi(t)dz(t) \quad (2.16)$$

with

$$\int_0^1 |\phi(t)|^2 ds < \infty \quad (2.17)$$

and $y(t)$ is uniformly bounded below. Then $E[y(t)] < \infty \forall t \in [0, 1]$ and $E[y(s)|\mathcal{F}_t] \leq y(t) \forall t \in [0, 1]$. That is, $y(t)$ is a supermartingale under P .

Proof. To prove that $y(t)$ is a supermartingale we have to show that $E[y(t)] < \infty \forall t \in [0, 1]$ and for any $t \leq s$, $E[y(s)|\mathcal{F}_t] \leq y(t)$. We will prove here that $E[y(1)] \leq y(0)$. The proof for arbitrary $t \leq s$ is similar.

Putting

$$\Phi(t) \equiv \int_0^t |\phi(s)|^2 ds,$$

let

$$T_n \equiv \inf(\{1\} \cup \{t \in [0, 1] : \Phi(t) \geq n\}),$$

the first time that $\Phi(t)$ is greater than or equal to n , or 1 if $\Phi(t)$ is always smaller than n . T_n is a stopping time. By the hypothesis that

$$\int_0^1 |\phi(t)|^2 ds < \infty,$$

we know $T_n \rightarrow 1$ a.s. In the (stochastic) time interval $[0, T_n]$ $y(t)$ is a martingale under P ; see Liptser and Shiriyayev [1977]. Thus

$$E[y(T_n)] = y(0) \quad \forall n. \quad (2.18)$$

As $T_n \rightarrow 1$ a.s., we know $y(T_n) \rightarrow y(1)$ a.s. Since $y(t)$ is bounded below uniformly across t , Fatou's lemma implies that

$$\begin{aligned} E[y(1)] &\leq \liminf_{n \rightarrow \infty} E[y(T_n)] \\ &= y(0), \end{aligned}$$

where the equality follows from (2.18). Hence $y(1)$ has a finite expectation under P and the expectation is smaller than $y(0)$. ■

It will be convenient to work with a normalized and re-invested wealth process $v(t)$ which uses $B(t)$ as numeraire and treats past consumption as if it had been reinvested in the locally riskless asset. Formally, we define

$$v(t) \equiv \frac{W(t)}{B(t)} + \int_{0-}^t \frac{1}{B(\tau)} dC(\tau), \quad (2.19)$$

where having the lower limit $0-$ (in place of 0) indicates to include the effect of any jump in C at zero in the integral, i.e.

$$\int_{0-}^t \frac{1}{B(\tau)} dC(\tau) = \frac{C(0)}{B(0)} + \int_0^t \frac{1}{B(\tau)} dC(\tau). \quad (2.20)$$

The following Lemma tells us that v is a driftless Itô integral under Q and that v is determined solely by risky investments and is independent of the “split” between consumption and the investment in the bond.

Lemma 2.2. *Let C be financed by $(\alpha, \theta) \in H$ with an initial wealth W_0 and let $W(t)$ be defined according to (2.5). Then, for all t ,*

$$v(t) = W_0 + \int_{0-}^t \theta(s)^\top \frac{\sigma(s)}{B(s)} dw^*(s). \quad (2.21)$$

Proof. We have that

$$dv(t) = d\left(\frac{W(t)}{B(t)}\right) + \frac{dC(t)}{B(t)},$$

(by definition of v)

$$= \frac{1}{B(t)} dW(t) - \frac{W(t)}{B(t)^2} dB(t) + \frac{dC(t)}{B(t)}$$

(by Itô's lemma)

$$= \frac{1}{B(t)} (\alpha(t)B(t)r(t)dt + \theta(t)^\top (\zeta(t)dt + \sigma(t)dw(t)) - dC(t))$$

$$- \frac{\alpha(t)B(t) + \theta(t)^\top (S(t) + \Delta D(t)) - \Delta C(t)}{B(t)} rdt + \frac{dC(t)}{B(t)}$$

(by definition of $B(t)$, (2.5) and (2.8))

$$= \frac{1}{B(t)} \theta(t)^\top ((\zeta(t) - r(t)S(t))dt + \sigma(t)dw(t))$$

(by cancellation and deletion of terms like $\Delta D(t)dt$ that integrate to zero)

$$= \theta(t)^\top \frac{\sigma(t)}{B(t)} dw^*(t) \quad (2.22)$$

(by Proposition 2.1). Because $v(0-) = W(0-)/B(0-) = W_0$, (2.21) follows directly from (2.22). ■

Lemma 2.2 tells us the local behavior of v under the budget constraints (2.8-2.11). Lemmas 2.1 and 2.2 together imply that “globally” v is a supermartingale.

