

# INFERENCE BASED ON CONDITIONAL MOMENT INEQUALITIES

By

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# Inference Based on Conditional Moment Inequalities

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## Abstract

In this paper, we propose an instrumental variable approach to constructing confidence sets (CS's) for the true parameter in models defined by conditional moment inequalities/equalities. We show that by properly choosing instrument functions, one can transform conditional moment inequalities/equalities into unconditional ones without losing identification power. Based on the unconditional moment inequalities/equalities, we construct CS's by inverting Cramér-von Mises-type or Kolmogorov-Smirnov-type tests. Critical values are obtained using generalized moment selection (GMS) procedures.

We show that the proposed CS's have correct uniform asymptotic coverage probabilities. New methods are required to establish these results because an infinite-dimensional nuisance parameter affects the asymptotic distributions. We show that the tests considered are consistent against all fixed alternatives and have power against some  $n^{-1/2}$ -local alternatives, though not all such alternatives. Monte Carlo simulations for three different models show that the methods perform well in finite samples.

*Keywords:* Asymptotic size, asymptotic power, conditional moment inequalities, confidence set, Cramér-von Mises, generalized moment selection, Kolmogorov-Smirnov, moment inequalities.

*JEL Classification Numbers:* C12, C15.

# 1 Introduction

This paper considers inference for parameters whose true values are restricted by conditional moment inequalities and/or equalities. The parameters need not be identified. Much of the literature on partially-identified parameters concerns unconditional moment inequalities, see the references given below. However, in many moment inequality models, the inequalities that arise are conditional moments given a vector of covariates  $X_i$ . In this case, the construction of a fixed number of unconditional moments requires an arbitrary selection of some functions of  $X_i$ . In addition, the selection of such functions leads to information loss that can be substantial. Specifically, the “identified set” based on a chosen set of unconditional moments can be noticeably larger than the identified set based on the conditional moments.<sup>1</sup> With moment inequalities there is a first-order loss in information when moving from conditional to unconditional moments—one loses identifying information. In contrast, with moment equality models, there is only a second-order loss when moving from conditional to unconditional moments—one increases the variance of an estimator and decreases the noncentrality parameter of a test.

This paper provides methods to construct CS’s for the true value of the parameter  $\theta$  by converting conditional moment inequalities into an infinite number of unconditional moment inequalities. This is done using weighting functions  $g(X_i)$ . We show how to construct a class  $\mathcal{G}$  of such functions such that there is no loss in information. We construct Cramér-von Mises-type (CvM) and Kolmogorov-Smirnov-type (KS) test statistics using a function  $S$  of the weighted sample moments, which depend on  $g \in \mathcal{G}$ . For example, the function  $S$  can be of the Sum, quasi-likelihood ratio (QLR), or Max form. The KS statistic is given by a supremum over  $g \in \mathcal{G}$ . The CvM statistic is given by an integral with respect to a probability measure  $Q$  on the space  $\mathcal{G}$  of  $g$  functions. Computation of the CvM test statistics can be carried out by truncation of an infinite sum or simulation of an integral. Asymptotic results are established for both exact and truncated/simulated versions of the test statistic.

For reasons explained below, the choice of critical values is important for all moment inequality tests, and especially so for conditional moment inequalities. Here we consider critical values based on generalized moment selection (GMS), as in Andrews and Soares

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<sup>1</sup>By “identified set,” we mean the set of parameter values that are consistent with the population moment inequalities/equalities, either unconditional or conditional, given the true distribution of the data.

(2010). For comparative purposes, we also provide results for subsampling critical values and “plug-in asymptotic” (PA) critical values. However, for reasons of accuracy of size and magnitude of power, we recommend GMS critical values over both subsampling and plug-in asymptotic critical values. The GMS critical values can be implemented using an asymptotic Gaussian distribution or the bootstrap.

The main contribution of this paper is to establish the properties of the CS’s described above. Our results apply to multiple moment inequalities and/or equalities and vector-valued parameters  $\theta$  with minimal regularity conditions on the conditional moment functions and the distribution of  $X_i$ . For example, no smoothness conditions or even continuity conditions are made on the conditional moment functions as functions of  $X_i$  and no conditions are imposed on the distribution of  $X_i$  (beyond the boundedness of  $2 + \delta$  moments of the moment functions). In consequence, the range of moment inequality models for which the methods are applicable is very broad.

The results of the paper are summarized as follows. The paper (i) develops critical values that take account of the issue of moment inequality slackness that arises in finite samples and uniform asymptotics, (ii) proves that the confidence sizes of the CS’s are correct asymptotically in a uniform sense, (iii) proves that the proposed CS’s yield no information loss (i.e., that the coverage probability for any point outside the identified set converges to zero as  $n \rightarrow \infty$ ), (iv) establishes asymptotic local power results for a certain class of  $n^{-1/2}$ -local alternatives, (v) extends the results to allow for the preliminary estimation of parameters that are identified given knowledge of the parameter of interest  $\theta$ , as occurs in some game theory examples, and (vi) extends the results to allow for time series observations. A companion paper, Andrews and Shi (2010), generalizes the CS’s and extends the asymptotic results to allow for an infinite number of conditional or unconditional moment inequalities, which makes the results applicable to tests of stochastic dominance and conditional stochastic dominance (see Lee and Whang (2009)).

The paper provides Monte Carlo simulation results for three models that exhibit different features. The first model is a quantile selection model. The parameter of interest is a nonparametric quantity, a conditional quantile. Selection effects yield the parameter to be unidentified. We introduce a quantile variant of Manski and Pepper’s (2000) monotone instrumental variables condition that provides conditional moment inequalities that bound the conditional quantile.<sup>2</sup> The second model is an interval

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<sup>2</sup>Papers in the literature that bound quantiles include Manski (1994), Lee and Melenberg (1998), and Blundell, Gosling, Ichimura, and Meghir (2007), among others. The condition employed here differs

outcome linear regression model with a bivariate parameter of interest. This model is considered in Manski and Tamer (2002).

The third model is a binary entry game model with multiple equilibria. It has ten parameters and a bivariate parameter of interest, which is the competitive effects vector. This model is related to models considered in Andrews, Berry, and Jia (2004), Beresteanu, Molchanov, and Molinari (2009), Galichon and Henry (2009b), and Ciliberto and Tamer (2009). In this model, the eight non-competitive effects parameters are estimated via a preliminary maximum likelihood estimator based on the number of entrants, similar to Berry (1992). These estimators are plugged into a set of moment conditions that includes two moment inequalities and two moment equalities. In this model the competitive effects are point identified and the moment inequalities are used to bring more information to bear. As far as we are aware, no other methods in the literature can handle a model of this sort.

The simulation results compare different forms of the test statistic: CvM versus KS, Sum versus QLR versus Max  $S$  function; and different forms of the critical value: GMS based on the asymptotic distribution (GMS/Asy), GMS based on the bootstrap (GMS/Bt), PA/Asy, PA/Bt, and subsampling (Sub). Coverage probabilities (CP's) for points in the identified set are computed and false coverage probabilities (FCP's) for points outside the identified set are computed.<sup>3</sup> In each model, we consider a basecase and variations on the basecase with different sample sizes, true data generating processes, and different GMS tuning parameters.

The CP's of all of the CS's in the first two models are quite good in the sense of being greater than or equal to .944 when the nominal level is .95 in all scenarios considered (except two with a Sub critical value). The CP's of CvM-based CS's with GMS critical values are quite close to .95 in one of two scenarios in the first model and in the second model and are around .98 in the other scenario in the first model. The latter over-coverage is not too surprising because non-similarity on the boundary in finite samples and asymptotically (in a uniform sense) is an inherent feature of good CS's in these contexts, as will be shown in future work. In the third model, CP's of the CS's vary across different true DGP's with CP's being greater than or equal to .95 in most cases except when the competitive effects parameters are large, in which case under-coverage of some CS's is as large as .037.

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from the conditions in these papers, although it is closely related to them, see Section 9.

<sup>3</sup>The FCP's are "coverage probability corrected," see Section 9 for details.

Across all three models, the simulation results show that the CvM-based CS's outperform the KS-based CS's in terms of FCP's. The Sum, QLR, and Max versions of the test statistics perform equally well in the first two models, while the Max version performs best in the third model, in terms of FCP's. In all three models, the GMS critical values outperform the PA and Sub critical values in terms of FCP's. The Asy and Bt versions of the GMS critical values perform similarly in the first two models. (The Bt critical values are not computed in the third model because they are expensive computationally.)

Variations on the basecase show a relatively low degree of sensitivity of the CP's and FCP's in most cases.

In sum, in the three models considered, the CvM/Max statistic coupled with the GMS/Asy critical value performed quite well in an absolute sense and best among the CS's considered. In the first two models, use of the Sum or QLR  $S$  function or the GMS/Bt critical value produced equally good results.

The literature related to this paper includes numerous papers dealing with unconditional moment inequality models, such as Andrews, Berry, and Jia (2004), Imbens and Manski (2004), Moon and Schorfheide (2006, 2009), Otsu (2006), Pakes, Porter, Ho, and Ishii (2006), Woutersen (2006), Bontemps, Magnac, and Maurin (2007), Canay (2010), Chernozhukov, Hong, and Tamer (2007), Andrews and Jia (2008), Beresteanu, Molchanov, and Molinari (2008), Beresteanu and Molinari (2008), Chiburis (2008), Guggenberger, Hahn, and Kim (2008), Romano and Shaikh (2008, 2010), Rosen (2008), Andrews and Guggenberger (2009), Andrews and Han (2009), Stoye (2009), Andrews and Soares (2010), Bugni (2010), and Canay (2010).

The literature on conditional moment inequalities is smaller and more recent. The present paper and the following papers have been written over more or less the same time period: Chernozhukov, Lee, and Rosen (2008), Fan (2008), Kim (2008), and Menzel (2008). An earlier paper than these by Galichon and Henry (2009a) considers a related testing problem with an infinite number unconditional moment inequalities of a particular type. The test statistic considered by Kim (2008) is the closest to that considered here. He considers subsampling critical values. The approach of Chernozhukov, Lee, and Rosen (2008) is different from that considered here. They consider tests based on nonparametric estimators such as kernels and sieves. Their results apply to scalar conditional lower and upper bounds on a parameter. Menzel's (2008) approach is different again. He investigates tests based on a finite number of moment inequalities in which

the number of inequalities increases with the sample size. None of the other papers in the literature that treat conditional moment inequalities provide contributions (ii)-(vi) listed above.

Papers that convert conditional moments into an infinite number of unconditional moments in point identified models include Bierens (1982), Bierens and Ploberger (1997), Chen and Fan (1999), Dominguez and Lobato (2004), and Khan and Tamer (2009), among others.

In addition to reporting a CS, it often is useful to report an estimated set. A CS accompanied by an estimated set reveals how much of the volume of the CS is due to randomness and how much is due to a large identified set. It is well-known that typical set estimators suffer from an inward-bias problem, e.g., see Haile and Tamer (2003) and Chernozhukov, Lee, and Rosen (2008). The reason is that an estimated boundary often behaves like the minimum or maximum of multiple random variables.

A simple solution to the inward-bias problem is to exploit the method of constructing median-unbiased estimators from confidence bounds with confidence level  $1/2$ , e.g., see Lehmann (1959, Sec. 3.5). The CS's in this paper applied with confidence level  $1/2$  are half-median-unbiased estimated sets. That is, the probability of including a point or any sequence of points in the identified set is greater than or equal to  $1/2$  with probability that converges to one. This property follows immediately from the uniform coverage probability results for the CS's. The level  $1/2$  CS, however, is not necessarily median-unbiased in two directions.<sup>4</sup> Nevertheless, this set is guaranteed not to be inward-median biased. Chernozhukov, Lee, and Rosen (2008) also provide bias reduction methods for set estimators.

The results of the paper are stated for the case where the parameter of interest,  $\theta$ , is finite-dimensional. However, all of the results except the local power results also hold for infinite-dimensional parameters  $\theta$ . Computation of a CS is noticeably more difficult in the infinite-dimensional case.

The CS's constructed in the paper provide model specification tests of the conditional moment inequality model. One rejects the model if a nominal  $1 - \alpha$  CS is empty. The results of the paper for CS's imply that this test has asymptotic size less than or equal to  $\alpha$  (with the inequality possibly being strict), e.g., see Andrews and Guggenberger

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<sup>4</sup>That is, the probability of including points outside the identified set is not necessarily less than or equal to  $1/2$  with probability that goes to one. This is because lower and upper confidence bounds on the boundary of an identified set do not necessarily coincide.



(2009) for details of the argument.

As noted above, the determination of appropriate critical values plays a major role in all moment inequality tests. This is because the null distribution of a test statistic depends greatly on the slackness, or lack thereof, of the different moment inequalities. The slackness represents a nuisance parameter that appears under the null hypothesis. With conditional moment inequalities, slackness comes in the form of a function, which is an infinite-dimensional parameter. In consequence, the issues that arise due to slackness are exacerbated in conditional moment inequality models compared to unconditional moment inequality models.

The effect of slackness in the moment inequalities causes a discontinuity in the pointwise asymptotic distribution of typical test statistics. This occurs because if a moment inequality is binding, then it affects the pointwise asymptotic distribution of the test statistic, but if the moment inequality is not binding, then the asymptotic distribution of the test statistic is the same as if this moment inequality did not enter the test statistic at all. However, in finite samples there is no discontinuity in the distribution of the test statistic. If a moment inequality is slack by a small amount, the finite sample distribution of the test statistic differs little from when it is binding. In finite samples what matters is how close or distant moment inequalities are to binding, not whether they are binding or not binding. The latter is a potentially misleading distinction obtained by focusing on pointwise asymptotics and is divorced from the finite-sample properties of the test statistic.

In the case of conditional moment inequalities, for some value(s) of  $x$ , an inequality that is binding for  $X_i = x$  is not binding for some value of  $x'$  that is arbitrarily close to  $x$ , provided the inequality is not binding for all  $x$  and is a smooth function of  $x$ . In consequence, one obtains an extreme form of discontinuity of the pointwise asymptotic distribution in which two moment inequalities are arbitrarily close to one another but pointwise asymptotics say that one inequality is irrelevant but the other is not.

The upshot of the discussion above is that pointwise asymptotics do not provide good approximations to the finite-sample properties of test statistics in moment inequality models, especially conditional models. The problem is that pointwise asymptotics do not provide uniform approximations. Depending on the sample size, different values of the “slackness function” cause problems—no matter how large is the sample size. In consequence, pointwise asymptotics fail to detect the potential problems. For issues concerning uniformity of asymptotics in other econometric models, see, e.g., Kabaila

(1995), Leeb and Pötscher (2005), Mikusheva (2007), and Andrews and Guggenberger (2010) (AG).

To ensure that a test (or CS) has good finite-sample size properties one needs to establish asymptotic results that hold uniformly over potential true distributions. AG and Andrews, Cheng, and Guggenberger (2009) (ACG) show that in certain problems one can establish uniform asymptotic results by determining the asymptotic behavior of a statistic and its critical value under particular drifting sequences of true distributions. These results apply to unconditional moment inequality models, see Andrews and Guggenberger (2009) and Andrews and Soares (2010). However, they do not apply to conditional moment inequality models. The reason is that the AG and ACG results require that the asymptotic distribution of the statistic only depends on a finite-dimensional parameter. In the unconditional moment inequality case, this is the vector of moment inequality slackness values.<sup>5</sup> However, with conditional moment inequalities, the nuisance parameter is a vector of slackness functions, which is infinite-dimensional.

The main technical contribution of this paper is to introduce a new method of proving uniformity results that applies to cases in which an infinite-dimensional nuisance parameter appears in the problem. The method is to establish an approximation to the sample size  $n$  distribution of the test statistic by a function of a Gaussian distribution where the function depends on the true slackness functions for the given sample size  $n$  and the approximation is uniform over all possible true slackness functions.<sup>6</sup> Then, one shows that the data-dependent critical value (the GMS critical value in the present case) is less than or equal to the  $1 - \alpha$  quantile of the given function of the Gaussian process with probability that goes to one uniformly over all potential true distributions. This establishes that the CS has correct size, greater than or equal to  $1 - \alpha$ , asymptotically and provides the justification for its use. Under a mild distributional continuity condition, one obtains that the asymptotic size equals  $1 - \alpha$ .

The remainder of the paper is organized as follows. Section 2 introduces the moment inequality/equality model. Section 3 specifies the class of test statistics that is considered. Section 4 defines GMS CS's. Section 5 establishes the uniform asymptotic

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<sup>5</sup>It also depends on the variance matrix of the moment functions, but the latter does not cause uniformity problems and is not an issue because it can be estimated consistently.

<sup>6</sup>Uniformity is obtained without any regularity conditions in terms of smoothness, uniform continuity, or even continuity of the conditional moment functions as functions of  $X_i$ . This is important because the slackness functions are normalized by an increasing function of  $n$  which typically would cause violation of uniform continuity or uniform bounds on the derivatives of smooth functions even if the underlying conditional moment inequality functions were smooth in  $X_i$ .

coverage properties of GMS and PA CS's. Section 6 establishes the consistency of GMS and PA tests against all fixed alternatives. This implies that GMS and PA CS's do not include any point outside the identified set with probability that goes to one. Section 7 shows that GMS and PA tests have power against some  $n^{-1/2}$ -local alternatives. Section 8 considers models in which preliminary consistent estimators of identified parameters are plugged into the moment inequalities/equalities. It also considers time series observations. Section 9 provides the Monte Carlo simulation results.

Appendix A provides proofs of the uniform asymptotic coverage probability results for GMS and PA CS's. Appendices B-E are given in the Supplement to this paper, Andrews and Shi (2009) (AS). Appendix B provides (i) results for KS tests and CS's, (ii) the extension of the results of the paper to truncated/simulated CvM tests and CS's, (iii) an illustration of the verification of the assumptions used for the local alternative results, (iv) an illustration of (serious) uniformity problems that arise with the Kolmogorov-Smirnov test unless the critical value is chosen carefully, (v) an illustration of problems with pointwise asymptotics, and (vi) asymptotic coverage probability results for subsampling CS's under drifting sequences of distributions. Appendix C gives proofs of the results stated in the paper, but not given in Appendix A. Appendix D provides proofs of the results stated in Appendix B. Appendix E provides a proof of some empirical process results that are used in Appendices A, C, and D. Appendix F provides some additional material concerning the Monte Carlo simulation results of Section 9.

## 2 Conditional Moment Inequalities/Equalities

### 2.1 Model

The conditional moment inequality/equality model is defined as follows. We suppose there exists a true parameter  $\theta_0 \in \Theta \subset R^{d_\theta}$  that satisfies the moment conditions:

$$\begin{aligned} E_{F_0}(m_j(W_i, \theta_0) | X_i) &\geq 0 \text{ a.s. } [F_{X,0}] \text{ for } j = 1, \dots, p \text{ and} \\ E_{F_0}(m_j(W_i, \theta_0) | X_i) &= 0 \text{ a.s. } [F_{X,0}] \text{ for } j = p + 1, \dots, p + v, \end{aligned} \quad (2.1)$$

where  $m_j(\cdot, \theta)$ ,  $j = 1, \dots, p + v$  are (known) real-valued moment functions,  $\{W_i = (Y_i', X_i')' : i \leq n\}$  are observed i.i.d. random vectors with distribution  $F_0$ ,  $F_{X,0}$  is the marginal distribution of  $X_i$ ,  $X_i \in R^{d_x}$ ,  $Y_i \in R^{d_y}$ , and  $W_i \in R^{d_w} (= R^{d_y + d_x})$ .

We are interested in constructing CS's for the true parameter  $\theta_0$ . However, we do not assume that  $\theta_0$  is point identified. Knowledge of  $E_{F_0}(m_j(W_i, \theta) | X_i)$  for all  $\theta \in \Theta$  does not necessarily identify  $\theta_0$ . Even knowledge of  $F_0$  does not necessarily point identify  $\theta_0$ .<sup>7</sup> The model, however, restricts the true parameter value to a set called the *identified set* (which could be a singleton). The identified set is

$$\Theta_{F_0} = \{\theta \in \Theta : (2.1) \text{ holds with } \theta \text{ in place of } \theta_0\}. \quad (2.2)$$

Let  $(\theta, F)$  denote generic values of the parameter and distribution. Let  $\mathcal{F}$  denote the parameter space for  $(\theta_0, F_0)$ . By definition,  $\mathcal{F}$  is a collection of  $(\theta, F)$  such that

- (i)  $\theta \in \Theta$ ,
- (ii)  $\{W_i : i \geq 1\}$  are i.i.d. under  $F$ ,
- (iii)  $E_F(m_j(W_i, \theta) | X_i) \geq 0$  a.s.  $[F_X]$  for  $j = 1, \dots, p$ ,
- (iv)  $E_F(m_j(W_i, \theta) | X_i) = 0$  a.s.  $[F_X]$  for  $j = p + 1, \dots, k$ ,
- (v)  $0 < Var_F(m_j(W_i, \theta)) < \infty$  for  $j = 1, \dots, k$ , and
- (vi)  $E_F |m_j(W_i, \theta) / \sigma_{F,j}(\theta)|^{2+\delta} \leq B$  for  $j = 1, \dots, k$ ,

for some  $B < \infty$  and  $\delta > 0$ , where  $k = p + v$ ,  $F_X$  is the marginal distribution of  $X_i$  under  $F$ , and  $\sigma_{F,j}^2(\theta) = Var_F(m_j(W_i, \theta))$ .<sup>8</sup> The  $k$ -vector of moment functions is denoted

$$m(W_i, \theta) = (m_1(W_i, \theta), \dots, m_k(W_i, \theta))'. \quad (2.4)$$

## 2.2 Confidence Sets

We are interested in CS's that cover the true value  $\theta_0$  with probability greater than or equal to  $1 - \alpha$  for  $\alpha \in (0, 1)$ . As is standard, we construct such CS's by inverting tests of the null hypothesis that  $\theta$  is the true value for each  $\theta \in \Theta$ . Let  $T_n(\theta)$  be a test statistic and  $c_{n,1-\alpha}(\theta)$  be a corresponding critical value for a test with nominal significance level

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<sup>7</sup>It makes sense to speak of a "true" parameter  $\theta_0$  in the present context because (i) there may exist restrictions not included in the moment inequalities/equalities in (2.1) that point identify  $\theta_0$ , but for some reason are not available or are not utilized, and/or (ii) there may exist additional variables not included in  $W_i$  which, if observed, would lead to point identification of  $\theta_0$ . Given such restrictions and/or variables, the true parameter  $\theta_0$  is uniquely defined even if it is not point identified by (2.1).

<sup>8</sup>Additional restrictions can be placed on  $\mathcal{F}$  and the results of the paper still hold. For example, one could specify that the support of  $X_i$  is the same for all  $F$  for which  $(\theta, F) \in \mathcal{F}$ .

$\alpha$ . Then, a nominal level  $1 - \alpha$  CS for the true value  $\theta_0$  is

$$CS_n = \{\theta \in \Theta : T_n(\theta) \leq c_{n,1-\alpha}(\theta)\}. \quad (2.5)$$

### 3 Test Statistics

#### 3.1 General Form of the Test Statistic

Here we define the test statistic  $T_n(\theta)$  that is used to construct a CS. We transform the conditional moment inequalities/equalities into equivalent unconditional moment inequalities/equalities by choosing appropriate weighting functions, i.e., instruments. Then, we construct a test statistic based on the unconditional moment conditions.

The unconditional moment conditions are of the form:

$$\begin{aligned} E_{F_0} m_j(W_i, \theta_0) g_j(X_i) &\geq 0 \text{ for } j = 1, \dots, p \text{ and} \\ E_{F_0} m_j(W_i, \theta_0) g_j(X_i) &= 0 \text{ for } j = p + 1, \dots, k, \text{ for } g = (g_1, \dots, g_k)' \in \mathcal{G}, \end{aligned} \quad (3.1)$$

where  $g = (g_1, \dots, g_k)'$  are instruments that depend on the conditioning variables  $X_i$  and  $\mathcal{G}$  is a collection of instruments. Typically  $\mathcal{G}$  contains an infinite number of elements.

The identified set  $\Theta_{F_0}(\mathcal{G})$  of the model defined by (3.1) is

$$\Theta_{F_0}(\mathcal{G}) = \{\theta \in \Theta : (3.1) \text{ holds with } \theta \text{ in place of } \theta_0\}. \quad (3.2)$$

The collection  $\mathcal{G}$  is chosen so that  $\Theta_{F_0}(\mathcal{G}) = \Theta_{F_0}$ , defined in (2.2). Section 3.3 provides conditions for this equality and gives examples of instrument sets  $\mathcal{G}$  that satisfy the conditions.

We construct test statistics based on (3.1). The sample moment functions are

$$\begin{aligned} \bar{m}_n(\theta, g) &= n^{-1} \sum_{i=1}^n m(W_i, \theta, g) \text{ for } g \in \mathcal{G}, \text{ where} \\ m(W_i, \theta, g) &= \begin{pmatrix} m_1(W_i, \theta) g_1(X_i) \\ m_2(W_i, \theta) g_2(X_i) \\ \vdots \\ m_k(W_i, \theta) g_k(X_i) \end{pmatrix} \text{ for } g \in \mathcal{G}. \end{aligned} \quad (3.3)$$

The sample variance-covariance matrix of  $n^{1/2}\bar{m}_n(\theta, g)$  is

$$\widehat{\Sigma}_n(\theta, g) = n^{-1} \sum_{i=1}^n (m(W_i, \theta, g) - \bar{m}_n(\theta, g)) (m(W_i, \theta, g) - \bar{m}_n(\theta, g))'. \quad (3.4)$$

The matrix  $\widehat{\Sigma}_n(\theta, g)$  may be singular or near singular with non-negligible probability for some  $g \in \mathcal{G}$ . This is undesirable because the inverse of  $\widehat{\Sigma}_n(\theta, g)$  needs to be consistent for its population counterpart uniformly over  $g \in \mathcal{G}$  for the test statistics considered below. In consequence, we employ a modification of  $\widehat{\Sigma}_n(\theta, g)$ , denoted  $\bar{\Sigma}_n(\theta, g)$ , such that  $\det(\bar{\Sigma}_n(\theta, g))$  is bounded away from zero. Different choices of  $\bar{\Sigma}_n(\theta, g)$  are possible. Here we use

$$\bar{\Sigma}_n(\theta, g) = \widehat{\Sigma}_n(\theta, g) + \varepsilon \cdot \text{Diag}(\widehat{\Sigma}_n(\theta, 1_k)) \text{ for } g \in \mathcal{G} \quad (3.5)$$

for some fixed  $\varepsilon > 0$ . Specifically, in the simulations in Section 9, we use  $\varepsilon = 5/100$ . By design,  $\bar{\Sigma}_n(\theta, g)$  is a linear combination of two scale equivariant functions and thus is scale equivariant. (That is, multiplying the moment functions  $m(W_i, \theta)$  by a diagonal matrix,  $D$ , changes  $\bar{\Sigma}_n(\theta, g)$  into  $D\bar{\Sigma}_n(\theta, g)D$ .) This yields a test statistic that is invariant to rescaling of the moment functions  $m(W_i, \theta)$ , which is an important property.

The test statistic  $T_n(\theta)$  is either a Cramér-von Mises-type (CvM) or Kolmogorov-Smirnov-type (KS) statistic. The CvM statistic is

$$T_n(\theta) = \int S(n^{1/2}\bar{m}_n(\theta, g), \bar{\Sigma}_n(\theta, g)) dQ(g), \quad (3.6)$$

where  $S$  is a non-negative function,  $Q$  is a weight function (i.e., probability measure) on  $\mathcal{G}$ , and the integral is over  $\mathcal{G}$ . The functions  $S$  and  $Q$  are discussed in Sections 3.2 and 3.4 below, respectively.

The Kolmogorov-Smirnov-type (KS) statistic is

$$T_n(\theta) = \sup_{g \in \mathcal{G}} S(n^{1/2}\bar{m}_n(\theta, g), \bar{\Sigma}_n(\theta, g)). \quad (3.7)$$

For brevity, in the text of the paper, the discussion focusses on CvM statistics and all results stated concern CvM statistics. Appendix B of AS gives detailed results for KS statistics.

## 3.2 Function S

To permit comparisons, we establish results in this paper for a broad family of functions  $S$  that satisfy certain conditions stated below. We now introduce three functions that satisfy these conditions. The first is the modified method of moments (MMM) or Sum function:

$$S_1(m, \Sigma) = \sum_{j=1}^p [m_j/\sigma_j]_-^2 + \sum_{j=p+1}^{p+v} [m_j/\sigma_j]^2, \quad (3.8)$$

where  $m_j$  is the  $j$ th element of the vector  $m$ ,  $\sigma_j^2$  is the  $j$ th diagonal element of the matrix  $\Sigma$ , and  $[x]_- = -x$  if  $x < 0$  and  $[x]_- = 0$  if  $x \geq 0$ .

The second function  $S$  is the quasi-likelihood ratio (QLR) function:

$$S_2(m, \Sigma) = \inf_{t=(t'_1, 0'_v)': t_1 \in R_{+, \infty}^p} (m - t)' \Sigma^{-1} (m - t). \quad (3.9)$$

The third function  $S$  is a “maximum” (Max) function. Used in conjunction with the KS form of the test statistic, this  $S$  function yields a pure KS-type test statistic:

$$S_3(m, \Sigma) = \max\{[m_1/\sigma_1]_-^2, \dots, [m_p/\sigma_p]_-^2, (m_{p+1}/\sigma_{p+1})^2, \dots, (m_{p+v}/\sigma_{p+v})^2\}. \quad (3.10)$$

The function  $S_2$  is more costly to compute than  $S_1$  and  $S_3$ .

Let  $m_I = (m_1, \dots, m_p)'$  and  $m_{II} = (m_{p+1}, \dots, m_k)'$ . Let  $\Delta$  be the set of  $k \times k$  positive-definite diagonal matrices. Let  $\mathcal{W}$  be the set of  $k \times k$  positive-definite matrices. Let (i)  $R_{[+\infty]}$ , (ii)  $R_+$ , and (iii)  $R_{+, \infty}$  denote the sets of scalars that are (i) real or  $+\infty$ , (ii) non-negative (and finite), and (iii) non-negative or  $+\infty$ , respectively. Let  $\mathcal{S} = \{(m, \Sigma) : m \in R_{[+\infty]}^p \times R^v, \Sigma \in \mathcal{W}\}$ , where  $R_{[+\infty]}^p = R_{[+\infty]} \times \dots \times R_{[+\infty]}$  with  $p$  copies.

We consider functions  $S$  that satisfy the following conditions.

**Assumption S1.**  $\forall (m, \Sigma) \in \mathcal{S}$ ,

- (a)  $S(Dm, D\Sigma D) = S(m, \Sigma) \forall D \in \Delta$ ,
- (b)  $S(m_I, m_{II}, \Sigma)$  is non-increasing in each element of  $m_I$ ,
- (c)  $S(m, \Sigma) \geq 0$ ,
- (d)  $S$  is continuous, and
- (e)  $S(m, \Sigma + \Sigma_1) \leq S(m, \Sigma)$  for all  $k \times k$  positive semi-definite matrices  $\Sigma_1$ .

