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Trading Off Generations: Infinitely-Lived Agent Versus OLG*

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Abstract: The prevailing literature discusses intergenerational trade-offs predominantly in infinitely-lived agent models despite the finite lifetime of individuals. We discuss these trade-offs in a continuous time OLG framework and relate the results to the infinitely-lived agent setting. We identify three shortcomings of the latter: First, underlying normative assumptions about social preferences cannot be deduced unambiguously. Second, the distribution among generations living at the same time cannot be captured. Third, the optimal solution may not be implementable in overlapping generations market economies. Regarding the recent debate on climate change, we conclude that it is indispensable to explicitly consider the generations' life cycles.

Keywords: climate change, discounting, infinitely-lived agents, intergenerational equity, overlapping generations, time preference

JEL-Classification: D63, H23, Q54

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1 Introduction

How much should a government invest in public infrastructure or in basic research? And how much should society invest in greenhouse gas mitigation? These decision problems exhibit two important common characteristics: a classical public good problem and an intergenerational equity trade-off. The public good problem requires governmental intervention to restore efficiency. However, doing so inevitably involves a distributional choice between current and future generations. Most models address the distributional component using an infinitely-lived agent (ILA) framework where the utility of the ILA is interpreted as a utilitarian social welfare function. The current literature offers two distinct answers how to specify the parameter values of this welfare function. The "positive" approach suggests a calibration-based procedure that attempts to avoid explicit normative assumptions. By contrast, the normative approach takes the standpoint that only ethical considerations are valid to address the intergenerational trade-off. These two approaches generally lead to significantly different results that can be traced back to the difference in the implied social discount rate.

Barro (1974) shows that, under appropriate assumptions on altruism implying operational bequests, finitely lived generations can be aggregated into a representative ILA. Recent empirical studies, however, indicate that the altruistic bequest motive is rather weak.[1] These findings suggest that an overlapping generations (OLG) model without altruistic bequests would better fit reality. If so, is it appropriate to use an ILA specification to discuss intergenerational equity trade-offs?

In this paper, we answer these questions by examining the relationship between an OLG model in continuous time and the standard ILA economy (i.e., Ramsey-Cass-Koopmans economy). We point out various short-comings of the ILA assumption and discuss positive and normative aspects of the distributional problem by disentangling the life plans of finitely-lived individuals from the long-run plans of a social planner. First, we construct an unregulated decentralized OLG economy of finitely-lived agents in continuous time. We determine the conditions for which a decentralized OLG economy is observationally equivalent to, i.e., exhibits the same macroeconomic observables as, an ILA economy. In particular, we explain why and how the preference parameters of the individual households in the decentralized OLG economy differ from those in the observationally equivalent ILA economy. Second, we introduce a social planner who maximizes the discounted

[1] See, e.g., Hurd (1987, 1989), Kopczuk and Lupton (2007), Laitner and Juster (1996), Laitner and Ohlsson (2001), Wilhelm (1996).

life time utilities of the OLG as, e.g., Calvo and Obstfeld (1988) or Burton (1993). We show that this utilitarian OLG economy is observationally equivalent to an appropriately chosen ILA economy. However, the distribution of consumption between old and young at any given point in time differs substantially from that of the decentralized economy if the rate of time preference (or generational discount rate) of the social planner is lower than that of the individual households. In this case the utilitarian OLG model implies a trade-off between equality among the generations living at the same time and equality of lifetime utilities between present and future generations. Third, we find that in the OLG context the ability to decentralize the social planner's solution is limited. In a constrained setting, in which age-discriminating taxes are not available to governments, the constrained social planner generally cannot achieve the first-best social optimum.

We apply our results to the recent debate on climate change mitigation. We identify the implicit normative assumptions in the positive approach to social discounting, as, for example, advocated by Nordhaus (2007). In particular, we point out that, in general, the positive approach to specify the social welfare function implicitly assumes that the time preference rate of the social planner exceeds the one of the individual households. This contrasts sharply with most of the ethical arguments found in the debate. The normative approach to social discounting, as followed by Stern (2007), employs an ILA model with a near zero rate of time preference expressing the desire to treat generations equally. However, the ILA model does not capture the distribution of consumption among generations alive at a given point in time. The utilitarian OLG model implies that a more equal treatment of lifetime utilities between present and future generations can come at the expense of a more unequal treatment of the generations alive at a given point in time – at least if individuals possess a positive rate of pure time preference.

There are several other papers that examine the relationship between ILA and OLG models. Aiyagari (1985) proved that under certain assumptions in discrete time an OLG model with two-period lived individuals is observationally equivalent to an ILA model. For the continuous time setting, Calvo and Obstfeld (1988) observed this equivalence for the social planner solution in an OLG setting with finitely lived agents. While the latter focus on time inconsistencies in fiscal policy, our focus is on intergenerational trade-offs. Formally, our OLG model most closely relates to d'Albis (2007) who examines the influence of demographic structure on capital accumulation and growth. In contrast to d'Albis (2007), we allow for exogenous technological change. In contrast to both of the above continuous time OLG models, we assume deterministic rather than stochastic life-times. Moreover, we provide an explicit mapping between the two frameworks with respect to the rates of time preference and the intertemporal elasticities of substitution. Our combination of continuous time and deterministic finite life-time also distinguishes our framework from the most widespread continuous time OLG framework by Blanchard (1985) and Yaari (1965), which features a constant probability of death and, thus, an infinite planning horizon. Several environmental economic applications, including Howarth (1998), Howarth (2000), Gerlagh and Keyzer (2001), Gerlagh and van der Zwaan (2000), and Stephan and Müller-Fürstenberger (1997), observed that ILA models can be calibrated to yield outcomes similar to OLG models. These papers use numerical simulations of integrated assessment models, whereas we derive the analytical relation between the decentralized OLG economy in continuous time and the ILA economy.

The paper is structured as follows. In Section 2, we develop a decentralized OLG model in continuous time. The ILA economy is introduced in Section 3. We derive conditions for observational equivalence of the decentralized OLG economy and an ILA model in Section 4. In Section 5, we examine the relationship between the latter and two social planner solutions, unconstrained and constrained. We apply our results to the recent debate on climate change mitigation in Section 6 and conclude in Section 7.

2 An OLG Growth Model in Continuous Time

We introduce an OLG exogenous growth model in continuous time and analyze the long-run individual and aggregate dynamics of a decentralized economy in market equilibrium.

2.1 Households

Consider a continuum of households, each living the finite time span T. All households exhibit the same intertemporal preferences irrespective of their time of birth $s \in (-\infty, \infty)$. We assume that if households are altruistic, their altruistic preferences are not sufficiently strong for an operative bequest motive. This allows us to abstract from altruism in individual preferences. As a consequence, all households maximize their own welfare U, which is the discounted stream of instantaneous utility derived from consumption during their lifetime

$$U(s) \equiv \int_s^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \exp\left[-\rho^H(t-s)\right] dt \ , \tag{1}$$

where $c(t,s)$ is the consumption at calender time t of households born at time s, σ^H is the constant intertemporal elasticity of substitution and ρ^H denotes the constant rate of (pure) time preference of the households. Each household is endowed with one unit of labor at any time alive, which is supplied inelastically to the labor market at wage $w(t)$. In addition, households may save and borrow assets $b(t,s)$ at the interest $r(t)$. The household's budget constraint is[2]

$$\dot{b}(t,s) = r(t)b(t,s) + w(t) - c(t,s) \ , \qquad t \in [s, s+T] \ . \tag{2}$$

Households are born without assets and are not allowed to be indebted at time of death. Thus, the following boundary conditions apply for all generations s

$$b(s,s) = 0 \ , \qquad b(s+T,s) \geq 0 \ . \tag{3}$$

Because of the non-operative bequest motive, intertemporal welfare U of a household born at time s always increases in consumption at time $s+T$. Thus, in the household optimum, the second boundary condition in equation (3) holds with equality.

Maximizing equation (1) for any given s subject to conditions (2) and (3) yields the well known Euler equation

$$\dot{c}(t,s) = \sigma^H \big[r(t) - \rho^H\big] c(t,s) \ , \qquad t \in [s, s+T] \ . \tag{4}$$

The behavior of a household born at time s is characterized by the system of differential equations (2) and (4) and the boundary conditions for the asset stock (3).

At any time $t \in (-\infty, \infty)$ the size of the population $N(t)$ increases at the constant rate $\nu \geq 0$. Normalizing the population at time $t = 0$ to unity implies the birth rate γ[3]

$$N(t) \equiv \exp[\nu t] \qquad \Rightarrow \qquad \gamma = \frac{\nu \exp[\nu T]}{\exp[\nu T] - 1} \ . \tag{5}$$

[2] Throughout the paper, partial derivatives are denoted by subscripts (e.g., $F_k(k,l) = \partial F(k,l)/\partial k$), derivatives with respect to calendar time t are denoted by dots and derivatives of functions depending on one variable only are denoted by primes.

[3] The equation is derived by solving $\int_{t-T}^{t} \gamma \exp[\nu s]\, ds = N(t)$, where $\gamma \exp[\nu s]$ denotes the cohort size of the generation born at time s. Observe that $\gamma \to 1/T$ for $\nu \to 0$ and $\gamma \to \nu$ for $T \to \infty$. Anticipating definition (12), we can also write $\gamma = 1/Q_T(\nu)$.

2.2 Firms

Consider a continuum of identical competitive firms $i \in [0,1]$. All firms produce a homogeneous consumption good under conditions of perfect competition from capital $k(t,i)$ and *effective labor* $A(t)l(t,i)$. $A(t)$ characterizes the technological level of the economy and grows exogenously at a constant rate ξ. Normalizing technological progress at $t=0$ to unity implies

$$A(t) \equiv \exp[\xi t] \ . \tag{6}$$

All firms have access to the same production technology $F(k(t,i), A(t)l(t,i))$, which exhibits constant returns to scale and positive but strictly decreasing marginal productivity with respect to both inputs capital and effective labor. Furthermore, F satisfies the Inada conditions.

Constant returns to scale of the production function and symmetry of the firms allow us to work with a representative firm whose decision variables are interpreted as aggregate variables. With minor abuse of notation, we introduce aggregate capital per effective labor, $k(t)$, and aggregate capital per capita, $\bar{k}(t)$,

$$k(t) \equiv \frac{\int_0^1 k(t,i)\,di}{A(t)\int_0^1 l(t,i)\,di} \ , \qquad \bar{k}(t) \equiv \frac{\int_0^1 k(t,i)\,di}{N(t)} \ . \tag{7}$$

In addition, we define the intensive form production function $f\big(k(t)\big) \equiv F\big(k(t),1\big)$.

Profit maximization of the representative firm yields for the wage $w(t)$ and the interest rate $r(t)$

$$w(t) = A(t)\left[f\big(k(t)\big) - f'\big(k(t)\big)k(t)\right] \ , \tag{8a}$$

$$r(t) = f'\big(k(t)\big) \ . \tag{8b}$$

2.3 Market Equilibrium and Aggregate Dynamics

In order to investigate the aggregate dynamics of the economy, we introduce aggregate household variables per effective labor by integrating over all living individuals and dividing by the product of technological level and the labor force of the economy. Analogously to equation (7) we define under slight abuse of notation per effective labor household variables, $x(t)$, and aggregate household variables per capita, $\bar{x}(t)$,

$$x(t) \equiv \frac{\int_{t-T}^{t} x(t,s)\gamma \exp[\nu s]\, ds}{A(t)\int_0^1 l(t,i)\, di} \ , \qquad \bar{x}(t) \equiv \frac{\int_{t-T}^{t} x(t,s)\gamma \exp[\nu s]\, ds}{N(t)} \ , \tag{9}$$

where $x(t,s)$ stands for the individual household variables consumption $c(t,s)$ and assets $b(t,s)$.

The economy consists of three markets: the labor market, the capital market and the consumption good market. We assume the economy to be in market equilibrium at all times t. In consequence, labor demand equals the population size, i.e., $\int_0^1 l(t,i)\, di = N(t)$, and capital in terms of effective labor equals aggregate assets in terms of effective labor, i.e., $k(t) = b(t)$. Then, the aggregate dynamics imply[4]

$$\frac{\dot{c}(t)}{c(t)} = \sigma^H \left[r(t) - \rho^H\right] - (\nu + \xi) - \frac{\Delta c(t)}{c(t)} \ , \tag{10a}$$

$$\dot{k}(t) = f(k(t)) - (\nu + \xi)k(t) - c(t) \ , \tag{10b}$$

where the term

$$\Delta c(t) \equiv \frac{\gamma \exp[\nu(t-T)]c(t,t-T) - \gamma \exp[\nu t]c(t,t)}{\exp[\nu t]\exp[\xi t]} \ . \tag{10c}$$

captures the difference in aggregate consumption per effective labor between the generation born and the generation dying at time t.

2.4 Steady State

Our analysis will concentrate on the long-run steady state growth path of the economy, in which both consumption per effective labor and capital per effective labor are constant over time, i.e., $c(t) = c^\star$, $k(t) = k^\star$. From equations (8) follows that in the steady state the interest rate $r(t) = r^\star \equiv f'(k^\star)$ is constant and the wage $w(t)$ grows at the rate of technological progress ξ. The wage relative to the technology level is constant in the steady state

$$w^\star \equiv \left.\frac{w(t)}{\exp[\xi t]}\right|_{k=k^\star} = \left[f(k^\star) - f'(k^\star)k^\star\right] \ . \tag{11}$$

[4] Note that $\dot{x}(t) = -(\nu+\xi)x(t) + \exp[-(\nu+\xi)t]\int_{t-T}^{t} \dot{x}(t,s)\gamma \exp[\nu s]\, ds + \gamma \left[x(t,t) - \frac{x(t,t-T)}{\exp[(\nu+\xi)T]}\right] \exp[-\xi t]$.

