Discussion paper

Pareto Optimal Insurance Policies in the Presence of Administrative Costs

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Abstract

In his classical article in *The American Economic Review*, Arthur Raviv (1979) examines Pareto optimal insurance contracts when there are ex-post insurance costs c induced by the indemnity I for loss x. Raviv's main result is that a necessary and sufficient condition for the Pareto optimal deductible to be equal to zero is $c'(I) = 0$ for all $I \geq 0$.

We claim that another type of cost function is called for in household insurance, caused by frequent but relatively small claims. If a fixed cost is incurred each time a claim is made, we obtain a non-trivial Pareto optimal deductible even if the cost function does not vary with the indemnity. This implies that when the claims are relatively small, it is not optimal for the insured to get a compensation since the costs outweighs the benefits, and a deductible will naturally occur.

We also discuss policies with an upper limit, and show that the insurer prefers such contracts, but the insured does not. In Raviv's paper it was also shown that policies with upper limits are dominated by policies with no upper limit, when there are ex-post costs to insurance. We show that the result is right, but the proof is wrong.

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I   Introduction

It seems broadly accepted that deductible policies give the best tradeoff between risk sharing and economizing on costly claim settlements. The presence of insurance costs are often considered as the "best" and most straightforward explanation of deductibles occurring in insurance contracts. There are other explanations, usually involving models of asymmetric information, like moral hazard (Holmstrőm (1979)) or adverse selection (Rothschild and Stiglitz (1976)). These models are much more complex than simply introducing ex-post costs in the classical model of risk sharing. In these models deductibles appear more or less as a by-product of the analysis. When e.g., moral hazard is present, it is socially optimal that the insured keeps more of the risk than when moral hazard is absent in order to get the incentives right. For example, when the insurer is risk neutral and the classical recipe is that full insurance is Pareto optimal, with moral hazard this is no longer the case. When there is adverse selection, the good risks can not be offered full coverage because of the presence of the bad risks. The latter, on the other hand, obtains full insurance when this is optimal. In both cases the insurance customers will end up taking more risk than in the neoclassical case. Whether this risk-sharing takes the form of a deductible, or as some other forms of coinsurance is not a central point.

The framework of Pareto optimal risk sharing between an insurer and an insurance buyer is built on Borch's classical theory (Borch (1960a-b), and Moffet (1979) was the first to formulate this problem in the neoclassical situation. Deductibles have also been analyzed in the framework of pure demand theory, such as in Arrow (1974), Schlesinger (1981) and Karni (1983). Raviv's analysis of Pareto optimal deductibles in the presence of insurance costs is the classical one, and is the first analysis connecting deductibles directly to these costs. For example are some of the results of Arrow clarified through the analysis of Raviv. Aase (2004-08) review various aspects of Pareto optimal risk sharing that involve deductibles, and Aase (2002) is a general review of risk sharing in insurance syndicates.

Borch (1990) divides insurance into three categories; life insurance, household insurance and business insurance. He notes that, for an insurer involved in household insurance up to one third of the total premium is used for administrative expenses. If the risk premium is approximately zero, this means that the loading γ is 50% in the standard premium formula $p = (1 + \gamma)E(I(X))$, which is large.

It should be fairly obvious that if all domestic claims are reported, caused by the relatively minor, but frequent accidents that occur in everyday life in the homes of ordinary, insured families, this would be prohibitively expen-

sive for the insurance industry to handle, let alone the mere logistics of the problem. This is where deductibles become important. In order to capture these costs, the cost function is assumed to be on the following form

$$C(I) = a\chi_{[I>0]} + c(I) \tag{1}$$

where

$$\chi_B = \begin{cases} 1, & \text{if } B \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

i.e., χ is the indicator function of the event B. Equation (1) means that whenever a claim is made, no matter how small, a cost $a > 0$ is incurred, with further costs determined by the function $c(\cdot)$ satisfying the standard conditions: $c(0) = b \geq 0, c'(I) \geq 0$, and $c''(I) \geq 0$ for all $I \geq 0$. Thus, even if the fixed costs $b = 0$, the function $C(\cdot)$ has a discontinuity in $I = 0$, with a positive jump size a.