It is worth pointing out that Assumption S1(d) requires  $S$  to be continuous in  $m$  at all points  $m$  in the extended vector space  $R_{[+\infty]}^p \times R^v$ , not only for points in  $R^{p+v}$ .

**Assumption S2.**  $S(m, \Sigma)$  is uniformly continuous in the sense that, for all  $m_0 \in R^k$  and all pd  $\Sigma_0$ ,  $\sup_{\mu \in R_+^p \times \{0\}^v} |S(m + \mu, \Sigma) - S(m_0 + \mu, \Sigma_0)| \rightarrow 0$  as  $(m, \Sigma) \rightarrow (m_0, \Sigma_0)$ .<sup>9</sup>

The following two assumptions are used only to establish the power properties of tests.

**Assumption S3.**  $S(m, \Sigma) > 0$  if and only if  $m_j < 0$  for some  $j = 1, \dots, p$  or  $m_j \neq 0$  for some  $j = p + 1, \dots, k$ , where  $m = (m_1, \dots, m_k)'$  and  $\Sigma \in \mathcal{W}$ .

**Assumption S4.** For some  $\chi > 0$ ,  $S(am, \Sigma) = a^\chi S(m, \Sigma)$  for all scalars  $a > 0$ ,  $m \in R^k$ , and  $\Sigma \in \mathcal{W}$ .

Assumptions S1-S4 are not restrictive as is shown by the following result.

**Lemma 1.** *The functions  $S_1$ ,  $S_2$ , and  $S_3$  satisfy Assumptions S1-S4.*

### 3.3 Instruments

When considering consistent specification tests based on conditional moment *equalities*, see Bierens (1982) and Bierens and Ploberger (1997), a wide variety of different types of functions  $g$  can be employed without loss of information, see Stinchcombe and White (1998). With conditional moment *inequalities*, however, it is much more difficult to distill the information in the moments because of the one-sided feature of the inequalities.

The collection of instruments  $\mathcal{G}$  needs to satisfy the following condition in order for the unconditional moments  $\{E_F m(W_i, \theta, g) : g \in \mathcal{G}\}$  to incorporate the same information as the conditional moments  $\{E_F(m(W_i, \theta) | X_i = x) : x \in R^{d_x}\}$ .

For any  $\theta \in \Theta$  and any distribution  $F$  with  $E_F \|m(W_i, \theta)\| < \infty$ , let

$$\begin{aligned} \mathcal{X}_F(\theta) = \{x \in R^{d_x} : E_F(m_j(W_i, \theta) | X_i = x) < 0 \text{ for some } j \leq p \text{ or} \\ E_F(m_j(W_i, \theta) | X_i = x) \neq 0 \text{ for some } j = p + 1, \dots, k\}. \end{aligned} \quad (3.11)$$

**Assumption CI.** For any  $\theta \in \Theta$  and distribution  $F$  for which  $E_F \|m(W_i, \theta)\| < \infty$  and

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<sup>9</sup>It is important that the supremum is only over  $\mu$  vectors with non-negative elements  $\mu_j$  for  $j \leq p$ . Without this restriction on the  $\mu$  vectors, Assumption S2 would not hold for typical  $S$  functions of interest.



$P_F(X_i \in \mathcal{X}_F(\theta)) > 0$ , there exists some  $g \in \mathcal{G}$  such that

$$\begin{aligned} E_F m_j(W_i, \theta) g_j(X_i) &< 0 \text{ for some } j \leq p \text{ or} \\ E_F m_j(W_i, \theta) g_j(X_i) &\neq 0 \text{ for some } j = p + 1, \dots, k. \end{aligned}$$

Note that CI abbreviates ‘‘conditionally identified.’’ The following simple Lemma indicates the importance of Assumption CI.

**Lemma 2.** *Assumption CI implies that  $\Theta_F(\mathcal{G}) = \Theta_F$  for all  $F$  with  $\sup_{\theta \in \Theta} E_F \|m(W_i, \theta)\| < \infty$ .*

Collections  $\mathcal{G}$  that satisfy Assumption CI contain non-negative functions whose supports are cubes, boxes, or bounded sets with other shapes whose supports are arbitrarily small, see below.

Next, we state a ‘‘manageability’’ condition that regulates the complexity of  $\mathcal{G}$ . It ensures that  $\{n^{1/2}(\bar{m}_n(\theta, g) - E_{F_n} \bar{m}_n(\theta, g)) : g \in \mathcal{G}\}$  satisfies a functional central limit theorem (FCLT) under drifting sequences of distributions  $\{F_n : n \geq 1\}$ . The latter is utilized in the proof of the uniform coverage probability results for the CS’s. The manageability condition is from Pollard (1990) and is defined and explained in Appendix E of AS.

**Assumption M.** (a)  $0 \leq g_j(x) \leq G(x) \forall x \in R^{d_x}, \forall j \leq k, \forall g \in \mathcal{G}$ , for some envelope function  $G(x)$ ,

(b)  $E_F G^{\delta_1}(X_i) \leq C$  for all  $F$  such that  $(\theta, F) \in \mathcal{F}$  for some  $\theta \in \Theta$ , for some  $C < \infty$ , and for some  $\delta_1 > 4/\delta + 2$ , where  $W_i = (Y_i', X_i) \sim F$  and  $\delta$  is as in the definition of  $\mathcal{F}$  in (2.3), and

(c) the processes  $\{g_j(X_{n,i}) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  are manageable with respect to the envelope function  $G(X_{n,i})$  for  $j = 1, \dots, k$ , where  $\{X_{n,i} : i \leq n, n \geq 1\}$  is a row-wise i.i.d. triangular array with  $X_{n,i} \sim F_{X,n}$  and  $F_{X,n}$  is the distribution of  $X_{n,i}$  under  $F_n$  for some  $(\theta_n, F_n) \in \mathcal{F}$  for  $n \geq 1$ .<sup>10</sup>

Now we give two examples of collections of functions  $\mathcal{G}$  that satisfy Assumptions CI and M. Appendix B of AS gives three additional examples, one of which is based on

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<sup>10</sup>The asymptotic results given below hold with Assumption M replaced by any alternative assumption that is sufficient to obtain the requisite empirical process results, see Assumption EP in Section 8.

B-splines.

**Example 1. (Countable Hypercubes).** Suppose  $X_i$  is transformed via a one-to-one mapping so that each of its elements lies in  $[0, 1]$ . There is no loss in information in doing so. Section 9 and Appendix B of AS provide examples of how this can be done.

Consider the class of indicator functions of cubes with side lengths  $(2r)^{-1}$  for all large positive integers  $r$  that partition  $[0, 1]^{d_x}$  for each  $r$ . This class is countable:

$$\begin{aligned} \mathcal{G}_{c-cube} &= \{g(x) : g(x) = 1(x \in C) \cdot 1_k \text{ for } C \in \mathcal{C}_{c-cube}\}, \text{ where} \\ \mathcal{C}_{c-cube} &= \left\{ C_{a,r} = \prod_{u=1}^{d_x} ((a_u - 1)/(2r), a_u/(2r)] \in [0, 1]^{d_x} : a = (a_1, \dots, a_{d_x})' \right. \\ &\quad \left. a_u \in \{1, 2, \dots, 2r\} \text{ for } u = 1, \dots, d_x \text{ and } r = r_0, r_0 + 1, \dots \right\} \end{aligned} \quad (3.12)$$

for some positive integer  $r_0$ .<sup>11</sup> The terminology “*c-cube*” abbreviates countable cubes. Note that  $C_{a,r}$  is a hypercube in  $[0, 1]^{d_x}$  with smallest vertex indexed by  $a$  and side lengths equal to  $(2r)^{-1}$ .

The class of countable cubes  $\mathcal{G}_{c-cube}$  leads to a test statistic  $T_n(\theta)$  for which the integral over  $\mathcal{G}$  reduces to a sum.

**Example 2 (Boxes).** Let

$$\begin{aligned} \mathcal{G}_{box} &= \{g : g(x) = 1(x \in C) \cdot 1_k \text{ for } C \in \mathcal{C}_{box}\}, \text{ where} \\ \mathcal{C}_{box} &= \left\{ C_{x,r} = \prod_{u=1}^{d_x} (x_u - r_u, x_u + r_u] \in R^{d_x} : x_u \in R, r_u \in (0, \bar{r}) \forall u \leq d_x \right\}, \end{aligned} \quad (3.13)$$

$x = (x_1, \dots, x_{d_x})'$ ,  $r = (r_1, \dots, r_{d_x})'$ ,  $\bar{r} \in (0, \infty]$ , and  $1_k$  is a  $k$ -vector of ones. The set  $\mathcal{C}_{box}$  contains boxes (i.e., hyper-rectangles or orthotopes) in  $R^{d_x}$  with centers at  $x \in R^{d_x}$  and side lengths less than  $2\bar{r}$ .

When the support of  $X_i$ , denoted  $Supp(X_i)$ , is a known subset of  $R^{d_x}$ , one can replace  $x_u \in R \forall u \leq d_x$  in (3.13) by  $x \in CH(Supp(X_i))$ , where  $CH(A)$  denotes the convex hull of  $A$ . Sometimes, it is convenient to transform the elements of  $X_i$  into  $[0, 1]$  via strictly increasing transformations as in Example 1 above. If the  $X_i$ 's are transformed in this way, then  $R$  in (3.13) is replaced by  $[0, 1]$ .

Both of the sets  $\mathcal{G}$  discussed above can be used with continuous and/or discrete

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<sup>11</sup>When  $a_u = 1$ , the left endpoint of the interval  $(0, 1/(2r)]$  is included in the interval.

regressors.

The following result establishes Assumptions CI and M for  $\mathcal{G}_{c-cube}$  and  $\mathcal{G}_{box}$ .

**Lemma 3.** *For any moment function  $m(W_i, \theta)$ , Assumptions CI and M hold with  $\mathcal{G} = \mathcal{G}_{c-cube}$  and  $\mathcal{G}_{box}$ .*

The proof of Lemma 3 is given in Appendix C of AS.

**Moment Equalities.** The sets  $\mathcal{G}$  introduced above use the same functions for the moment inequalities and equalities, i.e.,  $g$  is of the form  $g^* \cdot 1_k$ , where  $g^*$  is a real-valued function. It is possible to use different functions for the moment equalities than for the inequalities. One can take  $g = (g^{(1)'}, g^{(2)'})' \in \mathcal{G}^{(1)} \times \mathcal{G}^{(2)}$ , where  $g^{(1)}$  is an  $R^p$ -valued function in some set  $\mathcal{G}^{(1)}$  and  $g^{(2)}$  is an  $R^v$ -valued function in some set  $\mathcal{G}^{(2)}$ . Any “generically comprehensively revealing” class of functions  $\mathcal{G}^{(2)}$ , see Stinchcombe and White (1998), leads to a set  $\mathcal{G}$  that satisfies Assumption CI provided one uses a suitable class of functions  $\mathcal{G}^{(1)}$  (such as any of those defined above with  $1_k$  replaced by  $1_p$ ). For brevity, we do not provide further details.

### 3.4 Weight Function Q

The weight function  $Q$  can be any probability measure on  $\mathcal{G}$  whose support is  $\mathcal{G}$ . This support condition is needed to ensure that no functions  $g \in \mathcal{G}$ , which might have set-identifying power, are “ignored” by the test statistic  $T_n(\theta)$ . Without such a condition, a CS based on  $T_n(\theta)$  would not necessarily shrink to the identified set as  $n \rightarrow \infty$ . Section 6 below introduces the support condition formally and shows that the probability measures  $Q$  considered here satisfy it.

We now specify two examples of weight functions  $Q$ . Three others are specified in Appendix B of AS.

**Weight Function Q for  $\mathcal{G}_{c-cube}$ .** There is a one-to-one mapping  $\Pi_{c-cube} : \mathcal{G}_{c-cube} \rightarrow AR = \{(a, r) : a \in \{1, \dots, 2r\}^{d_x} \text{ and } r = r_0, r_0 + 1, \dots\}$ . Let  $Q_{AR}$  be a probability measure on  $AR$ . One can take  $Q = \Pi_{c-cube}^{-1} Q_{AR}$ . A natural choice of measure  $Q_{AR}$  is uniform on  $a \in \{1, \dots, 2r\}^{d_x}$  conditional on  $r$  combined with a distribution for  $r$  that has some probability mass function  $\{w(r) : r = r_0, r_0 + 1, \dots\}$ . This yields the test statistic to be

$$T_n(\theta) = \sum_{r=r_0}^{\infty} w(r) \sum_{a \in \{1, \dots, 2r\}^{d_x}} (2r)^{-d_x} S(n^{1/2} \bar{m}_n(\theta, g_{a,r}), \bar{\Sigma}_n(\theta, g_{a,r})), \quad (3.14)$$

where  $g_{a,r}(x) = 1(x \in C_{a,r}) \cdot 1_k$  for  $C_{a,r} \in \mathcal{C}_{c-cube}$ .

**Weight Function  $Q$  for  $\mathcal{G}_{box}$ .** There is a one-to-one mapping  $\Pi_{box} : \mathcal{G}_{box} \rightarrow \{(x, r) \in R^{d_x} \times (0, \bar{r})^{d_x}\}$ . Let  $Q^*$  be a probability measure on  $\{(x, r) \in R^{d_x} \times (0, \bar{r})^{d_x}\}$ . Then,  $\Pi_{box}^{-1}Q^*$  is a probability measure on  $\mathcal{G}_{box}$ . One can take  $Q = \Pi_{box}^{-1}Q^*$ . Any probability measure on  $R^{d_x} \times (0, \bar{r})^{d_x}$  whose support contains  $\mathcal{G}_{box}$  is a valid candidate for  $Q^*$ . If  $Supp(X_i)$  is known,  $R^{d_x}$  can be replaced by the convex hull of  $Supp(X_i)$ . One choice is to transform each regressor to lie in  $[0, 1]$  and to take  $Q^*$  to be the uniform distribution on  $[0, 1]^{d_x} \times (0, \bar{r})^{d_x}$ , i.e.,  $Unif([0, 1]^{d_x} \times (0, \bar{r})^{d_x})$ . In this case, the test statistic becomes

$$T_n(\theta) = \int_{[0,1]^{d_x}} \int_{(0,\bar{r})^{d_x}} S(n^{1/2}\bar{m}_n(\theta, g_{x,r}), \bar{\Sigma}_n(\theta, g_{x,r}))\bar{r}^{-d_x} dr dx, \quad (3.15)$$

where  $g_{x,r}(y) = 1(y \in C_{x,r}) \cdot 1_k$  and  $C_{x,r}$  denotes the box centered at  $x \in [0, 1]^{d_x}$  with side lengths  $2r \in (0, 2\bar{r})^{d_x}$ .

### 3.5 Computation of Sums, Integrals, and Suprema

The test statistics  $T_n(\theta)$  given in (3.14) and (3.15) involve an infinite sum and an integral with respect to  $Q$ . Analogous infinite sums and integrals appear in the definitions of the critical values given below. These infinite sums and integrals can be approximated by truncation, simulation, or quasi-Monte Carlo (QMC) methods. If  $\mathcal{G}$  is countable, let  $\{g_1, \dots, g_{s_n}\}$  denote the first  $s_n$  functions  $g$  that appear in the infinite sum that defines  $T_n(\theta)$ . Alternatively, let  $\{g_1, \dots, g_{s_n}\}$  be  $s_n$  i.i.d. functions drawn from  $\mathcal{G}$  according to the distribution  $Q$ . Or, let  $\{g_1, \dots, g_{s_n}\}$  be the first  $s_n$  terms in a QMC approximation of the integral wrt  $Q$ . Then, an approximate test statistic obtained by truncation, simulation, or QMC methods is

$$\bar{T}_{n,s_n}(\theta) = \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) S(n^{1/2}\bar{m}_n(\theta, g_\ell), \bar{\Sigma}_n(\theta, g_\ell)), \quad (3.16)$$

where  $w_{Q,n}(\ell) = Q(\{g_\ell\})$  when an infinite sum is truncated,  $w_{Q,n}(\ell) = s_n^{-1}$  when  $\{g_1, \dots, g_{s_n}\}$  are i.i.d. draws from  $\mathcal{G}$  according to  $Q$ , and  $w_{Q,n}(\ell)$  is a suitable weight when a QMC method is used. For example, in (3.14), the outer sum can be truncated at  $r_{1,n}$ , in which case,  $s_n = \sum_{r=r_0}^{r_{1,n}} (2r)^{d_x}$  and  $w_{Q,n}(\ell) = w(r)(2r)^{-d_x}$  for  $\ell$  such that  $g_\ell$  corresponds to  $g_{a,r}$  for some  $a$ . In (3.15), the integral over  $(x, r)$  can be replaced by an average over  $\ell = 1, \dots, s_n$ , the uniform density  $\bar{r}^{-d_x}$  deleted, and  $g_{x,r}$  replaced by  $g_{x_\ell, r_\ell}$ ,

where  $\{(x_\ell, r_\ell) : \ell = 1, \dots, s_n\}$  are i.i.d. with a  $Unif([0, 1]^{d_x} \times (0, \bar{r})^{d_x})$  distribution.

In Appendix B of AS, we show that truncation at  $s_n$ , simulation based on  $s_n$  simulation repetitions, or QMC approximation based on  $s_n$  terms, where  $s_n \rightarrow \infty$  as  $n \rightarrow \infty$ , is sufficient to maintain the asymptotic validity of the tests and CS's as well as the asymptotic power results under fixed alternatives and most of the results under  $n^{-1/2}$ -local alternatives.

The KS form of the test statistic requires the computation of a supremum over  $g \in \mathcal{G}$ . For computational ease, this can be replaced by a supremum over  $g \in \mathcal{G}_n$ , where  $\mathcal{G}_n \uparrow \mathcal{G}$  as  $n \rightarrow \infty$ , in the test statistic and in the definition of the critical value (defined below). The asymptotic results for KS tests given in Appendix B of AS show that the use of  $\mathcal{G}_n$  in place of  $\mathcal{G}$  does not affect the asymptotic properties of the test.

## 4 GMS Confidence Sets

### 4.1 GMS Critical Values

In this section, we define GMS critical values and CS's.

It is shown in Section 5 below that when  $\theta$  is in the identified set the “uniform asymptotic distribution” of  $T_n(\theta)$  is the distribution of  $T(h_n)$ , where  $h_n = (h_{1,n}, h_2)$ ,  $h_{1,n}(\cdot)$  is a function from  $\mathcal{G}$  to  $R_{+, \infty}^p \times \{0\}^v$  that depends on the slackness of the moment inequalities and on  $n$ , and  $h_2(\cdot, \cdot)$  is a  $k \times k$ -matrix-valued covariance kernel on  $\mathcal{G} \times \mathcal{G}$ . For  $h = (h_1, h_2)$ , define

$$T(h) = \int S(\nu_{h_2}(g) + h_1(g), h_2(g, g) + \varepsilon I_k) dQ(g), \quad (4.1)$$

where

$$\{\nu_{h_2}(g) : g \in \mathcal{G}\} \quad (4.2)$$

is a mean zero  $R^k$ -valued Gaussian process with covariance kernel  $h_2(\cdot, \cdot)$  on  $\mathcal{G} \times \mathcal{G}$ ,  $h_1(\cdot)$  is a function from  $\mathcal{G}$  to  $R_{+, \infty}^p \times \{0\}^v$ , and  $\varepsilon$  is as in the definition of  $\bar{\Sigma}_n(\theta, g)$  in (3.5).<sup>12</sup> The definition of  $T(h)$  in (4.1) applies to CvM test statistics. For the KS test statistic, one replaces  $\int \dots dQ(g)$  by  $\sup_{g \in \mathcal{G}} \dots$ .

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<sup>12</sup>The sample paths of  $\nu_{h_2}(\cdot)$  are concentrated on the set  $U_\rho^k(\mathcal{G})$  of bounded uniformly  $\rho$ -continuous  $R^k$ -valued functions on  $\mathcal{G}$ , where  $\rho$  is defined in Appendix A.

We are interested in tests of nominal level  $\alpha$  and CS's of nominal level  $1 - \alpha$ . Let

$$c_0(h, 1 - \alpha) (= c_0(h_1, h_2, 1 - \alpha)) \quad (4.3)$$

denote the  $1 - \alpha$  quantile of  $T(h)$ . If  $h_n = (h_{1,n}, h_2)$  was known, we would use  $c_0(h_n, 1 - \alpha)$  as the critical value for the test statistic  $T_n(\theta)$ . However,  $h_n$  is not known and  $h_{1,n}$  cannot be consistently estimated. In consequence, we replace  $h_2$  in  $c_0(h_{1,n}, h_2, 1 - \alpha)$  by a uniformly consistent estimator  $\widehat{h}_{2,n}(\theta)$  ( $= \widehat{h}_{2,n}(\theta, \cdot, \cdot)$ ) of the covariance kernel  $h_2$  and we replace  $h_{1,n}$  by a data-dependent GMS function  $\varphi_n(\theta)$  ( $= \varphi_n(\theta, \cdot)$ ) on  $\mathcal{G}$  that is constructed to be less than or equal to  $h_{1,n}(g)$  for all  $g \in \mathcal{G}$  with probability that goes to one as  $n \rightarrow \infty$ . Because  $S(m, \Sigma)$  is non-increasing in  $m_I$  by Assumption S1(b), where  $m = (m'_I, m'_{II})'$ , the latter property yields a test whose asymptotic level is less than or equal to the nominal level  $\alpha$ . (It is arbitrarily close to  $\alpha$  for certain  $(\theta, F) \in \mathcal{F}$ .) The quantities  $\widehat{h}_{2,n}(\theta)$  and  $\varphi_n(\theta)$  are defined below.

The nominal  $1 - \alpha$  GMS critical value is defined to be

$$c(\varphi_n(\theta), \widehat{h}_{2,n}(\theta), 1 - \alpha) = c_0(\varphi_n(\theta), \widehat{h}_{2,n}(\theta), 1 - \alpha + \eta) + \eta, \quad (4.4)$$

where  $\eta > 0$  is an arbitrarily small positive constant, e.g., .001. A nominal  $1 - \alpha$  GMS CS is given by (2.5) with the critical value  $c_{n,1-\alpha}(\theta)$  equal to  $c(\varphi_n(\theta), \widehat{h}_{2,n}(\theta), 1 - \alpha)$ .

The constant  $\eta$  is an *infinitesimal uniformity factor* (IUF) that is employed to circumvent problems that arise due to the presence of the infinite-dimensional nuisance parameter  $h_{1,n}$  that affects the distribution of the test statistic in both small and large samples. The IUF obviates the need for complicated and difficult-to-verify uniform continuity and strictly-increasing conditions on the large sample distribution functions of the test statistic.

The sample covariance kernel  $\widehat{h}_{2,n}(\theta)$  ( $= \widehat{h}_{2,n}(\theta, \cdot, \cdot)$ ) is defined by:

$$\begin{aligned} \widehat{h}_{2,n}(\theta, g, g^*) &= \widehat{D}_n^{-1/2}(\theta) \widehat{\Sigma}_n(\theta, g, g^*) \widehat{D}_n^{-1/2}(\theta), \text{ where} \\ \widehat{\Sigma}_n(\theta, g, g^*) &= n^{-1} \sum_{i=1}^n (m(W_i, \theta, g) - \overline{m}_n(\theta, g)) (m(W_i, \theta, g^*) - \overline{m}_n(\theta, g^*))' \text{ and} \\ \widehat{D}_n(\theta) &= \text{Diag}(\widehat{\Sigma}_n(\theta, 1_k, 1_k)). \end{aligned} \quad (4.5)$$

Note that  $\widehat{\Sigma}_n(\theta, g)$ , defined in (3.4), equals  $\widehat{\Sigma}_n(\theta, g, g)$  and  $\widehat{D}_n(\theta)$  is the sample variance-covariance matrix of  $n^{-1/2} \sum_{i=1}^n m(W_i, \theta)$ .

The quantity  $\varphi_n(\theta)$  is defined in Section 4.4 below.

## 4.2 GMS Critical Values for Approximate Test Statistics

When the test statistic is approximated via a truncated sum, simulated integral, or QMC quantity, as discussed in Section 3.4, the statistic  $T(h)$  in Section 4.1 is replaced by

$$\bar{T}_{s_n}(h) = \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) S(\nu_{h_2}(g_\ell) + h_1(g_\ell), h_2(g_\ell, g_\ell) + \varepsilon I_k), \quad (4.6)$$

where  $\{g_\ell : \ell = 1, \dots, s_n\}$  are the same functions  $\{g_1, \dots, g_{s_n}\}$  that appear in the approximate statistic  $\bar{T}_{n,s_n}(\theta)$ . We call the critical value obtained using  $\bar{T}_{s_n}(h)$  an approximate GMS (A-GMS) critical value.

Let  $c_{0,s_n}(h, 1 - \alpha)$  denote the  $1 - \alpha$  quantile of  $\bar{T}_{s_n}(h)$  for fixed  $\{g_1, \dots, g_{s_n}\}$ . The A-GMS critical value is defined to be

$$c_{s_n}(\varphi_n(\theta), \hat{h}_{2,n}(\theta), 1 - \alpha) = c_{0,s_n}(\varphi_n(\theta), \hat{h}_{2,n}(\theta), 1 - \alpha + \eta) + \eta. \quad (4.7)$$

This critical value is a quantile that can be computed by simulation as follows. Let  $\{\bar{T}_{s_n,\tau}(h) : \tau = 1, \dots, \tau_{reps}\}$  be  $\tau_{reps}$  i.i.d. random variables each with the same distribution as  $\bar{T}_{s_n}(h)$  and each with the same functions  $\{g_1, \dots, g_{s_n}\}$ , where  $h = (h_1, h_2)$  is evaluated at  $(\varphi_n(\theta), \hat{h}_{2,n}(\theta))$ . Then, the A-GMS critical value,  $c_{s_n}(\varphi_n(\theta), \hat{h}_{2,n}(\theta), 1 - \alpha)$ , is the  $1 - \alpha + \eta$  sample quantile of  $\{\bar{T}_{s_n,\tau}(\varphi_n(\theta), \hat{h}_{2,n}(\theta)) : \tau = 1, \dots, \tau_{reps}\}$  plus  $\eta$  for very small  $\eta > 0$  and large  $\tau_{reps}$ .

When constructing a CS, one carries out multiple tests with a different  $\theta$  value specified in the null hypothesis for each test. When doing so, we recommend using the same  $\{g_1, \dots, g_{s_n}\}$  functions for each  $\theta$  value considered (although this is not necessary for the asymptotic results to hold).

## 4.3 Bootstrap GMS Critical Values

Bootstrap versions of the GMS critical value in (4.4) and the A-GMS critical value in (4.7) can be employed. The bootstrap GMS critical value is

$$c^*(\varphi_n(\theta), \hat{h}_{2,n}^*(\theta), 1 - \alpha) = c_0^*(\varphi_n(\theta), \hat{h}_{2,n}^*(\theta), 1 - \alpha + \eta) + \eta, \quad (4.8)$$

where  $c_0^*(h, 1 - \alpha)$  is the  $1 - \alpha$  quantile of  $T^*(h)$  and  $T^*(h)$  is defined as in (4.1) but with  $\{\nu_{h_2}(g) : g \in \mathcal{G}\}$  and  $\widehat{h}_{2,n}(\theta)$  replaced by the bootstrap empirical process  $\{\nu_n^*(g) : g \in \mathcal{G}\}$  and the bootstrap covariance kernel  $\widehat{h}_{2,n}^*(\theta)$ , respectively. By definition,  $\nu_n^*(g) = n^{-1/2} \sum_{i=1}^n (m(W_i^*, \theta, g) - \overline{m}_n(\theta, g))$ , where  $\{W_i^* : i \leq n\}$  is an i.i.d. bootstrap sample drawn from the empirical distribution of  $\{W_i : i \leq n\}$ . Also,  $\widehat{h}_{2,n}^*(\theta, g, g^*)$ ,  $\widehat{\Sigma}_n^*(\theta, g, g^*)$ , and  $\widehat{D}_n^*(\theta)$  are defined as in (4.5) with  $W_i^*$  in place of  $W_i$ . Note that  $\widehat{h}_{2,n}^*(\theta, g, g^*)$  only enters  $c(\varphi_n(\theta), \widehat{h}_{2,n}^*(\theta), 1 - \alpha)$  via functions  $(g, g^*)$  such that  $g = g^*$ .

When the test statistic,  $\overline{T}_{n,s_n}(\theta)$ , is a truncated sum, simulated integral, or QMC quantity, a bootstrap A-GMS critical value can be employed. It is defined analogously to the bootstrap GMS critical value but with  $T^*(h)$  replaced by  $T_{s_n}^*(h)$ , where  $T_{s_n}^*(h)$  has the same definition as  $T^*(h)$  except that a truncated sum, simulated integral, or QMC quantity, appears in place of the integral with respect to  $Q$ , as in Section 4.2. The same functions  $\{g_1, \dots, g_{s_n}\}$  are used in all bootstrap critical value calculations as in the test statistic  $\overline{T}_{n,s_n}(\theta)$ .

#### 4.4 Definition of $\varphi_n(\theta)$

Next, we define  $\varphi_n(\theta)$ . As discussed above,  $\varphi_n(\theta)$  is constructed such that  $\varphi_n(\theta, g) \leq h_{1,n}(g) \forall g \in \mathcal{G}$  with probability that goes to one as  $n \rightarrow \infty$  uniformly over  $(\theta, F)$ . Let

$$\xi_n(\theta, g) = \kappa_n^{-1} n^{1/2} \overline{D}_n^{-1/2}(\theta, g) \overline{m}_n(\theta, g), \text{ where } \overline{D}_n(\theta, g) = \text{Diag}(\overline{\Sigma}_n(\theta, g)), \quad (4.9)$$

$\overline{\Sigma}_n(\theta, g)$  is defined in (3.5), and  $\{\kappa_n : n \geq 1\}$  is a sequence of constants that diverges to infinity as  $n \rightarrow \infty$ . The  $j$ th element of  $\xi_n(\theta, g)$ , denoted  $\xi_{n,j}(\theta, g)$ , measures the slackness of the moment inequality  $E_F m_j(W_i, \theta, g) \geq 0$  for  $j = 1, \dots, p$ .

Define  $\varphi_n(\theta, g) = (\varphi_{n,1}(\theta, g), \dots, \varphi_{n,p}(\theta, g), 0, \dots, 0)' \in R^k$  via, for  $j \leq p$ ,

$$\begin{aligned} \varphi_{n,j}(\theta, g) &= 0 && \text{if } \xi_{n,j}(\theta, g) \leq 1 \\ \varphi_{n,j}(\theta, g) &= \overline{h}_{2,n,j}(\theta, g)^{1/2} B_n && \text{if } \xi_{n,j}(\theta, g) > 1, \text{ where} \\ \overline{h}_{2,n}(\theta, g) &= \widehat{D}_n^{-1/2}(\theta) \overline{\Sigma}_n(\theta, g) \widehat{D}_n^{-1/2}(\theta) \text{ and } \overline{h}_{2,n,j}(\theta, g) = [\overline{h}_{2,n}(\theta, g)]_{jj}. \end{aligned} \quad (4.10)$$

**Assumption GMS1.** (a)  $\varphi_n(\theta, g)$  satisfies (4.10), where  $\{B_n : n \geq 1\}$  is a non-decreasing sequence of positive constants, and

(b) for some  $\zeta > 1$ ,  $\kappa_n - \zeta B_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

The constants  $\{B_n : n \geq 1\}$  in Assumption GMS1 need not diverge to infinity



for the GMS CS to have asymptotic size greater than or equal to  $1 - \alpha$ . However, for the GMS CS not to be asymptotically conservative,  $B_n$  must diverge to  $\infty$ , see Assumption GMS2(b) below. In the simulations in Section 9, we use  $\kappa_n = (0.3 \ln(n))^{1/2}$  and  $B_n = (0.4 \ln(n) / \ln \ln(n))^{1/2}$ , which satisfy Assumption GMS1.