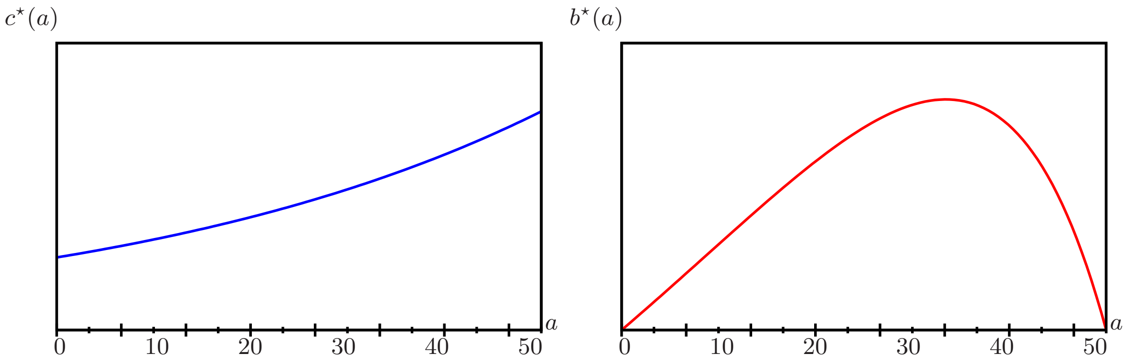

Figure 1: Steady state paths of consumption (left) and asset (right) for individual households over age.

For $T \in \mathbb{R}_{++}$ we define the function $Q_T : \mathbb{R} \to \mathbb{R}_+$ as

$$Q_T(r) \equiv \frac{1 - \exp[-rT]}{r} \ , \quad \forall\, r \neq 0 \ , \tag{12}$$

and $Q_T(0) \equiv T$. $Q_T(r)$ can be interpreted as the present value of an annuity received over T years, at the discount rate r. Properties of the function Q_T are summarized in Lemma 1 in appendix A.10. Expressing steady state consumption and wealth of individual households relative to the technology level returns functions that only depend on the household's *age* $a \equiv t - s$:

$$c^\star(a) \equiv \left. \frac{c(t,s)}{\exp[\xi t]} \right|_{k=k^\star} = w^\star \frac{Q_T(r^\star - \xi)}{Q_T(r^\star - \sigma^H(r^\star - \rho^H))} \exp\left[(\sigma^H(r^\star - \rho^H) - \xi)a\right] \ , \tag{13a}$$

$$\begin{aligned} b^\star(a) \equiv \left. \frac{b(t,s)}{\exp[\xi t]} \right|_{k=k^\star} &= w^\star Q_a(r^\star - \sigma^H(r^\star - \rho^H)) \ \exp[(r^\star - \xi)a] \\ &\times \left[\frac{Q_a(r^\star - \xi)}{Q_a(r^\star - \sigma^H(r^\star - \rho^H))} - \frac{Q_T(r^\star - \xi)}{Q_T(r^\star - \sigma^H(r^\star - \rho^H))} \right] \ . \end{aligned} \tag{13b}$$

Figure 1 illustrates these steady state paths for individual consumption and assets in terms of the technological level of the economy.[5] The individual consumption path grows exponentially over the lifetime of each generation. Individual household assets follow an inverted U-shape, i.e., households are born with no assets, accumulate assets in their youth and consume their wealth towards their death.

[5] The calculations use the following model specifications: $f(k) = k^\alpha$, $\alpha = 0.3$, $\rho = 3\%$, $\sigma = 1$, $\xi = 1.5\%$, $\nu = 0$, $T = 50$.

Applying the aggregation rule (9), we obtain for the aggregate values per effective labor

$$c^\star = w^\star \frac{Q_T(r^\star - \xi)}{Q_T(\nu)} \frac{Q_T(\nu + \xi - \sigma^H(r^\star - \rho^H))}{Q_T(r^\star - \sigma^H(r^\star - \rho^H))} \,, \tag{14a}$$

$$\begin{aligned} b^\star = & \frac{w^\star}{r^\star - \xi} \left[\frac{Q_T(\xi + \nu - r^\star)}{Q_T(\nu)} - 1 \right] - \frac{w^\star}{r^\star - \sigma^H(r^\star - \rho^H)} \\ & \times \frac{Q_T(r^\star - \xi)}{Q_T(\nu)} \frac{Q_T(\xi + \nu - r^\star) - Q_T(\xi + \nu - \sigma^H(r^\star - \rho^H))}{Q_T(r^\star - \sigma^H(r^\star - \rho^H))} \,. \end{aligned} \tag{14b}$$

The following proposition guarantees the existence of a non-trivial steady state for a large class of production functions, in particular, CES-production functions.

Proposition 1 (Existence of the steady state)
There exists a $k^\star > 0$ solving equations (8) and (14) with $b^\star = k^\star$ if

$$\lim_{k \to 0} -k f''(k) > \begin{cases} \sigma T \,, & \text{if } \sigma \in (0, 1] \\ T \,, & \text{if } \sigma > 1 \end{cases} \,. \tag{15}$$

The proof is given in the appendix.

In the proof of Proposition 1 we show that steady states may be equal to or larger than the golden rule capital stock k^{gr}, which is implicitly defined by $r^{gr} \equiv \nu + \xi = f'(k^{gr})$. As our aim is to compare the decentralized OLG with an ILA economy, we are particularly interested in steady states with $k^\star < k^{gr}$.[6]

Definition 1 (Decentralized OLG economy)

(i) *The set $\Gamma \equiv \{f, \xi, \nu, \sigma^H, \rho^H, T\}$ defines a decentralized OLG economy.*

(ii) *$\Gamma^\star \in \{\Gamma | \exists\, k^\star$ with $0 < k^\star < k^{gr}\}$ defines a decentralized OLG economy with a dynamically efficient capital stock $k^\star < k^{gr}$. For an economy $\Gamma^\star$ we refer by $k^\star$ and $r^\star$ to a steady state satisfying this condition.*

The following proposition shows the existence of dynamically efficient economies $\Gamma^\star$. Analogously to d'Albis (2007), we introduce the share of capital in output, $s(k)$, and the elasticity of substitution between capital and labor, $\epsilon(k)$,

$$s(k) \equiv \frac{k f'(k)}{f(k)} \,, \qquad \epsilon(k) \equiv -\frac{f(k) - f'(k)k}{k^2 f''(k)} \,. \tag{16}$$

[6] In the ILA economy only steady states $k^\star < k^{gr}$ may occur.

Proposition 2 (Existence and uniqueness of dynamically efficient steady states)
Given that condition (15) *holds, there exists a steady state with* $k^\star < k^{gr}$ *if*

$$\rho^H \geq \frac{\sigma^H - 1}{\sigma^H}\xi + \nu. \tag{17}$$

There exists exactly one $k^\star < k^{gr}$ *if*

$$s(k) \leq \epsilon(k) \qquad \text{and} \qquad \frac{d}{dk}\left(\frac{s(k)}{\epsilon(k)}\right) \geq 0\ , \tag{18a}$$

and, in case that $\sigma^H > 1$*,*

$$\rho^H < \frac{\sigma^H - 1}{\sigma^H}(\nu + \xi)\ . \tag{18b}$$

The proof is given in the appendix.

Although we cannot solve the implicit equation $k^\star = b^\star$ analytically and, therefore, cannot calculate the steady state interest rate $r^\star$, the following proposition determines a lower bound of the steady state interest rates in a dynamically efficient OLG economy.

Proposition 3 (Lower bound of steady state interest rate)
For any economy Γ^* *holds*

$$r^\star > \rho^H + \frac{\xi}{\sigma^H}.$$

The proof is given in the appendix.

The lower bound of the steady state interest rate in the decentralized OLG economy will play an important role for the comparison with the ILA economy.

3 Infinitely-Lived Agent Economy and Observational Equivalence

As intergenerational trade-offs are mostly discussed in ILA frameworks rather than in OLG models, we investigate how the macroeconomic observables of an OLG and ILA economy relate to each other. Therefore, we first introduce the ILA model and then define observational equivalence between two economies. Whenever we compare two different model structures in this paper we assume that population growth and the production side of the economy are identical.

Variables of the ILA model that are not exogenously fixed to its corresponding counterparts in the OLG model are indexed by a superscript ^{R}. The ILA model abstracts from individual generations' life cycles only considering *aggregate* consumption and asset holdings. In the ILA model optimal consumption and asset paths per capita are derived by maximizing the discounted stream of instantaneous utility of consumption per capita weighted by population size

$$U^R \equiv \int_0^\infty N(t) \frac{\bar{c}^R(t)^{1-\frac{1}{\sigma^R}}}{1-\frac{1}{\sigma^R}} \exp\left[-\rho^R t\right] dt \ , \tag{19}$$

subject to the budget constraint

$$\dot{b}^R(t) = \left[r^R(t) - \xi - \nu\right] b^R(t) + \frac{w^R(t)}{A(t)} - c^R(t) \ , \tag{20}$$

and the transversality condition

$$\lim_{t\to\infty} b(t) \exp\left[-\int_0^t r^R(t')\, dt' + (\xi+\nu)t\right] = 0 \ . \tag{21}$$

Maximizing (19) subject to (20) and (21) yields the well known Euler equation of the ILA model

$$\frac{\dot{c}^R(t)}{c^R(t)} = \sigma^R\left[r^R(t) - \rho^R\right] - \xi \ . \tag{22a}$$

Making use of equation (22a) we know that in a steady state the transversality condition translates into

$$\rho^R > \left(1 - \frac{1}{\sigma^R}\right) \xi + \nu \ . \tag{22b}$$

In the following we assume that the transversality condition is met. Note that it is the strict version for the Ramsey agent of the dynamic efficiency condition for the household in the OLG economy.

Assuming markets to be in equilibrium at all times (i.e., $\int_0^1 l(t,i)\, di = N(t)$ and $k^R(t) = b^R(t)$), the dynamics of the capital stock per effective labor in the ILA economy reads

$$\dot{k}^R(t) = f\big(k^R(t)\big) - (\nu+\xi) k^R(t) - c^R(t) \ , \tag{22c}$$

which is formally equivalent to the corresponding equation (10b) of the OLG economy.

To compare the different models we use the following definition:

Definition 2 (Observational equivalence)

(i) *Two economies A and B are* observationally equivalent *if coincidence in their current observable macroeconomic variables leads to coincidence of their future observable macroeconomic variables. Formally, if for any $c^A(0) = c^B(0)$ and $k^A(0) = k^B(0)$ it holds that $c^A(t) = c^B(t)$ and $k^A(t) = k^B(t)$ for all $t \geq 0$.*

(ii) *Two economies A and B are* observationally equivalent in steady state *if there exist $c^\star$ and $k^\star$ such that both economies are in a steady state.*

Note that observational equivalence in the steady state (ii) is weaker than general observational equivalence (i).

4 Decentralized OLG Versus Infinitely-Lived Agent Economy

Now, we investigate under what conditions a decentralized OLG economy, as outlined in Section 2, is observationally equivalent to an ILA economy, as defined in Section 3. The following proposition states the necessary and sufficient condition:

Proposition 4 (Decentralized OLG versus ILA economy)

(i) *A decentralized OLG economy $\Gamma^\star$ and an ILA economy are observationally equivalent if and only if for all $t \geq 0$ the following condition holds:*

$$\rho^R = \frac{\sigma^H}{\sigma^R}\,\rho^H + \left(1 - \frac{\sigma^H}{\sigma^R}\right) r(t) + \frac{1}{\sigma^R}\left[\frac{\Delta c(t)}{c(t)} + \nu\right] \,. \tag{23}$$

(ii) *For any decentralized OLG economy $\Gamma^\star$ there exists an ILA economy that is observationally equivalent in the steady state.*

The proof is given in the Appendix.

Proposition 4 states that any decentralized OLG economy $\Gamma^\star$ is – at least in the steady state – observationally equivalent to an ILA economy for an appropriate choice of (σ^R, ρ^R). Note that (σ^R, ρ^R) is, in general, not uniquely determined by (23).

If we assume that the intertemporal propensity to smooth consumption between two periods is the same for the households in the OLG and the ILA economy, i.e., $\sigma^H = \sigma^R$, we obtain the following corollary.

Corollary 1 (Identical intertemporal elasticity of substitution)
For $\sigma^R = \sigma^H$ condition (23) *reduces to*

$$\rho^R = \rho^H + \frac{1}{\sigma^R}\left[\frac{\Delta c(t)}{c(t)} + \nu\right] \ . \tag{24}$$

To understand why the pure rates of time preference in the ILA economy differs from the observationally equivalent OLG economy, we analyze the two terms in brackets on the right-hand side of equation (24). The first term in brackets captures the difference in consumption between the cohort dying and the cohort just born relative to aggregate consumption. The term is a consequence of the fact that every individual in the OLG model plans his own life cycle, saving while young and spending while old. If there is no population growth, i.e., $\nu = 0$ ($\gamma = 1/T$), individual consumption growth is higher than aggregate consumption growth and the term is always positive. More generally the following proposition states that the first term is positive if and only if individual consumption grows faster than aggregate consumption.[7]

Proposition 5 (Sign of $\boldsymbol{\Delta c(t)/c(t)}$)
For any decentralized OLG economy $\Gamma^\star$ $\Delta c(t)/c(t) > 0$ holds if and only if

$$\frac{\dot{c}(t,s)}{c(t,s)} > \frac{\dot{\bar{c}}(t)}{\bar{c}(t)} + \nu \quad \textit{for all } s \in [t-T, t] \ . \tag{25}$$

Proof: The equivalence between $\Delta c(t)/c(t) > 0$ and (25) is obtained by substituting the individual household's Euler equation (4) into the aggregate Euler equation (10a) and solving for $\Delta c(t)/c(t)$. □

Note that the right hand side of inequality (25) represents the growth rate of aggregate consumption.