Fixed costs not depending on claims made are measured by b in the above. The cost a is only triggered when the insurance customer actually makes a claim against the insurer. We then show that a necessary and sufficient condition for a Pareto optimal deductible to be equal to zero is that $a = 0$ and $c'(I) = 0$ for all I.

In other words, if $a > 0$, then a non-zero deductible $D > 0$ occurs even if $c'(I) = 0$ for all I. This aspect of cost accounting is accordingly not captured by the analysis in Raviv (1979). We claim it to be the important one related to administrative costs in household insurance.

Blazenko (1985) points out that there is an error is Raviv's proof of his main theorem cited above, but the result is correct. Section II develops the setting of the problem and the notation to be subsequently used. In Section III we analyze the optimality of policies with an upper limit in the pure supply theory of insurance. In section IV we point out an error made by Raviv (1979) in his proof that policies with an upper limit are dominated by policies with no upper limit and no deductibles, and present a corrected proof of this theorem. In section V we prove our main deductibles-result using the methodology of Blazenko. Section VI concludes.

II   Insurance with Costly Claim Settlement

The insured faces a random loss X with values $0 \leq x \leq M$, and probability density $f(x) > 0$. The indemnity to the insured is $I(x)$ if $X = x$, and the contract has premium p. The indemnity function is quite naturally constrained by

$$0 \leq I(x) \leq x \qquad \text{for any } x \geq 0, \tag{3}$$

implying that $I(0) = 0$. Costs of claim settlements are ex post, and given by (1) and (2). The insurer's utility function is v, where $v' > 0$ and $v'' \leq 0$, and final wealth is $w_v - I(x) + p - C(I(x))$ where w_v represents initial reserves. The insured's utility function is u, where $u' > 0$, $u'' < 0$, so the insured is strictly risk averse (otherwise he would not demand insurance). The insured's final wealth is $w_u - x + I(x) - p$, where w_u is the initial, risk-free part of wealth, and w_u and w_v are both positive constants.

Pareto optimal contracts (I, p) are generated as solutions of

$$\max_{I,p} Eu(w_u - X + I(X) - p) \text{ s.t. } Ev(w_v - I(X) - C(I(X)) + p) \geq k \quad (4)$$

As the constant k varies, the Pareto optimal frontier is generated. Using control theory (e.g., Seierstad and Sydsæter (1987)), the Hamiltonian of the problem is

$$\mathcal{H}(I, \lambda) = \Big(u(w_u - x + I(x) - p) + \lambda(v(w_v - I(x) - C(I(x)) + p) - k)\Big)f(x),$$

and the Lagrangian is

$$\mathcal{L}(I, \lambda, \mu_1(x), \mu_2(x)) = \mathcal{H}(I, \lambda) + \mu_1(x)I(x) + \mu_2(x)(x - I(x)).$$

If $I^*(x)$ denotes the optimal indemnity function, then

$$\mu_i(x) \geq 0 \quad \text{for all } x, i = 1, 2,$$
$$\mu_1(x) = 0 \quad \text{if } I^*(x) > 0,$$
$$\mu_2(x) = 0 \quad \text{if } I^*(x) < x, \text{ and}$$

$$\mu_1(x)I^*(x) = 0 \text{ for all } x, \text{ and } \mu_2(x)(x - I^*(x)) = 0 \text{ for all } x.$$

From this it follows that necessary conditions for a maximum with respect to the indemnity are

$$u'(w_u - x + I^*(x) - p) - \lambda v'\big(w_v - I^*(x) - C(I^*(x)) + p\big)\big(1 + c'(I^*(x))\big) = 0 \quad (5)$$

for all x such that $0 < I^*(x) < x$,

$$J(x) := u'(w_u - x - p) - \lambda v'\big(w_v - b + p\big)\big(1 + c'(0)\big) \leq 0 \quad\quad (6)$$

when $I^*(x) = 0$ for $x > 0$, and

$$K(x) := u'(w_u - p) - \lambda v'\big(w_v - x - a - c(x) + p\big)\big(1 + c'(x)\big) \geq 0 \quad (7)$$

when $I^*(x) = x > 0$. These conditions are also sufficient for a maximum when $u + \lambda v$ is concave in I. Due to the discontinuity of $C(I)$ in $I = 0$,