The multiplicand  $\bar{h}_{2,n,j}(\theta, g)^{1/2}$  in (4.10) is an “ $\varepsilon$ -adjusted” standard deviation estimator for the  $j$ th normalized sample moment based on  $g$ . It provides a suitable scaling for  $\varphi_n(\theta, g)$ .

## 4.5 “Plug-in Asymptotic” Confidence Sets

Next, for comparative purposes, we define plug-in asymptotic (PA) critical values. Subsampling critical values are defined and analyzed in Appendix B of AS. We strongly recommend GMS critical values over PA and subsampling critical values because (i) GMS tests are shown to be more powerful than PA tests asymptotically, see Comment 2 to Theorem 4 below, (ii) it should be possible to show that GMS tests have higher power than subsampling tests asymptotically and smaller errors in null rejection probabilities asymptotically by using arguments similar to those in Andrews and Soares (2010) and Bugni (2010), respectively, and (iii) the finite-sample simulations in Section 9 show better performance by GMS critical values than PA and subsampling critical values.

PA critical values are obtained from the asymptotic null distribution that arises when all conditional moment inequalities hold as equalities a.s. The PA critical value is

$$c(0_{\mathcal{G}}, \hat{h}_{2,n}(\theta), 1 - \alpha) = c_0(0_{\mathcal{G}}, \hat{h}_{2,n}(\theta), 1 - \alpha + \eta) + \eta, \quad (4.11)$$

where  $\eta$  is an arbitrarily small positive constant (i.e., an IUF),  $0_{\mathcal{G}}$  denotes the  $R^k$ -valued function on  $\mathcal{G}$  that is identically  $(0, \dots, 0)' \in R^k$ , and  $\hat{h}_{2,n}(\theta)$  is defined in (4.5). The nominal  $1 - \alpha$  PA CS is given by (2.5) with the critical value  $c_{n,1-\alpha}(\theta)$  equal to  $c(0_{\mathcal{G}}, \hat{h}_{2,n}(\theta), 1 - \alpha)$ .

Bootstrap PA, A-PA, and bootstrap A-PA critical values are defined analogously to their GMS counterparts in Sections 4.2 and 4.3.

## 5 Uniform Asymptotic Coverage Probabilities

In this section, we show that GMS and PA CS’s have correct uniform asymptotic coverage probabilities. The results of this section and those in Sections 6-8 below are

for CvM statistics based on integrals with respect to  $Q$ . Extensions of these results to A-CvM statistics and critical values are provided in Appendix B of AS. Appendix B also gives results for KS tests.

## 5.1 Notation

In order to establish uniform asymptotic coverage probability results, we now introduce notation for the population analogues of the sample quantities that appear in (4.5). Define

$$\begin{aligned}
h_{2,F}(\theta, g, g^*) &= D_F^{-1/2}(\theta)\Sigma_F(\theta, g, g^*)D_F^{-1/2}(\theta) \\
&= Cov_F \left( D_F^{-1/2}(\theta)m(W_i, \theta, g), D_F^{-1/2}(\theta)m(W_i, \theta, g^*) \right), \\
\Sigma_F(\theta, g, g^*) &= Cov_F(m(W_i, \theta, g), m(W_i, \theta, g^*)), \text{ and} \\
D_F(\theta) &= Diag(\Sigma_F(\theta, 1_k, 1_k)) (= Diag(Var_F(m(W_i, \theta)))). \tag{5.1}
\end{aligned}$$

To determine the asymptotic distribution of  $T_n(\theta)$ , we write  $T_n(\theta)$  as a function of the following quantities:

$$\begin{aligned}
h_{1,n,F}(\theta, g) &= n^{1/2}D_F^{-1/2}(\theta)E_F m(W_i, \theta, g), \\
h_{n,F}(\theta, g, g^*) &= (h_{1,n,F}(\theta, g), h_{2,F}(\theta, g, g^*)), \\
\widehat{h}_{2,n,F}(\theta, g, g^*) &= D_F^{-1/2}(\theta)\widehat{\Sigma}_n(\theta, g, g^*)D_F^{-1/2}(\theta), \\
\bar{h}_{2,n,F}(\theta, g) &= \widehat{h}_{2,n,F}(\theta, g, g) + \varepsilon\widehat{h}_{2,n,F}(\theta, 1_k, 1_k) (= D_F^{-1/2}(\theta)\bar{\Sigma}_n(\theta, g)D_F^{-1/2}(\theta)), \text{ and} \\
\nu_{n,F}(\theta, g) &= n^{-1/2} \sum_{i=1}^n D_F^{-1/2}(\theta)[m(W_i, \theta, g) - E_F m(W_i, \theta, g)]. \tag{5.2}
\end{aligned}$$

As defined, (i)  $h_{1,n,F}(\theta, g)$  is a  $k$ -vector of normalized means of the moment functions  $m(W_i, \theta, g)$  for  $g \in \mathcal{G}$ , which measure the slackness of the population moment conditions under  $(\theta, F)$ , (ii)  $h_{n,F}(\theta, g, g^*)$  contains the normalized means of  $D_F^{-1/2}(\theta)m(W_i, \theta, g)$  and the covariances of  $D_F^{-1/2}(\theta)m(W_i, \theta, g)$  and  $D_F^{-1/2}(\theta)m(W_i, \theta, g^*)$ , (iii)  $\widehat{h}_{2,n,F}(\theta, g, g^*)$  and  $\bar{h}_{2,n,F}(\theta, g)$  are hybrid quantities—part population, part sample—based on  $\widehat{\Sigma}_n(\theta, g, g^*)$  and  $\bar{\Sigma}_n(\theta, g)$ , respectively, and (iv)  $\nu_{n,F}(\theta, g)$  is the sample average of  $D_F^{-1/2}(\theta)m(W_i, \theta, g)$  normalized to have mean zero and variance that does not depend on  $n$ .

Note that  $\nu_{n,F}(\theta, \cdot)$  is an empirical process indexed by  $g \in \mathcal{G}$  with covariance kernel given by  $h_{2,F}(\theta, g, g^*)$ .

The normalized sample moments  $n^{1/2}\overline{m}_n(\theta, g)$  can be written as

$$n^{1/2}\overline{m}_n(\theta, g) = D_F^{1/2}(\theta)(\nu_{n,F}(\theta, g) + h_{1,n,F}(\theta, g)). \quad (5.3)$$

The test statistic  $T_n(\theta)$ , defined in (3.6), can be written as

$$T_n(\theta) = \int S(\nu_{n,F}(\theta, g) + h_{1,n,F}(\theta, g), \overline{h}_{2,n,F}(\theta, g))dQ(g). \quad (5.4)$$

Note the close resemblance between  $T_n(\theta)$  and  $T(h)$  (defined in (4.1)).

Let  $\mathcal{H}_1$  denote the set of all functions from  $\mathcal{G}$  to  $R_{+, \infty}^p \times \{0\}^v$ . Let

$$\begin{aligned} \mathcal{H}_2 &= \{h_{2,F}(\theta, \cdot, \cdot) : (\theta, F) \in \mathcal{F}\} \text{ and} \\ \mathcal{H} &= \mathcal{H}_1 \times \mathcal{H}_2. \end{aligned} \quad (5.5)$$

On the space of  $k \times k$ -matrix-valued covariance kernels on  $\mathcal{G} \times \mathcal{G}$ , which is a superset of  $\mathcal{H}_2$ , we use the metric  $d$  defined by

$$d(h_2^{(1)}, h_2^{(2)}) = \sup_{g, g^* \in \mathcal{G}} \|h_2^{(1)}(g, g^*) - h_2^{(2)}(g, g^*)\|. \quad (5.6)$$

For notational simplicity, for any function of the form  $b_F(\theta, g)$  for  $g \in \mathcal{G}$ , let  $b_F(\theta)$  denote the function  $b_F(\theta, \cdot)$  on  $\mathcal{G}$ . Correspondingly, for any function of the form  $b_F(\theta, g, g^*)$  for  $g, g^* \in \mathcal{G}$ , let  $b_F(\theta)$  denote the function  $b_F(\theta, \cdot, \cdot)$  on  $\mathcal{G}^2$ .

## 5.2 Uniform Asymptotic Distribution of the Test Statistic

The following Theorem provides a uniform asymptotic distributional result for the test statistic  $T_n(\theta)$ . It is used to establish uniform asymptotic coverage probability results for GMS and PA CS's.

**Theorem 1.** *Suppose Assumptions M, S1, and S2 hold. Then, for every compact subset  $\mathcal{H}_{2, \text{cpt}}$  of  $\mathcal{H}_2$ , all constants  $x_{h_{n,F}(\theta)} \in R$  that may depend on  $(\theta, F)$  and  $n$  through  $h_{n,F}(\theta)$ ,*

and all  $\delta > 0$ , we have

$$\begin{aligned} \text{(a)} \quad & \limsup_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2, F}(\theta) \in \mathcal{H}_{2, cpt}}} \left[ P_F(T_n(\theta) > x_{h_{n, F}(\theta)}) - P(T(h_{n, F}(\theta)) + \delta > x_{h_{n, F}(\theta)}) \right] \leq 0, \\ \text{(b)} \quad & \liminf_{n \rightarrow \infty} \inf_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2, F}(\theta) \in \mathcal{H}_{2, cpt}}} \left[ P_F(T_n(\theta) > x_{h_{n, F}(\theta)}) - P(T(h_{n, F}(\theta)) - \delta > x_{h_{n, F}(\theta)}) \right] \geq 0, \end{aligned}$$

where  $T(h) = \int S(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) dQ(g)$  and  $\nu_{h_2}(\cdot)$  is the Gaussian

process defined in (4.2).

**Comments. 1.** Theorem 1 gives a uniform asymptotic approximation to the distribution function of  $T_n(\theta)$ . Uniformity holds without *any* restrictions on the normalized mean (i.e., moment inequality slackness) functions  $\{h_{1, n, F_n}(\theta_n) : n \geq 1\}$ . In particular, Theorem 1 does not require  $\{h_{1, n, F_n}(\theta_n) : n \geq 1\}$  to converge as  $n \rightarrow \infty$  or to belong to a compact set. The Theorem does not require that  $T_n(\theta)$  has a unique asymptotic distribution under any sequence  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$ . These are novel features of Theorem 1.

**2.** The supremum and infimum in Theorem 1 are over compact sets of covariance kernels  $\mathcal{H}_{2, cpt}$ , rather than the parameter space  $\mathcal{H}_2$ . This is not particularly problematic because the potential asymptotic size problems that arise in moment inequality models are due to the pointwise discontinuity of the asymptotic distribution of the test statistic as a function of the means of the moment inequality functions, not as a function of the covariances between different moment inequalities.

**3.** Theorem 1 is proved using an almost sure representation argument and the bounded convergence theorem (BCT). The continuous mapping theorem does not apply because (i)  $T_n(\theta)$  does not converge in distribution uniformly over  $(\theta, F) \in \mathcal{F}$  and (ii)  $h_{1, n, F}(\theta, g)$  typically does not converge uniformly over  $g \in \mathcal{G}$  even in cases where it has a pointwise limit for all  $g \in \mathcal{G}$ .

### 5.3 An Additional GMS Assumption

The following assumption is not needed for GMS CS's to have uniform asymptotic coverage probability greater than or equal to  $1 - \alpha$ . It is used, however, to show that GMS CS's are not asymptotically conservative. (Note that typically GMS and PA CS's are asymptotically non-similar.) For  $(\theta, F) \in \mathcal{F}$  and  $j = 1, \dots, k$ , define  $h_{1, \infty, F}(\theta)$  to

have  $j$ th element equal to  $\infty$  if  $E_F m_j(W_i, \theta, g) > 0$  and  $j \leq p$  and 0 otherwise. Let  $h_{\infty, F}(\theta) = (h_{1, \infty, F}(\theta), h_{2, F}(\theta))$ .

**Assumption GMS2.** (a) For some  $(\theta_c, F_c) \in \mathcal{F}$ , the distribution function of  $T(h_{\infty, F_c}(\theta_c))$  is continuous and strictly increasing at its  $1 - \alpha$  quantile plus  $\delta$ , viz.,  $c_0(h_{\infty, F_c}(\theta_c), 1 - \alpha) + \delta$ , for all  $\delta > 0$  sufficiently small and  $\delta = 0$ ,

(b)  $B_n \rightarrow \infty$  as  $n \rightarrow \infty$ , and

(c)  $n^{1/2}/\kappa_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

Assumption GMS2(a) is not restrictive. For example, it holds for typical choices of  $S$  and  $Q$  for any  $(\theta_c, F_c)$  for which  $Q(\{g \in \mathcal{G} : h_{1, \infty, F_c}(\theta_c, g) = 0\}) > 0$ . Assumption GMS2(c) is satisfied by typical choices of  $\kappa_n$ , such as  $\kappa_n = (0.3 \ln n)^{1/2}$ .

## 5.4 Uniform Asymptotic Coverage Probability Results

The following Theorem gives uniform asymptotic coverage probability results for GMS and PA CS's.

**Theorem 2.** *Suppose Assumptions M, S1, and S2 hold and Assumption GMS1 also holds when considering GMS CS's. Then, for every compact subset  $\mathcal{H}_{2, \text{cpt}}$  of  $\mathcal{H}_2$ , GMS and PA confidence sets  $CS_n$  satisfy*

(a)  $\liminf_{n \rightarrow \infty} \inf_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2, F}(\theta) \in \mathcal{H}_{2, \text{cpt}}}} P_F(\theta \in CS_n) \geq 1 - \alpha$  and

(b) *if Assumption GMS2 also holds and  $h_{2, F_c}(\theta_c) \in \mathcal{H}_{2, \text{cpt}}$  (for  $(\theta_c, F_c) \in \mathcal{F}$  as in Assumption GMS2), then the GMS confidence set satisfies*

$$\lim_{\eta \rightarrow 0} \liminf_{n \rightarrow \infty} \inf_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2, F}(\theta) \in \mathcal{H}_{2, \text{cpt}}}} P_F(\theta \in CS_n) = 1 - \alpha,$$

where  $\eta$  is as in the definition of  $c(h, 1 - \alpha)$ .

**Comments. 1.** Theorem 2(a) shows that GMS and PA CS's have correct uniform asymptotic size over compact sets of covariance kernels. Theorem 2(b) shows that GMS CS's are at most infinitesimally conservative asymptotically. The uniformity results hold whether the moment conditions involve “weak” or “strong” instrumental variables.

**2.** An analogue of Theorem 2(b) holds for PA CS's if Assumption GMS2(a) holds and  $E_{F_c}(m_j(W_i, \theta_c) | X_i) = 0$  a.s. for  $j \leq p$  (i.e., if the conditional moment inequalities

hold as equalities a.s.) under some  $(\theta_c, F_c) \in \mathcal{F}$ .<sup>13</sup> However, the latter condition is restrictive—it fails in many applications.

3. Theorem 2 applies to CvM tests based on integrals with respect to a probability measure  $Q$ . Extensions to A-CvM and KS tests are given in Appendix B of AS.

4. Comments 1 and 2 to Theorem 1 also apply to Theorem 2.

## 6 Power Against Fixed Alternatives

We now show that the power of GMS and PA tests converges to one as  $n \rightarrow \infty$  for all fixed alternatives (for which the moment functions have  $2 + \delta$  moments finite). Thus, both tests are consistent tests. This implies that for any fixed distribution  $F_0$  and any parameter value  $\theta_*$  *not* in the identified set  $\Theta_{F_0}$ , the GMS and PA CS's do not include  $\theta_*$  with probability approaching one. In this sense, GMS and PA CS's based on  $T_n(\theta)$  fully exploit the conditional moment inequalities and equalities. CS's based on a finite number of unconditional moment inequalities and equalities do not have this property.

The null hypothesis is

$$\begin{aligned} H_0 : E_{F_0}(m_j(W_i, \theta_*)|X_i) &\geq 0 \text{ a.s. } [F_{X,0}] \text{ for } j = 1, \dots, p \text{ and} \\ E_{F_0}(m_j(W_i, \theta_*)|X_i) &= 0 \text{ a.s. } [F_{X,0}] \text{ for } j = p + 1, \dots, k, \end{aligned} \quad (6.1)$$

where  $\theta_*$  denotes the null parameter value and  $F_0$  denotes the fixed true distribution of the data. The alternative is  $H_1 : H_0$  does not hold. The following assumption specifies the properties of fixed alternatives (FA).

**Assumption FA.** The value  $\theta_* \in \Theta$  and the true distribution  $F_0$  satisfy: (a)  $P_{F_0}(X_i \in \mathcal{X}_{F_0}(\theta_*)) > 0$ , where  $\mathcal{X}_{F_0}(\theta_*)$  is defined in (3.11), (b)  $\{W_i : i \geq 1\}$  are i.i.d. under  $F_0$ , (c)  $Var_{F_0}(m_j(W_i, \theta_*)) > 0$  for  $j = 1, \dots, k$ , (d)  $E_{F_0}||m(W_i, \theta_*)||^{2+\delta} < \infty$  for some  $\delta > 0$ , and (e) Assumption M holds with  $F_0$  in place of  $F$  and  $F_n$  in Assumptions M(b) and M(c), respectively.

Assumption FA(a) states that violations of the conditional moment inequalities or equalities occur for the null parameter  $\theta_*$  for  $X_i$  values in some set with positive probability under  $F_0$ . Thus, under Assumption FA(a), the moment conditions specified in (6.1) do not hold. Assumptions FA(b)-(d) are standard i.i.d. and moment assumptions. As-

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<sup>13</sup>The proof follows easily from results given in the proof of Theorem 2(b).

sumption FA(e) holds for  $\mathcal{G}_{c-cube}$  and  $\mathcal{G}_{box}$  because  $\mathcal{C}_{c-cube}$  and  $\mathcal{C}_{box}$  are Vapnik-Cervonenkis classes of sets.

For  $g \in \mathcal{G}$ , define

$$\begin{aligned} m_j^*(g) &= E_{F_0} m_j(W_i, \theta_*) g_j(X_i) / \sigma_{F_0, j}(\theta_*) \text{ and} \\ \beta(g) &= \max\{-m_1^*(g), \dots, -m_p^*(g), |m_{p+1}^*(g)|, \dots, |m_k^*(g)|\}. \end{aligned} \quad (6.2)$$

Under Assumptions FA(a) and CI,  $\beta(g_0) > 0$  for some  $g_0 \in \mathcal{G}$ .

For a test based on  $T_n(\theta)$  to have power against all fixed alternatives, the weighting function  $Q$  cannot “ignore” any elements  $g \in \mathcal{G}$ , because such elements may have identifying power for the identified set. This requirement is captured in the following assumption, which is shown in Lemma 4 to hold for the two probability measures  $Q$  considered in Section 3.4.

Let  $F_{X,0}$  denote the distribution of  $X_i$  under  $F_0$ . Define the pseudo-metric  $\rho_X$  on  $\mathcal{G}$  by

$$\rho_X(g, g^*) = (E_{F_{X,0}} \|g(X_i) - g^*(X_i)\|^2)^{1/2} \text{ for } g, g^* \in \mathcal{G}. \quad (6.3)$$

Let  $\mathcal{B}_{\rho_X}(g, \delta)$  denote an open  $\rho_X$ -ball in  $\mathcal{G}$  centered at  $g$  with radius  $\delta$ .

**Assumption Q.** The support of  $Q$  under the pseudo-metric  $\rho_X$  is  $\mathcal{G}$ . That is, for all  $\delta > 0$ ,  $Q(\mathcal{B}_{\rho_X}(g, \delta)) > 0$  for all  $g \in \mathcal{G}$ .

The next result establishes Assumption Q for the probability measures  $Q$  on  $\mathcal{G}_{c-cube}$  and  $\mathcal{G}_{box}$  discussed in Section 3.4 above. Appendix B of AS provides analogous results for three other choices of  $Q$  and  $\mathcal{G}$ .

**Lemma 4.** *Assumption Q holds for the weight functions:*

(a)  $Q_a = \Pi_{c-cube}^{-1} Q_{AR}$  on  $\mathcal{G}_{c-cube}$ , where  $Q_{AR}$  is uniform on  $a \in \{1, \dots, 2r\}^{d_x}$  conditional on  $r$  and  $r$  has some probability mass function  $\{w(r) : r = r_0, r_0 + 1, \dots\}$  with  $w(r) > 0$  for all  $r$  and

(b)  $Q_b = \Pi_{box}^{-1} Unif([0, 1]^{d_x} \times (0, \bar{r})^{d_x})$  on  $\mathcal{G}_{box}$  with the centers of the boxes in  $[0, 1]^{d_x}$ .

**Comment.** The uniform distribution that appears in both specifications of  $Q$  in the Lemma could be replaced by another distribution and the results of the Lemma still hold provided the other distribution has the same support.

The following Theorem shows that GMS and PA tests are consistent against all fixed alternatives.

**Theorem 3.** Under Assumptions FA, CI, Q, S1, S3, and S4,

- (a)  $\lim_{n \rightarrow \infty} P_{F_0}(T_n(\theta_*) > c(\varphi_n(\theta_*), \widehat{h}_{2,n}(\theta_*), 1 - \alpha)) = 1$  and
- (b)  $\lim_{n \rightarrow \infty} P_{F_0}(T_n(\theta_*) > c(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha)) = 1.$

**Comment.** Theorem 3 implies the following for GMS and PA CS's: Suppose  $(\theta_0, F_0) \in \mathcal{F}$  for some  $\theta_0 \in \Theta$ ,  $\theta_*$  ( $\in \Theta$ ) is not in the identified set  $\Theta_{F_0}$  (defined in (2.2)), and Assumptions FA(c), FA(d), CI, M, S1, S3, and S4 hold, then for GMS and PA CS's we have

$$\lim_{n \rightarrow \infty} P_{F_0}(\theta_* \in CS_n) = 0.^{14} \tag{6.4}$$

## 7 Power Against $n^{-1/2}$ -Local Alternatives

In this section, we show that GMS and PA tests have power against certain, but not all,  $n^{-1/2}$ -local alternatives. This holds even though these tests fully exploit the information in the conditional moment restrictions, which is of an infinite-dimensional nature. These testing results have immediate implications for the volume of CS's, see Pratt (1961).

We show that a GMS test has asymptotic power that is greater than or equal to that of a PA test (based on the same test statistic) under all alternatives with strict inequality in certain scenarios. Although we do not do so here, arguments analogous to those in Andrews and Soares (2010) could be used to show that a GMS test's power is greater than or equal to that of a subsampling test with strictly greater power in certain scenarios.

For given  $\theta_{n,*} \in \Theta$  for  $n \geq 1$ , we consider tests of

$$\begin{aligned} H_0 : E_{F_n} m_j(W_i, \theta_{n,*}) &\geq 0 \text{ for } j = 1, \dots, p, \\ E_{F_n} m_j(W_i, \theta_{n,*}) &= 0 \text{ for } j = p + 1, \dots, k, \end{aligned} \tag{7.1}$$

and  $(\theta_{n,*}, F_n) \in \mathcal{F}$ , where  $F_n$  denotes the true distribution of the data. The null values  $\theta_{n,*}$  are allowed to drift with  $n$  or be fixed for all  $n$ . Drifting  $\theta_{n,*}$  values are of interest because they allow one to consider the case of a fixed identified set, say  $\Theta_0$ , and to derive the asymptotic probability that parameter values  $\theta_{n,*}$  that are not in the identified set, but drift toward it at rate  $n^{-1/2}$ , are excluded from a GMS or PA CS. In this scenario,

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<sup>14</sup>This holds because  $\theta_* \notin \Theta_{F_0}$  implies Assumption FA(a) holds,  $(\theta_0, F_0) \in \mathcal{F}$  implies Assumption FA(b) holds, and Assumption M with  $F = F_n = F_0$  implies Assumption FA(e) holds.



the sequence of true distributions are ones that yield  $\Theta_0$  to be the identified set, i.e.,  $F_n \in \mathcal{F}_0 = \{F : \Theta_F = \Theta_0\}$ .

The true parameters and distributions are denoted  $(\theta_n, F_n)$ . We consider the Kolmogorov-Smirnov metric on the space of distributions  $F$ .

The  $n^{-1/2}$ -local alternatives are defined as follows.

**Assumption LA1.** The true parameters and distributions  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$  and the null parameters  $\{\theta_{n,*} : n \geq 1\}$  satisfy:

(a)  $\theta_{n,*} = \theta_n + \lambda n^{-1/2}(1 + o(1))$  for some  $\lambda \in R^{d_\theta}$ ,  $\theta_{n,*} \in \Theta$ ,  $\theta_{n,*} \rightarrow \theta_0$ , and  $F_n \rightarrow F_0$  for some  $(\theta_0, F_0) \in \mathcal{F}$ ,

(b)  $n^{1/2}E_{F_n}m_j(W_i, \theta_n, g)/\sigma_{F_n,j}(\theta_n) \rightarrow h_{1,j}(g)$  for some  $h_{1,j}(g) \in R_{+, \infty}$  for  $j = 1, \dots, p$  and  $g \in \mathcal{G}$ ,

(c)  $d(h_{2,F_n}(\theta_n), h_{2,F_0}(\theta_0)) \rightarrow 0$  and  $d(h_{2,F_n}(\theta_{n,*}), h_{2,F_0}(\theta_0)) \rightarrow 0$  as  $n \rightarrow \infty$  (where  $d$  is defined in (5.6)),

(d)  $Var_{F_n}(m_j(W_i, \theta_{n,*})) > 0$  for  $j = 1, \dots, k$ , for  $n \geq 1$ , and

(e)  $\sup_{n \geq 1} E_{F_n}|m_j(W_i, \theta_{n,*})/\sigma_{F_n,j}(\theta_{n,*})|^{2+\delta} < \infty$  for  $j = 1, \dots, k$  for some  $\delta > 0$ .

**Assumption LA2.** The  $k \times d$  matrix  $\Pi_F(\theta, g) = (\partial/\partial\theta')[D_F^{-1/2}(\theta)E_F m(W_i, \theta, g)]$  exists and is continuous in  $(\theta, F)$  for all  $(\theta, F)$  in a neighborhood of  $(\theta_0, F_0)$  for all  $g \in \mathcal{G}$ .

For notational simplicity, we let  $h_2$  abbreviate  $h_{2,F_0}(\theta_0)$  throughout this section. Assumption LA1(a) states that the true values  $\{\theta_n : n \geq 1\}$  are  $n^{-1/2}$ -local to the null values  $\{\theta_{n,*} : n \geq 1\}$ . Assumption LA1(b) specifies the asymptotic behavior of the (normalized) moment inequality functions when evaluated at the true values  $\{\theta_n : n \geq 1\}$ . Under the true values, these (normalized) moment inequality functions are non-negative. Assumption LA1(c) specifies the asymptotic behavior of the covariance kernels  $\{h_{2,F_n}(\theta_n, \cdot, \cdot) : n \geq 1\}$  and  $\{h_{2,F_n}(\theta_{n,*}, \cdot, \cdot) : n \geq 1\}$ . Assumptions LA1(d) and LA1(e) are standard. Assumption LA2 is a smoothness condition on the normalized expected moment functions. Given the smoothing properties of the expectation operator, this condition is not restrictive.

Under Assumptions LA1 and LA2, we show that the moment inequality functions evaluated at the null values  $\{\theta_{n,*} : n \geq 1\}$  satisfy:

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{1/2}D_{F_n}^{-1/2}(\theta_{n,*})E_{F_n}m(W_i, \theta_{n,*}, g) &= h_1(g) + \Pi_0(g)\lambda \in R^k, \text{ where} \\ h_1(g) &= (h_{1,1}(g), \dots, h_{1,p}(g), 0, \dots, 0)' \in R^k \text{ and } \Pi_0(g) = \Pi_{F_0}(\theta_0, g). \end{aligned} \quad (7.2)$$

If  $h_{1,j}(g) = \infty$ , then by definition  $h_{1,j}(g) + y = \infty$  for any  $y \in R$ . We have  $h_1(g) + \Pi_0(g)\lambda \in R_{[+\infty]}^p \times R^v$ . Let  $\Pi_{0,j}(g)$  denote the  $j$ th row of  $\Pi_0(g)$  written as a column  $d_\theta$ -vector for  $j = 1, \dots, k$ .

The null hypothesis, defined in (7.1), does not hold (at least for  $n$  large) when the following assumption holds.

**Assumption LA3.** For some  $g \in \mathcal{G}$ ,  $h_{1,j}(g) + \Pi_{0,j}(g)'\lambda < 0$  for some  $j = 1, \dots, p$  or  $\Pi_{0,j}(g)'\lambda \neq 0$  for some  $j = p + 1, \dots, k$ .

Under the following assumption, if  $\lambda = \beta\lambda_0$  for some  $\beta > 0$  and some  $\lambda_0 \in R^{d_\theta}$ , then the power of GMS and PA tests against the perturbation  $\lambda$  is arbitrarily close to one for  $\beta$  arbitrarily large:

**Assumption LA3'.**  $Q(\{g \in \mathcal{G} : h_{1,j}(g) < \infty$  and  $\Pi_{0,j}(g)'\lambda_0 < 0$  for some  $j = 1, \dots, p$  or  $\Pi_{0,j}(g)'\lambda_0 \neq 0$  for some  $j = p + 1, \dots, k\}) > 0$ .

Assumption LA3' requires that either (i) the moment equalities detect violations of the null hypothesis for  $g$  functions in a set with positive  $Q$  measure or (ii) the moment inequalities are not too far from being binding, i.e.,  $h_{1,j}(g) < \infty$ , and the perturbation  $\lambda_0$  occurs in a direction that yields moment inequality violations for  $g$  functions in a set with positive  $Q$  measure.

Assumption LA3 is employed with the KS test. It is weaker than Assumption LA3'. It is shown in Appendix B of AS that if Assumption LA3 holds with  $\lambda = \beta\lambda_0$  (and some other assumptions), then the power of KS-GMS and KS-PA tests against the perturbation  $\lambda$  is arbitrarily close to one for  $\beta$  arbitrarily large.

In Appendix B of AS we illustrate the verification of Assumptions LA1-LA3 and LA3' in a simple example. In this example,  $v = 0$ ,  $h_{1,j}(g) < \infty \forall g \in \mathcal{G}$ , and  $\Pi_{0,j}(g) = -Eg(X_i) \forall g \in \mathcal{G}$ , so  $\Pi_{0,j}(g)'\lambda_0 < 0$  in Assumption LA3'  $\forall g \in \mathcal{G}$  with  $Eg(X_i) > 0$  for all  $\lambda_0 > 0$ .