The second term in brackets of equation (24) reflects that instantaneous utility in the ILA model is weighted by population size. Hence, for a growing population future consumption receives an increasing weight in the objective function. A corresponding weighting does not occur in the decentralized OLG economy, where all households only maximize own lifetime utility.

It follows immediately from Proposition 5 that for $\sigma^R = \sigma^H$ both effects in equation (24) together yield $\rho^R > \rho^H$ whenever $\dot{c}(t,s)/c(t,s) > \dot{\bar{c}}(t)/\bar{c}(t)$, i.e., individual consumption

[7] Equation (25) holds for all $s \in [t-T, t]$ if and only if it holds for some s.

growth dominates growth per capita. The following corollary shows that the latter condition always holds in the steady state and extends the analysis to the general case in which $\sigma^H \neq \sigma^R$.

Corollary 2 (Comparing time preference rates)
Suppose a decentralized OLG economy $\Gamma^\star$ *is observationally equivalent in the steady state to an ILA economy. Then the following statements hold:*

(i) $\sigma^R = \sigma^H \quad \Rightarrow \quad \rho^R > \rho^H$.

(ii) In general,

$$\rho^R > \rho^H \quad \Leftrightarrow \quad \sigma^R > \sigma^H \left[1 + \frac{1}{\xi}\left(\frac{\Delta c(t)}{c(t)} + \nu\right)\right]^{-1} . \tag{26}$$

The proof is given in the Appendix.

Equipping an ILA with a lower intertemporal substitutability than the household in the decentralized OLG economy would ceteris paribus increase the steady state interest rate in the ILA economy (as opposed to the situation whith coinciding elasticities). In order to match the same observed interest rate as before, the ILA's rate of time preference has to be lower. Thus, the time preference relation can flip around if picking the intertemporal elasticity of substitution of the ILA sufficiently below that of the household in the decentralized OLG economy (note that $[\cdot]^{-1} < 1$).

5 Utilitarian OLG Versus Infinitely-Lived Agent Economy

Consider an OLG economy, which is governed by a social planner maximizing a social welfare function. In this section, we investigate the conditions under which this economy is observationally equivalent to an ILA economy. We assume a utilitarian social welfare function in which the social planner trades off the weighted lifetime utility of different generations. The weight consists of two components. First, the lifetime utility of the generation born at time s is weighted by cohort size. Second, the social planner exhibits a social rate of time preference $\rho^S \geq 0$ at which he discounts the expected lifetime utility at birth for generations born in the future.[8]

[8] We examine the discounted utilitarian social welfare function of e.g. Burton (1993) and Calvo and Obstfeld (1988) as it represents the de facto standard in the economic literature. For a general criticism of discounted utilitarianism, as also employed in the climate change debate by Nordhaus (2007) and Stern (2007), see e.g. Sen and Williams (1982) and Asheim and Mitra (forthcoming). Calvo and

Assuming that the social planner maximizes social welfare from $t = 0$ onward, the social welfare function consists of two parts: (i) the weighted integral of the *remaining* lifetime utility of all generations alive at time $t = 0$, and (ii) the weighted integral of all future generations

$$\begin{aligned} W \equiv & \int_{-T}^{0} \left\{ \int_{0}^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \exp\left[-\rho^H(t-s)\right] dt \right\} \gamma \exp[\nu s] \exp[-\rho^S s] ds \\ & + \int_{0}^{\infty} \left\{ \int_{s}^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \exp\left[-\rho^H(t-s)\right] dt \right\} \gamma \exp[\nu s] \exp[-\rho^S s] ds \ . \end{aligned} \tag{27a}$$

The term in the first curly braces is the (remaining) lifetime utility $U(s)$ of a household born at time s, as given by equation (1), the functional form of which is a given primitive for the social planner. The term $\gamma \exp[\nu s]$ denotes the cohort size of the generation born at time s. Changing the order of integration and replacing $t - s$ by age a, we obtain

$$W = \int_{0}^{\infty} \left\{ \int_{0}^{T} \frac{c(t,t-a)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \gamma \exp\left[(\rho^S - \rho^H - \nu)a\right] da \right\} \exp\left[(\nu - \rho^S)t\right] dt \ . \tag{27b}$$

In the following, we consider two different scenarios. In the *unconstrained* utilitarian OLG economy, a social planner maximizes the social welfare function (27b) directly controlling investment and household consumption. Thus, the social planner is in command of a centralized economy. In contrast, in the *constrained* utilitarian OLG economy the social planner relies on a market economy, in which the households optimally control their savings and consumption maximizing their individual lifetime utility (1). In this second scenario, the social planner is constrained to influencing prices by a tax/subsidy regime in order to maximize the social welfare function (27b).

5.1 Unconstrained Utilitarian OLG Economy

We determine the unconstrained social planner's optimal allocation by maximizing (27b) subject to the budget constraint (10b) and the transversality condition

$$\lim_{t\to\infty} k(t) \exp\left[-\int_{0}^{t} f'(k(t'))\, dt' + (\xi + \nu)t\right] = 0 \ . \tag{28}$$

Obstfeld (1988) show that social welfare functions which do not treat all present and future generations symmetrically, i.e., discount lifetime utility to the same point of reference (here the date of birth), may lead to time-inconsistent optimal plans.

Following the approach of Calvo and Obstfeld (1988), we interpret the unconstrained social planner's optimization problem as two nested optimization problems. The first problem is obtained by defining

$$V(\bar{c}(t)) \equiv \max_{\{c(t,t-a)\}_{a=0}^{T}} \int_0^T \frac{c(t,t-a)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \gamma \exp\left[(\rho^S - \rho^H - \nu)a\right] da \ , \tag{29}$$

subject to

$$\int_0^T c(t,t-a)\gamma \exp[-\nu a] da \ \leq \ \bar{c}(t) \ . \tag{30}$$

The solution to this maximization problem is the social planner's optimal distribution of consumption between all generations alive at time t.

Proposition 6 (Optimal consumption distribution for given time *t*)
The optimal solution of the maximization problem (29) *subject to condition* (30) *is*

$$c(t,t-a) = \bar{c}(t) \frac{Q_T(\nu)}{Q_T(\nu + \sigma^H(\rho^H - \rho^S))} \exp\left[-\sigma^H(\rho^H - \rho^S)a\right] \ . \tag{31}$$

As a consequence, all households receive the same amount of consumption at time t irrespective of age for $\rho^H = \rho^S$, and receive less consumption the older (younger) they are at a given time t for $\rho^H > \rho^S$ ($\rho^H < \rho^S$).

The proof is given in the appendix.

Proposition 6 states that the difference between the households' rate of time preference ρ^H and the social rate of time preference ρ^S determines the social planner's optimal distribution of consumption across households of different age at some given time t. In particular, if $\rho^H > \rho^S$ the consumption profile with respect to age is qualitatively opposite to that of the decentralized solution at any time t, as following from the Euler equation (4) and illustrated in Figure 2.[9] That is, in the social planner's solution households receive less consumption the older they are, whereas they would consume more the older they are in the decentralized OLG economy. The intuition for this result is as follows. The social planner weighs the lifetime utility of every individual discounted to

[9] We do not take up a stance on the relationship between the individual and the social rate of time preference, but merely hint at the resulting consequences. This is in line with Burton (1993), who argues that "…they represent profoundly different concepts" (p. 121/122) and, thus, may differ. In fact, ρ^H trades off consumption today versus consumption tomorrow *within* each generation, while ρ^S trades off lifetime utilities *across* generations. If they are supposed to differ, then it is usually assumed that $\rho^H > \rho^S$ (see also Heinzel and Winkler 2007: Sec. 2).

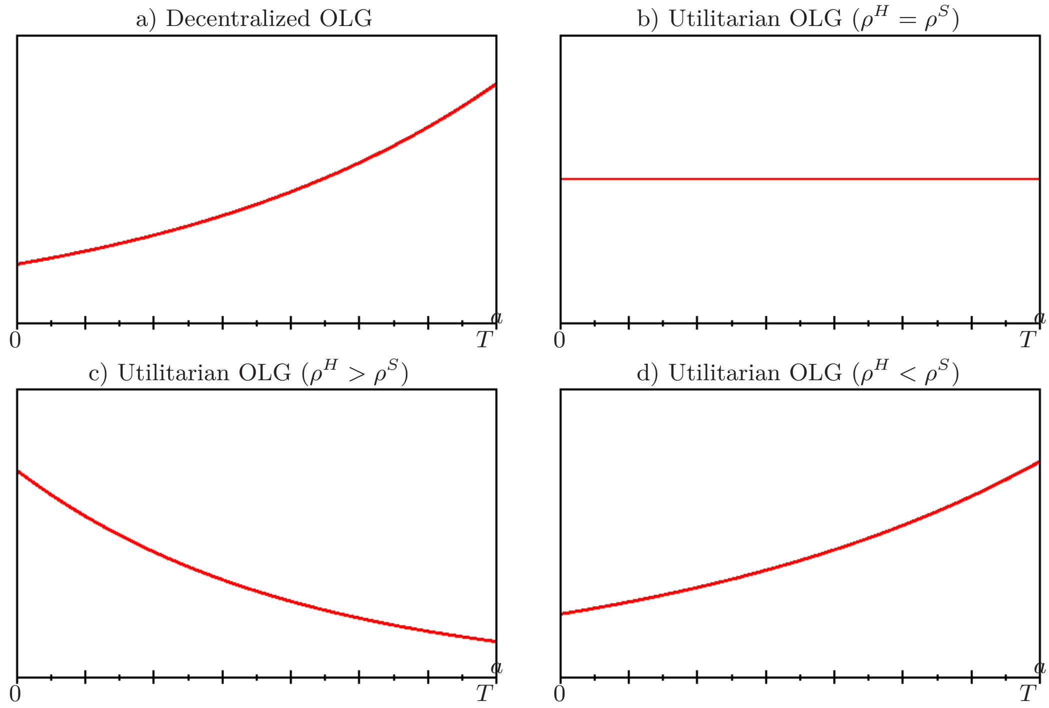

Figure 2: Distribution of consumption across all generations alive at given time t dependent on age a for the decentralized OLG and three different utilitarian OLGs.

the time of birth. Thus, the instantaneous utility at time t of those who are younger (born later) is discounted for a relatively longer time at the social planner's time preference (before birth) and for a relatively shorter time by the individual's time preference (after birth) than is the case for the instantaneous utility at time t of those who are older (born earlier). For $\rho^H > \rho^S$ the social planner's time preference is smaller and, thus, the young generation's utility at time t receives higher weight.

Proposition 6 shows that the standard approach of weighted intergenerational utilitarianism poses a trade-off between *intertemporal* generational equity and *intratemporal* generational equity to the social planner whenever households exhibit a positive rate of time preference. Lifetime utilities of today's and future generations receive equal weight if and only if the social rate of time preference is zero. But then, $\rho^H > \rho^S = 0$ implies that at each point in time the young enjoy higher consumption than the old. In contrast, an equal distribution of consumption among the generations alive is obtained if and only if social time preference matches individual time preference. However, a positive social rate of time preference comes at the expense of an unequal treatment of lifetime utilities of different generations. This trade-off only vanishes if the individuals' and the social planner's rates of time preference are both equal to zero. Such an equality trade-off can only be captured in an OLG model which explicitly considers the life cycles of different generations.

We now turn to the second part of the maximization problem, which optimizes $\bar{c}(t)$ over time. It is obtained by replacing the term in curly brackets in equation (27b) by the left hand side of equation (29) resulting in

$$\max_{\{\bar{c}(t)\}_{t=0}^{\infty}} \int_0^{\infty} V\big(\bar{c}(t)\big) \exp[\nu t] \exp\left[-\rho^S t\right] dt \ , \tag{32}$$

subject to the budget constraint (10b). Observe that problem (32) is formally equivalent to an ILA economy with the instantaneous utility function $V\big(\bar{c}(t)\big)$ and the time preference rate ρ^S.[10] We obtain $V\big(\bar{c}(t)\big)$ by inserting the optimal consumption profile (31) into equation (29) and carrying out the integration

$$V\big(\bar{c}(t)\big) = \left[\frac{Q_T\big(\nu + \sigma^H(\rho^H - \rho^S)\big)}{Q_T(\nu)}\right]^{\frac{1}{\sigma^H}} \frac{\bar{c}(t)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \ . \tag{33}$$

The social planner's maximization problem (32) is invariant under affine transformations of the objective function (33), in particular, under a multiplication with the inverse of the term in square brackets. Thus, problem (32) is identical to the optimization problem in the ILA economy when setting the intertemporal elasticity of substitution $\sigma^R = \sigma^H$ and the time preference rate $\rho^R = \rho^S$.

Proposition 7 (Unconstrained utilitarian OLG and ILA economy)
For an unconstrained utilitarian OLG economy, i.e., a social planner maximizing the social welfare function (27b) *subject to the budget constraint* (10b) *and the transversality condition* (28)*, the following statements hold:*

(i) An unconstrained utilitarian OLG economy is observationally equivalent to the ILA economy if and only if $\sigma^R = \sigma^H$ and $\rho^R = \rho^S$.

(ii) An unconstrained utilitarian OLG economy is observationally equivalent in the steady state to an ILA economy if and only if

$$\rho^R = \rho^S + \xi \, \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H} \ . \tag{34}$$

[10] Such an equivalence was already observed by Calvo and Obstfeld (1988).

The proof is given in the appendix.