H is only concave in I if zero is excluded. However, in the language of Lagrange's method, the "instantaneous kink" at zero is not enough to create a "duality gap" as long as a is not so large that the insurance costs outweigh the benefits of risk sharing. If this is not so, sufficiency of the above conditions are preserved for an inner solution.

The function $J(x)$ is continuous and increasing in x, while the function $K(x)$ is continuous and decreasing in x, meaning that either (6) or (7) holds, both can not hold for the same $x > 0$. If the quantity $L \geq 0$, where L is defined by

$$L := u'(w_u - p) - \lambda v'(w_v - b + p)\big(1 + c'(0)\big),$$

then (6) can not hold for any $x > 0$, and if $L \leq 0$, then (7) can not hold for any $x > 0$. Therefore the optimal solution is one with a deductible, or one with an upper limit. The deductible D and the upper limit B are defined by

$$u'(w_u - p - D) - \lambda v'(w_v - b + p)\big(1 + c'(0)\big) = 0, \tag{8}$$

and

$$u'(w_u - p) - \lambda v'(w_v - B - a - c(B) + p)\big(1 + c'(B)\big) = 0, \tag{9}$$

respectively. If we have a policy with a deductible, the optimal indemnity function depends on the deductible D through (8), and we denote I^* by by $I_D(x)$. If the policy is one with an upper limit, the optimal indemnity function depends on this limit B through (9), and we denote I^* by $I_B(x)$. If both $D = B = 0$, we call the optimal indemnity function $I_P(x)$. In the latter case it is determined from (5) for all $x \geq 0$ via a differential equation, with boundary condition $I_P(0) = 0$.

III Insurance Policies with an Upper Limit

In the pure demand theory of insurance, Arrow (1974) has shown that when the insurance customer's utility function u satisfies $u' > 0$ and $u'' < 0$, the solution to the problem

$$\max_{I(x) \geq 0} Eu(w - X + I(X) - p) \qquad \text{subject to } p = (1 + \gamma)E(I(X))$$

is a contract $I_D(x)$ with a deductible:

$$I_D(x) = \begin{cases} 0, & \text{if } x \leq D \\ x - D, & \text{if } x > D, \end{cases} \tag{10}$$

and $D > 0$ if and only if the loading $\gamma > 0$. Thus, in this framework full insurance is optimal when the premium is actuarially fair only. One way to

demonstrate this is to consider non-decreasing contracts $I(x) \geq 0$, and to observe that any deviations from the contract I_D satisfying $0 \leq I(x) \leq x$ represent a *mean preserving spread* in the wealth of the insured, in the sense of Stiglitz and Rothschild (1970). To use this line of proof, it is enough to assume $u'' \leq 0$.

The impression from results of this type is that contracts with a deductible are somehow "superior". However, and still in the absence of ex-post costs, by also bringing in the supply side, contracts with a deductible can not be Pareto optimal. Even if the premium p is actuarially unfair, the Pareto optimal deductible is zero, and if the insurer is risk neutral, full insurance is Pareto optimal. This follows from the following differential equation for the Pareto optimal indemnity function

$$\frac{\partial I(x)}{\partial x} = \frac{A_u(w_u - x + I(x) - p)}{A_u(w_u - x + I(x) - p) + A_v(w_v - I(x) + p)}, \qquad (11)$$

which, together with the boundary condition $I(0) = 0$ yields a unique solution for each p. When the premium p varies through a suitable range, this generates the Pareto frontier in (Eu, Ev)-space, since p now takes the role of the Lagrange multiplyer λ of the previous section in this regard. Here the functions A_u and A_v are the absolute risk aversions of the insured and the insurer respectively. From (11) we notice that when $v'' < 0$, then