Assumptions LA3 and LA3' can fail to hold even when the null hypothesis is violated. This typically happens if the true parameter/true distribution is fixed, i.e.,  $(\theta_n, F_n) = (\theta_0, F_0) \in \mathcal{F}$  for all  $n$  in Assumption LA1(a), the null hypothesis parameter  $\theta_{n,*}$  drifts with  $n$  as in Assumption LA1(a), and  $P_{F_0}(X_i \in \mathcal{X}_{zero}) = 0$ , where  $\mathcal{X}_{zero} = \{x \in R^{d_x} : E_{F_0}(m(W_i, \theta_0)|X_i = x) = 0\}$ . In such cases, typically  $h_{1,j}(g) = \infty \forall g \in \mathcal{G}$  (because the conditional moment inequalities are non-binding with probability one), Assumptions LA3 and LA3' fail, and Theorem 4 below shows that GMS and PA tests have trivial asymptotic power against such  $n^{-1/2}$ -local alternatives. For example, this occurs in the example of Section 12.5 in Appendix B of AS when  $P_{F_0}(X_i \in \mathcal{X}_{zero}) = 0$ .

As discussed in Section 12.5, asymptotic results based on a fixed true distribution provide poor approximations when  $P_{F_0}(X_i \in \mathcal{X}_{zero}) = 0$ . Hence, one can argue that it makes sense to consider local alternatives for sequences of true distributions  $\{F_n : n \geq 1\}$  for which  $h_{1,j}(g) < \infty$  for a non-negligible set of  $g \in \mathcal{G}$ , as in Assumption LA3', because such sequences are the ones for which the asymptotics provide good finite-sample approximations. For such sequences, GMS and PA tests have non-trivial power against  $n^{-1/2}$ -local alternatives, as shown in Theorem 4 below.

Nevertheless, local-alternative power results can be used for multiple purposes and for some purposes, one may want to consider local-alternatives other than those that satisfy Assumptions LA3 and LA3'.

The asymptotic distribution of  $T_n(\theta_{n,*})$  under  $n^{-1/2}$ -local alternatives is shown to be  $J_{h,\lambda}$ . By definition,  $J_{h,\lambda}$  is the distribution of

$$T(h_1 + \Pi_0\lambda, h_2) = \int S(\nu_{h_2}(g) + h_1(g) + \Pi_0(g)\lambda, h_2(g) + \varepsilon I_k) dQ(g), \quad (7.3)$$

where  $h = (h_1, h_2)$ ,  $\Pi_0$  denotes  $\Pi_0(\cdot)$ , and  $\nu_{h_2}(\cdot)$  is a mean zero Gaussian process with covariance kernel  $h_2 = h_{2,F_0}(\theta_0)$ . For notational simplicity, the dependence of  $J_{h,\lambda}$  on  $\Pi_0$  is suppressed.

Next, we introduce two assumptions, viz., Assumptions LA4 and LA5, that are used only for GMS tests in the context of local alternatives. The population analogues of  $\bar{\Sigma}_n(\theta, g)$  and its diagonal matrix are

$$\bar{\Sigma}_F(\theta, g) = \Sigma_F(\theta, g, g) + \varepsilon \Sigma_F(\theta, 1_k, 1_k) \text{ and } \bar{D}_F(\theta, g) = \text{Diag}(\bar{\Sigma}_F(\theta, g)), \quad (7.4)$$

where  $\Sigma_F(\theta, g, g)$  is defined in (5.1). Let  $\bar{\sigma}_{F,j}(\theta, g)$  denote the square-root of the  $(j, j)$  element of  $\bar{\Sigma}_F(\theta, g)$ .

**Assumption LA4.**  $\kappa_n^{-1} n^{1/2} E_{F_n} m_j(W_i, \theta_n, g) / \bar{\sigma}_{F_n,j}(\theta_n, g) \rightarrow \pi_{1,j}(g)$  for some  $\pi_{1,j}(g) \in R_{+,\infty}$  for  $j = 1, \dots, p$  and  $g \in \mathcal{G}$ .

In Assumption LA4 the functions are evaluated at the true value  $\theta_n$ , not at the null value  $\theta_{n,*}$ , and  $(\theta_n, F_n) \in \mathcal{F}$ . In consequence, the moment functions in Assumption LA4 satisfy the moment inequalities and  $\pi_{1,j}(g) \geq 0$  for all  $j = 1, \dots, p$  and  $g \in \mathcal{G}$ . Note that  $0 \leq \pi_{1,j}(g) \leq h_{1,j}(g)$  for all  $j = 1, \dots, p$  and all  $g \in \mathcal{G}$  (by Assumption LA1(b) and  $\kappa_n \rightarrow \infty$ .)

Let  $\pi_1(g) = (\pi_{1,1}(g), \dots, \pi_{1,p}(g), 0, \dots, 0)' \in R_{+,\infty}^p \times \{0\}^v$ . Let  $c_0(\varphi(\pi_1), h_2, 1-\alpha)$  denote

the  $1 - \alpha$  quantile of

$$\begin{aligned}
T(\varphi(\pi_1), h_2) &= \int S(\nu_{h_2}(g) + \varphi(\pi_1(g)), h_2(g) + \varepsilon I_k) dQ(g), \text{ where} \\
\varphi(\pi_1(g)) &= (\varphi(\pi_{1,1}(g)), \dots, \varphi(\pi_{1,p}(g)), 0, \dots, 0)' \in R^k \text{ and} \\
\varphi(x) &= 0 \text{ if } x \leq 1 \text{ and } \varphi(x) = \infty \text{ if } x > 1.
\end{aligned} \tag{7.5}$$

Let  $\varphi(\pi_1)$  denote  $\varphi(\pi_1(\cdot))$ . The probability limit of the GMS critical value  $c(\varphi_n(\theta), \widehat{h}_{2,n}(\theta), 1 - \alpha)$  is shown below to be  $c(\varphi(\pi_1), h_2, 1 - \alpha) = c_0(\varphi(\pi_1), h_2, 1 - \alpha + \eta) + \eta$ .

**Assumption LA5.** (a)  $Q(\mathcal{G}_\varphi) = 1$ , where  $\mathcal{G}_\varphi = \{g \in \mathcal{G} : \pi_{1,j}(g) \neq 1 \text{ for } j = 1, \dots, p\}$ , and

(b) the distribution function (df) of  $T(\varphi(\pi_1), h_2)$  is continuous and strictly increasing at  $x = c(\varphi(\pi_1), h_2, 1 - \alpha)$ , where  $h_2 = h_{2,F_0}(\theta_0)$ .

The value 1 that appears in  $\mathcal{G}_\varphi$  in Assumption LA5(a) is the discontinuity point of  $\varphi$ . Assumption LA5(a) implies that the  $n^{-1/2}$ -local power formulae given below do not apply to certain ‘‘discontinuity vectors’’  $\pi_1(\cdot)$ , but this is not particularly restrictive.<sup>15</sup> Assumption LA5(b) typically holds because of the absolute continuity of the Gaussian random variables  $\nu_{h_2}(g)$  that enter  $T(\varphi(\pi_1), h_2)$ .<sup>16</sup>

The following assumption is used only for PA tests.

**Assumption LA6.** The df of  $T(0_G, h_2)$  is continuous and strictly increasing at  $x = c(0_G, h_2, 1 - \alpha)$ , where  $h_2 = h_{2,F_0}(\theta_0)$ .

The probability limit of the PA critical value is shown to be  $c(0_G, h_2, 1 - \alpha) = c_0(0_G, h_2, 1 - \alpha + \eta) + \eta$ , where  $c_0(0_G, h_2, 1 - \alpha)$  denotes the  $1 - \alpha$  quantile of  $J_{(0_G, h_2), 0_{d_\theta}}$ .

**Theorem 4.** *Under Assumptions M, S1, S2, and LA1-LA2,*

(a)  $\lim_{n \rightarrow \infty} P_{F_n}(T_n(\theta_{n,*}) > c(\varphi_n(\theta_{n,*}), \widehat{h}_{2,n}(\theta_{n,*}), 1 - \alpha)) = 1 - J_{h,\lambda}(c(\varphi(\pi_1), h_2, 1 - \alpha))$   
*provided Assumptions GMS1, LA4, and LA5 also hold,*

<sup>15</sup> Assumption LA5(a) is not particularly restrictive because in cases where it fails, one can obtain lower and upper bounds on the local asymptotic power of GMS tests by replacing  $c(\varphi(\pi_1), h_2, 1 - \alpha)$  by  $c(\varphi(\pi_1-), h_2, 1 - \alpha)$  and  $c(\varphi(\pi_1+), h_2, 1 - \alpha)$ , respectively, in Theorem 4(a). By definition,  $\varphi(\pi_1-) = \varphi(\pi_1(\cdot)-)$  and  $\varphi(\pi_1(g)-)$  is the limit from the left of  $\varphi(x)$  at  $x = \pi_1(g)$ . Likewise  $\varphi(\pi_1+) = \varphi(\pi_1(\cdot)+)$  and  $\varphi(\pi_1(g)+)$  is the limit from the right of  $\varphi(x)$  at  $x = \pi_1(g)$ .

<sup>16</sup> If Assumption LA5(b) fails, one can obtain lower and upper bounds on the local asymptotic power of GMS tests by replacing  $J_{h,\lambda}(c(\varphi(\pi_1), h_2, 1 - \alpha))$  by  $J_{h,\lambda}(c(\varphi(\pi_1), h_2, 1 - \alpha)+)$  and  $J_{h,\lambda}(c(\varphi(\pi_1), h_2, 1 - \alpha)-)$ , respectively, in Theorem 4(a), where the latter are the limits from the left and right, respectively, of  $J_{h,\lambda}(x)$  at  $x = c(\varphi(\pi_1), h_2, 1 - \alpha)$ .

(b)  $\lim_{n \rightarrow \infty} P_{F_n}(T_n(\theta_{n,*}) > c(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_{n,*}), 1 - \alpha)) = 1 - J_{h,\lambda}(c(0_{\mathcal{G}}, h_2, 1 - \alpha))$  provided Assumption LA6 also holds, and

(c)  $\lim_{\beta \rightarrow \infty} [1 - J_{h,\beta\lambda_0}(c(\varphi(\pi_1), h_2, 1 - \alpha))] = \lim_{\beta \rightarrow \infty} [1 - J_{h,\beta\lambda_0}(c(0_{\mathcal{G}}, h_2, 1 - \alpha))] = 1$  provided Assumptions LA3', S3, and S4 hold.

**Comments.** **1.** Theorem 4(a) and 4(b) provide the  $n^{-1/2}$ -local alternative power function of the GMS and PA tests, respectively. Theorem 4(c) shows that the asymptotic power of GMS and PA tests is arbitrarily close to one if the  $n^{-1/2}$ -local alternative parameter  $\lambda = \beta\lambda_0$  is sufficiently large in the sense that its scale  $\beta$  is large.

**2.** We have  $c(\varphi(\pi_1), h_2, 1 - \alpha) \leq c(0_{\mathcal{G}}, h_2, 1 - \alpha)$  (because  $\varphi(\pi_1(g)) \geq 0$  for all  $g \in \mathcal{G}$  and  $S(m, \Sigma)$  is non-increasing in  $m_I$  by Assumption S1(b), where  $m = (m'_I, m'_{II})'$ ). Hence, the asymptotic local power of a GMS test is greater than or equal to that of a PA test. Strict inequality holds whenever  $\pi_1(\cdot)$  is such that  $Q(\{g \in \mathcal{G} : \varphi(\pi_1(g)) > 0\}) > 0$ . The latter typically occurs whenever the conditional moment inequality  $E_{F_n}(m_j(W_i, \theta_{n,*}) | X_i)$  for some  $j = 1, \dots, p$  is bounded away from zero as  $n \rightarrow \infty$  with positive  $X_i$  probability.

**3.** The results of Theorem 4 hold under the null hypothesis as well as under the alternative. The results under the null quantify the degree of asymptotic non-similarity of the GMS and PA tests.

**4.** Suppose the assumptions of Theorem 4 hold and each distribution  $F_n$  generates the same identified set, call it  $\Theta_0 = \Theta_{F_n} \forall n \geq 1$ . Then, Theorem 4(a) implies that the asymptotic probability that a GMS CS includes,  $\theta_{n,*}$ , which lies within  $O(n^{-1/2})$  of the identified set, is  $J_{h,\lambda}(c(\varphi(\pi_1), h_2, 1 - \alpha))$ . If  $\lambda = \beta\lambda_0$  and Assumptions LA3', S3, and S4 also hold, then  $\theta_{n,*}$  is not in  $\Theta_0$  (at least for  $\beta$  large) and the asymptotic probability that a GMS or PA CS includes  $\theta_{n,*}$  is arbitrarily close to zero for  $\beta$  arbitrarily large by Theorem 4(c). Analogous results hold for PA CS's.

## 8 Preliminary Consistent Estimation of Identified Parameters and Time Series

In this section, we consider the case in which the moment functions in (2.4) depend on a parameter  $\tau$  as well as  $\theta$  and a preliminary consistent estimator,  $\widehat{\tau}_n(\theta)$ , of  $\tau$  is available when  $\theta$  is the true value. (This requires that  $\tau$  is identified given the true value  $\theta$ .) For example, this situation often arises with game theory models, as in the third

model considered in Section 9 below. The parameter  $\tau$  may be finite dimensional or infinite dimensional. As pointed out to us by A. Aradillas-López, infinite-dimensional parameters arise as expectations functions in some game theory models. Later in the section, we also consider the case where  $\{W_i : i \leq n\}$  are time series observations.

Suppose the moment functions are of the form  $m_j(W_i, \theta, \tau)$  and the model specifies that (2.1) holds with  $m_j(W_i, \theta, \tau_F(\theta))$  in place of  $m_j(W_i, \theta)$  for  $j \leq k$  for some  $\tau_F(\theta)$  that may depend on  $\theta$  and  $F$ .

The normalized sample moment functions are of the form

$$n^{1/2}\overline{m}_n(\theta, g) = n^{-1/2} \sum_{i=1}^n m(W_i, \theta, \widehat{\tau}_n(\theta), g). \quad (8.1)$$

In the infinite-dimensional case,  $m(W_i, \theta, \widehat{\tau}_n(\theta), g)$  can be of the form  $m^*(W_i, \theta, \widehat{\tau}_n(W_i, \theta), g)$ , where  $\widehat{\tau}_n(W_i, \theta) : R^{d_w} \times \Theta \rightarrow R^{d_\tau}$  for some  $d_\tau < \infty$ .

Given (8.1), the quantity  $\Sigma_F(\theta, g, g^*)$  in (5.1) denotes the asymptotic covariance of  $n^{1/2}\overline{m}_n(\theta, \widehat{\tau}_n(\theta), g)$  and  $n^{1/2}\overline{m}_n(\theta, \widehat{\tau}_n(\theta), g^*)$  under  $(\theta, F)$ , rather than  $Cov_F(m(W_i, \theta, g), m(W_i, \theta, g^*))$ . Correspondingly,  $\widehat{\Sigma}_n(\theta, g, g^*)$  is not defined by (4.5) but is taken to be an estimator of  $\Sigma_F(\theta, g, g^*)$  that is consistent under  $(\theta, F)$ . With these adjusted definitions of  $\overline{m}_n(\theta, g)$  and  $\widehat{\Sigma}_n(\theta, g, g^*)$ , the test statistic  $T_n(\theta)$  and GMS or PA critical value  $c_{n,1-\alpha}(\theta)$  are defined in the same way as above.<sup>17</sup>

For example, when  $\tau$  is finite dimensional, the preliminary estimator  $\widehat{\tau}_n(\theta)$  is chosen to satisfy:

$$n^{1/2}(\widehat{\tau}_n(\theta) - \tau_F(\theta)) \rightarrow_d Z_F \text{ as } n \rightarrow \infty \text{ under } (\theta, F) \in \mathcal{F}, \quad (8.2)$$

for some normally distributed random vector  $Z_F$  with mean zero.

The normalized sample moments can be written as

$$\begin{aligned} n^{1/2}\overline{m}_n(\theta, g) &= D_F^{1/2}(\theta)(\nu_{n,F}(\theta, g) + h_{1,n,F}(\theta, g)), \text{ where} \\ \nu_{n,F}(\theta, g) &= n^{-1/2} \sum_{i=1}^n D_F^{-1/2}(\theta)[m(W_i, \theta, \widehat{\tau}_n(\theta), g) - E_F m(W_i, \theta, \tau_F(\theta), g)], \\ h_{1,n,F}(\theta, g) &= n^{1/2} D_F^{-1/2}(\theta) E_F m(W_i, \theta, \tau_F(\theta), g). \end{aligned} \quad (8.3)$$

In place of Assumption M, we use the following empirical process (EP) assumption.

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<sup>17</sup>When computing bootstrap critical values, one needs to bootstrap the estimator  $\widehat{\tau}_n(\theta)$  as well as the observations  $\{W_i : i \leq n\}$ .

Let  $\Rightarrow$  denote weak convergence. Let  $\{a_n : n \geq 1\}$  denote a subsequence of  $\{n\}$ .

**Assumption EP.** (a) For some specification of the parameter space  $\mathcal{F}$  that imposes the conditional moment inequalities and equalities and all  $(\theta, F) \in \mathcal{F}$ ,  $\nu_{n,F}(\theta, \cdot) \Rightarrow \nu_{h_{2,F}(\theta)}(\cdot)$  as  $n \rightarrow \infty$  under  $(\theta, F)$ , for some mean zero Gaussian process  $\nu_{h_{2,F}(\theta)}(\cdot)$  on  $\mathcal{G}$  with covariance kernel  $h_{2,F}(\theta)$  on  $\mathcal{G} \times \mathcal{G}$  and bounded uniformly  $\rho$ -continuous sample paths a.s. for some pseudo-metric  $\rho$  on  $\mathcal{G}$ .

(b) For any subsequence  $\{(\theta_{a_n}, F_{a_n}) \in \mathcal{F} : n \geq 1\}$  for which  $\lim_{n \rightarrow \infty} \sup_{g, g^* \in \mathcal{G}} \|h_{2, F_{a_n}}(\theta_{a_n}, g, g^*) - h_2(g, g^*)\| = 0$  for some  $k \times k$  matrix-valued covariance kernel on  $\mathcal{G} \times \mathcal{G}$ , we have (i)  $\nu_{a_n, F_{a_n}}(\theta_{a_n}, \cdot) \Rightarrow \nu_{h_2}(\cdot)$  and (ii)  $\sup_{g, g^* \in \mathcal{G}} \|\widehat{h}_{2, a_n, F_{a_n}}(\theta_{a_n}, g, g^*) - h_2(g, g^*)\| \rightarrow_p 0$  as  $n \rightarrow \infty$ .

The quantity  $\widehat{h}_{2, a_n, F_{a_n}}(\theta_{a_n}, g, g^*)$  is defined as in previous sections but with  $\widehat{\Sigma}_n(\theta, g, g^*)$  and  $\Sigma_F(\theta, g, g^*)$  defined as in this section.

With Assumption EP in place of Assumption M, the results of Theorem 2 hold when the GMS or PA CS depends on a preliminary estimator  $\widehat{\tau}_n(\theta)$ .<sup>18</sup> (The proof is the same as that given for Theorem 2 in Appendices A and C with Assumption EP replacing the results of Lemma A1.)

Next, we consider time series observations  $\{W_i : i \leq n\}$ . Let the moment conditions and sample moments be defined as in (2.3) and (3.3), but do not impose the definitions of  $\mathcal{F}$  and  $\widehat{\Sigma}_n(\theta, g)$  in (2.3) and (3.4). Instead, define  $\widehat{\Sigma}_n(\theta, g)$  in a way that is suitable for the temporal dependence properties of  $\{m(W_i, \theta, g) : i \leq n\}$ . For example,  $\widehat{\Sigma}_n(\theta, g)$  might need to be defined to be a heteroskedasticity and autocorrelation consistent (HAC) variance estimator. Or, if  $\{m(W_i, \theta) : i \leq n\}$  have zero autocorrelations under  $(\theta, F)$ , define  $\widehat{\Sigma}_n(\theta, g)$  as in (3.4). Given these definitions of  $\overline{m}_n(\theta, g)$  and  $\widehat{\Sigma}_n(\theta, g)$ , define the test statistic  $T_n(\theta)$  and GMS or PA critical value  $c_{n, 1-\alpha}(\theta)$  as in previous sections.<sup>19</sup>

Define  $\nu_{n,F}(\theta, g)$  as in (5.2). Now, with Assumption EP in place of Assumption M, the results of Theorem 2 hold with time series observations.

Note that Assumption EP also can be used when the observations are independent but not identically distributed.

<sup>18</sup>Equation (8.2) is only needed for this result in order to verify Assumption EP(a) in the finite-dimensional  $\tau$  case.

<sup>19</sup>With bootstrap critical values, the bootstrap employed needs to take account of the time series structure of the observations. For example, a block bootstrap does so.

## 9 Monte Carlo Simulations

### 9.1 Description of the Tests

In this section, we provide simulation evidence concerning the finite sample properties of the tests introduced in this paper. We consider three models: a quantile selection model, an interval outcome regression model, and an entry game model with multiple equilibria.

We compare different test statistics and critical values in terms of their coverage probabilities (CP's) for points in the identified set and their false coverage probabilities (FCP's) for points outside the identified set. Obviously, one wants FCP's to be as small as possible. FCP's are directly related to the power of the tests used to construct the CS and are related to the volume of the CS, see Pratt (1961).

The following test statistics are considered: (i) CvM/Sum, (ii) CvM/QLR, (iii) CvM/Max, (iv) KS/Sum, (v) KS/QLR, and (vi) KS/Max. In all three models countable hypercubes and truncated versions of the test statistics are employed. (More details are given below.) The weights  $\{w(r) : r = r_0, \dots\}$  employed by the CvM statistics, see (3.14), are proportional to  $(r^2 + 100)^{-1}$  for a cube with side-edge length indexed by  $r$ , for  $r = r_0, \dots$ . The number of boxes with side-edge length indexed by  $r$  is  $(2r)^{d_X}$ , where  $d_X$  denotes the dimension of the covariate  $X_i$ . The weights are normalized to sum to one, but this does not affect the results.

In all three models we consider the PA/Asy and GMS/Asy critical values. In the first two models we also consider the PA/Bt, GMS/Bt, and Sub critical values. The critical values are simulated using 5001 repetitions (for each original sample repetition).<sup>20</sup> The GMS critical value is based on  $\kappa_{n,bc} = (0.3 \ln(n))^{1/2}$ ,  $B_{n,bc} = (0.4 \ln(n) / \ln \ln(n))^{1/2}$ , and  $\varepsilon = 5/100$ , where *bc* abbreviates “basecase.” The same basecase values  $\kappa_{n,bc}$ ,  $B_{n,bc}$ , and  $\varepsilon$  are used in all three models. Additional results are reported for variations of these values. The subsample size is 20 when the sample size is 250. Results are reported for nominal 95% CS's. The number of simulation repetitions used to compute CP's and FCP's is 5000 for all cases. This yields a simulation standard error of .0031.

The reported FCP's are “CP corrected” by employing a critical value that yields a

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<sup>20</sup>The Sum, QLR, and Max statistics use the functions  $S_1$ ,  $S_2$ , and  $S_3$ , respectively. The PA/Asy and PA/Bt critical values are based on the asymptotic distribution and bootstrap, respectively, and likewise for the GMS/Asy and GMS/Bt critical values. The IUF  $\eta$  is set to 0 because its value, provided it is sufficiently small, has no effect in these models. Sub denotes a (non-recentered) subsampling critical value. The number of subsamples considered is 5001. They are drawn randomly without replacement.



CP equal to .95 at the closest point of the identified set if the CP at the closest point is less than .95. If the CP at the closest point is greater than .95, then no CP correction is carried out. The reason for this “asymmetric” CP correction is that CS’s may have CP’s greater than .95 for points in the identified set, even asymptotically, in the present context and one does not want to reward over-coverage of points in the identified set by CP correcting the critical values when making comparisons of FCP’s.

## 9.2 Quantile Selection Model

### 9.2.1 Description of the Model

In this model we are interested in the conditional  $\tau$ -quantile of a treatment response given the value of a covariate  $X_i$ . The results also apply to conditional quantiles of arbitrary responses that are subject to selection. We introduce a *quantile* monotone instrumental variable (QMIV) condition that is a variant of Manski and Pepper’s (2000) Monotone Instrumental Variable (MIV) condition. (The latter applies when the parameter of interest is a conditional *mean* of a treatment response.) A nice feature of the QMIV condition is that non-trivial bounds are obtained without assuming that the support of the response variable is bounded, which is restrictive in some applications. The nontrivial bounds result from the fact that the df’s that define the quantiles are naturally bounded between 0 and 1.

Other papers that bound quantiles using the natural bounds of df’s include Manski (1994), Lee and Melenberg (1998), and Blundell, Gosling, Ichimura, and Meghir (2007). The QMIV condition differs from the conditions in these papers, although it is closely related to them.<sup>21</sup>

The model set-up is quite similar to that in Manski and Pepper (2000). The observations are i.i.d. for  $i = 1, \dots, n$ . Let  $y_i(t) \in \mathcal{Y}$  be individual  $i$ ’s “conjectured” response variable given treatment  $t \in \mathcal{T}$ . Let  $T_i$  be the realization of the treatment for individual  $i$ . The observed outcome variable is  $Y_i = y_i(T_i)$ . Let  $X_i$  be a covariate whose support

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<sup>21</sup>Manski (1994, pp. 149-153) establishes the worst case quantile bounds, which do not impose any restrictions. Lee and Melenberg (1998, p. 30) and Blundell, Gosling, Ichimura, and Meghir (2007, pp. 330-331) provide quantile bounds based on the assumption of monotonicity in the selection variable  $T_i$  (which is binary in their contexts), which is a quantile analogue of Manski and Pepper’s (2000) monotone treatment selection condition, as well as bounds based on exclusion restrictions. In addition, Blundell, Gosling, Ichimura, and Meghir (2007, pp. 332-333) employ a monotonicity assumption that is close to the QMIV assumption, but their assumption is imposed on the whole conditional distribution of  $y_i(t)$  given  $X_i$ , rather than on a single conditional quantile, and they do not explicitly bound quantiles.

contains an ordered set  $\mathcal{X}$ . We observe  $W_i = (Y_i, X_i)$ . The parameter of interest,  $\theta$ , is the conditional  $\tau$ -quantile of  $y_i(t)$  given  $X_i = x_0$  for some  $t \in \mathcal{T}$  and some  $x_0 \in \mathcal{X}$ , which is denoted  $Q_{y_i(t)|X_i}(\tau|x_0)$ . We assume the conditional distribution of  $y_i(t)$  given  $X_i = x$  is absolutely continuous at its  $\tau$ -quantile for all  $x \in \mathcal{X}$ .

For examples, one could have: (i)  $y_i(t)$  is conjectured wages of individual  $i$  for  $t$  years of schooling,  $T_i$  is realized years of schooling, and  $X_i$  is measured ability or wealth, (ii)  $y_i(t)$  is conjectured wages when individual  $i$  is employed, say  $t = 1$ ,  $X_i$  is measured ability or wealth, and selection occurs due to elastic labor supply, (iii)  $y_i(t)$  is consumer durable expenditures when a durable purchase is conjectured,  $X_i$  is income or non-durable expenditures, and selection occurs because an individual may decide not to purchase a durable, and (iv)  $y_i(t)$  is some health response of individual  $i$  given treatment  $t$ ,  $T_i$  is the realized treatment, which may be non-randomized or randomized but subject to imperfect compliance, and  $X_i$  is some characteristic of individual  $i$ , such as weight, blood pressure, etc.

The quantile monotone IV assumption is as follows:

**Assumption QMIV.** The covariate  $X_i$  satisfies: for some  $t \in \mathcal{T}$  and all  $(x_1, x_2) \in \mathcal{X}^2$  such that  $x_1 \leq x_2$ ,

$$Q_{y_i(t)|X_i}(\tau|x_1) \leq Q_{y_i(t)|X_i}(\tau|x_2),$$

where  $\tau \in (0, 1)$ ,  $\mathcal{X}$  is some ordered subset of the support of  $X_i$ , and  $Q_{y_i(t)|X_i}(\tau|x)$  is the quantile function of  $y_i(t)$  conditional on  $X_i = x$ .<sup>22</sup>

This assumption may be suitable in the applications mentioned above.

Given Assumption QMIV, we have: for  $(x, x_0) \in \mathcal{X}^2$  with  $x \leq x_0$ ,

$$\begin{aligned} \tau &= P(y_i(t) \leq Q_{y_i(t)|X_i}(\tau|x) | X_i = x) \\ &\leq P(y_i(t) \leq \theta | X_i = x) \\ &= P(y_i(t) \leq \theta \ \& \ T_i = t | X_i = x) + P(y_i(t) \leq \theta \ \& \ T_i \neq t | X_i = x) \\ &\leq P(Y_i \leq \theta \ \& \ T_i = t | X_i = x) + P(T_i \neq t | X_i = x), \end{aligned} \tag{9.1}$$

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<sup>22</sup>The “ $\tau$ -quantile monotone IV” terminology follows that of Manski and Pepper (2000). Alternatively, it could be called a “ $\tau$ -quantile monotone covariate.”

Assumption QMIV can be extended to the case where additional (non-monotone) covariates arise, say  $Z_i$ . In this case, the QMIV condition becomes  $Q_{y_i(t)|Z_i, X_i}(\tau|z, x_1) \leq Q_{y_i(t)|Z_i, X_i}(\tau|z, x_2)$  when  $x_1 \leq x_2$  for all  $z$  in some subset  $\mathcal{Z}$  of the support of  $Z_i$ . Also, as in Manski and Pepper (2000), the assumption QMIV is applicable if  $\mathcal{X}$  is only a partially-ordered set.

where first equality holds by the definition of the  $\tau$ -quantile  $Q_{y_i(t)|X_i}(\tau|x)$ , the first inequality holds by Assumption QMIV, and the second inequality holds because  $Y_i = y_i(T_i)$  and  $P(A \cap B) \leq P(B)$ .

Analogously, for  $(x, x_0) \in \mathcal{X}^2$  with  $x \geq x_0$ ,

$$\begin{aligned}
\tau &= P(y_i(t) \leq Q_{y_i(t)|X_i}(\tau|x) | X_i = x) \\
&\geq P(y_i(t) \leq \theta | X_i = x) \\
&= P(y_i(t) \leq \theta \ \& \ T_i = t | X_i = x) + P(y_i(t) \leq \theta \ \& \ T_i \neq t | X_i = x) \\
&\geq P(Y_i \leq \theta \ \& \ T_i = t | X_i = x),
\end{aligned} \tag{9.2}$$

where the first inequality holds by Assumption QMIV and the second inequality holds because  $P(A) \geq 0$ .