Proposition 7 states that, maximizing the utilitarian social welfare function (27b) yields the same aggregate consumption and capital paths as maximizing the welfare (19) in the ILA model with $\sigma^R = \sigma^H$ and $\rho^R = \rho^S$. This result, however, does not imply that the unconstrained social planner problem can, in general, be replaced by an ILA model.

First, to derive the equivalence result, we have assumed a social planner who does *not* exhibit any preferences for smoothing lifetime utility across generations. The parameter σ^H in equation (33) stems from the individuals' preferences to smooth consumption within the lifetime of each generation. It is therefore a given primitive to the social planner. Thus, the only normative parameter the social planner may choose is the social time preference rate ρ^S. It remains an open question for future research whether a different welfare functional for the unconstrained utilitarian social planner exists that permits a normative choice of σ^S for the social planner and still delivers observational equivalence to an ILA model with $\rho^S = \rho^R$.

Second, in the ILA setting, the first-best solution can easily be decentralized, for example, via taxes that ensure the optimal path of the aggregate capital stock. However, this may not be the case for the unconstrained social planner's problem as the latter is also concerned about the intratemporal allocation of consumption across all generations alive at a certain point in time. Before, we investigate the decentralization of the social optimum in the next section, we compare the outcome of the OLG economy managed by the unconstrained social planner to that of a decentralized OLG economy. In all comparisons between a utilitarian and a decentralized OLG economy, we assume identical preferences of the individual households in both economies.

Proposition 8 (Unconstrained utilitarian OLG and decentralized OLG)

(i) For any economy Γ^* there exists an unconstrained utilitarian OLG that is observationally equivalent in the steady state. In such a steady state $\rho^S > \rho^H$.

(ii) In the steady state, an economy Γ^* and an unconstrained utilitarian OLG exhibit the same allocation of consumption across the generations alive at each point in time if and only if they are observationally equivalent in the steady state.

The proof is given in the appendix.

Remark: The converse of (i) is not true, as there exists no economy Γ^* that would be observationally equivalent to an unconstrained utilitarian OLG with $\rho^S < \rho^H$.

Proposition 8 implies that an unconstrained utilitarian OLG economy exhibits the same aggregate steady state as the decentralized OLG economy if and only if the intratemporal distribution of consumption between all generations alive coincide. For this to hold, the social planner's rate of time preference has to be higher than the individual households' rate of time preference.

5.2 Constrained Utilitarian OLG Economy

As seen in Proposition 8, the optimal solution of a social planner maximizing (27b) subject to the budget constraint (10b) and the transversality condition (28) is, in general, not identical to the outcome of a decentralized OLG economy.[11] Thus, the question arises whether and if so how the social optimum is implementable in a decentralized market economy. Calvo and Obstfeld (1988) show that it is possible to implement the social optimum by a transfer scheme discriminating by date of birth s and age a. Such a transfer scheme may be difficult to implement because of its administrative burden. In addition, it is questionable whether taxes and subsidies which are conditioned on age per se are politically viable.[12]

As a consequence, we consider a social planner that cannot discriminate transfers by age but may only influence prices via taxes and subsidies. In particular, we assume that the social planner may impose taxes/subsidies on capital and labor income. Let $\tau_r(t)$ and $\tau_w(t)$ denote the tax/subsidy on returns on savings and on labor income, respectively.[13] The individual households of the OLG economy base their optimal consumption and saving decisions on the effective interest rate $r^{\mathrm{e}}(t, \tau_r(t))$ and the effective wage $w^{\mathrm{e}}(t, \tau_w(t))$ defined by

$$r^{\mathrm{e}}(t, \tau_r(t)) = r(t) - \tau_r(t) \ , \tag{35a}$$

$$w^{\mathrm{e}}(t, \tau_w(t)) = w(t)[1 - \tau_w(t)] \ . \tag{35b}$$

Then, the individual budget constraint reads

$$\dot{b}^{\mathrm{e}}(t, s) = r^{\mathrm{e}}(t, \tau_r(t))\, b^{\mathrm{e}}(t, s) + w^{\mathrm{e}}(t, \tau_w(t)) - c^{\mathrm{e}}(t, s) \ . \tag{35c}$$

[11] Recall that we assume the individual preference parameters to be identical in both economies.

[12] See also the "Age Discrimination Act of 1975" for the US stating that "...no person in the United States shall, on the basis of age, be excluded from participation in, be denied the benefits of, or be subjected to discrimination under, any program or activity receiving Federal financial assistance." Note that programs like medicare use age as a proxy for the health condition and do not discriminate by age per se.

[13] Following the standard convention, $\tau_i(t)$ is positive if it is a tax and negative if it is a subsidy.

Given this budget constraint, individual households choose consumption paths which maximize lifetime utility (1). Thus, the optimal consumption path $c^{\mathrm{e}}(t, s, \{r(t'), \tau_r(t'), \tau_w(t')\}_{t'=s}^{s+T})$ is a function of the paths of the interest rate $r(t)$ and the taxes $\tau_r(t)$ and $\tau_w(t)$.

Note that for a given path of the interest rate and given tax/subsidy schemes $\{r(t), \tau_r(t), \tau_w(t)\}_{t=s}^{s+T}$ the individual household's optimal paths of consumption and assets can be characterized as in the decentralized OLG economy by (2) and (4) when using $r^{\mathrm{e}}(t, \tau_r(t))$ and $w^{\mathrm{e}}(t, \tau_w(t))$ instead of $r(t)$ and $w(t)$, respectively. Applying the aggregation rule (9) yields aggregate consumption per effective labor $c^{\mathrm{e}}(t, \{r(t'), \tau_r(t'), \tau_w(t')\}_{t'=t-T}^{t+T})$. To analyze observational equivalence between such a constrained utilitarian OLG economy and an ILA economy, we have to restrict redistribution to mechanisms which do not alter the aggregate budget constraint (10b) of the economy. We consider the following redistribution scheme which yields a balanced government budget at all times

$$\tau_w(t)w(t) = -\tau_r(t)\bar{b}(t) \ . \tag{35d}$$

Under these conditions the social optimum is, in general, not implementable.

Proposition 9 (Implementation of the social optimum)
The optimal solution of a social planner maximizing (27b) *subject to the budget constraint* (10b) *and the transversality condition* (28) *is not implementable by a tax/subsidy regime satisfying* (35) *unless this solution is identical to the outcome of the unregulated decentralized OLG economy* $\Gamma^{\star}$*.*

The proof is given in the appendix.

Proposition 9 states that a constrained social planner who can only impose a tax/subsidy regime on interest and wages cannot achieve the first-best social optimum. The intuition is that the constrained social planner can achieve the socially optimal aggregate levels of capital and consumption, but cannot implement the socially optimal intratemporal distribution of consumption across generations living at the same time. The only exception occurs if the social optimum happens to be identical to the outcome of the unregulated OLG economy. In this case, there is no need for the social planner to interfere and, thus, it does not matter whether the social planner can freely re-distribute consumption among generations or is constrained to a self-financing tax/subsidy scheme. In all other cases, the constrained social planner will choose a tax path such as to achieve a second-best optimum. In consequence, Proposition 9 questions the validity of the ILA model in deriving distributional policy advice for a democratic government that, most likely, is not able to redistribute by age between the generations alive.

6 Stern vs. Nordhaus – A Critical Review of Choosing the Social Rate of Time Preference

A prime example for questions of intergenerational equity is the mitigation of anthropogenic climate change, as most of its costs accrue today while the benefits spread over decades or even centuries. The question of optimal greenhouse gas abatement has been analyzed in integrated assessment models combining an ILA economy with a climate model. Interpreting the ILA's utility function (19) as a utilitarian social welfare function, intergenerational equity concerns are closely related to the choice of intertemporal elasticity of substitution σ^R and the rate of time preference ρ^R. This is illustrated well by Nordhaus (2007), who compares two runs of his open source integrated assessment model DICE-2007. The first run uses his preferred specifications $\sigma^R = 0.5$ and $\rho^R = 1.5\%$. The second run employs $\sigma^R = 1$ and $\rho^R = 0.1\%$, which are the parameter values chosen by Stern (2007). These different parameterizations cause a difference in the optimal reduction rate of emissions in the period 2010–2019 of 14% versus 53% and a difference in the optimal carbon tax of 35$ versus 360$ per ton C.

In the following, we apply our results derived in Sections 4 and 5 to critically review recent approaches to the evaluation of climate change mitigation scenarios. We focus on the Stern (2007) review, which we consider a paradigm for the normative approach, and its critique by Nordhaus (2007), which we consider representative for the positive approach. We find that neither Nordhaus' (2007) positive nor Stern's (2007) normative approach spell out all the normative assumptions hidden in their respective descriptions of the intergenerational allocation problem. We identify the implicit assumptions and the shortcomings in the current debate and argue that our analysis lays out a more comprehensive foundation for approaching the valuation problem in the integrated assessment of climate change.

6.1 The "positive" approach

The majority of economists in the climate change debate takes an observation-based approach to social discounting. This view is exemplarily laid out in Nordhaus' (2007) critical review of the Stern (2007) review of climate change. Individual preferences towards climate change mitigation cannot be observed directly in market transactions because of the public good characteristic of greenhouse gas abatement. However, we observe everyday investment decisions on capital markets that carry information on intertemporal preferences. In particular, we observe the market interest rate and the steady state growth rate of the economy. The positive approach transfers this information into pairs of intertemporal elasticity of substitution σ^R and pure time preference ρ^R in an ILA economy. This ILA economy is interpreted as a utilitarian social planner model and confronts the climate problem in an integrated assessment model.

Our paper provides the tools to critically re-examine the positive approach explicitly accounting for the finite lifespan of individuals living in an OLG economy. In Section 4 we found that, indeed, there exists an observationally equivalent ILA economy for the decentralized OLG economy. However, we also showed that the rate of time preference of the ILA does not reflect the actual time preference rate of the (homogeneous) individuals in the decentralized OLG economy. The ILA model overestimates the rate of time preference for two reasons. First, the ILA model assumes that every individual plans for an infinite future when taking their market decisions. However, the households in the OLG economy only plan for their own lifespan when revealing their preferences on the market. Interpreting these decisions as if being taken with an infinite time horizon overstates the actual pure time preference. Second, the ILA model assumes that the representative consumer accounts for population growth by giving more weight to the welfare of the larger future population. If the households in the OLG economy dismiss this farsighted altruistic reasoning, the ILA approach once more overestimates individual time preference rates.[14]

The second step of the positive approach passes the preferences of the 'imaginary' ILA on to a social planner. This step calls for an explicit justification, given that we have shown these preferences to differ from those of the individuals in the economy. In particular, calibrating social planner preferences to those in an ILA model is not innocuous. The social planner model is used to extrapolate action beyond observation. The particular way extrapolation takes place determines what action is optimal. Obviously, carrying over the households' ρ^H and σ^H to the social planner and increasing his time horizon to address time scales of climate change would imply a very different policy recommenda-

[14] Of course, if bequests would be operative and the Ricardian equivalence holds then the interest rate conveys the long run opportunity cost of investments exceeding the lifetime of individual households. If the bequest motive is not operative – as suggested by a number of empirical studies (e.g., Hurd 1987, Hurd 1989, Laitner and Ohlsson 2001) – our setting applies.

tion than does the procedure of calibration and extrapolation followed by the positive approach of, e.g., Nordhaus (2007). A problem with the positive approach is that the normative content of the preferences assigned to the social planner is concealed. They are a combined estimate of agents' preferences and macroeconomic characteristics driven by life cycles. If we would like to capture only current observed preferences a clear cut approach would terminate the time horizon of the social planner T periods into the future, use individual households' rates of time preference, and introduce a weight that reflects the current individuals still alive at a given point of time in the future. If we acknowledge that climate change is a problem where individuals agree to adopt time horizons that exceed their own lifetime, we can adopt an infinite planning horizon. Then, however, on what grounds is it justified to increase social planner impatience over that of the individuals? The same question arises in the context of increasing impatience in order to match the fact that observed individuals in the decentralized OLG economy do not take account of future population growth. If one considers it adequate to endow the social planner with a welfare function giving more weight to larger (future) populations, then why would one increase the time preference rate to crowd out this effect? We do not provide an answer to these normative questions, but point to the normative content of the positive approach and its possible normative inconsistencies. Moreover, Proposition 4 and Corollary 1 provide a starting point for extracting an individual household's time preference from the macroscopic ILA description if desired.

A numerical illustration shows how the inferred ILA preferences differ from actual household preferences. For our exogenous parameters underlying the economies (excluding preference parameters) we choose a capital share $\alpha = .3$, a rate of technological progress $\xi = 2\%$, a rate of population growth $\nu = 0\%$, a life time $T = 50$ and an interest rate $r = 5.5\%$ close to Nordhaus (2007). Assuming the elasticities $\sigma^H = \sigma^R = .5$ as in Nordhaus (2008) latest version of DICE, the ILA model implies a rate of time preference of the representative household (and social planner) of $\rho^R = 1.5\%$, while the individuals of the decentralized OLG economy exhibit a time preference of $\rho^H = -5.3\%$.[15] The surprising finding of a negative rate of time preference questions the plausibility of the above specifications. A simple sensitivity check suggests that increasing the intertemporal elasticity of substitution is most promising for resolving the negativity puzzle. The literature estimating intertemporal substitutability in approaches that disentangle intertemporal substitutability from risk aversion precisely suggests such an increase. If we follow Vissing-Jørgensen and Attanasio (2003) building on Epstein and Zin (1991)

[15] The calculation solves equation (14b) or, alternatively, $F(5.5\%) = J(5.5\%)$ in the notation introduced in the proof of proposition 1.

and Campbell (1996) the authors give us a best guess of $\sigma = 1.5$, which is also part of a parameter constellation best explaining the equity premium puzzle. Employing their estimate for the households in the decentralized OLG and the observationally equivalent ILA economy we find $\rho^H = 1.9\%$ as opposed to $\rho^R = 4.2\%$. Then, a social planner equipped with the corresponding household preferences would go along with a steady state interest rate $r^\star = \rho^H + \frac{\xi}{\sigma} = 1.9\% + \frac{2\%}{1.5} = 3.2\%$ rather than $r^\star = 5.5\%$. Such a 2.3% difference in the social discount rate in the cost benefit analysis of climate change has a major effect on the social costs of carbon and optimal abatement efforts.[16] Finally, let us give a numerical example adopting the wide-spread assumption of logarithmic utility ($\sigma = 1$) which is also used in the Stern (2007) review. Interestingly, we find $\rho^H = 0.1\%$, which is precisely the rate chosen for the ILA model in the Stern (2007) review.