$$0 < I'(x) < x \qquad \text{for all } x \geq 0, \qquad (12)$$

and together with $I(0) = 0$ and the mean value theorem, it follows that

$$0 < I(x) < x \qquad \text{for all } x > 0,$$

verifying that full insurance is not Pareto optimal when both parties are strictly risk averse. Notice that the natural restriction $0 \leq I(x) \leq x$ is not binding at the optimum for any $x > 0$. From this it follows that neither contracts with a deductible, nor contracts with an upper limit are Pareto optimal, since both these contracts would violate the requirement (12) for some x. When the insurer is risk neutral, then $I(x) = x$ so full insurance is optimal, regardless of the value of p, actuarially fair or not[1]. One would, however, only expect to observe contracts that are also individually rational for both parties, i.e., contracts that are in the core.

These normative conclusions are in agreement with the observation that consumers show a propensity for low or no deductible insurance policies

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[1]If $p \geq (1 + \gamma)EI(X)$ is added as a constraint, e.g., Arrow (1970), a deductible will arise, but this does not really generate Pareto optimal policies.

against small to moderate risks. Considering auto insurance as a proxy for insurance against such risks, Pashigian, Schkade and Menefee (1966) find that out of a sample, from 1962, of more than 0.8 million insured drivers, 53.8 percent chose the lowest deductible and 45.7 percent chose the next lowest. Cummins and Weisbart (1977) report that a proposal in Pennsylvania to raise the minimum auto insurance deductible from $ 50 to $ 100 during the 1970s was ultimately withdrawn after massive consumer outcry, even though such legislation could have saved consumers millions of dollars each year. Similar attitudes to risk are reported in medical insurance, which is another proxy for moderate risks (the U.S. Bureau of Labor Statistics (1999)).

Let us turn to the pure supply-side theory of insurance. Here we consider an insurer with utility function v, where $v' > 0$ and $v'' \leq 0$, and risk-free reserves w, facing the problem

$$\max_{I(x) \leq x} Ev(w - I(X) + p) \qquad \text{subject to } p = (1 + \gamma)E(I(X)). \qquad (13)$$

We can then show

Theorem 1 *When the insurer selects to offer insurance contracts $(I(x), p)$, the contract $I^*(x)$ solving (13) is one with an upper limit B: $I^*(x) = I_B(x)$ where*

$$I_B(x) = \begin{cases} x, & \text{if } x \leq B \\ B, & \text{if } x > B. \end{cases} \qquad (14)$$

If the loading $\gamma = 0$ and $v'' < 0$, then $B = 0$.

<u>Proof.</u> Since $v' > 0$, the solution to (13) is the same as the solution to the problem with the inequality constraint $p \leq (1 + \gamma)E(I(X))$, because the insurer wants more premiums to less. Using control theory, the Hamiltonian of this latter problem is

$$\mathcal{H}(I; \lambda) = v(w - I(x) + p) + \lambda((1 + \gamma)I(x) - p))f(x),$$

where $\lambda > 0$ is a constant, and the Lagrangian is

$$\mathcal{L}(I; \mu, \lambda) = \mathcal{H}(I; \lambda) + \mu(x)(x - I(x)),$$

where $\mu(x) \geq 0$, $\mu(x) = 0$ if $I^*(x) < x$, and $\mu(x)(x - I^*(x)) = 0$ for all x, where I^* denotes the optimal contract. From the maximum principle it follows that the necessary and sufficient first order conditions are found as follows:

$$\mathcal{H}(I^*; \lambda) \geq \mathcal{H}(I; \lambda) \qquad \text{for all } I \text{ such that } I(x) < x,$$

which leads to

$$v'(w - I^*(x) + p) = \lambda(1 + \gamma) \qquad \text{when } I^*(x) < x. \qquad (15)$$

Furthermore

$$\frac{\partial \mathcal{L}(I^*; \mu, \lambda)}{\partial I} = 0 \qquad \text{for all } x,$$

which means that

$$v'(w - x + p) - \lambda(1 + \gamma) = -\frac{\mu(x)}{f(x)} \leq 0 \qquad \text{when } I^*(x) = x > 0. \qquad (16)$$

Since the function $v'(w - x + p)$ is increasing in x, it is clear that there is some $B \geq 0$ for which $v'(w - B + p) = \lambda(1 + \gamma)$ and (16) holds true whenever $x \leq B$. From (15) it is clear that when $x > B$ then $I^*(x) = B$, from which the contract (14) follows.