The inequalities in (9.1) and (9.2) impose sharp bounds on  $\theta$ . They can be rewritten as conditional moment inequalities:

$$\begin{aligned}
E(1(X_i \leq x_0)[1(Y_i \leq \theta, T_i = t) + 1(T_i \neq t) - \tau] | X_i) &\geq 0 \text{ a.s. and} \\
E(1(X_i \geq x_0)[\tau - 1(Y_i \leq \theta, T_i = t)] | X_i) &\geq 0 \text{ a.s.}
\end{aligned} \tag{9.3}$$

For the purposes of the Monte Carlo simulations, we consider the following data generating process (DGP):

$$\begin{aligned}
y_i(1) &= \mu(X_i) + \sigma(X_i) u_i, \text{ where } \partial\mu(x)/\partial x \geq 0 \text{ and } \sigma(x) \geq 0, \\
T_i &= 1\{\varphi(X_i) + \varepsilon_i \geq 0\}, \text{ where } \partial\varphi(x)/\partial x \geq 0, \\
X_i &\sim Unif[0, 2], \ (\varepsilon_i, u_i) \sim N(0, I_2), \ X_i \perp (\varepsilon_i, u_i), \\
Y_i &= y_i(T_i), \text{ and } t = 1.
\end{aligned} \tag{9.4}$$

The variable  $y_i(0)$  is irrelevant (because  $Y_i$  enters the moment inequalities in (9.3) only through  $1(Y_i \leq \theta, T_i = t)$ ) and, hence, is left undefined.

Under the DGP above,  $X_i$  satisfies the QMIV assumption for any  $\tau \in (0, 1)$ . We consider the median:  $\tau = 0.5$ . We focus on the conditional median of  $y_i(1)$  given  $X_i = 1.5$ , i.e.,  $\theta = Q_{y_i(1)|X_i}(0.5|1.5)$  and  $x_0 = 1.5$ .

Some algebra shows that the conditional moment inequalities in (9.3) imply:

$$\begin{aligned}\theta &\geq \underline{\theta}(x) := \mu(x) + \sigma(x) \Phi^{-1} \left( 1 - [2\Phi(\varphi(x))]^{-1} \right) \text{ for } x \leq 1.5 \text{ and} \\ \theta &\leq \bar{\theta}(x) := \mu(x) + \sigma(x) \Phi^{-1} \left( [2\Phi(\varphi(x))]^{-1} \right) \text{ for } x \geq 1.5.\end{aligned}\tag{9.5}$$

We call  $\underline{\theta}(x)$  and  $\bar{\theta}(x)$  the lower and upper bound functions on  $\theta$ , respectively. The identified set for the quantile selection model is

$$\left[ \sup_{x \leq x_0} \underline{\theta}(x), \inf_{x \geq x_0} \bar{\theta}(x) \right].\tag{9.6}$$

The shape of the lower and upper bound functions depends on the shape of the  $\varphi$ ,  $\mu$ , and  $\sigma$  functions. We consider two specifications, one that yields flat bound functions and the other that yields kinky bound functions.

Under the flat bound DGP,  $\mu(x) = 2$ ,  $\sigma(x) = 1$ , and  $\varphi(x) = 1 \forall x \in [0, 2]$ . In this case,

$$\begin{aligned}\underline{\theta}(x) &= 2 + \Phi^{-1} \left( 1 - [2\Phi(1)]^{-1} \right) \text{ for } x \leq 1.5 \text{ and} \\ \bar{\theta}(x) &= 2 + \Phi^{-1} \left( [2\Phi(1)]^{-1} \right) \text{ for } x \geq 1.5.\end{aligned}\tag{9.7}$$

Figure 1 shows the flat bound functions. The solid line is the lower bound function  $\underline{\theta}(x)$ , and the dashed line is the upper bound function  $\bar{\theta}(x)$ . Note that  $\underline{\theta}(x)$  is defined only for  $x \in [0, 1.5]$  and  $\bar{\theta}(x)$  only for  $x \in [1.5, 2]$ .

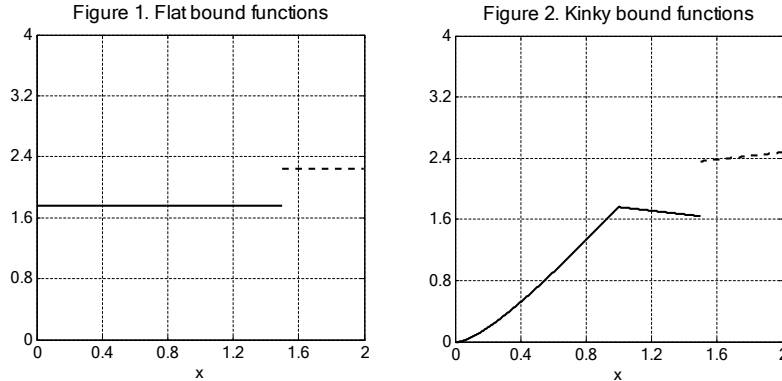
Under the kinky bound DGP,  $\mu(x) = 2(x \wedge 1)$ ,  $\sigma(x) = x$ ,  $\varphi(x) = x \wedge 1$ .<sup>23</sup> In this case,

$$\begin{aligned}\underline{\theta}(x) &= 2(x \wedge 1) + x \cdot \Phi^{-1} \left( 1 - [2\Phi(x \wedge 1)]^{-1} \right) \text{ for } x \leq 1.5 \text{ and} \\ \bar{\theta}(x) &= 2(x \wedge 1) + x \cdot \Phi^{-1} \left( [2\Phi(x \wedge 1)]^{-1} \right) \text{ for } x \geq 1.5.\end{aligned}\tag{9.8}$$

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<sup>23</sup>The kinky shaped  $\mu$  and  $\varphi$  functions are the same as in the simulation example in Chernozhukov, Lee, and Rosen (2008).

Figure 2 shows the kinky bound functions.



### 9.2.2 $g$ Functions

The  $g$  functions employed by the test statistics are indicator functions of hypercubes in  $[0, 1]$ , i.e., intervals. It is not assumed that the researcher knows that  $X_i \sim U[0, 2]$ . Hence, the regressor  $X_i$  is transformed via a general method to lie in  $(0, 1)$ . This method takes the transformed regressor to be  $\Phi((X_i - \bar{X}_n)/\sigma_{X,n})$ , where  $\bar{X}_n$  and  $\sigma_{X,n}$  are the sample mean and standard deviations of  $X_i$  and  $\Phi(\cdot)$  is the standard normal df. The hypercubes have side-edge lengths  $(2r)^{-1}$  for  $r = r_0, \dots, r_1$ , where  $r_0 = 1$  and the basecase value of  $r_1$  is 7. The basecase number of hypercubes is 56. We also report results for  $r_1 = 5, 9$ , and 11, which yield 30, 90, and 132 hypercubes, respectively.

### 9.2.3 Simulation Results

Tables I-III report CP's and CP-corrected FCP's for a variety of test statistics and critical values for a range of cases. The CP's are for the lower endpoint of the identified interval in Tables I-III. (Appendix F of AS provides additional results for the upper endpoints.) FCP's are for points below the lower endpoint.<sup>24</sup>

Table I provides comparisons of different test statistics when each statistic is coupled with PA/Asy and GMS/Asy critical values. Table II provides comparisons of the

<sup>24</sup>Note that the DGP is the same for FCP's as for CP's, just the value  $\theta$  that is to be covered is different. For the lower endpoint of the identified set and the flat bound, FCP's are computed for  $\theta$  equal to  $\underline{\theta}(1) - 0.25 \times \text{sqrt}(250/n)$ . For the lower endpoint with the kinky bound, FCP's are computed for  $\theta$  equal to  $\underline{\theta}(1) - 0.58 \times \text{sqrt}(250/n)$ . These points are chosen to yield similar values for the FCP's across the different cases considered.

Table I. Quantile Selection Model: Basecase Test Statistic Comparisons

(a) Coverage Probabilities							
	Statistic:	CvM/Sum	CvM/QLR	CvM/Max	KS/Sum	KS/QLR	KS/Max
DGP	Crit Val						
Flat Bound	PA/Asy	.979	.979	.976	.972	.972	.970
	GMS/Asy	.953	.953	.951	.963	.963	.960
Kinky Bound	PA/Asy	.999	.999	.999	.994	.994	.994
	GMS/Asy	.983	.983	.983	.985	.985	.984
(b) False Coverage Probabilities (coverage probability corrected)							
Flat Bound	PA/Asy	.51	.50	.48	.68	.67	.66
	GMS/Asy	.37	.37	.37	.60	.60	.59
Kinky Bound	PA/Asy	.65	.65	.62	.68	.68	.67
	GMS/Asy	.35	.35	.34	.53	.53	.52

\* These results are for the lower endpoint of the identified interval.

PA/Asy, PA/Bt, GMS/Asy, GMS/Bt, and Sub critical values for the CvM/Max and KS/Max test statistics. Table III provides robustness results for the CvM/Max and KS/Max statistics coupled with GMS/Asy critical values. The Table III results show the degree of sensitivity of the results to (i) the sample size,  $n$ , (ii) the number of cubes employed, as indexed by  $r_1$ , (iii) the choice of  $(\kappa_n, B_n)$  for the GMS/Asy critical values, and (iv) the value of  $\varepsilon$ , upon which the variance estimator  $\bar{\Sigma}_n(\theta, g)$  depends. Table III also reports results for confidence intervals with nominal level .5, which yield asymptotically half-median unbiased estimates of the lower endpoint.

Table I shows that all tests have CP's greater than or equal to .95 with flat and kinky bound DGP's. The PA/Asy critical values lead to noticeably larger over-coverage than the GMS/Asy critical values. The GMS/Asy critical values lead to CP's that are close to .95 with the flat bound DGP and larger than .95 with the kinky bound DGP. The CP results are not sensitive to the choice of test statistic function: Sum, QLR, or Max. They are only marginally sensitive to the choice of test statistic form: CvM or KS.

The FCP results of Table I show (i) a clear advantage of CvM-based CI's over

Table II. Quantile Selection Model: Basecase Critical Value Comparisons\*

(a) Coverage Probabilities						
	Critical Value:	PA/Asy	PA/Bt	GMS/Asy	GMS/Bt	Sub
DGP	Statistic					
Flat Bound	CvM/Max	.976	.977	.951	.950	.983
	KS/Max	.970	.973	.960	.959	.942
Kinky Bound	CvM/Max	.999	.999	.983	.982	.993
	KS/Max	.994	1.00	.984	.982	.950

(b) False Coverage Probabilities (coverage probability corrected)						
Flat Bound	CvM/Max	.48	.49	.37	.36	.57
	KS/Max	.66	.69	.59	.57	.69
Kinky Bound	CvM/Max	.62	.64	.34	.33	.47
	KS/Max	.67	.72	.52	.50	.47

\* These results are for the lower endpoint of the identified interval.

KS-based CI's, (ii) a clear advantage of GMS/Asy critical values over PA/Asy critical values, and (iii) little difference between the test statistic functions: Sum, QLR, and Max. These results hold for both the flat and kinky bound DGP's.

Table II compares the critical values PA/Asy, PA/Bt, GMS/Asy, GMS/Bt, and Sub. The results show little difference in terms of CP's and FCP's between the Asy and Bt versions of the PA and GMS critical values in most cases. The GMS critical values noticeably out-perform the PA critical values in terms of FCP's. For the CvM/Max statistic, which is the better statistic of the two considered, the GMS critical values also noticeably out-perform the Sub critical values in terms of FCP's.

Table III provides results for the CvM/Max and KS/Max statistics coupled with the GMS/Asy critical values for several variations of the basecase. The table shows that these CS's perform quite similarly for different sample sizes, different numbers of cubes, and a smaller constant  $\varepsilon$ .<sup>25</sup> There is some sensitivity to the magnitude of the GMS

<sup>25</sup>The  $\theta$  value at which the FCP's are computed differs from the lower endpoint of the identified set by a distance that depends on  $n^{-1/2}$ . Hence, Table III suggests that the "local alternatives" that give equal FCP's decline with  $n$  at a rate that is slightly faster than  $n^{-1/2}$  over the range  $n = 100$  to  $1000$ .

Table III. Quantile Selection Model with Flat Bound: Variations on the Basecase\*

Case	(a) Coverage Probabilities		(b) False Cov Probs (CPcor)		
	Statistic:	CvM/Max	KS/Max	CvM/Max	KS/Max
	Crit Val:	GMS/Asy	GMS/Asy	GMS/Asy	GMS/Asy
Basecase ( $n = 250, r_1 = 7, \varepsilon = 5/100$ )		.951	.960	.37	.59
$n = 100$		.957	.968	.40	.64
$n = 500$		.954	.955	.36	.58
$n = 1000$		.948	.948	.34	.57
$r_1 = 5$		.949	.954	.36	.56
$r_1 = 9$		.951	.963	.37	.61
$r_1 = 11$		.951	.966	.37	.63
$(\kappa_n, B_n) = 1/2(\kappa_{n,bc}, B_{n,bc})$		.948	.954	.38	.58
$(\kappa_n, B_n) = 2(\kappa_{n,bc}, B_{n,bc})$		.967	.968	.38	.63
$\varepsilon = 1/100$		.949	.957	.37	.64
$\alpha = .5$		.518	.539	.03	.08
$\alpha = .5$ & $n = 500$		.513	.531	.03	.07

\* These results are for the lower endpoint of the identified interval.

tuning parameters,  $(\kappa_n, B_n)$ —doubling their values increases CP's, but halving their values does not decrease their CP's below .95. Across the range of cases considered the CvM-based CS's out perform the KS-based CS's in terms of FCP's and are comparable in terms of CP's.

The last two rows of Table III show that a CS based on  $\alpha = .5$  provides a good choice for an estimator of the identified set. For example, the lower endpoint estimator based on the CvM/Max-GMS/Asy CS with  $\alpha = .5$  is close to being median-unbiased. It is less than the lower bound with probability is .518 and exceeds it with probability .482 when  $n = 250$ .

In conclusion, we find that the CS based on the CvM/Max statistic with the GMS/Asy critical value performs best in the quantile selection models considered. Equally good are the CS's that use the Sum or QLR statistic in place of the Max statistic and the GMS/Bt critical value in place of the GMS/Asy critical value. The CP's and FCP's of the CvM/Max-GMS/Asy CS are quite good over a range of sample sizes.



## 9.3 Interval Outcome Regression Model

### 9.3.1 Description of Model

This model has been considered by Manski and Tamer (2002, Sec. 4.5). It is a regression model where the outcome variable  $Y_i^*$  is partially observed:

$$Y_i^* = \theta_1 + X_i\theta_2 + U_i, \text{ where } E(U_i|X_i) = 0 \text{ a.s., for } i = 1, \dots, n. \quad (9.9)$$

One observes  $X_i$  and an interval  $[Y_{L,i}, Y_{U,i}]$  that contains  $Y_i^*$ :  $Y_{L,i} = [Y_i]$  and  $Y_{U,i} = [Y_i] + 1$ , where  $[x]$  denotes the integer part of  $x$ . Thus,  $Y_i^* \in [Y_{L,i}, Y_{U,i}]$ .

It is straightforward to see that the following conditional moment inequalities hold in this model:

$$\begin{aligned} E(\theta_1 + X_i\theta_2 - Y_{L,i}|X_i) &\geq 0 \text{ a.s. and} \\ E(Y_{U,i} - \theta_1 - X_i\theta_2|X_i) &\geq 0 \text{ a.s.} \end{aligned} \quad (9.10)$$

In the simulation experiment, we take the true parameters to be  $(\theta_1, \theta_2) = (1, 1)$  (without loss of generality),  $X_i \sim U[0, 1]$ , and  $U_i \sim N(0, 1)$ . We consider a basecase sample size of  $n = 250$ , as well as  $n = 100, 500$ , and  $1000$ .

The parameter  $\theta = (\theta_1, \theta_2)$  is not identified. The identified set is a parallelogram in  $(\theta_1, \theta_2)$  space with vertices at  $(.5, 1), (.5, 2), (1.5, 0)$ , and  $(1.5, 1)$ . (Appendix F of AS provides a figure that illustrates the identified set.) By symmetry, CP's of CS's are the same for the points  $(.5, 1)$  and  $(1.5, 1)$ . Also, they are the same for  $(.5, 2)$  and  $(1.5, 0)$ . We focus on CP's at the corner point  $(.5, 1)$ , which is in the identified set, and at points close to  $(.5, 1)$  but outside the identified set.<sup>26</sup> The corner point  $(.5, 1)$  is of interest because it is a point in the identified set where CP's of CS's typically are strictly less than one. Due to the features of the model, the CP's of CS's typically equal one (or essentially equal one) at interior points, non-corner boundary points, and the corner points  $(.5, 2)$  and  $(1.5, 0)$ .

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<sup>26</sup>Specifically, the  $\theta$  values outside the identified set are given by  $\theta_1 = 0.5 - 0.075 \times (500/n)^{1/2}$  and  $\theta_2 = 1.0 - 0.050 \times (500/n)^{1/2}$ . These  $\theta$  values are selected so that the FCP's of the CS's take values in an interesting range for all values of  $n$  considered.

### 9.3.2 $g$ Functions

The  $g$  functions employed by the test statistics are indicator functions of hypercubes in  $[0, 1]$ . It is not assumed that the researcher knows that  $X_i \sim U[0, 1]$  and so the regressor  $X_i$  is transformed via the same method as in the quantile selection model to lie in  $(0, 1)$ .

### 9.3.3 Simulation Results

Tables IV-VI provide results for the interval outcome regression model that are analogous to the results in Tables I-III for the quantile selection model. In spite of the differences in the models—the former is linear and parametric with a bivariate parameter, while the latter is nonparametric with a scalar parameter—the results are similar.

Table IV shows that the CvM/Max statistic combined with the GMS/Asy critical value has CP's that are very close to the nominal level .95. Its FCP's are noticeably lower than those for CS's that use the KS form or PA-based critical values. The CvM/Sum-GMS/Asy and CvM/QLR-GMS/Asy CS's perform equally well as the Max version. Table V shows that the results for the Asy and Bt versions of the critical values are quite similar for the CvM/Max-GMS CS, which is the best CS. The Sub critical value yields substantial under-coverage for the KS/Max statistic. The Sub critical values are dominated by the GMS critical values in terms of FCP's.

Table VI shows that the CS's do not exhibit much sensitivity to the sample size or the number of cubes employed. It also shows that at the non-corner boundary point  $\theta = (1.0, 0.5)$  and the corner point  $\theta = (1.5, 0)$ , all CP's are (essentially) equal to one.<sup>27</sup> Lastly, Table VI shows that the lower endpoint estimator based on the CvM/Max-GMS/Asy CS with  $\alpha = .5$  is close to being median-unbiased, as in the quantile selection model. It is less than the lower bound with probability is .472 and exceeds it with probability .528 when  $n = 250$ .

We conclude that the preferred CS for this model is of the CvM form, combined with the Max, Sum, or QLR function, and uses a GMS critical value, either Asy or Bt.

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<sup>27</sup>This is due to the fact that the CP's at these points are linked to their CP's at the corner point  $\theta = (0.5, 1.0)$  given the linear structure of the model. If the CP is reduced at the two former points (by reducing the critical value), the CP at the latter point is very much reduced and the CS does not have the desired size.

Table IV. Interval Outcome Regression Model: Basecase Test Statistic Comparisons

(a) Coverage Probabilities							
Critical Value	Statistic:	CvM/Sum	CvM/QLR	CvM/Max	KS/Sum	KS/QLR	KS/Max
	PA/Asy	.990	.993	.990	.989	.990	.989
	GMS/Asy	.950	.950	.950	.963	.963	.963
(b) False Coverage Probabilities (coverage probability corrected)							
	PA/Asy	.62	.66	.61	.78	.80	.78
	GMS/Asy	.37	.37	.37	.61	.61	.61

Table V. Interval Outcome Regression Model: Basecase Critical Value Comparisons

(a) Coverage Probabilities						
Statistic	Critical Value:	PA/Asy	PA/Bt	GMS/Asy	GMS/Bt	Sub
	CvM/Max	.990	.995	.950	.941	.963
	KS/Max	.989	.999	.963	.953	.890
(b) False Coverage Probabilities (coverage probability corrected)						
	CvM/Max	.61	.69	.37	.38	.45
	KS/Max	.78	.96	.61	.54	.66

## 9.4 Entry Game Model

### 9.4.1 Description of the Model

This model is a complete information simultaneous game (entry model) with two players and  $n$  i.i.d. plays of the game. We consider Nash equilibria in pure strategies. Due to the possibility of multiple equilibria, the model is incomplete. In consequence,

Table VI. Interval Outcome Regression Model: Variations on the Basecase

Case	(a) Coverage Probabilities		(b) False Cov Probs (CPcor)		
	Statistic:	CvM/Max	KS/Max	CvM/Max	KS/Max
	Crit Val:	GMS/Asy	GMS/Asy	GMS/Asy	GMS/Asy
Basecase ( $n = 250, r_1 = 7, \varepsilon = 5/100$ )		.950	.963	.37	.61
$n = 100$		.949	.970	.39	.66
$n = 500$		.950	.956	.37	.60
$n = 1000$		.954	.955	.37	.60
$r_1 = 5$ (30 cubes)		.949	.961	.37	.59
$r_1 = 9$ (90 cubes)		.951	.965	.37	.63
$r_1 = 11$ (132 cubes)		.950	.968	.38	.64
$(\kappa_n, B_n) = 1/2(\kappa_{n,bc}, B_{n,bc})$		.944	.961	.40	.62
$(\kappa_n, B_n) = 2(\kappa_{n,bc}, B_{n,bc})$		.958	.973	.39	.65
$\varepsilon = 1/100$		.946	.966	.39	.69
$(\theta_1, \theta_2) = (1.0, 0.5)$		.999	.996	.91	.92
$(\theta_1, \theta_2) = (1.5, 0.0)$		1.000	.996	.99	.97
$\alpha = .5$		.472	.481	.03	.08
$\alpha = .5$ & $n = 500$		.478	.500	.03	.07

two conditional moment inequalities and two conditional moment equalities arise. Andrews, Berry, and Jia (2004), Beresteanu, Molchanov, and Molinari (2009), Galichon and Henry (2009b), and Ciliberto and Tamer (2009) also consider moment inequalities and equalities in models of this sort.

We consider the case where the two players' utility/profits depend linearly on vectors of covariates,  $X_{i,1}$  and  $X_{i,2}$ , with corresponding parameters  $\tau_1$  and  $\tau_2$ . A scalar parameter  $\theta_1$  indexes the competitive effect on player 1 of entry by player 2. Correspondingly,  $\theta_2$  indexes the competitive effect on player 2 of entry by player 1.

Specifically, for player  $b = 1, 2$ , player  $b$ 's utility/profits are given by

$$\begin{aligned}
 &X'_{i,b}\tau_b + U_{i,b} \text{ if the other player does not enter and} \\
 &X'_{i,b}\tau_b - \theta_b + U_{i,b} \text{ if the other player enters,}
 \end{aligned} \tag{9.11}$$

where  $U_{i,b}$  is an idiosyncratic error known to both players, but unobserved by the econometrician. The random variables observed by the econometrician are the covariates  $X_{i,1} \in R^4$  and  $X_{i,2} \in R^4$  and the outcome variables  $Y_{i,1}$  and  $Y_{i,2}$ , where  $Y_{i,b}$  equals 1 if player  $b$  enters and 0 otherwise for  $b = 1, 2$ . The unknown parameters are  $\theta = (\theta_1, \theta_2)' \in [0, \infty)^2$ , and  $\tau = (\tau_1', \tau_2')' \in R^8$ . Let  $Y_i = (Y_{i,1}, Y_{i,2})$  and  $X_i = (X_{i,1}', X_{i,2}')'$ .

The covariate vector  $X_{i,b}$  equals  $(1, X_{i,b,2}, X_{i,b,3}, X_i^*)' \in R^4$ , where  $X_{i,b,2}$  has a Bern( $p$ ) distribution with  $p = 1/2$ ,  $X_{i,b,3}$  has a  $N(0, 1)$  distribution,  $X_i^*$  has a  $N(0, 1)$  distribution and is the same for  $b = 1, 2$ . The idiosyncratic error  $U_{i,b}$  has a  $N(0, 1)$  distribution. All random variables are independent of each other. Except when specified otherwise, the equilibrium selection rule (ESR) employed is a maximum profit ESR (which is unknown to the econometrician). That is, if  $Y_i$  could be either  $(1, 0)$  or  $(0, 1)$  in equilibrium, then it is  $(1, 0)$  if player 1's monopoly profit exceeds that of player 2 and is  $(0, 1)$  otherwise. We also provide some results for a "player 1 first" ESR in which  $Y_i = (1, 0)$  whenever  $Y_i$  could be either  $(1, 0)$  or  $(0, 1)$  in equilibrium.

The moment inequality functions are

$$\begin{aligned} m_1(W_i, \theta, \tau) &= P(X_{i,1}'\tau_1 + U_{i,1} \geq 0, X_{i,2}'\tau_2 - \theta_2 + U_{i,2} \leq 0 | X_i) - 1(Y_i = (1, 0)) \\ &= \Phi(X_{i,1}'\tau_1)\Phi(-X_{i,2}'\tau_2 + \theta_2) - 1(Y_i = (1, 0)) \text{ and} \\ m_2(W_i, \theta, \tau) &= P(X_{i,1}'\tau_1 - \theta_1 + U_{i,1} \leq 0, X_{i,2}'\tau_2 + U_{i,2} \geq 0 | X_i) - 1(Y_i = (0, 1)), \\ &= \Phi(-X_{i,1}'\tau_1 + \theta_1)\Phi(X_{i,2}'\tau_2) - 1(Y_i = (0, 1)). \end{aligned} \tag{9.12}$$

We have  $E(m_1(W_i, \theta_0, \tau_0) | X_i) \geq 0$  a.s., where  $\theta_0$  and  $\tau_0$  denote the true values, because given  $X_i$  a necessary condition for  $Y_i = (1, 0)$  is  $X_{i,1}'\tau_1 + U_{i,1} \geq 0$  and  $X_{i,2}'\tau_2 - \theta_2 + U_{i,2} \leq 0$ . However, this condition is not sufficient for  $Y_i = (1, 0)$  because some sample realizations with  $Y_i = (0, 1)$  also may satisfy this condition. An analogous argument leads to  $E(m_2(W_i, \theta_0, \tau_0) | X_i) \geq 0$  a.s.

The two moment equality functions are

$$\begin{aligned} m_3(W_i, \theta, \tau) &= 1(Y_i = (1, 1)) - P(X_{i,1}'\tau_1 - \theta_1 + U_{i,1} \geq 0, X_{i,2}'\tau_2 - \theta_2 + U_{i,2} \geq 0 | X_i), \\ &= 1(Y_i = (1, 1)) - \Phi(X_{i,1}'\tau_1 - \theta_1)\Phi(X_{i,2}'\tau_2 - \theta_2), \text{ and} \\ m_4(W_i, \theta, \tau) &= 1(Y_i = (0, 0)) - P(X_{i,1}'\tau_1 + U_{i,1} \leq 0, X_{i,2}'\tau_2 + U_{i,2} \leq 0 | X_i) \\ &= 1(Y_i = (0, 0)) - \Phi(-X_{i,1}'\tau_1)\Phi(-X_{i,2}'\tau_2). \end{aligned} \tag{9.13}$$

We employ a preliminary estimator of  $\tau$  given  $\theta$ , as in Section 8. In particular, we

use the probit ML estimator  $\widehat{\tau}_n(\theta) = (\widehat{\tau}_{n,1}(\theta)', \widehat{\tau}_{n,2}(\theta)')'$  of  $\tau = (\tau_1', \tau_2')'$  given  $\theta$  based on the observations  $\{(1(Y_i = (0, 0))), 1(Y_i = (1, 1)), X_{i,1}, X_{i,2}) : i \leq n\}$ .<sup>28</sup>

The model described above is point identified because  $\tau$  is identified by the second moment equality  $m_4(W_i, \theta, \tau)$  and  $\theta$  is identified by the first moment equality  $m_3(W_i, \theta, \tau)$  given that  $\tau$  is identified. However, additional information about  $\theta$  and  $\tau$  is provided by the moment inequalities in (9.12), which we exploit by the methods employed here.

### 9.4.2 g Functions

We take the functions  $g$  to be hypercubes in  $R^2$ . They are functions of the 2-vector  $X_i^* = (X_{i,1}^{*'}, X_{i,2}^{*'})' = (X_{i,1}'\widehat{\tau}_{n,1}(\theta), X_{i,2}'\widehat{\tau}_{n,2}(\theta))'$ . The vector  $X_i^*$  is transformed first to have sample mean equal to zero and sample variance matrix equal to  $I_2$  (by multiplication by the inverse of the upper-triangular Cholesky decomposition of the sample covariance matrix of  $X_i^*$ ). Then, it is transformed to lie in  $[0, 1]^2$  by applying the standard normal df  $\Phi(\cdot)$  element by element.

The hypercubes have side-edge lengths  $(2r)^{-1}$  for  $r = r_0, \dots, r_1$ , where  $r_0 = 1$  and the basecase value of  $r_1$  is 3. The basecase number of hypercubes is 56. We also report results for  $r_1 = 2$  and 4, which yield 20 and 120 hypercubes, respectively.

### 9.4.3 Simulation Results

Tables VII and VIII provide results for the entry game model. Results are provided for GMS/Asy critical values only because (i) PA/Asy critical values are found to provide identical results and (ii) bootstrap and subsampling critical values are computationally quite costly because they require computation of the bootstrap or subsample ML estimator for each repetition of the critical value calculations.

Table VII provides CP's and FCP's for competitive effect  $\theta$  values ranging from  $(0, 0)$  to  $(3, 1)$ .<sup>29</sup> Table VII shows that the CP's for all CS's vary as  $\theta$  varies with values ranging from .913 to .987. The QLR-based CS's tend to have higher CP's than the Sum- and Max-based CS's. The CvM/Max statistic dominates all other statistics except the CvM/QLR statistic in terms of FCP's. In addition, CvM/Max dominates CvM/QLR—in most cases by a substantial margin—except for  $\theta = (2, 2)$  or  $(3, 1)$ . Hence, CvM/Max

<sup>28</sup>See Appendix F of AS for the specification of the log likelihood function and its gradient.

<sup>29</sup>The  $\theta$  values for which FCP's are computed are given by  $\theta_1 - .1 \times \text{sqrt}(500/n)$  and  $\theta_2 - .1 \times \text{sqrt}(500/n)$ , where  $(\theta_1, \theta_2)$  is the true parameter vector.

Table VII. Entry Game Model: Test Statistic Comparisons for Different Competitive Effects Parameters  $(\theta_1, \theta_2)$

(a) Coverage Probabilities							
Case	Statistic:	CvM/Sum	CvM/QLR	CvM/Max	KS/Sum	KS/QLR	KS/Max
$(\theta_1, \theta_2) = (0, 0)$		.979	.972	.980	.977	.975	.985
$(\theta_1, \theta_2) = (1, 0)$		.961	.980	.965	.959	.983	.972
$(\theta_1, \theta_2) = (1, 1)$		.961	.985	.961	.955	.985	.962
$(\theta_1, \theta_2) = (2, 0)$		.935	.982	.935	.944	.984	.952
$(\theta_1, \theta_2) = (2, 1)$		.943	.974	.940	.953	.987	.947
$(\theta_1, \theta_2) = (3, 0)$		.921	.975	.915	.938	.935	.984
$(\theta_1, \theta_2) = (2, 2)$		.928	.942	.913	.943	.972	.922
$(\theta_1, \theta_2) = (3, 1)$		.928	.950	.918	.949	.973	.932

(b) False Coverage Probabilities (coverage probability corrected)							
$(\theta_1, \theta_2) = (0, 0)$		.76	.99	.59	.91	.99	.83
$(\theta_1, \theta_2) = (1, 0)$		.60	.99	.42	.83	.66	.99
$(\theta_1, \theta_2) = (1, 1)$		.62	.96	.41	.82	.99	.58
$(\theta_1, \theta_2) = (2, 0)$		.51	.83	.35	.66	.96	.47
$(\theta_1, \theta_2) = (2, 1)$		.57	.57	.38	.69	.82	.44
$(\theta_1, \theta_2) = (3, 0)$		.49	.41	.36	.61	.43	.64
$(\theta_1, \theta_2) = (2, 2)$		.59	.34	.39	.65	.42	.49
$(\theta_1, \theta_2) = (3, 1)$		.57	.27	.39	.65	.47	.44

is clearly the best statistic in terms of FCP's. The CP's of the CvM/Max CS are good for many  $\theta$  values, but they are low for relatively large  $\theta$  values. For  $\theta = (3, 0)$ ,  $(2, 2)$ , and  $(3, 1)$ , its CP's are .915, .913, and .918, respectively. This is a "small" sample effect—for  $n = 1000$ , this CS has CP's for these three cases equal to .934, .951, and .952, respectively.