We have pointed out above that the positive approach does not avoid implicit normative assumptions. These assumptions are important when the ILA model, which is calibrated to match decentralized market equilibria, is used for policy recommendations. In this paragraph, we compare the ILA model to our full-fletched model of the unconstrained utilitarian planner developed in section 5.1, which makes normative assumptions explicit. Such a comparison is justified by Proposition 7 asserting that both models give rise to observationally equivalent equilibria. From Proposition 8 we know that, if we make the OLG structure explicit and calibrate the unconstrained utilitarian OLG economy to the decentralized unregulated market equilibrium, the condition $\rho^S > \rho^H$ has to hold.[17] Therefore, interpreting an ILA economy that reflects the decentralized market equilibrium as an observationally equivalent utilitarian OLG economy necessarily involves the assumption that the intergenerational time preference rate of the social planner is higher than individual time preference rate. This assumption stands in sharp contrast to most of the literature on intergenerational ethics.

[16] Having chosen $T = 50$ years in our simulation is based on the assumption that T is the time an individual is active in the market rather than actual life time. To illustrate the sensitivity with respect to the time horizon let us consider $T = 75$ instead. Then, for $\sigma^H = \sigma^R = .5$ we still find a negative rate of pure time preference $\rho^H = -1.6\%$ and using $\sigma^H = \sigma^R = 1.5$ we obtain $\rho^H = 3.1\%$ as opposed to $\rho^R = 4.2\%$. This results in $r^\star = 3.1\% + \frac{2\%}{1.5} = 4.4\%$ rather than $r^\star = 5.5\%$.

[17] Note that the social welfare function (27b) we considered does not include any preferences for smoothing lifetime utility of different generations over time. Of course, such functional forms are conceivable but it is not clear whether and how such a utilitarian OLG economy translates into an observationally equivalent ILA economy.

6.2 The normative approach

The normative approach to social discounting aims at treating all generations alike and, therefore, argues that a positive rate of time preference is non-ethical. This view is supported by a number of authors including Ramsey (1928), Pigou (1932), Harrod (1948), Koopmans (1965), Solow (1974), Broome and Schmalensee (1992) and Cline (1992). The Stern (2007) review of climate change effectively uses a zero rate of time preference. It adopts a parameter value $\rho^R = 0.1\%$ in order to capture a small but positive probability that society becomes extinct.[18]

In a normative approach to social discounting it seems more natural to jump straight to an ILA model. By normatively justified assumptions the social planner exhibits an infinite planning horizon and particular values of the time preference rate and the intertemporal elasticity of substitution. It is obvious, however, that the ILA model cannot capture any distinction or interaction between intergenerational weighting and individual time preference. Nevertheless, Proposition 7 shows that a social planner fully controlling an OLG economy is observationally equivalent to an ILA economy if the parameters σ^R and ρ^R are appropriately chosen. In particular, the intertemporal path of aggregate consumption does not depend on the individual rate of time preference ρ^H, but only on the social planner's rate of time preference ρ^S. In fact, the time preference rate of the social planner coincides with the rate of time preference ρ^R of the observationally equivalent ILA economy. This finding provides some support for Stern's (2007) normative approach to intergenerational equity in the ILA model. However, the shortcut of setting up an ILA economy exhibits a number of caveats as questions of intergenerational equity are more complex than the ILA model reveals.

First, according to Proposition 7, the interpretation of the time preference rate of the ILA economy as the time preference rate of a social planner in an observationally equivalent social planner OLG economy ($\rho^R = \rho^S$) requires that the intertemporal elasticity of substitution in the ILA economy be equal to that of the individual households in the OLG economy, i.e., $\sigma^R = \sigma^H$. This, however, implies that the intertemporal elasticity of substitution is a primitive to the social planner and cannot be chosen to match particular normative considerations.[19]

[18] Strictly speaking this is not time preference, but Yaari (1965) shows the equivalence of discounting because of a constant probability of death/extinction and a corresponding rate of time preference.

[19] It is important to emphasize that we consider a specific utilitarian welfare function (27b) without intergenerational smoothing of lifetime utility.

Second, interpreting the ILA economy as a utilitarian social planner OLG neglects the *intratemporal* allocation of consumption across all generations alive at each point in time. The utilitarian OLG model allows us to explicitly analyze the social planner's optimal intratemporal distribution of consumption. As shown in Proposition 6, it depends on the difference between the social planner's and the individual households' rates of time preference. Usually, it is assumed that the normatively chosen social rate of time preference ρ^S is smaller than the individual rate of time preference ρ^H.[20] According to Proposition 6, in this case the oldest generation receives least consumption while the newborns get most among all generations alive (see Figure 2 part c), while it would be the other way round in the decentralized OLG economy (see Figure 2 part a). As a consequence, the standard approach of discounted utilitarianism poses a trade-off between *intertemporal* and *intratemporal* generational equity to the social planner whenever households exhibit a positive rate of pure time preference. It is, therefore, not obvious which type of generational equity the aim of 'treating all generations alike' should refer to.

Third, apart from the question whether consumption discrimination by age is justified on ethical grounds, it is questionable whether it is implementable. In Proposition 9 we show that, in general, a social planner whose policy instruments are limited to non-age-discriminating taxes and subsidies cannot implement the first-best solution. In fact, the first-best social optimum can only be achieved in the special case that it coincides with the outcome of the decentralized OLG economy without any regulatory intervention. Thus, the ILA economy, interpreted as an unconstrained social planner model, cannot capture this second best aspect of optimal policies.

7 Conclusions

Although the lifetime of individuals is finite, intergenerational trade-offs are most often discussed within ILA frameworks, which are interpreted as a utilitarian social welfare function. In this paper, we analyzed to what extend this interpretation is justified, in particular, if we assume a non-operative bequest motive.

We examined under which conditions an ILA economy is observationally equivalent to (i) a decentralized OLG economy and (ii) an OLG economy in which a social planner maximizes a utilitarian welfare function. We found that preference parameters differ in the decentralized OLG and the observationally equivalent ILA economy. In general, pure

[20] This assumption seems particularly reasonable if ρ^S is close to zero. With respect to the Stern review, it implies that the individual households' time preference rates exceed $\rho^S = 0.1\%$.

time preference of an ILA planner is higher than pure time preference of the households in the observationally equivalent OLG economy. Moreover, in a normative setting, a utilitarian social planner faces a trade-off between intergenerational and intragenerational equity that cannot be captured in the ILA model. Finally, even if the optimal intertemporal allocation of the economy's aggregates coincide between the ILA economy and the utilitarian social planner controlling an OLG economy, the limited implementability of the first best allocation can only be observed and discussed in the OLG context. We applied our results to the recent dispute between Stern (2007) and Nordhaus (2007) in the discussion on the mitigation of climate change and concluded that the ILA model cannot adequately address important aspects of intergenerational trade-offs. In consequence, we argue to explicitly consider the generations' life cycles.

Our analysis employs two central assumptions. First, we assume selfish individual households. Although several empirical studies suggest that altruistic bequest motives are rather weak, extending the model to include different degrees of altruism is an interesting venue for future research. Second, part of our analysis assumes a specific utilitarian social welfare function. Although commonplace in the literature, this assumption drives some of our results, such as the trade-off between intra- and intergenerational equity. In particular, discounted utilitarianism in general has been questioned as an appropriate approach to deal with questions of intergenerational equity (e.g., Asheim and Mitra forthcoming). It will be interesting to explore how other welfare functions relate to the one we have chosen and to the standard ILA model.

A Appendix

A.1 Proof of Proposition 1

We prove the existence of a non-trivial steady state, i.e. $k^\star \neq 0$. For this purpose, we re-write equation (14b) for $r^* \notin \{\xi, \nu + \xi\}$ as[21]

$$b^\star = \frac{w^\star}{r^\star - \nu - \xi} \left\{ \frac{Q_T(r^\star - \xi)}{Q_T(\nu)} \frac{Q_T\big(\nu + \xi - \sigma^H(r^\star - \rho^H)\big)}{Q_T\big(r^\star - \sigma^H(r^\star - \rho^H)\big)} - 1 \right\} . \tag{A.1}$$

We define the function $J : \mathbb{R} \to \mathbb{R}$ by

$$J(r) \equiv \frac{Q_T(r - \xi)}{Q_T(\nu)} \frac{Q_T\big(\nu + \xi - \sigma^H(r - \rho^H)\big)}{Q_T\big(r - \sigma^H(r - \rho^H)\big)} , \quad \forall\, r \in \mathbb{R} \tag{A.2}$$

for which Lemma 2 in Appendix A.10 summarizes some useful properties. Defining moreover

$$\phi(k) \equiv \frac{f(k) - f'(k)k}{f'(k) - \nu - \xi} \big[J\big(f'(k)\big) - 1\big] . \tag{A.3}$$

the steady state is given by the solution of the equation

$$k = \phi(k) , \tag{A.4}$$

or equivalently

$$\frac{f(k) - (\nu + \xi)k}{f(k) - f'(k)k} = J\left(f'(k)\right) . \tag{A.5}$$

Note that the transformation of equation (A.4) to (A.5) is only valid for $r^\star \neq r^{gr} \equiv \nu + \xi$ implying $k \neq k^{gr} \equiv f'^{-1}(r^{gr})$. Thus, the case $r^\star = \nu + \xi$ will deserve special attention.

We discuss solutions to equation (A.5) in terms of the interest rate r instead of the capital stock k. Therefore, we define

$$F(r) \equiv \frac{f\big(k(r)\big) - (\nu + \xi)k(r)}{f\big(k(r)\big) - f'\big(k(r)\big)k(r)} , \tag{A.6}$$

[21] The equivalence of equation (14b) and (A.1) is easily verified by multiplying over the terms in the denominator and expanding the resulting expressions. In addition, the domain of the functions making up the right hand side of equations (14b) and (A.1) can be extended to $r^* \in \{\xi, \nu + \xi\}$ by limit. Both right hand side functions are continuous and coincide for these points. Thus, the two equations are equivalent for all $r^\star$.

where $k(r) = f'^{-1}(r)$, which is well defined due to the strict monotonicity of $f'(k)$. Observe that $k'(r) = 1/f''(k(r))$. The derivative of F with respect to r yields:

$$F'(r) = \frac{f'(k(r)) - (\nu + \xi)}{f''(k(r))\left[f(k(r)) - f'(k(r))k(r)\right]} + \frac{k(r)\left[f(k(r)) - (\nu + \xi)k(r)\right]}{\left[f(k(r)) - f'(k(r))k(r)\right]^2} \ . \quad \text{(A.7)}$$

We seek a steady state interest rate $r^\star$. Under the assumption $r^\star \neq r^{gr}$ it is a solution of the equation $F(r^\star) = J(r^\star)$.

To analyze the case $r = r^{gr}$ respectively $k = k^{gr}$ we can define $\phi(k)$ continuously by limit[22]

$$\lim_{k \to k^{gr}} \phi(k) = \left[(f(k^{gr}) - f'(k^{gr})k^{gr}\right] J'(f'(k^{gr})) \in (0, \infty) \ , \quad \text{(A.8)}$$

where we use l'Hospital's rule (recognizing that $J(f'(k^{gr})) = 1$). For the extended domain we find that equation (A.4) becomes

$$\begin{aligned} k^{gr} &= \lim_{k \to k^{gr}} \phi(k) = \left[f(k^{gr}) - f'(k^{gr})\,k^{gr}\right] J'(f'(k^{gr})) \\ \Leftrightarrow J'(r^{gr}) &= \frac{k^{gr}}{f(k^{gr}) - r^{gr}k^{gr}} = F'(r^{gr}) \ . \end{aligned} \quad \text{(A.9)}$$

Thus for $r = r^{gr}$ we have a steady state if $J'(r^{gr}) = F'(r^{gr})$ rather than when $J(r) = F(r)$ as for all other interest rates.

Coming back to equation (A.5) we find for $r = r^{gr} \equiv \nu + \xi$ that

$$F(r^{gr}) = 1 = J(r^{gr}) \ . \quad \text{(A.10)}$$

We can distinguish three different cases depending on whether $F'(r^{gr})$ is (i) equal, (ii) smaller or (iii) greater than $J'(r^{gr})$.

(i) $F'(r^{gr}) = J'(r^{gr})$
As shown in the preceding paragraph, in this case there exists a steady state $k^\star = k^{gr}$.

(ii) $F'(r^{gr}) < J'(r^{gr})$
In this case, there exists a steady state $k^\star > k^{gr}$ and $r^\star < r^{gr}$ (inefficient steady state),

[22] See footnote 21 for the reason why a solution to equation (A.4) in the limit is also a solution to equation (14b).

because $J(0) > 0$ and

$$\lim_{r\to 0} F(r) = \lim_{r\to 0} \frac{\overbrace{f(k(r))/k(r) - (\nu+\xi)}^{\to -(\nu+\xi)}}{\underbrace{f(k(r))/k(r) - r}_{\to 0}} = -\infty \ . \tag{A.11}$$

Note that $\lim_{r\to 0} f(k(r))/k(r) = 0$ because f is strictly concave and satisfies Inada conditions. Because of equation (A.11) the $F(r)$ curve has to cross the $J(r)$ curve from below somewhere in the interval $(0, r^{gr})$. The intersection defines $r^\star$ and, thus, $k^\star$.