If the insurer is strictly risk averse and the premium is actuarially fair, no contract is offered, or $B = 0$. $\qquad \square$

We may now go on and find the optimal B given that the contact is one with an upper bound. We limit ourselves to the following:

Theorem 2 *In the present framework, the optimal upper bound $B > 0$ if and only if $\gamma > 0$.*

<u>Proof.</u> We use the notation

$$g(B) := Ev(w - I_B(X) + p(B)),$$

where the premium

$$p(B) = (1 + \gamma)EI_B(X) = (1 + \gamma)\left(\int_0^B x f(x)dx + BP[X > B] \right),$$

and $P[C]$ denotes the probability of the event C. From this we get that

$$\frac{\partial p(B)}{\partial B} = (1 + \gamma)P[X > B] > 0$$

so the premium p is an increasing function of the upper limit B, as expected. Moreover

$$\frac{\partial g(B)}{\partial B} = P[X > B]\left((1+\gamma) \int_0^M v'(w - I_B(x) + p(B))f(x)dx - v'(w - B + p(B)) \right).$$

The integral can be written

$$\int_0^B v'(w - x + p(B))f(x)dx + \int_B^M v'(w - B + p(B))f(x)dx,$$

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and using the mean value theorem for integrals, it follows that

$$\int_0^B v'(w-x+p(B))f(x)dx = v'(w-\theta+p(B))\int_0^B f(x)dx \qquad \text{for some } \theta \in [0, B],$$

while

$$\int_B^M v'(w - B + p(B))f(x)dx = v'(w - B + p(B))P[X > B].$$

From this we have

$$\frac{\partial g(B)}{\partial B} = P[X > B]\Big\{(1+\gamma)\big(v'(w-\theta+p(B))P[X \le B]+v'(w-B+p(B))P[X > B]\big)$$

$$-v'(w - B + p(B))\Big\},$$

and setting B equal to zero finally gives

$$\frac{\partial g(B)}{\partial B}\big|_{B=0} = P[X > 0]\,\gamma\,v'(w + p(0)) > 0 \Leftrightarrow \gamma > 0,$$

which proves the theorem. $\qquad \square$

In the reinsurance business excess of loss (XL) contracts are common. These are a combination of contracts with a deductible, and contracts with an upper bound: The ceding company takes part of the risk itself up to a certain value D, then cedes the remaining risk to a reinsurer, except that there is some upper bound B beyond which the reinsurer is not responsible. Consider a simple example:

Example 1. An insurer with reserves $w = 3$ offers insurance against a loss X with probability distribution given in Table 1. Consider a contract with an upper bound $B = 1$ and loading $\gamma = 0.1$. The insurer's wealth W_B is then $(2.73; \frac{2}{3}, 3.73; \frac{1}{3})$. If the insurer instead offers a contract with a deductible D at the same premium as above, then $D = 0.5$, and the insurer's wealth W_D is instead given by the distribution $(2.33; \frac{1}{3}, 3.23; \frac{1}{3}, 3.73; \frac{1}{3})$. It is easy to see that the random wealth W_D is a mean preserving spread of W_B, so all risk averters will prefer to offer the policy with the upper bound B to the one with deductible D. Here it is seen that $W_D = W_B + \varepsilon$ in distribution, where the conditional distribution of ε given $W_B = 2.73$ is $(-0.5; \frac{1}{2}, 0.5; \frac{1}{2})$. Since the insurer prefers the certain outcome 2.73 to the lottery $2.73 + \varepsilon$, the fact that W_B is preferred to W_D actually follows from the substitution axiom. $\square$