Table VIII provides results for variations on the basecase  $\theta$  value of  $(1, 1)$  for the CvM/Max and KS/Max statistics combined with GMS/Asy critical values. The CP's and FCP's of the CvM/Max CS increase with  $n$ . They are not sensitive to the number of hypercubes. There is some sensitivity to the magnitude of  $(\kappa_n, B_n)$ , but it is relatively small. There is noticeable sensitivity of the CP of the KS/Max CS to  $\varepsilon$ , but less so for

Table VIII. Entry Game Model: Variations on the Basecase  $(\theta_1, \theta_2) = (1, 1)$

Case	(a) Coverage Probabilities		(b) False Cov Probs (CPcor)		
	Statistic:	CvM/Max	KS/Max	CvM/Max	KS/Max
	Crit Val:	GMS/Asy	GMS/Asy	GMS/Asy	GMS/Asy
Basecase ( $n = 500, r_1 = 3, \varepsilon = 5/100$ )		<b>.961</b>	<b>.962</b>	<b>.41</b>	<b>.58</b>
$n = 250$		.948	.963	.39	.56
$n = 1000$		.979	.968	.52	.65
$r_1 = 2$ (20 cubes)		.962	.956	.41	.55
$r_1 = 4$ (120 cubes)		.962	.964	.42	.59
$(\kappa_n, B_n) = 1/2(\kappa_{n,bc}, B_{n,bc})$		.954	.959	.39	.57
$(\kappa_n, B_n) = 2(\kappa_{n,bc}, B_{n,bc})$		.967	.962	.42	.58
$\varepsilon = 1/100$		.926	.873	.32	.66
Reg'r Variances = 2		.964	.968	.54	.71
Reg'r Variances = 1/2		.963	.966	.29	.43
Player 1 First Eq Sel Rule		.955	.957	.39	.57
$\alpha = .5$		.610	.620	.05	.13
$\alpha = .5$ & $n = 1000$		.695	.650	.06	.16

the CvM/Max CS. There is relatively little sensitivity of CP's to changes in the DGP via changes in the regressor variances (of  $X_{i,b,2}$  and  $X_{i,b,3}$  for  $b = 1, 2$ ) or a change in the equilibrium selection rule to player 1 first.

The last two rows of Table VIII provide results for estimators of the identified set based on CS's with  $\alpha = .5$ . The two CS's considered are half-median unbiased. For example, the CvM/Max-GMS/Asy CS with  $\alpha = .5$  covers the true value with probability .610, which exceeds .5, when  $n = 500$ .

In conclusion, in the entry game model we prefer the CvM/Max-GMS/Asy CS over other CS's considered because of its the clear superiority in terms of FCP's even though it under-covers somewhat for large values of the competitive effects vector  $\theta$ .



## 10 Appendix A

In this Appendix, we prove Theorems 1 and 2(a). Proofs of the other results stated in the paper are given in Appendix C in AS.

The following Lemma is used in the proofs of Theorems 1, 2, 3, and 4. It establishes a functional CLT and uniform LLN for certain independent non-identically distributed empirical processes.

Let  $h_2$  denote a  $k \times k$ -matrix-valued covariance kernel on  $\mathcal{G} \times \mathcal{G}$  (such as an element of  $\mathcal{H}_2$ ).

**Definition SubSeq( $h_2$ ).**  $SubSeq(h_2)$  is the set of subsequences  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\}$ , where  $\{a_n : n \geq 1\}$  is some subsequence of  $\{n\}$ , for which

$$(i) \lim_{n \rightarrow \infty} \sup_{g, g^* \in \mathcal{G}} \|h_{2, F_{a_n}}(\theta_{a_n}, g, g^*) - h_2(g, g^*)\| = 0,$$

(ii)  $\theta_{a_n} \in \Theta$ , (iii)  $\{W_i : i \geq 1\}$  are i.i.d. under  $F_{a_n}$ , (iv)  $Var_{F_{a_n}}(m_j(W_i, \theta_{a_n})) > 0$  for  $j = 1, \dots, k$ , for  $n \geq 1$ , (v)  $\sup_{n \geq 1} E_{F_{a_n}} |m_j(W_i, \theta_{a_n}) / \sigma_{F_{a_n}, j}(\theta_{a_n})|^{2+\delta} < \infty$  for  $j = 1, \dots, k$ , for some  $\delta > 0$ , and (vi) Assumption M holds with  $F_{a_n}$  in place of  $F$  and  $F_n$  in Assumptions M(b) and M(c), respectively.

The sample paths of the Gaussian process  $\nu_{h_2}(\cdot)$ , which is defined in (4.2) and appears in the following Lemma, are bounded and uniformly  $\rho$ -continuous a.s. The pseudo-metric  $\rho$  on  $\mathcal{G}$  is a pseudo-metric commonly used in the empirical process literature:

$$\rho^2(g, g^*) = tr(h_2(g, g) - h_2(g, g^*) - h_2(g^*, g) + h_2(g^*, g^*)). \quad (10.1)$$

For  $h_2(\cdot, \cdot) = h_{2, F}(\theta, \cdot, \cdot)$ , where  $(\theta, F) \in \mathcal{F}$ , this metric can be written equivalently as

$$\begin{aligned} \rho^2(g, g^*) &= E_F \|D_F^{-1/2}(\theta)[\tilde{m}(W_i, \theta, g) - \tilde{m}(W_i, \theta, g^*)]\|^2, \text{ where} \\ \tilde{m}(W_i, \theta, g) &= m(W_i, \theta, g) - E_F m(W_i, \theta, g). \end{aligned} \quad (10.2)$$

**Lemma A1.** For any subsequence  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\} \in SubSeq(h_2)$ ,

(a)  $\nu_{a_n, F_{a_n}}(\theta_{a_n}, \cdot) \Rightarrow \nu_{h_2}(\cdot)$  as  $n \rightarrow \infty$  (as processes indexed by  $g \in \mathcal{G}$ ), and

(b)  $\sup_{g, g^* \in \mathcal{G}} \|\widehat{h}_{2, a_n, F_{a_n}}(\theta_{a_n}, g, g^*) - h_2(g, g^*)\| \rightarrow_p 0$  as  $n \rightarrow \infty$ .

**Comments.** 1. The proof of Lemma A1 is given in Appendix E of AS. Part (a) is proved by establishing the manageability of  $\{m(W_i, \theta_{a_n}, g) - E_{F_{a_n}} m(W_i, \theta_{a_n}, g) : g \in \mathcal{G}\}$

and by establishing a functional central limit theorem (FCLT) for  $R^k$ -valued i.n.i.d. empirical processes with the pseudo-metric  $\rho$  by using the FCLT in Pollard (1990, Thm. 10.2) for real-valued empirical processes. Part (b) is proved using a maximal inequality given in Pollard (1990, (7.10)).

**2.** To obtain uniform asymptotic coverage probability results for CS's, Lemma A1 is applied with  $(\theta_{a_n}, F_{a_n}) \in \mathcal{F}$  for all  $n \geq 1$  and  $h_2 \in \mathcal{H}_2$ . In this case, conditions (ii)-(vi) in the definition of  $SubSeq(h_2)$  hold automatically by the definition of  $\mathcal{F}$ . To obtain power results under fixed and local alternatives, Lemma A1 is applied with  $(\theta_{a_n}, F_{a_n}) \notin \mathcal{F}$  for all  $n \geq 1$  and  $h_2$  may or may not be in  $\mathcal{H}_2$ .

**Proof of Theorem 1.** First, we prove part (a). Let  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$  be a sequence for which  $h_{2, F_n}(\theta_n) \in \mathcal{H}_{2, cpt}$  for all  $n \geq 1$  and the term in square brackets in Theorem 1(a) evaluated at  $(\theta_n, F_n)$  differs from its supremum over  $(\theta, F) \in \mathcal{F}$  with  $h_{2, F}(\theta) \in \mathcal{H}_{2, cpt}$  by  $\delta_n$  or less, where  $0 < \delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Such a sequence always exists. To prove part (a), it suffices to show that part (a) holds with the supremum deleted and with  $(\theta, F)$  replaced by  $(\theta_n, F_n)$ .

By the compactness of  $\mathcal{H}_{2, cpt}$ , given any subsequence  $\{u_n : n \geq 1\}$  of  $\{n\}$ , there exists a subsubsequence  $\{a_n : n \geq 1\}$  for which  $d(h_{2, F_{a_n}}(\theta_{a_n}), h_{2, 0}) \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\theta_0 \in \Theta$ , where  $d$  is defined in (5.6), and some  $h_{2, 0} \in \mathcal{H}_{2, cpt}$ . This and  $(\theta_{a_n}, F_{a_n}) \in \mathcal{F}$  for all  $n \geq 1$  implies that  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\} \in SubSeq(h_2)$ .

Now, by Lemma A1, we have

$$\begin{pmatrix} \nu_{a_n, F_{a_n}}(\theta_{a_n}, \cdot) \\ \widehat{h}_{2, a_n, F_{a_n}}(\theta_{a_n}, \cdot) \end{pmatrix} \Rightarrow \begin{pmatrix} \nu_{h_{2, 0}}(\cdot) \\ h_{2, 0}(\cdot) \end{pmatrix} \text{ as } n \rightarrow \infty \quad (10.3)$$

as stochastic processes on  $\mathcal{G}$ , where  $\widehat{h}_{2, a_n, F_{a_n}}(\theta_{a_n}, g) = \widehat{h}_{2, a_n, F_{a_n}}(\theta_{a_n}, g, g)$  and  $h_{2, 0}(g) = h_{2, 0}(g, g)$ .

Given this, by the almost sure representation theorem, e.g., see Pollard (1990, Thm. 9.4), there exists a probability space and random quantities  $\tilde{\nu}_{a_n}(\cdot)$ ,  $\tilde{h}_{2, a_n}(\cdot)$ ,  $\tilde{\nu}_0(\cdot)$ , and  $\tilde{h}_2(\cdot)$  defined on it such that (i)  $(\tilde{\nu}_{a_n}(\cdot), \tilde{h}_{2, a_n}(\cdot))$  has the same distribution as  $(\nu_{a_n, F_{a_n}}(\theta_{a_n}, \cdot), \widehat{h}_{2, a_n, F_{a_n}}(\theta_{a_n}, \cdot))$ , (ii)  $(\tilde{\nu}_0(\cdot), \tilde{h}_2(\cdot))$  has the same distribution as  $(\nu_{h_{2, 0}}(\cdot), h_{2, 0}(\cdot))$ , and

$$(iii) \sup_{g \in \mathcal{G}} \left\| \begin{pmatrix} \tilde{\nu}_{a_n}(g) \\ \tilde{h}_{2, a_n}(g) \end{pmatrix} - \begin{pmatrix} \tilde{\nu}_0(g) \\ \tilde{h}_2(g) \end{pmatrix} \right\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ a.s.} \quad (10.4)$$

Because  $h_{2,0}(\cdot)$  is deterministic, condition (ii) implies that  $\tilde{h}_2(\cdot) = h_{2,0}(\cdot)$  a.s.

Define

$$\begin{aligned}\tilde{h}_{2,a_n}^\varepsilon(\cdot) &= \tilde{h}_{2,a_n}(\cdot) + \varepsilon \cdot \text{Diag}(\tilde{h}_{2,a_n}(\mathbf{1}_k)), \\ \tilde{T}_{a_n} &= \int S(\tilde{\nu}_{a_n}(g) + h_{1,a_n,F_{a_n}}(\theta_{a_n}, g), \tilde{h}_{2,a_n}^\varepsilon(g)) dQ(g), \\ h_{2,0}^\varepsilon(\cdot) &= h_{2,0}(\cdot) + \varepsilon I_k, \text{ and} \\ \tilde{T}_{a_n,0} &= \int S(\tilde{\nu}_0(g) + h_{1,a_n,F_{a_n}}(\theta_{a_n}, g), h_{2,0}^\varepsilon(g)) dQ(g).\end{aligned}\tag{10.5}$$

By construction,  $\tilde{T}_{a_n}$  and  $T_{a_n}(\theta_{a_n})$  have the same distribution, and  $\tilde{T}_{a_n,0}$  and  $T(h_{a_n,F_{a_n}}(\theta_{a_n}))$  have the same distribution for all  $n \geq 1$ .

Hence, to prove part (a), it suffices to show that

$$A = \limsup_{n \rightarrow \infty} \left[ P_{F_{a_n}}(\tilde{T}_{a_n} > x_{h_{a_n,F_{a_n}}}(\theta_{a_n})) - P(\tilde{T}_{a_n,0} + \delta > x_{h_{a_n,F_{a_n}}}(\theta_{a_n})) \right] \leq 0.\tag{10.6}$$

Below we show that

$$\tilde{T}_{a_n} - \tilde{T}_{a_n,0} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ a.s.}\tag{10.7}$$

Let

$$\begin{aligned}\tilde{\Delta}_n &= 1(\tilde{T}_{a_n,0} + (\tilde{T}_{a_n} - \tilde{T}_{a_n,0}) > x_{h_{a_n,F_{a_n}}}(\theta_{a_n})) - 1(\tilde{T}_{a_n,0} + \delta > x_{h_{a_n,F_{a_n}}}(\theta_{a_n})) \\ &= \tilde{\Delta}_n^+ - \tilde{\Delta}_n^-, \text{ where} \\ \tilde{\Delta}_n^+ &= \max\{\tilde{\Delta}_n, 0\} \in [0, 1] \text{ and } \tilde{\Delta}_n^- = \max\{-\tilde{\Delta}_n, 0\} \in [0, 1].\end{aligned}\tag{10.8}$$

By (10.7) and  $\delta > 0$ ,  $\lim_{n \rightarrow \infty} \tilde{\Delta}_n^+ = 0$  a.s. Hence, by the BCT,

$$\begin{aligned}\lim_{n \rightarrow \infty} E_{F_{a_n}} \tilde{\Delta}_n^+ &= 0 \text{ and} \\ A &= \limsup_{n \rightarrow \infty} E_{F_{a_n}} \tilde{\Delta}_n = \limsup_{n \rightarrow \infty} E_{F_{a_n}} \tilde{\Delta}_n^+ - \liminf_{n \rightarrow \infty} E_{F_{a_n}} \tilde{\Delta}_n^- \\ &= -\liminf_{n \rightarrow \infty} E_{F_{a_n}} \tilde{\Delta}_n^- \leq 0.\end{aligned}\tag{10.9}$$

Hence, (10.6) holds and the proof of part (a) is complete, except for (10.7).

To prove part (b), analogous results to (10.6), (10.8), and (10.9) hold by analogous arguments.

It remains to show (10.7). We do so by fixing a sample path  $\omega$  and using the bounded

convergence theorem (because  $\tilde{T}_{a_n}$  and  $\tilde{T}_{a_n,0}$  are both integrals over  $g \in \mathcal{G}$  with respect to the measure  $Q$ ). Let  $\tilde{\Omega}$  be the collection of all  $\omega \in \Omega$  such that  $(\tilde{\nu}_{a_n}(g), \tilde{h}_{2,a_n}(g))(\omega)$  converges to  $(\tilde{\nu}_0(g), h_{2,0}(g))(\omega)$  uniformly over  $g \in \mathcal{G}$  as  $n \rightarrow \infty$  and  $\sup_{g \in \mathcal{G}} \|\tilde{\nu}_0(g)(\omega)\| < \infty$ . By (10.4) and  $\tilde{h}_2(\cdot) = h_{2,0}(\cdot)$  a.s.,  $P(\tilde{\Omega}) = 1$ . Consider a fixed  $\omega \in \tilde{\Omega}$ . By Assumption S2 and (10.4), for all  $g \in \mathcal{G}$ ,

$$\sup_{\mu \in \mathbb{R}_+^p \times \{0\}^v} \left| S \left( \tilde{\nu}_{a_n}(g)(\omega) + \mu, \tilde{h}_{2,a_n}^\varepsilon(g)(\omega) \right) - S \left( \tilde{\nu}_0(g)(\omega) + \mu, h_{2,0}^\varepsilon(g) \right) \right| \rightarrow 0 \quad (10.10)$$

as  $n \rightarrow \infty$  a.s. Thus, for all  $g \in \mathcal{G}$  and all  $\omega \in \tilde{\Omega}$ ,

$$\begin{aligned} & S \left( \tilde{\nu}_{a_n}(g)(\omega) + h_{1,a_n,F_{a_n}}(\theta_{a_n}, g), \tilde{h}_{2,a_n}^\varepsilon(g)(\omega) \right) \\ & \quad - S \left( \tilde{\nu}_0(g)(\omega) + h_{1,a_n,F_{a_n}}(\theta_{a_n}, g), h_{2,0}^\varepsilon(g) \right) \\ & \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (10.11)$$

Next, we show that for fixed  $\omega \in \tilde{\Omega}$  the first summand on the left-hand side of (10.11) is bounded by a constant. Let  $0 < \chi < 1$ . By (10.4), there exists  $N < \infty$  such that for all  $n \geq N$ ,

$$\sup_{g \in \mathcal{G}} \|\tilde{\nu}_{a_n}(g)(\omega) - \tilde{\nu}_0(g)(\omega)\| < \chi \text{ and } \left\| \text{Diag}(\tilde{h}_{2,a_n}(1_k))(\omega) - I_k \right\| < \chi \quad (10.12)$$

using the fact that  $\text{Diag}(h_{2,0}(1_k)) = I_k$  by construction. Let  $B_\chi(\omega) = \sup_{g \in \mathcal{G}} \|\tilde{\nu}_0(g)(\omega)\| + \chi$ . Then, for all  $n \geq N$ ,

$$\sup_{g \in \mathcal{G}} \|\tilde{\nu}_{a_n}(g)(\omega)\| \leq B_\chi(\omega) < \infty. \quad (10.13)$$

First, consider the case where no moment equalities are present, i.e.,  $v = 0$  and  $k = p$ . In this case, for  $n \geq N$ , we have: for all  $g \in \mathcal{G}$ ,

$$\begin{aligned} 0 & \leq S(\tilde{\nu}_{a_n}(g)(\omega) + h_{1,a_n,F_{a_n}}(\theta_{a_n}, g), \tilde{h}_{2,a_n}^\varepsilon(g)(\omega)) \\ & \leq S(\tilde{\nu}_{a_n}(g)(\omega), \tilde{h}_{2,a_n}^\varepsilon(g)(\omega)) \\ & \leq S(-B_\chi(\omega)1_p, \varepsilon \cdot \text{Diag}(\tilde{h}_{2,a_n}(1_p))) \\ & \leq S(-B_\chi(\omega)1_p, \varepsilon(1 - \chi)I_p), \end{aligned} \quad (10.14)$$

where the first inequality holds by Assumption S1(c), the second inequality holds by

Assumption S1(b) and  $h_{1,a_n,F_{a_n}}(\theta_{a_n}, g) \geq 0_p$  (which holds because  $(\theta_{a_n}, F_{a_n}) \in \mathcal{F}$ ), the third inequality holds by Assumption S1(b) and (10.13) as well as by Assumption S1(e) and the definition of  $\tilde{h}_{2,a_n}^\varepsilon(g)(\omega)$  in (10.5), and the last inequality holds by Assumption S1(e) and (10.12). For fixed  $\omega \in \tilde{\Omega}$ , the constant  $S(-B_\chi(\omega)1_p, \varepsilon(1-\chi)I_p)$  bounds the first summand on the left-hand side of (10.11) for all  $n \geq N$ .

For the case where  $v > 0$ , the third inequality in (10.14) needs to be altered because  $S(m, \Sigma)$  is not assumed to be non-increasing in  $m_{II}$ , where  $m = (m'_I, m'_{II})'$ . In this case, for the bound with respect to the last  $v$  elements of  $\tilde{\nu}_{a_n}(g)(\omega)$ , denoted by  $\tilde{\nu}_{a_n,II}(g)(\omega)$ , we use the continuity condition on  $S(m, \Sigma)$ , i.e., Assumption S1(d), which yields uniform continuity of  $S(-B_\chi(\omega)1_p, m_{II}, \varepsilon(1-\chi)I_k)$  over the compact set  $\{m_{II} : \|m_{II}\| \leq B_\chi(\omega) < \infty\}$  and delivers a finite bound because  $\sup_{g \in \mathcal{G}, n \geq 1} \|\tilde{\nu}_{a_n,II}(g)(\omega)\| \leq B_\chi(\omega)$ .

By an analogous but simpler argument, for fixed  $\omega \in \tilde{\Omega}$ , the second summand on the left-hand side of (10.11) is bounded by a constant.

Hence, the conditions of the BCT hold and for fixed  $\omega \in \tilde{\Omega}$ ,  $\tilde{T}_{a_n}(\omega) - \tilde{T}_{a_n,0}(\omega) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus, (10.7) holds and the proof is complete.  $\square$

For GMS CS's, Theorem 2(a) follows immediately from the following three Lemmas. The PA critical value is a GMS critical value with  $\varphi_n(x) = 0$  for all  $x \in R$  and this function  $\varphi_n(x)$  satisfies Assumption GMS1 (though not Assumption GMS2(b)). Hence, Theorem 2(a) for GMS CS's covers PA CS's.

**Lemma A2.** *Suppose Assumptions M, S1, and S2 hold. Then, for every compact subset  $\mathcal{H}_{2,cpt}$  of  $\mathcal{H}_2$  and all  $\delta > 0$ ,*

$$\limsup_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} P_F(T_n(\theta) > c_0(h_{n,F}(\theta), 1 - \alpha) + \delta) \leq \alpha.$$

**Lemma A3.** *Suppose Assumptions M, S1, and GMS1 hold. Then, for every compact subset  $\mathcal{H}_{2,cpt}$  of  $\mathcal{H}_2$ ,*

$$\lim_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} P_F \left( c(\varphi_n(\theta), \hat{h}_{2,n}(\theta), 1 - \alpha) < c(h_{1,n,F}(\theta), \hat{h}_{2,n}(\theta), 1 - \alpha) \right) = 0.$$

**Lemma A4.** *Suppose Assumptions M, S1, and S2 hold. Then, for every compact*

subset  $\mathcal{H}_{2,cpt}$  of  $\mathcal{H}_2$  and for all  $0 < \delta < \eta$  (where  $\eta$  is as in the definition of  $c(h, 1 - \alpha)$ ),

$$\lim_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} P_F \left( c(h_{1,n,F}(\theta), \widehat{h}_{2,n}(\theta), 1 - \alpha) < c_0(h_{1,n,F}(\theta), h_{2,F}(\theta), 1 - \alpha) + \delta \right) = 0.$$

The following Lemma is used in the proof of Lemma A4.

**Lemma A5.** *Suppose Assumptions M, S1, and S2 hold. Let  $\{h_{2,n} : n \geq 1\}$  and  $\{h_{2,n}^* : n \geq 1\}$  be any two sequences of  $k \times k$ -valued covariance kernels on  $\mathcal{G} \times \mathcal{G}$  such that  $d(h_{2,n}, h_{2,n}^*) \rightarrow 0$  and  $d(h_{2,n}, h_{2,0}) \rightarrow 0$  for some  $k \times k$ -valued covariance kernel  $h_{2,0}$  on  $\mathcal{G} \times \mathcal{G}$ . Then, for all  $\eta_1 > 0$  and all  $\delta > 0$ ,*

$$\liminf_{n \rightarrow \infty} \inf_{h_1 \in \mathcal{H}_1} [c_0(h_1, h_{2,n}, 1 - \alpha + \eta_1) + \delta - c_0(h_1, h_{2,n}^*, 1 - \alpha)] \geq 0.$$

**Proof of Lemma A2.** For all  $\delta > 0$ , we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} P_F (T_n(\theta) > c_0(h_{n,F}(\theta), 1 - \alpha) + \delta) \\ & \leq \limsup_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} [P_F (T_n(\theta) > c_0(h_{n,F}(\theta), 1 - \alpha) + \delta) \\ & \quad - P(T(h_{n,F}(\theta)) > c_0(h_{n,F}(\theta), 1 - \alpha))] \\ & \quad + \limsup_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} P(T(h_{n,F}(\theta)) > c_0(h_{n,F}(\theta), 1 - \alpha)) \\ & \leq 0 + \alpha, \end{aligned} \tag{10.15}$$

where the second inequality holds by Theorem 1(a) with  $x_{h_{n,F}(\theta)} = c_0(h_{n,F}(\theta), 1 - \alpha) + \delta$  and by the definition of the quantile  $c_0(h_{n,F}(\theta), 1 - \alpha)$  of  $T(h_{n,F}(\theta))$ .  $\square$

**Proof of Lemma A3.** Let  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$  be a sequence for which  $h_{2,F_n}(\theta_n) \in \mathcal{H}_{2,cpt}$  and the probability in the statement of the Lemma evaluated at  $(\theta_n, F_n)$  differs from its supremum over  $(\theta, F) \in \mathcal{F}$  (with  $h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}$ ) by  $\delta_n$  or less, where  $0 < \delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Such a sequence always exists. It suffices to show

$$\lim_{n \rightarrow \infty} P_{F_n} \left( c(\varphi_n(\theta_n), \widehat{h}_{2,n}(\theta_n), 1 - \alpha) < c(h_{1,n,F_n}(\theta_n), \widehat{h}_{2,n}(\theta_n), 1 - \alpha) \right) = 0. \tag{10.16}$$

By the compactness of  $\mathcal{H}_{2,cpt}$ , given any subsequence  $\{u_n : n \geq 1\}$  of  $\{n\}$ , there

exists a subsubsequence  $\{a_n : n \geq 1\}$  for which  $d(h_{2,F_{a_n}}(\theta_{a_n}), h_{2,0}) \rightarrow 0$  as  $n \rightarrow \infty$  for some  $h_{2,0} \in \mathcal{H}_{2,cpt}$ . This and  $(\theta_{a_n}, F_{a_n}) \in \mathcal{F}$  for all  $n \geq 1$  implies that  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\} \in SubSeq(h_2)$ . Hence, it suffices to show

$$\lim_{n \rightarrow \infty} P_{F_{a_n}} \left( c(\varphi_{a_n}(\theta_{a_n}), \widehat{h}_{2,a_n}(\theta_{a_n}), 1 - \alpha) < c(h_{1,a_n,F_{a_n}}(\theta_{a_n}), \widehat{h}_{2,a_n}(\theta_{a_n}), 1 - \alpha) \right) = 0 \quad (10.17)$$

for  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\} \in SubSeq(h_2)$ .

By Lemma A1(a), for  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\} \in SubSeq(h_2)$ , we have

$$\nu_{a_n, F_{a_n}}(\theta_{a_n}, \cdot) \Rightarrow \nu_{h_{2,0}}(\cdot) \text{ as } n \rightarrow \infty. \quad (10.18)$$

We now show that for all sequences  $\tau_n \rightarrow \infty$  as  $n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} P_{F_{a_n}} \left( \sup_{g \in \mathcal{G}, j \leq p} |\nu_{a_n, F_{a_n}, j}(\theta_{a_n}, g)| > \tau_{a_n} \right) = 0, \quad (10.19)$$

where  $\nu_{a_n, F_{a_n}, j}(\theta_{a_n}, g)$  denotes the  $j$ th element of  $\nu_{a_n, F_{a_n}}(\theta_{a_n}, g)$ . We show this by noting that (10.18) and the continuous mapping theorem give:  $\forall \tau > 0$ ,

$$\lim_{n \rightarrow \infty} P_{F_{a_n}} \left( \sup_{g \in \mathcal{G}, j \leq p} |\nu_{a_n, F_{a_n}, j}(\theta_{a_n}, g)| > \tau \right) = P \left( \sup_{g \in \mathcal{G}, j \leq p} |\nu_{h_{2,0}, j}(g)| > \tau \right), \quad (10.20)$$

where  $\nu_{h_{2,0}, j}(g)$  denotes the  $j$ th element of  $\nu_{h_{2,0}}(g)$ . In addition, the sample paths of  $\nu_{h_{2,0}, j}(\cdot)$  are bounded a.s., which yields  $1 \left( \sup_{g \in \mathcal{G}, j \leq p} |\nu_{h_{2,0}, j}(g)| > \tau \right) \rightarrow 0$  as  $\tau \rightarrow \infty$  a.s. Hence, by the bounded convergence theorem,

$$\lim_{\tau \rightarrow \infty} P \left( \sup_{g \in \mathcal{G}, j \leq p} |\nu_{h_{2,0}, j}(g)| > \tau \right) = 0. \quad (10.21)$$

Equations (10.20) and (10.21) imply (10.19).

Next, we have

$$\begin{aligned} \xi_{a_n}(\theta_{a_n}, g) &= \kappa_{a_n}^{-1} \left( \overline{D}_{a_n}^{-1/2}(\theta_{a_n}, g) D_{F_{a_n}}^{1/2}(\theta_{a_n}) \right) a_n^{1/2} D_{F_{a_n}}^{-1/2}(\theta_{a_n}) \overline{m}_{a_n}(\theta_{a_n}, g) \\ &= \kappa_{a_n}^{-1} \text{Diag}^{-1/2}(\overline{h}_{2,a_n, F_{a_n}}(\theta_{a_n}, g)) \text{Diag}^{1/2}(\overline{h}_{2,a_n}(\theta_{a_n}, g)) \\ &\quad \times \text{Diag}^{-1/2}(\overline{h}_{2,a_n}(\theta_{a_n}, g)) (\nu_{a_n, F_{a_n}}(\theta_{a_n}, g) + h_{1,a_n, F_{a_n}}(\theta_{a_n}, g)) \\ &= \kappa_{a_n}^{-1} ((I_k + o_p(1)) \text{Diag}^{-1/2}(\overline{h}_{2,a_n}(\theta_{a_n}, g)) (\nu_{a_n, F_{a_n}}(\theta_{a_n}, g) + h_{1,a_n, F_{a_n}}(\theta_{a_n}, g))), \end{aligned} \quad (10.22)$$

where the second equality holds by the definitions of  $\bar{h}_{2,a_n,F_{a_n}}(\theta_{a_n}, g)$ ,  $\nu_{a_n,F_{a_n}}(\theta_{a_n}, g)$ , and  $h_{1,a_n,F_{a_n}}(\theta_{a_n}, g)$  in (5.2) and  $\bar{D}_n(\theta, g) = \text{Diag}(\bar{\Sigma}_n(\theta, g))$  and the third equality holds by Lemma A1(b) using the fact that  $\bar{h}_{2,a_n,F_{a_n}}(\theta_{a_n}, g)$  is a function of  $\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n}, g)$ , see (5.2), and Definition SubSeq( $h_2$ ).