Lemma 2.3. *Under the nonnegative wealth constraint $W(t) \geq 0$, v is a supermartingale under Q .*

Proof. Because consumption is nonnegative, nonnegative wealth implies that v is also nonnegative. Therefore, the result is immediate from Lemmas 2.1 and 2.2. ■

Proposition 2.2. (asset pricing) *Given the nonnegative wealth constraint $W(t) \geq 0$, the expectation under Q of the total discounted value of consumption is no larger than initial wealth. Formally, if the consumption plan C is financed by (α, θ) with initial wealth W_0 ,*

$$E^* \left[\int_{0-}^1 \frac{1}{B(t)} dC(t) \right] \leq W_0. \quad (2.23)$$

Proof. We have that

$$E^* \left[\int_{0-}^1 \frac{1}{B(s)} dC(s) \right] \leq E^*[v(1)]$$

(by definition of v and nonnegativity of $W(1)/B(1)$)

$$\leq v(0-) = W_0,$$

by Lemma 2.3, definition of v , and (2.10). ■

The fact that (2.23) contains an inequality is due to the possibility of “suicidal strategies” that throw away money, such as running a doubling strategy in reverse (Harrison and Pliska [1981]). For practical economic purposes, all strategies having a strict inequality in (2.23) can be ignored (provided that preferences are strictly increasing). This is the justification for referring to Proposition 2.2 as a pricing result (in spite of the inequality).

The following theorem is the main result of this section. It shows that a lower bound on wealth precludes arbitrage even in the absence of any other restrictions on the trading strategies. An implication of this theorem is that an arbitrage opportunity must have a corresponding wealth process that is unbounded from below.

Theorem 2.1. (absence of arbitrage) *Given the nonnegative wealth constraint $W(t) \geq 0$, no arbitrage opportunities exist.*

Proof. Assume to the contrary that there is an arbitrage opportunity C financed by $(\alpha, \theta) \in H$. Proposition 2.2 shows that

$$E^* \left[\int_{0-}^1 \frac{1}{B(t)} dC(t) \right] \leq 0, \quad (2.24)$$

because the initial cost of an arbitrage opportunity is zero. Next note that $C(1)$ is nonnegative and not identically zero. By the assumption that $B(t) > 0$, we know that

$$E^* \left[\int_{0-}^1 \frac{1}{B(t)} dC(t) \right] > 0,$$

a contradiction to (2.24). ■

The traditional definition of arbitrage we are using is not the only possible choice for our model. Implicit in the notion of arbitrage is the idea that it can be undertaken at arbitrary scale. In models like ours in which the set of feasible consumption streams is not a linear space (because of the nonnegative wealth constraint), the availability of a net trade may depend on the starting point and the scale of the net trade. Because the nonnegative wealth constraint is scale independent, our sort of arbitrage opportunities could (if available) be undertaken at any scale. However, there may be other net trades that could be made at arbitrary scale starting from some strategy (such as holding the bond) that one would want to label as arbitrage opportunities. If that were the case, we would have a different definition of arbitrage. Fortunately, our proof of the absence of arbitrage would still work in these cases, since the pricing relation implies that increasing scale would eventually violate the budget constraint.

To relate Proposition 2.2 and Theorem 2.1 to previous literature, Harrison and Pliska [1981] and Duffie and Huang [1985] have used an integrability condition to rule out doubling strategies. The integrability condition is

$$E^* \left[\left(\int_0^1 \frac{|\theta(t)^\top \sigma(t)|^2}{B(t)^2} dt \right)^{p/2} \right] < \infty, \quad (2.25)$$

for some $p \in [1, \infty)$. This condition can be used instead of the non-negative wealth constraint in Proposition 2.2 and Theorem 2.1. Furthermore, pricing as in Proposition 2.2 is with equality. Before we proceed, we record a technical lemma.

Lemma 2.4. *Let θ satisfy the integrability condition (2.25) for some $p \in [1, \infty)$. Then*

$$\int_0^t \theta(s)^\top \frac{\sigma(s)}{B(s)} dw^*(s) \quad t \in [0, 1] \quad (2.26)$$

is an L^p -martingale under Q , that is, it is a martingale under Q with

$$E^* \left[\left| \int_0^t \theta(s)^\top \frac{\sigma(s)}{B(s)} dw^*(s) \right|^p \right] < \infty \quad (2.27)$$

for all t .