(iii) $F'(r^{gr}) > J'(r^{gr})$
In this case, $J(r) < F(r)$ for $r > r^{gr}$ in a sufficiently small neighborhood around r^{gr}. We show that there exists $r^\star > r^{gr}$ such that $J(r)$ crosses $F(r)$ from below at $r = r^\star$. First, we show that

$$\lim_{r\to\infty} \frac{J'(r)}{J(r)} > \lim_{r\to\infty} \frac{F'(r)}{F(r)} \ . \tag{A.12}$$

We know from part (iii) and (v) of Lemma 2 that

$$\lim_{r\to\infty} \frac{J'(r)}{J(r)} = \begin{cases} \sigma T \ , & \text{if } \sigma \in (0,1] \\ T \ , & \text{if } \sigma > 1 \end{cases} \ . \tag{A.13}$$

In addition, we obtain for

$$\lim_{r\to\infty} \frac{F'(r)}{F(r)} = \lim_{r\to\infty} \left[\underbrace{\frac{r - (\nu+\xi)}{f''(k(r))\left[f(k(r)) - (\nu+\xi)k(r)\right]}}_{\leq 0} + \frac{k(r)}{f(k(r)) - rk(r)} \right] . \tag{A.14}$$

The sign of the underbraced term holds as the numerator goes to $+\infty$, $f''(k(r)) < 0$ and $f(k(r)) - (\nu+\xi)k(r)$ goes to $+0$. Thus the limit of the first term lies between $-\infty$ and 0. Thus, a sufficient condition for (A.12) to hold is

$$\begin{aligned} \lim_{r\to\infty} \frac{k(r)}{f(k(r)) - rk(r)} &= \lim_{k\to 0} \frac{k}{f(k) - f'(k)k} \\ &= \lim_{k\to 0} \frac{1}{-f''(k)k} < \begin{cases} \sigma T \ , & \text{if } \sigma \in (0,1] \\ T \ , & \text{if } \sigma > 1 \end{cases} \ , \end{aligned} \tag{A.15}$$

which is ensured to hold by condition (15).

Second, observe that condition (A.12) implies

$$\lim_{r\to\infty} \frac{d}{dr} \ln\left(\frac{J(r)}{F(r)}\right) \in \{\mathbb{R}_{++}, \infty\} \ . \tag{A.16}$$

Thus, there exist r_0 and $\epsilon > 0$ such that

$$\lim_{r\to\infty} \ln\left(\frac{J(r)}{F(r)}\right) \geq \lim_{r\to\infty} \ln\left(\frac{J(r_0)}{F(r_0)}\right) + \epsilon(r - r_0) = \infty \tag{A.17}$$

and, in consequence, there exists a finite $r = r^\star$ where $J(r)$ crosses $F(r)$ from below.

A.2 Proof of Proposition 2

Given that condition (15) holds, we know from the proof of Proposition 1 that there exists a steady state with $k^\star < k^{gr}$ if $F'(r^{gr}) > J'(r^{gr})$. As $F'(r^{gr}) > 0$, a sufficient condition for $F'(r^{gr}) > J'(r^{gr})$ to hold is that

$$J'(r^{gr}) = q(\nu) - q((\nu + \xi)(1 - \sigma^H) + \sigma^H \rho^H) \leq 0. \tag{A.18}$$

As $q'(r) > 0$ due to part (v) of Lemma 1, condition (A.18) holds if and only if (17) is satisfied.

We now derive sufficient conditions such that there exists only one steady state $k^\star < k^{gr}$. Suppose that condition (15) holds, which guarantees existence of a dynamically efficient steady state. There exists only one steady state interest rate $r^\star$ with $r^\star > r^{gr}$ if and only if

$$\begin{aligned} &F'(r)|_{r=r^\star} < J'(r)|_{r=r^\star} \ , \quad \forall\, r^\star > r^{gr} \\ &\qquad \Leftrightarrow \left.\frac{F'(r)}{F(r)}\right|_{r=r^\star} < \left.\frac{J'(r)}{J(r)}\right|_{r=r^\star} \ , \quad \forall\, r^\star > r^{gr} \ . \end{aligned} \tag{A.19}$$

The second line holds, as $F(r) = J(r)$ for all $r = r^\star$. A sufficient condition for (A.19) to hold is that

$$\frac{d}{dr}\left(\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star}\right) < 0 \qquad \wedge \qquad \frac{d}{dr}\left(\left.\frac{J'(r)}{J(r)}\right|_{r=r^\star}\right) > 0 \ , \quad \forall\, r^\star > r^{gr} \ . \tag{A.20}$$

>From part (ii) and (iv) of Lemma 2 we know that the second condition holds for all

$r > r^{gr}$ if, in case that $\sigma > 1$, also condition (18b) holds.

$$\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star} = \left.\left[\frac{r-\nu-\xi}{f''(k(r))\,[f(k(r))-(\nu+\xi)k(r)]} + \frac{k(r)}{f(k(r))-rk(r)}\right]\right|_{r=r^\star} \tag{A.21a}$$

$$= \left.\left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{F(r)}\right) + \frac{k(r)}{f(k(r))-rk(r)}\right]\right|_{r=r^\star} \tag{A.21b}$$

$$= \left.\left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{J(r)}\right) + \frac{k(r)}{f(k(r))-rk(r)}\right]\right|_{r=r^\star} \tag{A.21c}$$

$$= \underbrace{\frac{k(r)}{f(k(r))-rk(r)}}_{\equiv g_1(r)}\left.\left[1-\left(1-\frac{1}{J(r)}\right)\underbrace{\frac{f(k(r))-rk(r)}{-k^2(r)f''(k(r))}}_{\equiv g_2(r)}\right]\right|_{r=r^\star} . \tag{A.21d}$$

>From the second to the third line we employed $F(r) = J(r)$ for all $r = r^\star$. We show in the following that $g_1'(r) \leq 0$ and $g_2'(r) \geq 0$ are sufficient for $\frac{d}{dr}\left(\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star}\right) < 0$.

First, observe from equation (A.3) that $J(r^\star) > 1$ for all $r^\star > r^{gr}$. As $J(r)$ is U-shaped on $r \in (r^{rg}, \infty)$ because of part (ii) and (iv) of Lemma 2 and $J(r^{gr}) = 1$, this implies that $J'(r^\star) > 0$ for all $r^\star > r^{gr}$.

Second, we show that $\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star} > 0$ for all $r^\star > r^{gr}$ if $g_2'(r) \geq 0$. Observe that

$$\lim_{r^\star\to\infty}\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star} = \lim_{r\to\infty}\left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{J(r)}\right) + \frac{k(r)}{f(k(r))-rk(r)}\right] \tag{A.22a}$$

$$= \lim_{r\to\infty}\left[\frac{1}{k(r)f''(k(r))} + \frac{k(r)}{f(k(r))-rk(r)}\right] \tag{A.22b}$$

$$= \lim_{r\to\infty}\left[\frac{1}{k(r)f''(k(r))} - \frac{1}{k(r)f''(k(r))}\right] = 0 . \tag{A.22c}$$

In addition, we know that $g_1(r) > 0$ for all $r > 0$ and

$$\lim_{r\to\infty} g_1(r) = \lim_{r\to\infty}\frac{1}{k(r)f''(k(r))} > 0 . \tag{A.23}$$

The latter implies together with equation (A.22)

$$\lim_{r\to\infty} g_2(r)\left(1-\frac{1}{J(r)}\right) = 1 . \tag{A.24}$$

As $g_2(r)\left(1-\frac{1}{J(r)}\right)$ equals zero at $r = r^{gr}$ and is monotonically increasing in r for $g_2'(r) \geq 0 = 0$, this implies that $F'(r)/F(r)|_{r=r^\star} > 0$ for all $r^\star > r^{gr}$. Then, we obtain for $g_1'(r) \leq 0$ and $g_2'(r) \geq 0$

$$\frac{d}{dr}\left(\frac{F'(r)}{F(r)}\bigg|_{r=r^\star}\right) = g_1'(r)\left[1-\left(1-\frac{1}{J(r)}\right)g-2(r)\right] - g_1(r)g_2(r)\frac{J'(r)}{J^2(r)}$$
$$- g_1(r)g_2'(r)\left(1-\frac{1}{J(r)}\right) < 0\ . \tag{A.25}$$

The conditions $s(k) \geq \epsilon(k)$ and $\frac{d}{dk}\left(\frac{s(k)}{\epsilon(k)}\right)$ are sufficient for $g_1'(r) \leq 0$ and $g_2'(r) \geq 0$. $\square$

A.3 Proof of Proposition 3

We show that $\sigma(r^\star - \rho^H) - \xi > 0$ is a necessary condition for aggregate assets $b^\star$ to be strictly positive in a dynamically efficient steady state, i.e., $(\sigma^H, \rho^H) \in \Gamma_{\Psi,T}$. As $b^\star = k^\star$ holds, this implies that for $k^\star > 0$ the steady state real interest rate must exceed $\rho^H + \frac{\xi}{\sigma}$.

The household's wealth, as given by equation (13b), can be re-written to yield

$$b^\star(a) = \frac{w^\star}{r^\star - \xi}\{\theta \exp\left[(\sigma(r^\star - \rho^H) - \xi)a\right] + (1-\theta)\exp[(r^\star - \xi)a] - 1\}\ , \tag{A.26}$$

with

$$\theta = \frac{1 - \exp[-(r^\star - \xi)T]}{1 - \exp[-(r^\star - \sigma^H(r^\star - \rho))T]}\ . \tag{A.27}$$

Assuming a dynamically efficient steady states implies that $r^\star - \xi > 0$ and we obtain from (A.27)

$$\theta \begin{cases} < 1, & \text{if } \sigma(r^\star - \rho^H) - \xi < 0 \\ = 1, & \text{if } \sigma(r^\star - \rho^H) - \xi = 0 \\ > 1, & \text{if } \sigma(r^\star - \rho^H) - \xi > 0 \end{cases}\ . \tag{A.28}$$

Thus, we can directly infer from (A.26) that $b^\star(a) = 0$ for all $a \in [0, T]$ for $\sigma(r^\star - \rho^H) - \xi = 0$. As all households hold no assets, the aggregate capital stock equals zero. To show that $\sigma(r^\star - \rho^H) - \xi < 0$ precludes strictly positive capital stocks, we analyze the second derivative of $b^\star(a)$

$$\frac{d^2\, b^\star(a)}{d\, a^2} = \frac{w^\star}{r^\star - \xi}\{\theta(\sigma(r^\star - \rho^H) - \xi)^2 \exp\left[(\sigma(r^\star - \rho^H) - \xi)a\right]$$
$$+ (1-\theta)(r^\star - \xi)^2 \exp[(r^\star - \xi)a]\}\ . \tag{A.29}$$

For $\sigma(r^\star - \rho^H) - \xi < 0$, $\theta < 1$ holds, which implies that $\frac{d^2 b^\star(a)}{d\,a^2} > 0$. Hence, the household's wealth profile is strictly convex. Together with the boundary conditions $b^\star(0) = 0 = b^\star(T)$ this implies that all households possess non-positive wealth at all times. This, in turn, precludes $k^\star > 0$.

Further, it is obvious from (A.26) and (A.29) that $\sigma(r^\star - \rho^H) - \xi > 0$ does not contradict strictly positive wealth of the individual households and, therefore, is a necessary condition for $k^\star > 0$. $\square$

A.4 Proof of Proposition 4

(i) Both economies exhibit the same technology and rate of population growth by assumption and, thus, the market equilibria on the capital and the labor market imply that the equations of motion for the aggregate capital per effective labor (22c) and (10b) coincide. The remaining difference in the macroeconomic system dynamics is governed by the Euler equations (10a) and (22a) and by the transversality condition (21).

"$\Rightarrow$": Suppose the two economies are observationally equivalent, i.e., coincidence in the initial levels of consumption and capital imply coincidence at all future times. For this to hold the Euler equations (10a) and (22a) have to coincide giving rise to (23).

"$\Leftarrow$": If condition (23) holds, then also the Euler equations (10a) and (22a) coincide and the system dynamics of both economies is governed by the same system of two ordinary first order differential equations. The solution is uniquely determined by some initial conditions on c and k. Thus, if the two economies coincide in the levels of consumption and capital at one point in time they also do so for all future times. In consequence, the two economies are observationally equivalent. Moreover, the capital stock is an equilibrium of $\Gamma^\star$ implying $k^\star < k^{gr}$. As a consequence, the transversality condition for the ILA economy is satisfied and, thus, the described path is indeed an optimal solution.

(ii) Let $r^\star$ be the steady state interest rate of $\Gamma^\star$. Thus, all combinations of (ρ^R, σ^R) which satisfy

$$r^\star = \rho^R + \frac{\xi}{\sigma^R} \ , \tag{A.30}$$

yield ILA economies which are observationally equivalent in the steady state. As for all $\Gamma^\star$, $r^\star < r^{gr}$ holds, also the transversality condition (21) is satisfied. $\square$

A.5 Proof of Corollary 2

(i) For the steady state, equation (10a) returns $\frac{1}{\sigma^H}\left[\frac{\Delta c(t)}{c(t)}+\nu\right]=r(t)-\rho^H-\frac{\xi}{\sigma^H}$ which, by Proposition 3, is strictly positive. Thus, by equation (24) $\rho^R-\rho^H>0$.