Let  $\tau_n = (\kappa_n/\zeta - B_n)/2$ . By Assumption GMS1(b),  $\tau_n = (\kappa_n - \zeta B_n)/2\zeta \rightarrow \infty$  as  $n \rightarrow \infty$ . Also,

$$(\kappa_n/\zeta - \tau_n) - B_n = (\kappa_n/\zeta + B_n)/2 - B_n = \tau_n \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (10.23)$$

For  $\tau_n$  defined in this way, we have

$$\begin{aligned} & P_{F_{a_n}} \left( c(\varphi_{a_n}(\theta_{a_n}), \widehat{h}_{2,a_n}(\theta_{a_n}), 1 - \alpha) < c(h_{1,a_n,F_{a_n}}(\theta_{a_n}), \widehat{h}_{2,a_n}(\theta_{a_n}), 1 - \alpha) \right) \\ & \leq P_{F_{a_n}} \left( \varphi_{a_n,j}(\theta_{a_n}, g) > h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g) \text{ for some } j \leq p, \text{ some } g \in \mathcal{G} \right) \\ & \leq P_{F_{a_n}} \left( \begin{array}{l} \xi_{a_n,j}(\theta_{a_n}, g) > 1 \ \& \ h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g) < \bar{h}_{2,a_n,F_{a_n},j}(\theta_{a_n}, g)^{1/2} B_{a_n} \\ \text{for some } j \leq p, \text{ some } g \in \mathcal{G} \end{array} \right) \\ & \leq P_{F_{a_n}} \left( \begin{array}{l} (1 + o_p(1)) [\bar{h}_{2,a_n,F_{a_n},j}^{-1/2}(\theta_{a_n}, g) \nu_{a_n,F_{a_n},j}(\theta_{a_n}, g) + \bar{h}_{2,a_n,F_{a_n},j}^{-1/2}(\theta_{a_n}, g) h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g)] \\ > \kappa_{a_n} \ \& \ h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g) < \bar{h}_{2,a_n,F_{a_n},j}(\theta_{a_n}, g)^{1/2} B_{a_n} \text{ for some } j \leq p, \text{ some } g \in \mathcal{G} \end{array} \right) \\ & \leq P_{F_{a_n}} \left( \begin{array}{l} (1 + o_p(1)) [\tau_{a_n} + \bar{h}_{2,a_n,F_{a_n},j}^{-1/2}(\theta_{a_n}, g) h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g)] > \kappa_{a_n} \ \& \\ \bar{h}_{2,a_n,F_{a_n}}^{-1/2}(\theta_{a_n}, g) h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g) < B_{a_n} \text{ for some } j \leq p, \text{ some } g \in \mathcal{G} \end{array} \right) \\ & \quad + P_{F_{a_n}} \left( \sup_{g \in \mathcal{G}, j \leq p} |\bar{h}_{2,a_n,F_{a_n},j}^{-1/2}(\theta_{a_n}, g) \nu_{a_n,F_{a_n},j}(\theta_{a_n}, g)| > \tau_{a_n} \right) \\ & \leq P_{F_{a_n}} \left( \begin{array}{l} \bar{h}_{2,a_n,F_{a_n}}^{-1/2}(\theta_{a_n}, g) h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g) > \kappa_{a_n}/\zeta - \tau_{a_n} \ \& \\ \bar{h}_{2,a_n,F_{a_n}}^{-1/2}(\theta_{a_n}, g) h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g) < B_{a_n} \text{ for some } j \leq p, \text{ some } g \in \mathcal{G} \end{array} \right) + o(1) \\ & = o(1), \end{aligned} \quad (10.24)$$

where the first inequality holds because  $c_0(h, 1 - \alpha + \eta)$  and  $c(h, 1 - \alpha)$  are non-increasing in the first  $p$  elements of  $h_1$  by Assumption S1(b), the second inequality holds because  $(\theta_{a_n}, F_{a_n}) \in \mathcal{F}$  implies that  $h_{1,a_n,F_{a_n},j}(\theta_{a_n}, g) \geq 0 \ \forall j \leq p, \forall g \in \mathcal{G}$  and Assumption GMS1(a) implies that (i)  $\varphi_{a_n,j}(\theta_{a_n}, g) = 0 \leq h_{1,a_n,F_{a_n}}(\theta_{a_n}, g)$  whenever  $\xi_{a_n,j}(\theta_{a_n}, g) \leq 1$  and (ii)  $\varphi_{a_n,j}(\theta_{a_n}, g) \leq \bar{h}_{2,a_n,F_{a_n},j}(\theta_{a_n}, g)^{1/2} B_{a_n}$  a.s.  $\forall j \leq p, \forall g \in \mathcal{G}$ , the third inequality holds by (10.22), the fourth inequality holds because  $P(A) \leq P(A \cap B) + P(B^c)$ , the last inequality holds with probability that goes to one as  $n \rightarrow \infty$  (wp $\rightarrow$  1) because  $\kappa_{a_n}/(1 + o_p(1)) > \kappa_{a_n}/\zeta$  wp $\rightarrow$  1 for  $\zeta > 1$  and using (10.19) with  $\tau_{a_n}$  replaced by



$\varepsilon^{1/2}\tau_{a_n}/2$  because  $\widehat{h}_{2,a_n,F_{a_n},j}^{-1/2}(\theta_{a_n}, g) \leq \varepsilon^{-1/2}h_{2,j}^{-1/2}(1_k, 1_k)(1 + o_p(1)) = \varepsilon^{-1/2}(1 + o_p(1))$  by Lemma A1(b) and (5.2), and the equality holds using (10.23).

Hence, (10.17) holds and the Lemma is proved.  $\square$

**Proof of Lemma A4.** The result of the Lemma is equivalent to

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} P_F \left( c_0(h_{1,n,F}(\theta), \widehat{h}_{2,n}(\theta), 1 - \alpha + \eta) \right. \\ \left. < c_0(h_{1,n,F}(\theta), h_{2,F}(\theta), 1 - \alpha) - \varepsilon^* \right) = 0, \end{aligned} \quad (10.25)$$

where  $\varepsilon^* = \eta - \delta > 0$ . By considering a sequence  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$  that is within  $\delta_n \rightarrow 0$  of the supremum in (10.25) for all  $n \geq 1$ , it suffices to show that

$$\begin{aligned} \lim_{n \rightarrow \infty} P_{F_n} \left( c_0(h_{1,n,F_n}(\theta_n), \widehat{h}_{2,n}(\theta_n), 1 - \alpha + \eta) \right. \\ \left. < c_0(h_{1,n,F_n}(\theta_n), h_{2,F_n}(\theta_n), 1 - \alpha) - \varepsilon^* \right) = 0. \end{aligned} \quad (10.26)$$

Given any subsequence  $\{u_n\}$  of  $\{n\}$ , there exists a subsubsequence  $\{a_n\}$  such that  $d(h_{2,F_{a_n}}(\theta_{a_n}), h_{2,0}) \rightarrow 0$  as  $n \rightarrow \infty$  for some  $h_{2,0} \in \mathcal{H}_{2,cpt}$  because  $h_{2,F_n}(\theta_n) \in \mathcal{H}_{2,cpt}$ . Hence, it suffices to show that (10.26) holds with  $a_n$  in place of  $n$ .

The condition  $d(h_{2,F_{a_n}}(\theta_{a_n}), h_{2,0}) \rightarrow 0$  and  $(\theta_n, F_n) \in \mathcal{F}$  for all  $n \geq 1$  imply that  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\} \in \text{SubSeq}(h_{2,0})$ . Hence, by Lemma A1(b),  $d(\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n}), h_{2,0}) \rightarrow_p 0$  as  $n \rightarrow \infty$ . Furthermore,

$$\begin{aligned} & \widehat{h}_{2,a_n}(\theta_{a_n}, g, g^*) \\ &= \widehat{D}_{a_n}^{-1/2}(\theta_{a_n}) \widehat{\Sigma}_{a_n}(\theta_{a_n}, g, g^*) \widehat{D}_{a_n}^{-1/2}(\theta_{a_n}) \\ &= \text{Diag}(\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n}, 1_k))^{-1/2} \widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n}, g, g^*) \text{Diag}(\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n}, 1_k))^{-1/2}. \end{aligned} \quad (10.27)$$

Hence,  $d(\widehat{h}_{2,a_n}(\theta_{a_n}), h_{2,0}) \rightarrow_p 0$  as  $n \rightarrow \infty$ . Given this, using the almost sure representation theorem as above, we can construct  $\{\tilde{h}_{2,a_n}(g, g^*) : g, g^* \in \mathcal{G}\}$  such that  $d(\tilde{h}_{2,a_n}, h_{2,0}) \rightarrow 0$  as  $n \rightarrow \infty$  a.s. and  $\tilde{h}_{2,a_n}$  and  $\widehat{h}_{2,a_n}(\theta_{a_n})$  have the same distribution under  $(\theta_{a_n}, F_{a_n})$  for all  $n \geq 1$ .

For fixed  $\omega$  in the underlying probability space such that  $d(\tilde{h}_{2,a_n}(\cdot, \cdot)(\omega), h_{2,0}) \rightarrow 0$  as  $n \rightarrow \infty$ , Lemma A5 with  $h_{2,n} = \tilde{h}_{2,a_n}(\omega)$  ( $= \tilde{h}_{2,a_n}(\cdot, \cdot)(\omega)$ ),  $h_{2,n}^* = h_{2,F_{a_n}}(\theta_{a_n})$ ,  $h_{2,0} = h_{2,0}$ ,

and  $\eta_1 = \eta$  gives: for all  $\delta > 0$ ,

$$\liminf_{n \rightarrow \infty} \left[ c_0(h_{1,a_n, F_{a_n}}(\theta_{a_n}), \tilde{h}_{2,a_n}(\omega), 1 - \alpha + \eta) + \delta - c_0(h_{1,a_n, F_{a_n}}(\theta_{a_n}), h_{2, F_{a_n}}(\theta_{a_n}), 1 - \alpha) \right] \geq 0. \quad (10.28)$$

Equation (10.28) holds a.s. This implies that (10.26) holds with  $a_n$  in place of  $n$  because (i)  $\tilde{h}_{2,a_n}$  and  $\widehat{h}_{2,a_n}(\theta_{a_n})$  have the same distribution for all  $n \geq 1$  and (ii) for any sequence of sets  $\{A_n : n \geq 1\}$ ,  $P(A_n \text{ ev.}) (= P(\cup_{m=1}^{\infty} \cap_{k=m}^{\infty} A_k)) = 1$  (where ev. abbreviates eventually) implies that  $P(A_n) \rightarrow 1$  as  $n \rightarrow \infty$ .  $\square$

**Proof of Lemma A5.** Below we show that for  $\{h_{2,n}\}$  and  $\{h_{2,n}^*\}$  as in the statement of the Lemma, for all constants  $x_{h_1, h_{2,n}^*} \in R$  that may depend on  $h_1 \in \mathcal{H}_1$  and  $h_{2,n}^*$ , and all  $\delta > 0$ ,

$$\limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} \left[ P(T(h_1, h_{2,n}) \leq x_{h_1, h_{2,n}^*}) - P(T(h_1, h_{2,n}^*) \leq x_{h_1, h_{2,n}^*} + \delta) \right] \leq 0. \quad (10.29)$$

Note that this result is similar to those of Theorem 1.

We use (10.29) to obtain: for all  $\delta > 0$  and  $\eta_1 > 0$ ,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} P(T(h_1, h_{2,n}) \leq c_0(h_1, h_{2,n}^*, 1 - \alpha) - \delta) \\ & \leq \limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} \left[ P(T(h_1, h_{2,n}) \leq c_0(h_1, h_{2,n}^*, 1 - \alpha) - \delta) \right. \\ & \quad \left. - P(T(h_1, h_{2,n}^*) \leq c_0(h_1, h_{2,n}^*, 1 - \alpha) - \delta/2) \right] \\ & \quad + \limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} P(T(h_1, h_{2,n}^*) \leq c_0(h_1, h_{2,n}^*, 1 - \alpha) - \delta/2) \\ & \leq 0 + 1 - \alpha \\ & < 1 - \alpha + \eta_1, \end{aligned} \quad (10.30)$$

where the second inequality holds by (10.29) with  $\delta/2$  in place of  $\delta$  and  $x_{h_1, h_{2,n}^*} = c_0(h_1, h_{2,n}^*, 1 - \alpha) - \delta$  and by the definition of the  $1 - \alpha$  quantile of  $T(h_1, h_{2,n}^*)$ .

We now use (10.30) to show by contradiction that the result of the Lemma holds. Suppose the result of the Lemma does not hold. Then, there exist constants  $\delta > 0$  and  $\varepsilon^* > 0$ , a subsequence  $\{a_n : n \geq 1\}$ , and a sequence  $\{h_{1,a_n} \in \mathcal{H}_1 : n \geq 1\}$  such that

$$\lim_{n \rightarrow \infty} \left[ c_0(h_{1,a_n}, h_{2,a_n}, 1 - \alpha + \eta_1) + \delta - c_0(h_{1,a_n}, h_{2,a_n}^*, 1 - \alpha) \right] \leq -\varepsilon^* < 0. \quad (10.31)$$

Using this and (10.30), we have

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} P(T(h_{1,a_n}, h_{2,a_n}) \leq c_0(h_{1,a_n}, h_{2,a_n}, 1 - \alpha + \eta_1) + \delta) \\
& \leq \limsup_{n \rightarrow \infty} P(T(h_{1,a_n}, h_{2,a_n}) \leq c_0(h_{1,a_n}, h_{2,a_n}^*, 1 - \alpha) - \varepsilon^*/2) \\
& \leq \limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} P(T(h_1, h_{2,a_n}) \leq c_0(h_1, h_{2,a_n}^*, 1 - \alpha) - \varepsilon^*/2) \\
& < 1 - \alpha + \eta_1,
\end{aligned} \tag{10.32}$$

where the first inequality holds by (10.31) and the last inequality holds by (10.30) with  $\varepsilon^*/2$  in place of  $\delta$ .

Equation (10.32) is a contradiction to (10.31) because the left-hand side quantity in (10.32) (without the  $\limsup_{n \rightarrow \infty}$ ) is greater than or equal to  $1 - \alpha + \eta_1$  for all  $n \geq 1$  by the definition of the  $1 - \alpha + \eta_1$  quantile  $c_0(h_{1,a_n}, h_{2,a_n}, 1 - \alpha + \eta_1)$  of  $T(h_{1,a_n}, h_{2,a_n})$ . This completes the proof of the Lemma except for establishing (10.29).

To establish (10.29), we write

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} \left[ P(T(h_1, h_{2,n}) \leq x_{h_1, h_{2,n}^*}) - P(T(h_1, h_{2,n}^*) \leq x_{h_1, h_{2,n}^*} + \delta) \right] \\
& \leq \limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} \left[ P(T(h_1, h_{2,n}) \leq x_{h_1, h_{2,n}^*}) - P(T(h_1, h_{2,0}) \leq x_{h_1, h_{2,n}^*} + \delta/2) \right] \\
& + \limsup_{n \rightarrow \infty} \sup_{h_1 \in \mathcal{H}_1} \left[ P(T(h_1, h_{2,0}) \leq x_{h_1, h_{2,n}^*} + \delta/2) - P(T(h_1, h_{2,n}^*) \leq x_{h_1, h_{2,n}^*} + \delta) \right].
\end{aligned} \tag{10.33}$$

The first summand on the right-hand side of (10.33) is less than or equal to 0 by the same argument as used to prove Theorem 1(a) with  $\nu_{a_n, F_{a_n}}(\theta_{a_n}, \cdot)$  replaced by  $\nu_{h_{2,a_n}}(\cdot)$  in (10.3), where  $\nu_{h_{2,a_n}}(\cdot)$  is defined in (4.2), because  $d(h_{2,a_n}, h_{2,0}) \rightarrow 0$  as  $n \rightarrow \infty$  implies that the Gaussian processes  $\nu_{h_{2,a_n}}(\cdot) \Rightarrow \nu_{h_{2,0}}(\cdot)$  as  $n \rightarrow \infty$ . This argument uses Assumption S2.

Similarly, the second summand on the right-hand side of (10.33) is less than or equal to 0 by an argument analogous to that for Theorem 1(b). Hence, (10.29) is established, which completes the proof.  $\square$

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Supplemental Material  
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# 11 Outline

This Supplement includes five appendices.

The first appendix, Appendix B, provides a number of supplemental results to the main paper. These include:

- (i) results for Kolmogorov-Smirnov (KS) and approximate Cramér von Mises (A-CvM) tests and CS's in Section 12.1,
- (ii) three additional examples of collections  $\mathcal{G}$  and probability measures  $Q$  that satisfy Assumptions CI, M, FA(e), and Q in Section 12.2,
- (iii) an illustration of the verification of Assumptions LA1-LA3 in Section 12.3,
- (iv) an illustration of some uniformity issues that arise with infinite-dimensional nuisance parameters in Section 12.4,
- (v) an illustration of problems with pointwise asymptotics in Section 12.5, and
- (vi) coverage probability results for subsampling tests and CS's under drifting sequences of distributions in Section 12.6.

Appendix C provides proofs of the results that are stated in the main paper but are not proved in Appendix A. These include:

- (i) the proofs of Lemmas 2 and 3 and Theorem 2(b) in Section 13.1,
- (ii) the proofs of Lemma 4 and Theorem 3 concerning fixed alternatives in Section 13.2,
- (iii) the proof of Theorem 4 concerning local power in Section 13.3, and
- (iv) the proof of Lemma 1 concerning the verification of Assumptions S1-S4 in Section 13.4.

Appendix D provides proofs of the results stated in Appendix B. These include:

- (i) the proofs of Kolmogorov-Smirnov and approximate Cramér von Mises results in Section 14.1,
- (ii) the proof of Lemma B2 in Section 14.2,
- (iii) the proofs of Theorems B4 and B5 regarding uniformity issues in Section 14.3, and
- (iv) the proofs of the subsampling results in Section 14.4.

Appendix E proves Lemma A1 which is stated in Appendix A of the main paper.

Appendix F provides some additional material concerning the Monte Carlo simulation results.

## 12 Appendix B

### 12.1 Kolmogorov-Smirnov and Approximate CvM Tests and CS's

In this Section, we provide results for Kolmogorov-Smirnov (KS) and approximate CvM (A-CvM) tests and CS's defined in Sections 3.1 and 4.2, respectively. A-CvM tests are Cramér-von Mises-type tests in which the test statistic is an infinite sum that is truncated to include only the first  $s_n$  functions  $\{g_1, \dots, g_{s_n}\}$  or the test statistic is an integral with respect to the measure  $Q$  and the integral is approximated by a (possibly weighted) average over the functions  $\{g_1, \dots, g_{s_n}\}$ , which are obtained by simulation or by a quasi-Monte Carlo (QMC) method. The same functions  $\{g_1, \dots, g_{s_n}\}$  are used for the test statistic and the critical value. In the case of simulated functions, the probabilistic results given here are for fixed (i.e., non-random) functions  $\{g_1, \dots, g_{s_n}\}$ . If  $\{g_1, \dots, g_{s_n}\}$  are obtained via i.i.d. draws from  $Q$ , then the probability results are made conditional on the observed functions  $\{g_1, \dots, g_{s_n}\}$  for  $n \geq 1$ .

We show that (i) KS and A-CvM CS's have uniform asymptotic coverage probabilities that are greater than or equal to their nominal level  $1 - \alpha$ , (ii) KS and A-CvM tests have asymptotic power equal to one for all fixed alternatives, and (iii) KS and A-CvM tests have asymptotic power that is arbitrarily close to one for a broad array of  $n^{-1/2}$ -local alternatives whose localization parameter is arbitrarily large.

We consider a slightly more general KS statistic than that defined in (3.7):

$$T_n(\theta) = \sup_{g \in \mathcal{G}_n} S(n^{1/2} \bar{m}_n(\theta, g), \bar{\Sigma}_n(\theta, g)), \quad (12.1)$$

where  $\mathcal{G}_n \subset \mathcal{G}$ .

For KS tests and CS's, we make use of the following assumptions.

**Assumption KS.**  $\mathcal{G}_n \uparrow \mathcal{G}$  as  $n \rightarrow \infty$ .

Let  $\mathcal{W}_{bd}$  denote a subset of  $\mathcal{W}$  (the set of  $k \times k$  positive definite matrices) containing matrices whose eigenvalues are bounded away from zero and infinity.

**Assumption S2'.**  $S(m, \Sigma)$  is uniformly continuous in the sense that for all bounded

sets  $\mathcal{M}$  in  $R^k$  and all sets  $\mathcal{W}_{bd}$

$$\sup_{\mu \in R_+^p \times \{0\}^v} \sup_{\substack{m, m_0 \in \mathcal{M}: \\ \|m - m_0\| \leq \delta}} \sup_{\substack{\Sigma, \Sigma_0 \in \mathcal{W}_{bd}: \\ \|\Sigma - \Sigma_0\| \leq \delta}} |S(m + \mu, \Sigma) - S(m_0 + \mu, \Sigma_0)| \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

The following Lemma shows that Assumption S2' is not restrictive.

**Lemma B1.** *The functions  $S_1, S_2,$  and  $S_3$  satisfy Assumption S2'.*

The following assumption is a strengthening of Assumptions LA1(b) and LA2.

**Assumption LA2'.** (a) For all  $B < \infty$ ,  $\sup_{g \in \mathcal{G}: h_1(g) \leq B} \|h_{1,n,F_n}(\theta_n, g) - h_1(g)\| \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\theta_n, F_n$ , and  $h_1(g)$  are as in Assumption LA1, and

(b) the  $k \times d$  matrix  $\Pi_F(\theta, g) = (\partial/\partial\theta')[D_F^{-1/2}(\theta)E_F m(W_i, \theta, g)]$  exists and satisfies: for all sequences  $\{\delta_n : n \geq 1\}$  such that  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ ,

$$\sup_{\|\theta - \theta_0\| \leq \delta_n} \sup_{g \in \mathcal{G}} \|\Pi_{F_n}(\theta, g) - \Pi_{F_0}(\theta, g)\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } \sup_{g \in \mathcal{G}} \|\Pi_{F_0}(\theta_0, g)\| < \infty,$$

where  $\theta_0, F_0$ , and  $F_n$  are as in Assumption LA1.

Assumption LA2'(a) only requires uniform convergence of  $h_{1,n,F_n}(\theta_n, g)$  to  $h_1(g)$  over  $\{g \in \mathcal{G} : h_1(g) \leq B\}$  because uniform convergence over  $g \in \mathcal{G}$  typically does not hold. Assumption LA2' is not restrictive.

For A-CvM tests and CS's, we use Assumptions S2', LA2', and the following assumptions, which hold automatically in the case of an approximate test statistic that is a truncated sum with  $s_n \rightarrow \infty$ .

**Assumption A1.** The functions  $\{g_1, \dots, g_{s_n}\}$  for  $n \geq 1$  are fixed (i.e., non-random) and  $s_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Assumption A2.** The functions  $\{g_1, g_2, \dots\}$  satisfy:

$$\sum_{\ell=1}^{s_n} w_{Q,n}(\ell) S(m^*(g_\ell), h_{2,F_0}(\theta_*, g_\ell) + \varepsilon I_k) \rightarrow \int S(m^*(g), h_{2,F_0}(\theta_*, g) + \varepsilon I_k) dQ(g) \text{ as } n \rightarrow \infty,$$

where  $m^*(g) = (m_1^*(g), \dots, m_k^*(g))'$ ,  $m_j^*(g) = E_{F_0} m_j(W_i, \theta_*) g_j(X_i) / \sigma_{F_0,j}(\theta_*)$ ,  $\theta_*$  and  $F_0$  are defined as in Assumption FA,  $w_{Q,n}(\ell) = Q(\{g_\ell\})$  in the case of an approximate test statistic that is truncated sum,  $w_{Q,n}(\ell) = n^{-1}$  in the case of an approximate test

statistic that is a simulated integral, and  $w_{Q,n}(\ell)$  is a suitable weight when a test statistic is approximated by a QMC method.

**Assumption A3.** The functions  $\{g_1, g_2, \dots\}$  satisfy: for some sequence of constants  $\{B_c^* < \infty : c = 1, 2, \dots\}$  such that  $B_c^* \rightarrow \infty$  as  $c \rightarrow \infty$ ,

$$\begin{aligned} & \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) 1(h_1(g_\ell) < B_c^*) S(\Pi_0(g_\ell) \lambda_0, h_2(g_\ell) + \varepsilon I_k) \\ \rightarrow & \int 1(h_1(g) < B_c^*) S(\Pi_0(g) \lambda_0, h_2(g) + \varepsilon I_k) dQ(g) \text{ as } n \rightarrow \infty, \end{aligned}$$

where  $\Pi_0(g) = \Pi_{F_0}(\theta_0, g)$ ,  $h_2(g) = h_{2,F_0}(\theta_0, g)$ , and  $\theta_0$  and  $F_0$  are defined as in Assumption LA1.

Assumptions A1-A3 are not restrictive because (i) they hold automatically if the approximate test statistic is a truncated sum and (ii) if the approximate test statistic is a simulated integral and  $\{g_1, g_2, \dots\}$  are i.i.d. with distribution  $Q$  and  $s_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then they hold conditional on  $\{g_1, g_2, \dots\}$  with probability one.

The following result establishes that nominal  $1 - \alpha$  KS and A-CvM CS's have uniform asymptotic coverage probability greater than or equal to  $1 - \alpha$ .

**Theorem B1.** *Suppose Assumptions M, S1, and S2' hold and Assumption GMS1 holds when considering GMS CS's. Then, for every compact subset  $\mathcal{H}_{2,cpt}$  of  $\mathcal{H}_2$ , KS-GMS, KS-PA, A-CvM-GMS, and A-CvM-PA confidence sets  $CS_n$  satisfy*

$$\liminf_{n \rightarrow \infty} \inf_{\substack{(\theta, F) \in \mathcal{F}: \\ h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}} P_F(\theta \in CS_n) \geq 1 - \alpha.$$

**Comments. 1.** Assumptions KS and A1 are not needed in Theorem B1.

**2.** Theorem B1 is an analogue of Theorem 2(a) for CS's based on KS and A-CvM statistics. It is proved by making adjustments to the proof of Theorem 2(a). An analogue of Theorem 2(b) is not given here because the proof of Theorem 2(b) does not go through with KS or A-CvM test statistics. The proof of Theorem 2(b) utilizes the bounded convergence theorem which applies only if the test statistic is an integral with respect to some measure  $Q$ . The continuous mapping theorem cannot be applied because the convergence of  $h_{1,n,F_n}(\theta_n, g)$  to  $h_{1,\infty,F_0}(\theta_0, g)$  is not uniform over  $g \in \mathcal{G}$  for many sequences  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$ , where  $(\theta_n, F_n) \rightarrow (\theta_0, F_0)$ .

The next result shows that KS and A-CvM tests have asymptotic power equal to one against all fixed alternatives. This implies that any parameter value outside the identified set is included in a KS or A-CvM CS with probability that goes to zero as  $n \rightarrow \infty$ , see the Comment to Theorem 3.

**Theorem B2.** *Suppose Assumptions FA, CI, Q, S1, S3, and S4 hold, Assumption KS holds when considering the KS test, and Assumptions A1 and A2 hold when considering A-CvM tests. Then, the KS-GMS and KS-PA tests satisfy the results of Theorem 3 concerning power under fixed alternatives. In addition, A-CvM-GMS and A-CvM-PA tests, respectively, satisfy*

- (a)  $\lim_{n \rightarrow \infty} P_{F_0}(\overline{T}_{n,s_n}(\theta_*) > c_{s_n}(\varphi_n(\theta_*), \widehat{h}_{2,n}(\theta_*), 1 - \alpha)) = 1$  and
- (b)  $\lim_{n \rightarrow \infty} P_{F_0}(\overline{T}_{n,s_n}(\theta_*) > c_{s_n}(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha)) = 1.$

The following result is for  $n^{-1/2}$ -local alternatives.

**Theorem B3.** *Suppose Assumptions M, S1-S4, S2', LA1, and LA2' hold, Assumptions KS and LA3 hold when considering the KS test, and Assumptions A1, A3, and LA3' hold when considering A-CvM tests. Let  $\theta_{n,*} = \theta_{n,*}(\beta) = \theta_n + \beta\lambda_0 n^{-1/2}(1 + o(1))$  be as in Assumption LA1(a) with  $\lambda = \beta\lambda_0$  for some  $\beta > 0$  and  $\lambda_0 \in R^{d_\theta}$ . Then, under  $n^{-1/2}$ -local alternatives, the A-CvM-GMS and A-CvM-PA tests, respectively, satisfy*

- (a)  $\lim_{\beta \rightarrow \infty} \lim_{n \rightarrow \infty} P_{F_n}(\overline{T}_{n,s_n}(\theta_{n,*}(\beta)) > c_{s_n}(\varphi_n(\theta_{n,*}(\beta)), \widehat{h}_{2,n}(\theta_{n,*}(\beta)), 1 - \alpha)) = 1$  provided Assumption GMS1 also holds,
- (b)  $\lim_{\beta \rightarrow \infty} \lim_{n \rightarrow \infty} P_{F_n}(\overline{T}_{n,s_n}(\theta_{n,*}(\beta)) > c_{s_n}(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_{n,*}(\beta)), 1 - \alpha)) = 1,$  and
- (c) KS-GMS and KS-PA tests satisfy parts (a) and (b), respectively, with  $\overline{T}_{n,s_n}(\theta_{n,*}(\beta))$  replaced by  $T_n(\theta_{n,*}(\beta))$  and with the subscript  $s_n$  on  $c_{s_n}(\cdot, \cdot, \cdot)$  deleted.

**Comment.** Theorem B3 shows that KS and A-CvM tests have power arbitrarily close to one for the same  $n^{-1/2}$ -local alternatives as Cramér-von Mises tests that are based on integrals with respect to a probability measure  $Q$ .

## 12.2 Instruments and Weight Functions

In this section we provide three additional examples of instruments  $\mathcal{G}$  and weight functions  $Q$  that satisfy Assumptions CI, M, F(e), and Q. We also specify non-data-dependent methods for transforming a regressor to lie in  $[0, 1]$ .

If  $x \in R$  is known to lie in an open, closed, or half-open interval denoted by  $[c, d]$ ,

where  $-\infty \leq c \leq d \leq \infty$ , then one can transform  $x$  into  $[0, 1]$  via

$$\begin{aligned} t(x) &= \frac{x-c}{d-c} & \text{if } c > -\infty \ \& \ d < \infty, & \quad t(x) &= \frac{e^x}{1+e^x} & \text{if } c = -\infty \ \& \ d = \infty, \\ t(x) &= \frac{e^{x-c}-1}{1+e^{x-c}} & \text{if } c > -\infty \ \& \ d = \infty, & \quad t(x) &= \frac{2e^{x-d}}{1+e^{x-d}} & \text{if } c = -\infty \ \& \ d < \infty. \end{aligned} \quad (12.2)$$

Alternatively, a vector  $X_i$  can be transformed first to have sample mean equal to zero and sample variance matrix equal to  $I_{d_x}$  (by multiplication by the inverse of the upper-triangular Cholesky decomposition of the sample covariance matrix of  $X_i$ ). Then, it can be transformed to lie in  $[0, 1]^{d_x}$  by applying the standard normal df  $\Phi(\cdot)$  element by element. This method is employed in Section 9.4.