Proof. See Jacod [1979]. ■

Theorem 2.2. *Let C be financed by $(\alpha, \theta) \in H$ satisfying the integrability condition (2.25) for some $p \in [1, \infty)$. Then*

$$E^* \left[\int_{0-}^1 \frac{1}{B(t)} dC(t) \right] \leq W_0. \quad (2.28)$$

That is, the discounted value of consumption is less than the initial wealth. The inequality becomes an equality if $W(1) = 0$. Furthermore, there is no arbitrage.

Proof. The proof essentially follows the proofs of Proposition 2.2 and Theorem 2.1. By Lemma 2.3, the process v is an L^p -martingale under Q . Thus

$$E^* \left[\frac{W(1)}{B(1)} + \int_{0^-}^1 \frac{1}{B(t)} dC(t) \right] = W_0.$$

By (2.11) and positivity of $B(1)$, the first assertion follows. The second assertion is obvious. The proof of no arbitrage is identical to that of Theorem 2.1, except that we use the first part of this theorem instead of Proposition 2.2. ■

3. Nonnegative Wealth and Feasible Consumption Streams

In Section 2, we showed that a nonnegative wealth constraint can substitute for an integrability condition in a proof of the absence of arbitrage when there is an equivalent martingale measure. This result is interesting because the nonnegative wealth constraint is economically motivated while the integrability condition is difficult to interpret economically. In this section, we show that in a wide variety of situations, the set of feasible consumption plans is essentially the same under either condition. For $p \in [1, \infty)$, we will use $L^p(Q)$ to denote the space of consumption plans C such that

$$E \left[\left| \int_{0^-}^1 \frac{1}{B(t)} dC(t) \right|^p \right] < \infty. \quad (3.1)$$

We consider separately the special case with complete markets and the general case in which markets may be incomplete. We provide a separate proof for the special case of complete markets because the result for complete markets is slightly stronger than the general result and the proof is much simpler than the general proof. Here is a characterization of when markets are complete.

Lemma 3.1. *The measure Q is unique if and only if $\sigma(t)$ is nonsingular ν -a.e. Suppose Q is unique and let $C \in L^p(Q)$ for some $p \in (1, \infty)$. Then there exists $(\alpha, \theta) \in H$ with θ satisfying (2.25) for the same p that finances C with $W(1) = 0$ and an initial cost*

$$W_0 = E^* \left[\int_{0^-}^1 \frac{1}{B(t)} dC(t) \right]. \quad (3.2)$$

The wealth process for this strategy is

$$W(t)/B(t) = E^* \left[\int_t^1 \frac{1}{B(s)} dC(s) \middle| \mathcal{F}_t \right]. \quad (3.3)$$

Proof. A measure Q is completely determined by its Random–Nikodym derivative η with respect to P . From the proof of Proposition 2.1 we can see that η in turn is determined by κ . From (2.14), κ is uniquely determined if and only if $\sigma(t)$ is nonsingular ν -a.e. In such event,

$$\kappa(t) = -\sigma(t)^{-1}(\zeta(t) - r(t)S(t)).$$

Next let $C \in L^p(Q)$ for some $p \in (1, \infty)$. By the martingale representation theorem (see Jacod [1979]), there exists an N -dimensional process $\phi(t)$ with

$$E^* \left[\left(\int_0^1 |\phi(t)|^2 dt \right)^{\frac{p}{2}} \right] < \infty$$

such that

$$\int_{0-}^1 \frac{1}{B(t)} dC(t) = E^* \left[\int_{0-}^1 \frac{1}{B(t)} dC(t) \right] + \int_0^t \phi(s)^\top dw^*(s).$$

Now define

$$\theta(t) \equiv \phi(t)^\top \sigma(t)^{-1} B(t).$$

It is easily verified that θ satisfies the integrability condition.

Now set $W(1) = 0$. Lemmas 2.2 and 2.3 imply that

$$\frac{W(t)}{B(t)} = E^* \left[\int_t^1 \frac{1}{B(s)} dC(s) \middle| \mathcal{F}_t \right].$$

That is, wealth at any time t is equal to the expected present value (using B for discounting). Now, defining $\alpha(t)$ by inverting (2.6), it is then straightforward to verify that (α, θ) finances C with an initial wealth

$$W_0 \equiv E^* \left[\int_{0-}^1 \frac{1}{B(t)} dC(t) \right]$$

and a final wealth $W(1) = 0$. ■

From Lemma 3.1 we learn that any $C \in L^p(Q)$, $1 < p < \infty$, is financed by some trading strategy satisfying the p -integrability condition when there exists a unique equivalent martingale measure. In such event, we say that the markets are complete. Here is our equivalence theorem for complete markets.