This example indicates that we could have constructed an alternative proof of Theorem 1 by searching among non-decreasing contracts I, and verifying that the associated random wealth W_I is a mean preserving spread of W_B. It may be noted that we have also included variable costs in the above example, and the conclusion still holds.
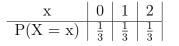
x	0	1	2
P(X = x)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 1: Probability distribution of X

IV Upper Limit Policies and Insurance Costs

In the previous section we have seen that in the pure theory of insurance supply, policies with an upper limit have a certain optimality property in that the insurer prefers to offer such contracts to all other contracts having the same premium. When the insurance customer is also brought into the model, we have seen that Pareto optimal contracts are not of this type, nor does Pareto optimal contracts contain a deductible.

In the present section we show that in the presence of insurance costs, constrained Pareto optimal contracts do not contain an upper bound. In Theorem 2 of Raviv (1979) this result is proved by comparing the slopes of the indifference curves for the insured and the insurer in p, B space. In doing so, Raviv employs two different relations for $\frac{dp}{dB}$ depending upon which indifference curve is held constant. While there can be many different connections between p and B, there is only one relation for this derivative for any given p, B, derived from equation (9). To seek a further relationship for this derivative is accordingly inappropriate.

First notice that with Pareto optimal contracts with an upper limit B, if they were to exist, B would not serve as a cap on compensations as in (14) of Theorem 1. It means that $I_B(x) := I(x) = x$ when $x \leq B$, and is given as a solution to the differential equation

$$\frac{dI(x)}{dx} = \frac{A_u(W_u)}{A_u(W_u) + A_v(W_v)((1 + c'(I)) + c''(I)/(1 + c'(I))} \tag{17}$$

when $x > B$, where $W_u = w_u - x + I_B(x) - p$ and $W_v = w_v - I_B(x) - C(I_B(x)) + p$, which follows from differentiating the first order condition (5) with respect to x. Since $\frac{dI(x)}{dx} > 0$, such policies imply risk sharing for losses above the upper limit B.

Next notice that from the relationship (9) we obtain

$$\frac{dp(B)}{dB} = \frac{A_v(w_v + p - B - a - c(B))(1 + c'(B)) + c''(B)/(1 + c'(B))}{A_u(w_u - p) + A_v(w_v + p - B - a - c(B))} \tag{18}$$

which shows that $p(B)$ is an increasing function of B under our assumptions on the variable cost function c.

The insured's expected utility with an upper limit is denoted by $\tilde{u}(B)$ and is given by

$$\tilde{u}(B) = \int_0^B u(w_u - p(B))f(x)dx + \int_B^M u(w_u - x + I_B(x) - p(B))f(x)dx,$$

and the derivative of this with respect to B is

$$
\begin{aligned}
\frac{d\tilde{u}(B)}{dB} = &-\frac{dp(B)}{dB}\int_0^B u'(w_u - p(B))f(x)dx \\
&+ \int_B^M u(w_u - x + I_B(x) - p(B))(-\frac{dp(B)}{dB} + \frac{\partial I_B(x)}{\partial B})f(x)dx.
\end{aligned}
\tag{19}
$$

The insurer's expected utility with an upper limit is

$$
\begin{aligned}
\tilde{v}(B) = &\int_0^B v(w_v - x + p(B) - a - c(x))f(x)dx \\
&+ \int_B^M v(w_v - I_B(x) - +p(B) - a - c(I_B(x)))f(x)dx,
\end{aligned}
$$

and the derivative with respect to B is

$$
\begin{aligned}
\frac{d\tilde{v}(B)}{dB} = &\frac{dp(B)}{dB}\int_0^B v'(w_v - x + p(B) - a - c(x))f(x)dx \\
&+ \int_B^M v(w_v - I_B(x) + p(B) - a - c(I_C(x))) \cdot \\
&(\frac{dp(B)}{dB} - \frac{\partial I_B(x)}{\partial B} - c'(I_B(x))\frac{\partial I_B(x)}{\partial B})f(x)dx.
\end{aligned}
\tag{20}
$$