(ii) From the respective Euler equations (10a) and (22a) we obtain the condition that

$$r-\frac{\xi}{\sigma^R}=\rho^R>\rho^H=r-\frac{1}{\sigma^H}\left[\frac{\Delta c(t)}{c(t)}+\nu+\xi\right] \tag{A.31}$$

$$\Leftrightarrow \frac{\sigma^H}{\sigma^R}<\frac{1}{\xi}\left[\frac{\Delta c(t)}{c(t)}+\nu+\xi\right] \tag{A.32}$$

which is equivalent to equation (26). □

A.6 Proof of Proposition 6

The optimization problem (29) subject to condition (30) is equivalent to a resource extraction model (or an isoperimetrical control problem). We denote consumption at time t of an individual of age a by $\mathcal{C}(a)\equiv c(t,t-a)$ and define the stock of consumption left to distribute among those older than age a by

$$y(a)=\bar{c}(t)-\int_0^a \mathcal{C}(a')\gamma\exp[-\nu a']\,da' \ . \tag{A.33}$$

Then, the problem of optimally distributing between the age groups is equivalent to optimally 'extracting' the consumption stock over age (instead of time). The equation of motion of the stock is $\frac{dy}{da}=-\mathcal{C}(a)\gamma\exp[-\nu a]$, the terminal condition is $y(T)\geq 0$, and the present value Hamiltonian reads

$$\mathcal{H}=\frac{\mathcal{C}(a)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}}\gamma\exp\left[(\rho^S-\rho^H-\nu)a\right]-\lambda(a)\mathcal{C}(a)\gamma\exp[-\nu a] \ , \tag{A.34}$$

where $\lambda(a)$ denotes the co-state variable of the stock y. The first order conditions yield

$$\lambda(a)=\mathcal{C}(a)^{-\frac{1}{\sigma^H}}\exp\left[\left(\rho^S-\rho^H\right)a\right] \ , \tag{A.35a}$$

$$\dot{\lambda}(a)=0 \ , \tag{A.35b}$$

which imply that

$$\mathcal{C}(a)=\mathcal{C}(0)\exp\left[\sigma^H(\rho^S-\rho^H)a\right] \ . \tag{A.36}$$

As $\lambda(T)$ is obviously not zero, transversality implies that $y(T) = 0$. Therefore, we obtain from equation (A.33), acknowledging $Q_T(\nu) = 1/\gamma$,

$$\mathcal{C}(0) = \bar{c}(t) \; \frac{Q_T(\nu)}{Q_T(\nu + \sigma^H(\rho^H - \rho^S))} \; , \tag{A.37}$$

which, together with equation (A.36), returns equation (31). □

A.7 Proof of Proposition 7

(i) The equivalence of the unconstrained social planner problem and of the optimization problem in the ILA economy pointed out in relation to equations (32) and (33) implies the Euler equation of the unconstrained social planner economy

$$\frac{\dot{c}(t)}{c(t)} = \sigma^H \left[r(t) - \rho^S \right] - \xi \; . \tag{A.38}$$

For both economies the Euler equation implies that a time varying consumption rate also implies a time varying interest rate (and obviously so does a time varying capital stock).

For observational equivalence to hold, consumption and interest rate of the unconstrained utilitarian OLG economy have to coincide with that of the ILA economy, implying the following equality of the Euler equations

$$\begin{aligned} \sigma^H \left[r(t) - \rho^S \right] - \xi = \sigma^R \left[r(t) - \rho^R \right] - \xi \\ \Leftrightarrow \sigma^R \rho^R - \sigma^H \rho^S = (\sigma^R - \sigma^H) r(t) \; . \end{aligned} \tag{A.39}$$

For a time varying interest rate this equation can only be satisfied if $\sigma^R = \sigma^H$ and $\rho^H = \rho^S$.

If $\sigma^R = \sigma^H$ and $\rho^H = \rho^S$ hold, the equivalence of the two problems was explained in relation to equations (32) and (33).

(ii) Existence of an observationally equivalent ILA economy implies that, first, the ILA economy has to be in a steady state as well and, second, that the steady state Euler equations have to coincide implying

$$\begin{aligned} & r = \rho^R - \frac{\xi}{\sigma^R} = \rho^S - \frac{\xi}{\sigma^H} \\ & \quad \Rightarrow \rho^R - \rho^S = \xi \; \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H} \; . \end{aligned}$$

The same reasoning applies when starting from the ILA economy steady state and assuming an observationally equivalent unconstrained utilitarian OLG economy.

If equation (34) is satisfied and the unconstrained utilitarian OLG economy is in a steady state, equation (A.38) implies

$$r^S = \rho^S + \frac{\xi}{\sigma^H} \; . \tag{A.40}$$

Using equation (34) to substitute ρ^S on the right hand side yields

$$r^S = \rho^R - \xi \; \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H} + \frac{\xi}{\sigma^H} = \rho^R + \frac{\xi}{\sigma^R} = r^R \; . \tag{A.41}$$

Thus, also the ILA economy is in a steady state (see Section 3) with coinciding interest rate. As the interest rates coincide, so does the capital stock and so do the consumption paths. Starting with the ILA steady state with interest rate r^R yields a coinciding unconstrained utilitarian OLG steady state by the same procedure. □

A.8 Proof of Proposition 8

(i) According to the proof of Proposition 7, the Euler equation of the unconstrained social planner solution is (A.38). In a steady state with interest rate $r^\star$ it is satisfied for any (obviously non-empty) set of preference parameters σ^H and ρ^S satisfying

$$\rho^S + \frac{\xi}{\sigma^H} = r^\star \; . \tag{A.42}$$

Moreover, by virtue of Proposition 3, $\rho^S = r^\star - \frac{\xi}{\sigma^H} > \rho^H$ holds. Note that for all decentralized economies $\Gamma^\star$ $r^\star < r^{gr}$. Hence, the same reasoning as in the proof of Proposition 4 can be applied to make sure that the budget constraints of the decentralized OLG and the unconstrained utilitarian social planner OLG coincide. The condition $r^\star < r^{gr}$ also implies that the social planner's transversality condition is satisfied.

(ii) Using (31), we can write the intratemporal allocation of consumption across the generations alive in steady state in the unconstrained utilitarian OLG as

$$c_S^\star(a) = \frac{c(t,t-a)}{\exp[\xi t]} = c^\star \frac{Q_T(\nu)}{Q_T(\nu + \sigma^H(\rho^H - \rho^S))} \exp[-\sigma^H(\rho^H - \rho^S)a] \; . \tag{A.43}$$

The intratemporal allocation of consumption in the decentralized OLG economy is given by (13a) and can be written as

$$c_d^\star(a) = c^\star \frac{Q_T(\nu)}{Q_T(\nu + \xi - \sigma^H(r_d^\star - \rho^H))} \exp[(\sigma^H(r_d^\star - \rho^H) - \xi)a] \; , \tag{A.44}$$

where $r_d^\star$ is the steady state interest rate of the decentralized OLG in which the households exhibit the same preference parameters as in the unconstrained utilitarian OLG economy.

$\Rightarrow$: Suppose that the allocation of consumption across all generations alive at each point is identical. For this to be the case, the following two equations have to hold simultaneously for all $a \in [0, T]$

$$\exp[-\sigma^H(\rho^H - \rho^S)a] = \exp[(\sigma^H(r_d^\star - \rho^H) - \xi)a] \; , \tag{A.45a}$$

$$\sigma^H(\rho^H - \rho^S) = \xi - \sigma^H(r_d^\star - \rho^H) \; . \tag{A.45b}$$

Minor mathematical transformations show that this only holds for

$$\rho^s = r_d^\star - \frac{\xi}{\sigma_H} \; . \tag{A.46}$$

This is the condition for the unconstrained utilitarian OLG and the decentralized OLG to be observationally equivalent in steady state.

$\Leftarrow$: Now suppose that the unconstrained utilitarian OLG and the decentralized OLG are observationally equivalent in steady state, i.e., equation (A.46) is satisfied.

Inserting ρ^S as given by (A.46) into (A.43) yields

$$c_S^\star(a) = c^\star \frac{Q_T(\nu)}{Q_T(\nu + \xi - \sigma^H(r_d^\star - \rho^H))} \exp[(\sigma^H(r_d^\star - \rho^H) - \xi)a] \; , \tag{A.47}$$

which is identical to (A.44). Hence, observational equivalence in steady state is also sufficient for identical allocations across the generations alive in both economies. $\square$

A.9 Proof of Proposition 9

We show that the constrained social planner can implement the steady state social optimum with a tax/subsidy regime on interest and wages only if the steady states of the first-best optimum and the decentralized OLG economy coincide. This implies that the first-best solution is, in general, not implementable, as every first-best solution converges to a non-implementable steady state.

We show that for a given steady state, the intratemporal distribution of consumption coincides in the constrained and the unconstrained utilitarian OLG economy if and only if $\tau_r^\star = 0$. To see this consider an unconstrained utilitarian OLG economy in steady state. The household problem in the constrained utilitarian OLG economy is identical to the household problem in the decentralized economy if we substitute $r(t)$ by $r^{\mathrm{e}}(t)$ and $w(t)$ by $w^{\mathrm{e}}(t)$. Solving for individual consumption and wealth in the steady states yields analogously to equations (13a) and (13b):

$$c^{\mathrm{e}\star}(a) \equiv \left.\frac{c^{\mathrm{e}}(t,s)}{\exp[\xi t]}\right|_{k=k^\star} = w^{\mathrm{e}\star}\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}\exp\left[(\sigma^H(r^{\mathrm{e}\star}-\rho^H)-\xi)a\right] , \tag{A.48a}$$

$$\begin{aligned} b^{\mathrm{e}\star}(a) \equiv \left.\frac{b^{\mathrm{e}}(t,s)}{\exp[\xi t]}\right|_{k=k^\star} &= w^{\mathrm{e}\star}Q_a(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H))\ \exp[(r^{\mathrm{e}\star}-\xi)a] \\ &\times\left[\frac{Q_a(r^{\mathrm{e}\star}-\xi)}{Q_a(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}-\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}\right] , \end{aligned} \tag{A.48b}$$

where $r^{\mathrm{e}\star} = r^{\mathrm{e}}(t)$ and $w^{\mathrm{e}\star} = w^{\mathrm{e}}(t)/\exp[\xi t]$, both evaluated at the steady state. Following the aggregation rule (9), we derive for aggregate steady state consumption and wealth:

$$c^{\mathrm{e}\star} = w^{\mathrm{e}\star}\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T(\nu)}\frac{Q_T(\nu+\xi-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}{Q_T(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H))} , \tag{A.49a}$$

$$\begin{aligned} b^{\mathrm{e}\star} = \frac{w^{\mathrm{e}\star}}{r^{\mathrm{e}\star}-\xi}&\left[\frac{Q_T(\xi+\nu-r^{\mathrm{e}\star})}{Q_T(\nu)}-1\right]-\frac{w^{\mathrm{e}\star}}{r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)} \\ &\times\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T(\nu)}\frac{Q_T(\xi+\nu-r^{\mathrm{e}\star})-Q_T(\xi+\nu-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}{Q_T(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H))} . \end{aligned} \tag{A.49b}$$

Inserting equation (A.49a) into equation (A.48a), we obtain the following intratemporal distribution of consumption

$$c^{e\star}(a) = c^{e\star} \frac{Q_T(\nu)}{Q_T(\nu + \xi - \sigma^H(r^{e\star} - \rho^H))} \exp\left[(\sigma^H(r^{e\star} - \rho^H) - \xi)a\right] \ . \tag{A.50}$$

By virtue of equation (31), however, the steady state intertemporal distribution of consumption in the social optimum yields:

$$c^\star(a) = c^\star \frac{Q_T(\nu)}{Q_T(\nu - \sigma^H(\rho^S - \rho^H))} \exp\left[(\sigma^H(\rho^S - \rho^H))a\right] \ . \tag{A.51}$$

Aggregate equivalence requires that $c^{e\star} = c^\star$. Distributional equivalence at a point in time requires moreover that equation (A.50) and equation (A.51) coincide. Together these conditions imply that $\sigma^H(r^{e\star} - \rho^H) - \xi = \sigma^H(\rho^S - \rho^H) \Leftrightarrow r^{e\star} = \rho^S + \frac{\xi}{\sigma^H}$. Thus, by equation (A.42), it must be $r^{e\star} = r^\star$ and therefore $\tau_r^\star = 0$. □

A.10 Characteristics of the functions characterizing the steady state capital stock

Lemma 1
The function $Q_T(r)$ defined in (12) satisfies:

(i) $Q_T(r) > 0$ *for all* $r \in \mathbb{R}$,

(ii) $Q'_T(r) < 0$ *for all* $r \in \mathbb{R}$.

The function

$$q(r) \equiv \frac{Q'_T(r)}{Q_T(r)} = \frac{T}{\exp(rT) - 1} - \frac{1}{r} \ , \tag{A.52}$$

satisfies

(iii) $q(r) < 0$ *for all* $r \in \mathbb{R}$,

(iv) $\lim_{r\to\infty} q(r) = 0$ *and* $\lim_{r\to-\infty} q(r) = -T$,

(v) $q'(r) = q'(-r) > 0$ *for all* $r \in \mathbb{R}$,

(vi) $q'(r) > z^2 q'(zr)$ *for all* $r \in \mathbb{R}$, $z \in (0, 1)$,

(vii) $y^2 q'(yr) > z^2 q'(zr)$ *for all* $r \in \mathbb{R}$, $y > z \geq 1$,

(viii) $q''(r) < 0$ *for all* $r \in \mathbb{R}_{++}$.