Theorem 3.1. (equivalence of nonnegative wealth and the integrability condition when the markets are complete) Fix $p \in (1, \infty)$ and initial wealth W_0 . The set of feasible consumption plans in $L^p(Q)$ is the same under nonnegative wealth ($W(t) \geq 0$) and the integrability condition (2.25).

Proof. We first claim that any strategy satisfying the integrability condition satisfies the nonnegative wealth constraint. This can be seen from (3.2). Since a consumption plan is nonnegative

and nondecreasing, the left-hand side of (3.2) must be nonnegative. Therefore any consumption plan in $L^p(Q)$ financed by a strategy satisfying a p -integrability condition must be available with a nonnegative wealth constraint. Conversely, let $C \in L^p(Q)$ be financed by $(\alpha, \theta) \in H$ satisfying a nonnegative wealth constraint with an initial wealth W_0 . Suppose first that

$$W_0 < \hat{W}_0 \equiv E^* \left[\int_0^1 \frac{1}{B(t)} dC(t) \right].$$

Lemma 3.1 shows that there exists a strategy (α, θ) satisfying the p -integrability condition that finances C with an initial wealth \hat{W}_0 and a zero final wealth. Using the same risky investment strategy θ and a position in the locally riskless bond equal to $\alpha(t) + W_0 - \hat{W}_0$ can finance the given consumption plan C with a final wealth equal to

$$W(1) = (W_0 - \hat{W}_0)B(1).$$

This strategy certainly satisfies the p -integrability condition and finances C . Next, if $W_0 = \hat{W}_0$. Then Lemma 3.1 shows that there exists a strategy satisfying the p -integrability condition that finances C with $W(1) = 0$.

■

Remark 3.1. When $p = 1$, Lemma 3.1 fails because there exist consumption plans in $L^1(Q)$ that are not financed by strategies satisfying the integrability condition. It is still true that all feasible strategies satisfying the integrability condition also satisfy the nonnegative wealth constraint, and the two consumption sets are the same up to closure, as will be demonstrated in Theorem 3.2. Therefore, the consumption set for nonnegative wealth is larger than the set under integrability in a way that may solve some closure problems.

Theorem 3.1 says that when markets are complete, any consumption plan in $L^p(Q)$ for some $p \in (1, \infty)$ financed by a strategy satisfying the nonnegative wealth constraint is financed by a strategy satisfying the p -integrability condition that possibly throws away some strictly positive final wealth. Conversely, a strategy satisfying the p -integrability condition must also satisfy the nonnegative constraint.

The second equivalence theorem deals with the case where the markets may not be complete. In this case we have been unable to show that the two sets are the same (although they may always be the same – we just don't know). Our result is that the two sets are the same up to closure, with the consumption set under nonnegative wealth no smaller than the consumption set under the integrability condition. Therefore, as in the case $p = 1$ for complete markets, we can only say

that the integrability condition is at least as attractive, since it has some potential of solving (or partially solving) a closure problem.

Theorem 3.2. *(equivalence of nonnegative wealth and the integrability condition when markets may not be complete) Fix $p \in [1, \infty)$ and initial wealth W_0 . The set of feasible consumption plans in $L^p(Q)$ is no smaller under nonnegative wealth ($W(t) \geq 0$) than under the integrability condition (2.25), and up to closure the two sets are the same.*

Proof. The proof that any strategy satisfying a p -integrability condition must also satisfy the nonnegative wealth constraint is the same as in Theorem 3.1. So we need to prove that any consumption plan in $L^p(Q)$ financed by a strategy satisfying the nonnegative wealth constraint is in the closure of the set of consumption plans financed by strategies satisfying the p -integrability condition.

If we can show that $v(1)$ is essentially uniformly smaller across states for the nonnegative wealth strategy than for some strategy satisfying the integrability condition, we will be done. We can simply follow the same consumption withdrawal strategy in both cases and the strategy satisfying the integrability condition will have nonnegative residual wealth at the end. (Recall from Lemma 2.2 that v depends only on θ and not the mix between C and α .)