Provided contracts are of the upper upper limit type, problem (4) is completed by solving

$$\max_B \left(\tilde{u}(B) + \lambda\tilde{v}(B)\right). \tag{21}$$

We then have the following

Theorem 3 *If the variable costs c satisfy $c'(I) > 0$ with positive probability, then the Pareto optimal contracts are not of the upper limit type. If $c'(I) = 0$ for all I, then $B = 0$ is the Pareto optimal upper limit.*

Proof. The derivative of the objective function in (21) with respect to B is

$$-\frac{dp}{dB}\int_0^B\left[u'(w_u-p)-\lambda v'(w_v-x+p-a-c(x))\right]f(x)dx$$

$$+\int_B^M\left[u'(w_u-x+I(x)-p)\right.$$

$$-\lambda v'(w_v-I(x)+p-a-c(I(x)))(1+c'(I(x)))\left]\frac{\partial I(x)}{\partial B}f(x)dx\right.$$

$$-\frac{dp}{dB}\int_B^M\left[u'(w_u-x+I(x)-p)\right.$$

$$-\lambda v'(w_v-I(x)+p-a-c(I(x)))\left]f(x)dx,\right.$$

which follows from the expressions (19) and (20). Here $I(x)$ is given by (17). If $c'(I) > 0$ with positive probability, the second term in square brackets is zero from (5). The third term in square brackets is then strictly positive with positive probability, and since $\frac{dp(B)}{dB} > 0$, the third term is strictly negative. The first term in square brackets is greater than or equal to zero from (7), so the first term is smaller than or equal to zero. As a consequence, the derivative of the objective function is strictly negative for all B. Therefore the Pareto optimal contracts can not be of of the upper limit type when $c'(I) > 0$ with positive probability.

When $c'(I) = 0$ for all I both the second and the third term in square brackets are zero for all $B \geq 0$ from (5). The first term is also zero when $B = 0$, implying that the Pareto optimal upper limit is $B = 0$. $\qquad\square$

Intuitively, an increase in B from zero has the effect of increasing insurance coverage for all losses which, in turn, increases the dead-weight loss due to increased insurance costs and therefore is suboptimal.

V Pareto Optimal Deductibles in the Presence of Costs

When there are no ex-post costs, we know from Section III that Pareto optimal contracts have no deductibles. From Raviv (1979) and Blazenko (1985) we know that when there are variable costs $c(I)$, then the Pareo optimal deductible is zero if and only if $c'(I) = 0$ for all I.

In this section we show that when the cost function is given by (1), then we need to add to this that $a = 0$ as well. So for example, when $a > 0$ and $c'(I) = 0$ for all I there is a non-zero deductible $D > 0$. As we have argued in the introduction, this term really captures the essence of costs

in the household insurance business, and it is rather intuitive that optimal contracts entail deductibles in this situation.

We proceed as follows: First notice that from the relationship (8) we obtain

$$\frac{dp(D)}{dD} = -\frac{A_u(w_u - p - D)}{A_u(w_u - p - D) + A_v(w_v + p - b)} \qquad (22)$$

which shows that $p(D)$ is a decreasing function of D, as expected. The insured's expected utility with deductible D is denoted by $\bar{u}(D)$ and is given by

$$\bar{u}(D) = \int_0^D u(w_u) - x - p(D)f(x)dx + \int_D^M u(w_u - x + I_D(x) - p(D))f(x)dx,$$

and the derivative of this with respect to D is

$$
\begin{aligned}
\frac{d\bar{u}(D)}{dD} = &-\frac{dp(D)}{dD}\int_0^D u'(w_u - x - p(D))f(x)dx \\
&+ \int_D^M u(w_u - x + I_D(x) - p(D))(-\frac{dp(D)}{dD} + \frac{\partial I_D(x)}{\partial D})f(x)dx.
\end{aligned}
\qquad (23)
$$