Proof: (i) Obviously, $Q_T(r) > 0$ for all $r \neq 0$. In addition, $\lim_{r\to 0} Q_T(r) = T > 0$.
(ii) We obtain

$$Q_T'(r) = -\frac{1 - \exp[-rT](1 + rT)}{r^2} \ .$$

For all $r \neq 0$:

$$Q_T'(r) < 0 \quad \Leftrightarrow \quad \exp[-rT](1 + rT) < 1 \quad \Leftrightarrow \quad 1 + rT < \exp[rT] \ .$$

The last inequality holds as $x + 1 < \exp[x]$ for all $x \in \mathbb{R}$. In addition, $\lim_{r\to 0} Q_T'(r) = -\frac{T^2}{2} < 0$.
(iii) Follows directly from items (i) and (ii).
(iv) Follows directly from the definition (A.52).
(v) We obtain:

$$q'(r) = -\frac{1}{r^2} - \frac{T^2 \exp[-rT]}{(1 - \exp[-rT])^2} = \frac{1}{r^2} - \frac{T^2}{2(\cosh[rT] - 1)} \ .$$

For all $r \neq 0$:

$$q'(r) > 0 \quad \Leftrightarrow \quad 2(\cosh[rT] - 1) > r^2T^2 \quad \Leftrightarrow \quad \cosh[rT] > 1 + \frac{r^2T^2}{2} \ .$$

The last inequality holds as $\cosh[x] > 1 + \frac{x^2}{2}$ for all $x \in R$. In addition, $\lim_{r\to 0} q'(r) = \frac{T^2}{12} > 0$.
(vi) The statement holds if and only if:

$$\begin{aligned} q'(r) - z^2 q'(zr) = \frac{z^2T^2}{2(\cosh[zrT] - 1)} - \frac{T^2}{2(\cosh[rT] - 1)} > 0 \\ \Leftrightarrow z^2(\cosh[rT] - 1) > \cosh[zrT] - 1 \ . \end{aligned}$$

To see that the last inequality holds, we employ the infinite series expansion of $\cosh[x]$:

$$\begin{aligned} z^2(\cosh[x] - 1) - (\cosh[zx] - 1) &= z^2 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - 1 \right) - \left(\sum_{n=0}^{\infty} \frac{(zx)^{2n}}{(2n)!} - 1 \right) \\ &= z^2 \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{(zx)^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \left(z^2 - z^{2n} \right) > 0 \ . \end{aligned}$$

The inequality holds, as the first summand is zero and all other terms are strictly positive for all $z \in (0,1)$. (vii) The statement holds if and only if:

$$y^2 q'(yr) - z^2 q'(zr) = \frac{z^2T^2}{2(\cosh[zrT]-1)} - \frac{y^2T^2}{2(\cosh[yrT]-1)} > 0$$
$$\Leftrightarrow z^2(\cosh[yrT]-1) > y^2\cosh[zrT]-1 \ .$$

Employing the infinite series expansion of $\cosh[x]$, we obtain

$$z^2(\cosh[yx]-1) - y^2(\cosh[zx]-1) = z^2\left(\sum_{n=0}^{\infty}\frac{(yx)^{2n}}{(2n)!}-1\right) - y^2\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-1\right)$$
$$= z^2\sum_{n=1}^{\infty}\frac{(yx)^{2n}}{(2n)!} - y^2\sum_{n=1}^{\infty}\frac{(yx)^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}z^2y^2\left(y^{2(n-1)}-z^{2(n-1)}\right) > 0 \ .$$

The inequality holds, as the first summand is zero and all other terms are strictly positive for all $y > z \geq 1$.
(viii) We obtain:

$$q''(r) = -\frac{2}{r^3} + \frac{2T^3\sinh[rT]}{(2\cosh[rT]-2)^2} = -2T^3\left(\frac{1}{(rT)^3} + \frac{\sinh[rT]}{(2\cosh[rT]-2)^2}\right)$$

Then, the statement holds if and only if $(\cosh[x]-2)^2 > x^3\sinh[x]$. To see this, we employ the infinite series expansion of $\cosh[x]$ and $\sinh[x]$

$$\left(2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-2\right)^2 - x^3\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \left(2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\right)^2 - \sum_{n=0}^{\infty}\frac{x^{2n+4}}{(2n+1)!}$$
$$= 4\left(\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\right)^2 - \sum_{n=0}^{\infty}\frac{x^{2n+4}}{(2n+1)!}$$

Both series exhibit all even powers of x starting with x^4:

$$x^4\left(\frac{4}{2!2!}-1\right) + x^6\left(\frac{2\cdot 4}{2!4!}-\frac{1}{3!}\right) + x^8\left(\frac{2\cdot 4}{2!6!}+\frac{4}{4!4!}-\frac{1}{5!}\right) + \cdots \geq 0 \ .$$

The inequality holds as the first term is zero and all other terms are strictly positive for all $x \in \mathbb{R}_{++}$. □

Lemma 2
For all $\xi, \nu \in \mathbb{R}_{++}$ the function J defined in (A.2) *satisfies*

(i) $J(r) > 0$.

For all $\xi, \nu \in \mathbb{R}_{++}$ and $\sigma^H \in (0,1]$ the function J satisfies *(ii)* $\frac{d}{dr}\left(\frac{J'(r)}{J(r)}\right) > 0$ *for all* $r \geq \xi$*,*

(iii) $\lim_{r\to\infty} \frac{J'(r)}{J(r)} = \sigma^H T$*.*

For all $\xi, \nu \in \mathbb{R}_{++}$ *and* $\sigma^H > 1$ *the function* J *satisfies*

(iv) $\frac{d}{dr}\left(\frac{J'(r)}{J(r)}\right) > 0$ *for all* $r \geq \nu + \xi$ *and* $\rho^H < \frac{\sigma^H - 1}{\sigma^H}(\nu + \xi)$*,*

(v) $\lim_{r\to\infty} \frac{J'(r)}{J(r)} = T$*.*

Proof: (i) Follows immediately from $Q_T(r) > 0$ for all $r \in \mathbb{R}$ as shown in Lemma 1.
(ii) Using the definition (A.52), we obtain

$$\frac{J'(r)}{J(r)} = q(r-\xi) - \sigma^H q\left(\nu + \xi - \sigma^H\left(r - \rho^H\right)\right) - \left(1 - \sigma^H\right) q\left(r - \sigma^H\left(r - \rho^H\right)\right) \ , \tag{A.53a}$$

and

$$\begin{aligned} M(r) \equiv \frac{d}{dr}\left(\frac{J'(r)}{J(r)}\right) &= \frac{J''(r)}{J(r)} - \left(\frac{J'(r)}{J(r)}\right)^2 \qquad \text{(A.53b)} \\ &= q'(r-\xi) + \left(\sigma^H\right)^2 q'\left(\nu + \xi - \sigma^H\left(r-\rho^H\right)\right) - \left(1-\sigma^H\right)^2 q'\left(r - \sigma^H\left(r - \rho^H\right)\right) \ . \end{aligned}$$

For $\sigma^H \in (0, 1]$ set $x = r - \xi$ and restrict attention to all $x \geq 0$

$$\begin{aligned} M(x) &= q'(x) + \left(\sigma^H\right)^2 q'\left(\nu + \left(1-\sigma^H\right)\xi - \sigma^H\left(x - \rho^H\right)\right) \\ &\quad - \left(1-\sigma^H\right)^2 q'\left(\left(1-\sigma^H\right)x + \left(1-\sigma^H\right)\xi + \sigma^H\rho^H\right) \\ &> q'(x) - \left(1-\sigma^H\right)^2 q'\left(\left(1-\sigma^H\right)x + \left(1-\sigma^H\right)\xi + \sigma^H\rho^H\right) \\ &\geq q'(x) - \left(1-\sigma^H\right)^2 q'\left(\left(1-\sigma^H\right)x\right) \geq 0 \ . \end{aligned}$$

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vi) of Lemma 1.
(iii) Follows directly from equation (A.53a) and part (iv) of Lemma 1.
(iv) For $\sigma^H > 1$ and $\rho^H < \frac{\sigma^H-1}{\sigma^H}(\nu + \xi)$ consider only $r \geq \nu + \xi$

$$\begin{aligned} M(r) &= q'(r-\xi) + \left(\sigma^H\right)^2 q'\left(\sigma^H r - \sigma^H\rho^H - (\nu+\xi)\right) - \left(\sigma^H - 1\right)^2 q'\left(\left(\sigma^H - 1\right)r + \sigma^H r\right) \\ &> \left(\sigma^H\right)^2 q'\left(\sigma^H r - \sigma^H\rho^H - (\nu+\xi)\right) - \left(\sigma^H - 1\right)^2 q'\left(\left(\sigma^H - 1\right)r + \sigma^H r\right) \\ &> \left(\sigma^H\right)^2 q'\left(\sigma^H r\right) - \left(\sigma^H - 1\right)^2 q'\left(\left(\sigma^H - 1\right)r\right) \geq 0 \end{aligned}$$

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vii) of Lemma 1.

(v) Follows directly from equation (A.53a) and part (iv) of Lemma 1. □

References

S. R. Aiyagari. Observational equivalence of the overlapping generations and the discounted dynamic programming frameworks for one-sector growth. *Journal of Economic Theory*, 35: 201–21, 1985.

G. Asheim and T. Mitra. Sustainability and discounted utilitarianism in models of economic growth. *Mathematical Social Science*, forthcoming.

R. J. Barro. Are government bonds net wealth? *Journal of Political Economy*, 82:1095–117, 1974.

O. J. Blanchard. Debt, deficits, and finite horizons. *Journal of Political Economy*, 93:223–47, 1985.

J. Broome and R. Schmalensee. *Counting the Cost of Global Warming*. White Horse Press, Cambridge, 1992.

P. S. Burton. Intertemporal preferences and intergenerational equity considerations in optimal resource harvesting. *Journal of Environmental Economics and Management*, 24:119–32, 1993.

G. A. Calvo and M. Obstfeld. Optimal time-consistent fiscal policy with finite lifetimes. *Econometrica*, 56:411–32, 1988.

J. Y. Campbell. Understanding risk and return. *Journal of Political Economy*, 104(2):298–345, 1996.

W. R. Cline. *The Economics of Global Warming*. Institute for International Economics, Washington D.C., 1992.

H. d'Albis. Demographic structure and capital accumulation. *Journal of Economic Theory*, 132: 411–34, 2007.

L. G. Epstein and S. E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of Political Economy*, 99(2):263–86, 1991.

R. Gerlagh and M. A. Keyzer. Sustainability and the intergenerational distribution of natural resouce entitlements. *Journal of Public Economics*, 79:315–41, 2001.

R. Gerlagh and B. C.C. van der Zwaan. Overlapping generations versus infinitely-lived agent: The case of global warming. In R. Howarth and D. Hall, editors, *The long-term economics of climate change*, volume 3, pages 301–27. JAI Press, Stamford, 2000.

R. F. Harrod. *Towards a Dynamic Economics*. Macmillan, London, 1948.

C. Heinzel and R. Winkler. The role of environmental and technology policies in the transition to a low-carbon energy industry. Working Paper No. 07/71, CER-ETH – Center of Economic Research at ETH Zurich, 2007.

R. B. Howarth. An overlapping generations model of climate-economy interactions. *Scandinavian Journal of Economics*, 100:575–91, 1998.

R. B. Howarth. Climate change and the representative agent. *Environmental and Resource Economics*, 15:135–48, 2000.

M. D. Hurd. Savings of the elderly and desired bequests. *American Economic Review*, 77: 298–312, 1987.

M. D. Hurd. Mortality risk and bequests. *Econometrica*, 57:779–814, 1989.

T. C. Koopmans. On the concept of optimal economic growth. In *Study week on the econometric approach to development planning*, volume 28 of *Pontificiae Academiae Scientiarum scripta varia*, pages 225–66. North Holland, Amsterdam, 1965.

W. Kopczuk and J. P. Lupton. To leave or not to leave: The distribution of bequest motives. *Review of Economic Studies*, 74:207–35, 2007.

J. Laitner and F. T. Juster. New evidence on altruism: A study of TIAA-CREF retirees. *American Economic Review*, 86(4):893–908, 1996.

J. Laitner and H. Ohlsson. Bequest motives: A comparison of Sweden and the United State. *Journal of Public Economics*, 79:205–36, 2001.

W. Nordhaus. *A Question of Balance: Economic Modeling of Global Warming*. Yale University Press, New Haven, 2008.

W. D. Nordhaus. A review of the Stern review on the economics of climate change. *Journal of Economic Literature*, 45:686–702, 2007.

A. C. Pigou. *The Economics of Welfare*. Macmillan, London, 1932.

F. Ramsey. A mathematical theory of savings. *Economic Journal*, 38:543–59, 1928.

A. Sen and B. Williams, editors. *Utilitarianism and Beyond*. Cambridge University Press, Cambridge and New York, 1982.

R. M. Solow. Intergenerational equity and exhaustible resources. *Review of Economic Studies*, 41:29–45, 1974.

G. Stephan and G. Müller-Fürstenberger. Overalpping generations or infinitely-lived agents. Intergenerational altruism and the economics of global warming. *Environmental and Resource Economics*, 10:27–40, 1997.

N. Stern. *The Economics of Climate Change – The Stern Review*. Cambridge University Press, Cambridge, 2007.

A. Vissing-Jørgensen and O. P. Attanasio. Stock-market participation, intertemporal substitution, and risk-aversion. *American Economic Review*, 93(2):383–391, 2003.

M. O. Wilhelm. Bequest behavior and the effect of heirs' earnings: Testing the altruistic model of bequests. *American Economic Review*, 86(4):874–92, 1996.

M. E. Yaari. Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies*, 32:137–50, 1965.