Let X be the space of $v(1)$ generated by strategies satisfying the p -integrability condition (2.25) with an initial wealth W_0 , imposing the budget constraint including $W(1) \geq 0$ (which is (2.11)), but without imposing any further requirement that $v(1) \geq 0$ or $W(t) \geq 0$ for $t < 1$. It is easily verified that X is an affine subspace of $L^p(\Omega, \mathcal{F}, Q)$ in the norm topology and is therefore convex. Fix some $\hat{v}(1)$ generated by a strategy satisfying the nonnegative wealth constraint. It suffices to show that $\hat{v}(1) \in cl(X - L_+^p(\Omega, \mathcal{F}, Q))$, where cl indicates the closure and $L_+^p(\Omega, \mathcal{F}, P)$ denotes the positive orthant of $L^p(\Omega, \mathcal{F}, P)$. This follows because $(X - L_+^p(\Omega, \mathcal{F}, Q)) \cap L_+^p(\Omega, \mathcal{F}, Q)$ is the set of feasible $v(1)$ under integrability once we allow for the free disposal of residual wealth at the end.

To show that $\hat{v}(1) \in cl(X - L_+^p(\Omega, \mathcal{F}, Q))$, we use a proof by contradiction: suppose not. In our proof by contradiction we are supposing that $\hat{v}(1)$ is not an element of $cl(X - L_+^p(\Omega, \mathcal{F}, Q))$. Because $cl(X - L_+^p(\Omega, \mathcal{F}, Q))$ is closed and convex, and because a set containing a single point is compact and convex, a standard separation theorem (Rudin [1973], Theorem 3.4(b)) tells us that there exists a linear functional $\lambda \in L^q(\Omega, \mathcal{F}, Q)$ (where $1/p + 1/q = 1$) such that

$$E^*[\lambda \hat{v}(1)] > E^*[\lambda(v(1) - \delta)] \quad (3.4)$$

for all $v(1) \in X$ and $\delta \in L_+^p(\Omega, \mathcal{F}, Q)$. Clearly, $\lambda \geq 0$ (or else (3.4) could not hold for all $\delta \in L_+^p(\Omega, \mathcal{F}, Q)$), and $\lambda \neq 0$ (or else (3.4) could not have a strict inequality). Also, since X is an affine subspace, $E^*[\lambda v(1)]$ must be the same constant K for all $v(1) \in X$. As we show shortly, $\hat{v}(1)$ is the

almost-sure limit of elements of X , and therefore $E^*[\lambda\hat{v}(1)] \leq K$ by Fatou's lemma, contradicting (3.4).

We have yet to prove that $\hat{v}(1)$ is the almost-sure limit of elements of X . Let $\hat{\theta}$ be the risky strategy that corresponds to $\hat{v}(1)$. Define

$$\Sigma(t) \equiv \int_0^t \frac{|\hat{\theta}(s)^\top \sigma(s)|^2}{B(s)^2} ds$$

and

$$T_n \equiv \inf(\{1\} \cup \{t \in [0, 1] : \Sigma(t) \geq n\}).$$

Define $v_n(\omega, t) = v(\omega, t \wedge T_n(\omega))$. (The symbol \wedge indicates the minimum.) In other words, v_n is the same as v until the cumulative variance of v , in units of the locally riskless bond, equals n , and constant thereafter. Each v_n corresponds to the same portfolio strategy as $\hat{v}(1)$ until the integral defining Σ reaches n , and corresponds to holding only the locally riskless bond afterwards. But this integral is the same as the integral in the integrability condition (2.25), which is therefore satisfied for the strategy underlying v_n (since the integral is bounded uniformly by n). Therefore $v_n(1) \in X$. But $\Sigma(1) < \infty$ is implied by (2.7) and the fact that $B(t)$ is uniformly bounded below. Therefore, $v_n(1)$ converges to $v(1)$ almost surely. But each $v_n(1)$ is an element of X . Therefore we have shown that $\hat{v}(1)$ is the almost-sure limit of elements of X , which is all we had left to show. ■

The conclusion of Theorem 3.2 is weaker than that of Theorem 3.1. It says that the consumption sets under nonnegative wealth and integrability are the same up to closure and that the consumption set under nonnegative wealth is no smaller. The case $p = 1$ no longer requires special treatment because the closure problem now exists for all p . In terms of the proof, the problem for $p > 1$ is that while X is closed and free disposal $-L_+^p(\Omega, \mathcal{F}, Q)$ is closed, we do not know that $X - L_+^p(\Omega, \mathcal{F}, Q)$ is closed.

4. Concluding remarks

In the dynamic choice problems considered above, the commodity spaces are specified under the martingale measure. This is unnatural. We demonstrated that solution sets under the nonnegative wealth constraint and under the integrability condition coincide. But we did not deal with whether the solution sets are empty. We refer readers to Cox and Huang [1987a, 1987b] and Pages [1987] for an extensive treatment of these two issues.

The nonnegative wealth constraint is more intuitively understandable than the integrability condition. They, however, are functionally indistinguishable for a large class of economic problems!

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