The insurer's expected utility with a deductible is

$$
\begin{aligned}
\bar{v}(D) = &\int_0^D v(w_v + p(D) - c(0))f(x)dx \\
&+ \int_D^M v(w_v - I_D(x) + p(D) - C(I_D(x)))f(x)dx,
\end{aligned}
$$

and the derivative with respect to D is

$$
\begin{aligned}
\frac{d\bar{v}(D)}{dD} = &\frac{dp(D)}{dD}\int_0^D v'(w_v + p(D) - c(0))f(x)dx \\
&+ \int_D^M v(w_v - I_D(x) + p(D) - a - c(I_D(x))) \cdot \\
&\quad (\frac{dp(D)}{dD} - \frac{\partial I_D(x)}{\partial D} - c'(I_D(x))\frac{\partial I_D(x)}{\partial D})f(x)dx \\
&+ v(w_v + p(D) - c(0))f(D) - v(w_v + p(D) - a - c(0))f(D).
\end{aligned}
\qquad (24)
$$

Provided Pareto optimal contracts contain a non-negative deductible, problem (4) is completed by solving

$$\max_D \big(\bar{u}(D) + \lambda\bar{v}(D)\big). \qquad (25)$$

We then have the following

13

Theorem 4 *A necessary and sufficient condition for the Pareto optimal deductible D to be equal to zero is $c'(I) = 0$ for all I and $a = 0$.*

<u>Proof.</u> The derivative of the objective function in (25) evaluated at $D = 0$ is

$$
\begin{aligned}
&-\frac{dp(0)}{dD} \int_0^M \left[u'(w_u - x + I_D(x) - p) \right. \\
&\left. -\lambda v'(w_v - I_D(x) + p - a - c(I_D(x))) \right] f(x) dx \\
&+ \int_0^M \left[u'(w_u - x + I_D(x) - p) \right. \\
&\left. -\lambda v'(w_v - I_D(x) + p - a - c(I_D(x)))(1 + c'(I_D(x))) \right] \frac{\partial I_D(x)}{\partial D} f(x) dx \\
&+ v(w_v + p(0) - c(0)) f(0) - v(w_v + p(0) - a - c(0)) f(0),
\end{aligned}
$$

which follows from the expressions (23) and (24). Here $I_D(x)$ is given by (17). If $c'(I) > 0$ with positive probability, the second term in square brackets is zero from (5). The first term in square brackets is then strictly positive with positive probability, and since $\frac{dp(0)}{dD} < 0$, the first term is strictly positive. The last term is greater than or equal to zero if $a \geq 0$, since v is increasing. As a consequence, the derivative of the objective function is strictly positive evaluated at $D = 0$. Therefore the Pareto optimal deductible is not zero.

When $c'(I) = 0$ for all I both the first and the second term in square brackets are zero, from (5), and the last term is zero only if $a = 0$. In this case the Pareto optimal deductible is zero. If $a > 0$ the Pareto optimal deductible is not zero.

When evaluating the derivative of the objective function at any D, the additional term to the above expression is

$$
-\frac{dp(D)}{dD} \int_0^D \left[u'(w_u - x - p) - \lambda v'(w_v + p - b)) \right] f(x) dx
$$

which may be negative from (6) depending on the cost function (e.g., $c'(0) > 0$). Thus, unlike the situation with an upper limit, this derivative may change sign. If the objective is maximized at $D = M$ the costs of claim settlement overwhelm the advantages of risk sharing. $\qquad \square$

VI Conclusions

When there are no ex-post costs, the pure theory of insurance demand implies that contracts with a deductible are preferred by the insured among contracts with the same premium. In the pure theory of insurance supply we have

demonstrated that insurers prefer to offer contracts with an upper bound to any other contract with the same premium. When Pareto optimal contracts are considered, on the other hand, neither of these contract forms are optimal.

When there are ex-post costs, it is still the case that policies with an upper limit are not part of the solution. When there are fixed costs triggered whenever a claim is made, deductibles appear in the Pareto optimal policies even if there are no variable costs. When there are no such fixed costs, a deductible arises whenever the costs are variable.

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