**Example 3. (B-splines).** A collection of B-splines provides a set  $\mathcal{G}$  that satisfies Assumptions CI and M for those  $(\theta, F)$  for which  $E_F(m_j(W_i, \theta) | X_i = x)$  is a continuous function of  $x$  for all  $j \leq k$ . The regressors are transformed to lie in  $[0, 1]^{d_x}$ . We consider normalized cubic B-splines with equally-spaced knots on  $[0, 1]^{d_x}$ . (B-splines of other orders also could be considered.) The class of normalized cubic B-splines is a countable set defined by

$$\begin{aligned} \mathcal{G}_{B\text{-spline}} &= \{g(x) : g(x) = B_C(x) \cdot 1_k \text{ for } C \in \mathcal{C}_{B\text{-spline}}\}, \text{ where} \\ \mathcal{C}_{B\text{-spline}} &= \left\{ C_{a,r}^* = \prod_{u=1}^{d_x} [((a_u - 1)/(2r), (a_u + 3)/(2r))] \cap [0, 1] \in [0, 1]^{d_x} : a = (a_1, \dots, a_{d_x})' \right. \\ &\quad \left. a_u \in \{-2, -1, \dots, 2r\} \text{ for } u = 1, \dots, d_x \text{ and } r = r_0, r_0 + 1, \dots \right\} \text{ and} \\ B_{C_{a,r}^*}(x) &= 1(x \in C_{a,r}^*) \\ &\quad \times \prod_{u=1}^{d_x} \begin{cases} y_u^3/6 & \text{for } x_u \in ((a_u - 1)/(2r), a_u/(2r)] \\ (-3y_u^3 + 12y_u^2 - 12y_u + 4)/6 & \text{for } x_u \in (a_u/(2r), (a_u + 1)/(2r)] \\ (-3z_u^3 + 12z_u^2 - 12z_u + 4)/6 & \text{for } x_u \in ((a_u + 1)/(2r), (a_u + 2)/(2r)] \\ z_u^3/6 & \text{for } x_u \in ((a_u + 2)/(2r), (a_u + 3)/(2r)] \\ 0 & \text{otherwise,} \end{cases} \\ &\quad x = (x_1, \dots, x_{d_x})', \ y_u = 2rx_u - (a_u - 1), \text{ and } z_u = 4 - y_u \text{ for } u = 1, \dots, d_x, \end{aligned} \quad (12.3)$$

for some positive integer  $r_0$ , see Schumaker (2007, p. 136). If  $d_x = 1$ , a B-spline in  $\mathcal{G}_{B\text{-spline}}$  has finite support given by the union of four consecutive subintervals each of length  $(2r)^{-1}$ . If  $d_x \geq 1$ , a cubic B-spline in  $\mathcal{G}_{B\text{-spline}}$  has support on a  $d_x$ -dimensional

hypercube in  $[0, 1]^{d_x}$  with edges of length  $4 \cdot (2r)^{-1}$ .

Note that a bounded continuous product kernel with bounded support could be used in place of B-splines in Example 3.

**Weight Function Q for  $\mathcal{G}_{B-spline}$ .** There is a one-to-one mapping  $\Pi_{B-spline} : \mathcal{G}_{B-spline} \rightarrow AR^*$ , where  $AR^*$  is defined as  $AR$  is defined in Section 3.4 but with  $\{-2, -1, \dots, 2r\}^{d_x}$  in place of  $\{1, \dots, 2r\}^{d_x}$ . We take  $Q = \Pi_{B-spline}^{-1} Q_{AR^*}$ , where  $Q_{AR^*}$  is a probability measure on  $AR^*$ . For example, the uniform distribution on  $a \in \{-2, -1, \dots, 2r\}^{d_x}$  conditional on  $r$  and some discrete mass function  $\{w(r) : r = r_0, r_0 + 1, \dots\}$  on  $r$  gives the test statistic:

$$T_n(\theta) = \sum_{r=r_0}^{\infty} w(r) \sum_{a \in \{-2, -1, \dots, 2r\}^{d_x}} (2r+3)^{-d_x} S(n^{1/2} \bar{m}_n(\theta, g_{a,r}), \bar{\Sigma}_n(\theta, g_{a,r})), \quad (12.4)$$

where  $g_{a,r}(x) = B_{C_{a,r}^*}(x) \cdot 1_k$  for  $C_{a,r}^* \in \mathcal{C}_{B-spline}$

**Example 4 (Data-dependent Boxes).** Next, we consider a class of functions  $\mathcal{G}_{box,dd}$  that is designed to be applied with a data-dependent weight function  $Q$  defined below. Because this  $Q$  only puts positive weight on center-points  $x$  that are in the support of  $X_i$ , it turns out to be necessary to consider boxes with different left and right edge lengths as measured from the ‘‘center’’ point. (See footnote 30 below for an explanation.)

We define

$$\mathcal{G}_{box,dd} = \{g : g(x) = 1(x \in C) \cdot 1_k \text{ for } C \in \mathcal{C}_{box,dd}\}, \text{ where} \quad (12.5)$$

$$\mathcal{C}_{box,dd} = \left\{ C_{x,r_1,r_2} = \prod_{u=1}^{d_x} (x_u - r_{1,u}, x_u + r_{2,u}] : x \in \text{Supp}_{F_{X,0}}(X_i), r_{1,u}, r_{2,u} \in (0, \bar{r}) \forall u \leq d_x \right\}$$

for some  $\bar{r} \in (0, \infty]$ ,  $x = (x_1, \dots, x_{d_x})'$ ,  $r_1 = (r_{1,1}, \dots, r_{1,d_x})'$ ,  $r_2 = (r_{2,1}, \dots, r_{2,d_x})'$ , and  $\text{Supp}_{F_{X,0}}(X_i)$  denotes the support of  $X_i$  when  $F_0$  is the true distribution.

**Data-dependent Q for  $\mathcal{G}_{box,dd}$ .** There is a one-to-one mapping  $\Pi_{box,dd} : \mathcal{G}_{box,dd} \rightarrow \{(x, r_1, r_2) \in \text{Supp}_{F_{X,0}}(X_i) \times (0, \bar{r})^{2d_x}\}$ . Thus, for any probability measure  $Q^*$  on  $\{(x, r_1, r_2) \in \text{Supp}_{F_{X,0}}(X_i) \times (0, \bar{r})^{2d_x}\}$ ,  $(\Pi_{box,dd})^{-1} Q^*$  is a valid probability measure on  $\mathcal{G}_{box,dd}$ . In this case, the inverse mapping  $(\Pi_{box,dd})^{-1}$  is  $(\Pi_{box,dd})^{-1}[x, r_1, r_2] = g_{x,r_1,r_2}(\cdot) =$



$1(\cdot \in C_{x,r_1,r_2}) \cdot 1_k$ . Let

$$Q_{F_{X,0}}^* = F_{X,0} \times \text{Unif} \left( \left( \prod_{u=1}^{d_x} (0, \sigma_{X,u} \bar{r}) \right)^2 \right), \text{ where}$$

$$\sigma_{X,u}^2 = \text{Var}_{F_{X,0}}(X_{i,u}) \text{ for } u = 1, \dots, d_x \quad (12.6)$$

and  $F_{X,0}$  denotes the true distribution of  $X_i$ .<sup>30</sup> The scale factors  $\sigma_{X,1}, \dots, \sigma_{X,d_x}$  are included here to make  $Q_{F_{X,0}}^*$  equivariant to location and scale changes in  $X_i$ . Of course,  $F_{X,0}$  and  $\{\sigma_{X,u}^2 : u \leq d_x\}$  are unknown, so they need to be replaced by estimators. The distribution  $F_{X,0}$  can be estimated by the empirical distribution of  $X_i$  based on a subsample of size  $b_n$  of  $\{X_i : i \leq n\}$ , denoted by  $\widehat{F}_{X,b_n}(\cdot)$ . Here we use the empirical distribution based on a subsample, rather than the whole sample, because the computational costs are large when  $b_n = n$  and  $n$  is large.<sup>31</sup> The variances  $\{\sigma_{X,u}^2 : u \leq d_x\}$  can be estimated by the sample variances based on  $\{X_i : i \leq n\}$ , denoted by  $\{\widehat{\sigma}_{X,n,u}^2 : u = 1, \dots, d_x\}$ . In this case, the test statistic is

$$T_n(\theta) = \int_{R^{d_x}} \int_{\left(\prod_{u=1}^{d_x} (0, \widehat{\sigma}_{X,n,u} \bar{r})\right)^2} S(n^{1/2} \bar{m}_n(\theta, g_{x,r_1,r_2}), \bar{\Sigma}_n(\theta, g_{x,r_1,r_2})) \times \prod_{u=1}^{d_x} (\widehat{\sigma}_{X,n,u} \bar{r})^{-2} dr_1 dr_2 d\widehat{F}_{X,m_n}(x) \quad (12.7)$$

$$= b_n^{-1} \sum_{i=1}^{b_n} \int_{\left(\prod_{u=1}^{d_x} (0, \widehat{\sigma}_{X,n,u} \bar{r})\right)^2} S(n^{1/2} \bar{m}_n(\theta, g_{X_i,r_1,r_2}), \bar{\Sigma}_n(\theta, g_{X_i,r_1,r_2})) dr_1 dr_2 \prod_{u=1}^{d_x} (\widehat{\sigma}_{X,n,u} \bar{r})^{-2},$$

where  $g_{x,r_1,r_2}$  is as above.

When an approximate test statistic  $\bar{T}_{n,s_n}(\theta)$  that is a simulated integral is employed,

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<sup>30</sup>One might think that a natural data-dependent measure  $Q$  is  $Q^s = \Pi_{box}^{-1}(F_{X,0} \times \text{Unif}((0, \bar{r})^{d_x}))$ , defined on  $\mathcal{G}_{box}^s$ , where  $\mathcal{G}_{box}^s$  is defined as  $\mathcal{G}_{box}$  is defined in (3.13) but with  $R$  replaced by  $\text{Supp}(X_i)$ . However, such a  $Q$  does not necessarily have support that contains  $\mathcal{G}_{box}^s$  and, hence, the resulting test may not have power against all fixed alternatives. See the following paragraph for details. It is for this reason that  $\mathcal{G}_{box,dd}$  is defined to contain boxes that are asymmetric about their center points.

The probability distribution  $Q^s$  on  $\mathcal{G}_{box}^s$ , does not necessarily satisfy Assumption Q. To see why, consider a simple example with  $d_x = 1$  and  $k = 1$ . Suppose  $X_i$  takes only four values: 0, 1, 2, 3 each with probability 1/4 and  $\bar{r} > 1$ . Then, for  $g_{1,1}(x) = 1(x \in (0, 2]) \in \mathcal{G}_{box}^s$ , we have  $\mathcal{B}(g_{1,1}, \delta) = \{g_{1,1}\}$ . This holds because if  $\omega > 0$ ,  $g_{1,1+\omega}(0) = 1$  but  $g_{1,1}(0) = 0$ ; if  $\omega < 0$ ,  $g_{1,1+\omega}(2) = 0$  but  $g_{1,1}(2) = 1$ ; if  $\omega > 0$ ,  $g_{2,1+\omega}(3) = 1$  but  $g_{1,1}(3) = 0$ ; and if  $\omega < 0$ ,  $g_{2,1+\omega}(1) = 0$  but  $g_{1,1}(1) = 1$ . The set  $\{g_{1,1}\}$  has zero  $Q^s$  measure. So,  $Q^s$  does not satisfy Assumption Q.

<sup>31</sup>Also, it is easier to establish the asymptotic validity of this procedure when  $b_n/n \rightarrow 0$  as  $n \rightarrow \infty$ .

see (3.16) in Section 3.5, it is defined as in (12.7) but with the integral over  $(r_1, r_2)$  replaced by an average over  $\ell = 1, \dots, s_n$ , the density  $\prod_{u=1}^{d_x} (\hat{\sigma}_{X,n,u}\bar{r})^{-2}$  deleted, and  $g_{X_i, r_1, r_2}$  replaced by  $g_{X_i, r_{1,\ell}, r_{2,\ell}}$ , where  $\{(r_{1,\ell}, r_{2,\ell}) : \ell = 1, \dots, s_n\}$  are i.i.d. with a  $Unif(\prod_{u=1}^{d_x} (0, \hat{\sigma}_{X,n,u}\bar{r}))^2$  distribution. Alternatively, in this case, one can take  $b_n = s_n$ , delete the integral over  $(r_1, r_2)$ , delete the density  $\prod_{u=1}^{d_x} (\hat{\sigma}_{X,n,u}\bar{r})^{-2}$ , and replace  $g_{X_i, r_1, r_2}$  by  $g_{X_i, r_{1,i}, r_{2,i}}$ , where  $\{(r_{1,i}, r_{2,i}) : i = 1, \dots, s_n\}$  are as above.

**Example 5. (Continuous/Discrete Regressors).** The collections  $\mathcal{G}_{c-cube}$  and  $\mathcal{G}_{box}$  (defined in the main paper) and  $\mathcal{G}_{B-spline}$  and  $\mathcal{G}_{box,dd}$  (defined here) can be used with continuous and/or discrete regressors. However, one can design  $\mathcal{G}$  to exploit the known support of discrete regressors. Suppose  $X_i = (X'_{1,i}, X'_{2,i})'$ , where  $X_{1,i} \in R^{d_{x,1}}$  is a continuous random vector and  $X_{2,i} \in R^{d_{x,2}}$  is a discrete random vector that takes values in a countable set  $D = \{x_{2,1}, x_{2,2}, \dots\}$ , where  $x_{2,u} \in R^{d_{x,2}}$  for all  $u \geq 1$ . Define the set  $\mathcal{G}_{c/d}$  by

$$\mathcal{G}_{c/d} = \{g : g = g_1 g_2, g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_D\}, \quad (12.8)$$

where  $x = (x'_1, x'_2)'$ ,  $g_1$  is an  $R^k$ -valued function of  $x_1$ ,  $g_2$  is an  $R$ -valued function of  $x_2$ ,  $\mathcal{G}_1 = \mathcal{G}_{c-cube}, \mathcal{G}_{box}, \mathcal{G}_{B-spline}$ , or  $\mathcal{G}_{box,dd}$ , with  $x$  and  $d_x$  replaced by  $x_1$  and  $d_{x,1}$ , respectively, and  $\mathcal{G}_D = \{g_d : g_d(x_2) = 1_{\{d\}}(x_2)\}$  for  $d \in D$ .

**Weight Function Q for  $\mathcal{G}_{c/d}$ .** When  $\mathcal{G}$  is of the form  $\mathcal{G}_{c/d}$ , it is natural to take  $Q$  to be of the form  $Q_1 \times Q_D$ , where  $Q_1$  is a probability measure on  $\mathcal{G}_1$ , such as any of those considered above with  $x_1$  in place of  $x$ , and  $Q_D$  is a probability measure on  $D$ . If  $D$  is a finite set, then one may take  $Q_D$  to be uniform. For example, when  $\mathcal{G}_1 = \mathcal{G}_{box}$  and  $Q_D$  is uniform, the test statistic is

$$T_n(\theta) = \frac{1}{\#D} \sum_{d \in D} \int_{[0,1]^{d_{x,1}}} \int_{(0,\bar{r})^{d_{x,1}}} S(n^{1/2}\bar{m}_n(\theta, g_{x_1,r}g_d), \bar{\Sigma}_n(\theta, g_{x_1,r}g_d)) \bar{r}^{-d_x} dr dx_1, \quad (12.9)$$

where  $\#D$  denotes the number of elements in  $D$  and  $x_1 \in R^{d_{x,1}}$ . When  $\mathcal{G}_1 = \mathcal{G}_{c-cube}$  or  $\mathcal{G}_{B-spline}$ ,  $T_n(\theta)$  is a combination of the formulae given above.

The following result establishes Assumptions CI, M, and FA(e) for  $\mathcal{G}_{B-spline}, \mathcal{G}_{box,dd}$ , and  $\mathcal{G}_{c/d}$  and Assumption Q for the weight functions  $Q$  on these sets.

**Lemma B2.** (a) *For any moment function  $m(W_i, \theta)$ , Assumptions CI and M hold with  $\mathcal{G} = \mathcal{G}_{B-spline}$  for all  $(\theta, F)$  for which  $E_F(m_j(W_i, \theta) | X_i = x)$  is a continuous function of  $x$  for all  $j \leq k$ .*

(b) For any moment function  $m(W_i, \theta)$ , Assumptions CI and M hold with  $\mathcal{G} = \mathcal{G}_{\text{box}, dd}$ .

(c) For any moment function  $m(W_i, \theta)$ , Assumptions CI and M hold with  $\mathcal{G} = \mathcal{G}_{c/d}$ , where  $\mathcal{G}_1 = \mathcal{G}_{c\text{-cube}}, \mathcal{G}_{\text{box}}, \mathcal{G}_{B\text{-spline}}$ , or  $\mathcal{G}_{\text{box}, dd}$ , with  $(x, d_x)$  replaced by  $(x_1, d_{x,1})$  and in the case of  $\mathcal{G}_1 = \mathcal{G}_{B\text{-spline}}$  Assumption CI and M only hold for  $(\theta, F)$  for which  $E_F(m_j(W_i, \theta) | X_{i,1} = x_1, X_{2,i} = d)$  is a continuous function of  $x_1 \in [0, 1]^{d_{x,1}} \forall d \in D, \forall j \leq k$ .

(d) Assumption FA(e) holds for  $\mathcal{G}_{B\text{-spline}}, \mathcal{G}_{\text{box}, dd}$ , and  $\mathcal{G}_{c/d}$ .

(e) Assumption Q holds for the weight function  $Q_c = \Pi_{B\text{-spline}}^{-1} Q_{AR^*}$  on  $\mathcal{G}_{B\text{-spline}}$ , where  $Q_{AR^*}$  is uniform on  $a \in \{-2, -1, \dots, 2r\}^{d_x}$  conditional on  $r$  and  $r$  has some probability mass function  $\{w(r) : r = r_0, r_0 + 1, \dots\}$  with  $w(r) > 0$  for all  $r$ .

(f) Assumption Q holds for the weight function  $Q_d = (\Pi_{\text{box}, dd})^{-1} Q_{F_{X,0}^*}$ , where  $Q_{F_{X,0}^*} = (F_{X,0} \times \text{Unif}((\Pi_{u=1}^{d_x}(0, \sigma_{X,u}\bar{r}))^2))$  on  $\mathcal{G}_{\text{box}, dd}$ .

(g) Assumption Q holds for the weight function  $Q_e = Q_1 \times Q_D$  on  $\mathcal{G}_{c/d}$ , where  $Q_1$  is a probability measure on  $\mathcal{G}_1$  equal to any of the distributions  $Q$  on  $\mathcal{G}$  considered in part (e), part (f), or in Lemma 4 but with  $x_1$  in place of  $x$ ,  $D$  is a finite set, and  $Q_D = \text{Unif}(D)$ .

**Comment.** The uniform distribution that appears in parts (e)-(g) of the Lemma could be replaced by another distribution and the results of the Lemma still hold provided the other distribution has the same support. For example, in part (g), Assumption Q holds when  $D$  is a countably infinite set and  $Q_D$  is a probability measure whose support is  $D$ .

## 12.3 Example: Verification of Assumptions

### LA1-LA3 and LA3'

Here we verify Assumptions LA1-LA3 and LA3' in a simple example for purposes of illustration. These assumptions are the main assumptions employed with local alternatives.

**Example.** Suppose  $W_i = (Y_i, X_i)' \in R^2$  and there is a single moment inequality function  $m(W_i, \theta) = Y_i - \theta$  and no moment equalities, i.e.,  $p = 1$  and  $v = 0$ . Suppose the true parameters/distributions  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$  and the null values  $\{\theta_{n,*} \in \Theta, : n \geq 1\}$  satisfy: (i)  $\theta_n \rightarrow \theta_0$  and  $F_n \rightarrow F_0$  (under the Kolmogorov metric) for some  $(\theta_0, F_0) \in \mathcal{F}$ , (ii)  $\theta_{n,*} = \theta_n + \lambda n^{-1/2}$  for some  $\lambda > 0$ , (iii)  $Y_i = \theta_n + \mu(X_i)n^{-1/2} + U_i$ , (iv)  $\mu(x) \geq 0, \forall x \in R$ , and (v) under all  $F$  such that  $(\theta, F) \in \mathcal{F}$  for some  $\theta \in \Theta$ ,  $(X_i, U_i)$  are i.i.d.

with distribution that does not depend on  $F$ ,  $X_i$  and  $U_i$  are independent,  $E_F U_i = 0$ ,  $Var_F(U_i) = 1$ ,  $Var_F(X_i) \in (0, \infty)$ , and  $E_F |U_i|^{2+\delta} + E_F |\mu(X_i)|^{2+\delta} < \infty$  for some  $\delta > 0$ , and  $\sup_{g \in \mathcal{G}} E_F(1 + \mu^2(X_i))(1 + g^2(X_i)) < \infty$ .

We show that in this example Assumptions LA1 and LA2 hold, Assumption LA3 holds if  $\lambda$  is sufficiently large, and Assumption LA3' holds if  $\mathcal{G}$  and  $Q$  satisfy Assumptions CI and Q, respectively.

By (v), we can write  $E_F g(X_i) = E g(X_i)$  and  $E_F \mu(X_i)g(X_i) = E \mu(X_i)g(X_i)$ .

Assumption LA1(a) holds by (i) and (ii). Assumption LA1(b) holds by the following calculations:

$$\begin{aligned} n^{1/2} E_{F_n} m(W_i, \theta_n, g) &= n^{1/2} E_{F_n} (U_i + \mu(X_i)n^{-1/2})g(X_i) = h_1(g), \text{ where} \\ h_1(g) &= E \mu(X_i)g(X_i) \in [0, \infty) \text{ and} \\ \sigma_{F_n}^2(\theta_n) &= Var_{F_n}(Y_i) = Var_{F_n}(U_i + \mu(X_i)n^{-1/2}) = 1 + n^{-1}Var_{F_n}(\mu(X_i)) \rightarrow 1. \end{aligned} \tag{12.10}$$

To show Assumption LA1(c), we have

$$\begin{aligned} E_{F_n} Y_i^2 g(X_i)g^*(X_i) &= E_{F_n} (\theta_n + \mu(X_i)n^{-1/2} + U_i)^2 g(X_i)g^*(X_i) \\ &\rightarrow E_{F_0} (\theta_0 + U_i)^2 g(X_i)g^*(X_i) \\ &= E_{F_0} Y_i^2 g(X_i)g^*(X_i) \text{ as } n \rightarrow \infty, \end{aligned} \tag{12.11}$$

uniformly over  $g, g^* \in \mathcal{G}$ , using (i), (iii), and (v). Here we have used  $Y_i = \theta_0 + U_i$  under  $F_0$ . This holds because  $F_n \rightarrow F_0$  by (ii), which implies that  $P_{F_n}(Y_i \leq y) \rightarrow P_{F_0}(Y_i \leq y)$  for all continuity points  $Y_i$ , but direct calculations show that  $P_{F_n}(Y_i \leq y) = P(\theta_n + \mu(X_i)n^{-1/2} + U_i \leq y) \rightarrow P(\theta_0 + U_i \leq y)$  for all continuity points  $y$  of  $U_i + \theta_0$  and, hence,  $Y_i = \theta_0 + U_i$  under  $F_0$ .

Next, we write

$$\begin{aligned} &E_{F_n} m(W_i, \theta_n, g)m(W_i, \theta_n, g^*) \\ &= E_{F_n} Y_i^2 g(X_i)g^*(X_i) - \theta_n E[E_{F_n}(Y_i|X_i)(g(X_i) + g^*(X_i))] + \theta_n^2 E g(X_i)g^*(X_i) \\ &= E_{F_n} Y_i^2 g(X_i)g^*(X_i) - \theta_n E[(\theta_n + \mu(X_i)n^{-1/2})(g(X_i) + g^*(X_i))] \\ &\quad + \theta_n^2 E g(X_i)g^*(X_i) \\ &= E_{F_0} Y_i^2 g(X_i)g^*(X_i) - \theta_0^2 E g(X_i) - \theta_0^2 E g^*(X_i) + \theta_0^2 E g(X_i)g^*(X_i) + o(1) \\ &= E_{F_0} m(W_i, \theta_0, g)m(W_i, \theta_0, g^*) + o(1), \end{aligned} \tag{12.12}$$

where  $o(1)$  holds uniformly over  $g, g^* \in \mathcal{G}$ , using (12.11), (i), (iii), and (v). In addition,  $E_{F_n} m(W_i, \theta_n, g) = o(1)$  and  $E_{F_0} m(W_i, \theta_0, g) = o(1)$  uniformly over  $g \in \mathcal{G}$  by (12.10) and (v). Hence, the first part of Assumption LA1(c) holds. The second part of Assumption LA1(c) holds by the same argument with  $\theta_{n,*}$  in place of  $\theta_n$ .

Assumption LA1(d) holds because  $Var_{F_n}(m_j(W_i, \theta_{n,*})) = Var_{F_n}(m_j(W_i, \theta_n)) > 0$ . Assumption LA1(e) holds using (v) and the above expression for  $\sigma_{F_n}^2(\theta_n)$ .

Assumption LA2 holds because  $\Pi_F(\theta, g)$  does not depend on  $(\theta, F)$  by the following calculations and (v):  $\forall F$  such that  $(\theta, F) \in \mathcal{F}$  and  $\forall g \in \mathcal{G}$ ,

$$\begin{aligned} \Pi_F(\theta, g) &= (\partial/\partial\theta)[D_F^{-1/2}(\theta)E_F m(W_i, \theta, g)] \\ &= \sigma_F^{-1}(\theta)(\partial/\partial\theta)E_F(Y_i - \theta)g(X_i) = -\sigma_F^{-1}(\theta)Eg(X_i), \end{aligned} \quad (12.13)$$

where the second equality holds because  $D_F(\theta) = \sigma_F^2(\theta) = Var_F(Y_i)$  does not depend on  $\theta$ .

We have:  $\Pi_0(g) = \Pi_{F_0}(\theta_0, g) = -Eg(X_i)$  by (12.13) and  $\sigma_{F_0}^2(\theta_0) = 1$ . Hence, in Assumption LA3,  $h_1(g) + \Pi_0(g)\lambda = E\mu(X_i)g(X_i) - Eg(X_i)\lambda$ , which is negative whenever  $\lambda > E\mu(X_i)g(X_i)/Eg(X_i)$ . Hence, if the null value  $\theta_{n,*}$  deviates from the true value  $\theta_n$  by enough (i.e., if  $n^{1/2}(\theta_{n,*} - \theta_n) = \lambda$  is large enough), then the null hypothesis is violated for all  $n$  and Assumption LA3 holds.

Next, we show that Assumption LA3' holds provided Assumptions CI and Q hold. We have: (a)  $\Pi_0(g) = -Eg(X_i)$ , (b)  $h_1(g) < \infty \forall g \in \mathcal{G}$  by (12.10) using (v), and (c)  $\lambda_0 = \lambda/\beta > 0$  because  $\lambda > 0$  by (ii) and  $\beta > 0$  by definition. Hence, the condition of Assumption LA3' reduces to

$$Q(\{g \in \mathcal{G} : Eg(X_i) > 0\}) > 0. \quad (12.14)$$

Suppose  $Eg^*(X_i) > 0$  for some  $g^* \in \mathcal{G}$ . (This is a very weak requirement on  $\mathcal{G}$  and is implied by Assumption CI, see below.) Let  $\delta_1 = Eg^*(X_i) > 0$ . Then, using the metric  $\rho_X$  defined in Section 6, for any  $g \in \mathcal{G}$  with  $\rho_X(g, g^*) < \delta_1$ , we have  $Eg(X_i) > 0$  because otherwise  $g(X_i) = 0$  a.s. and  $\delta_1 > \rho_X(g, g^*) = (Eg^*(X_i)^2)^{1/2} \geq Eg^*(X_i) = \delta_1$ , which is a contradiction. Thus,  $Eg(X_i) > 0$  for all  $g \in \mathcal{B}_{\rho_X}(g^*, \delta_1)$ , where  $\mathcal{B}_{\rho_X}(g^*, \delta_1)$  is the open  $\rho_X$ -ball in  $\mathcal{G}$  centered at  $g^*$  with radius  $\delta_1$ . By Assumption Q,  $Q(\mathcal{B}_{\rho_X}(g^*, \delta_1)) > 0$ . Hence, (12.14) holds and Assumption LA3' is verified.

Lastly, we show that Assumption CI implies that  $Eg^*(X_i) > 0$  for some  $g^* \in \mathcal{G}$ . For

all  $\theta > \theta_0$ , we have

$$\begin{aligned}\mathcal{X}_{F_0}(\theta) &= \{x \in R : E_{F_0}(m_j(W_i, \theta) | X_i = x) < 0\} \\ &= \{x \in R : \theta_0 - \theta < 0\} = R,\end{aligned}\tag{12.15}$$

where the second equality holds because  $Y_i = \theta_0 + U_i$  under  $F_0$ , and so,  $E_{F_0}(m_j(W_i, \theta) | X_i = x) = E_{F_0}(Y_i - \theta | X_i = x) = \theta_0 - \theta$ .

By (12.15),  $P_{F_0}(X_i \in \mathcal{X}_{F_0}(\theta)) = P_{F_0}(X_i \in R) = 1 > 0$ . Hence, by Assumption CI, there exists  $g^* \in \mathcal{G}$  such that  $E_{F_0}m(W_i, \theta)g^*(X_i) = E(\theta_0 - \theta)g^*(X_i) < 0$  for  $\theta > \theta_0$ . That is,  $Eg^*(X_i) > 0$ .

## 12.4 Uniformity Issues with Infinite-Dimensional Nuisance Parameters

This section illustrates one of the subtleties that arises when considering the uniform asymptotic behavior of a test or CS in a scenario in which a test statistic exhibits a “discontinuity in its asymptotic distribution” and an infinite-dimensional nuisance parameter affects the asymptotic behavior of the test statistic.

In many testing problems, the asymptotic distribution of a KS-type statistic is determined by establishing the weak convergence of some underlying stochastic process and applying the continuous mapping theorem. This yields the asymptotic distribution to be the supremum of the limit process. In the context of conditional moment inequalities with drifting sequences of distributions, this method does not work. The reason is that the normalized mean function of the underlying stochastic process, i.e.,  $h_{1,n,F_n}(\theta_n, g)$ , often (in fact, usually) does not converge uniformly over  $g \in \mathcal{G}$  to its pointwise limit, i.e.,  $h_1(g)$ , and, hence, stochastic equicontinuity fails.<sup>32</sup>

We show by counter-example that the asymptotic distribution under drifting sequences of null distributions of a KS statistic, where the “sup” is over  $g \in \mathcal{G}$ , does not necessarily equal the supremum of the limiting process indexed by  $g \in \mathcal{G}$  that is determined by the finite-dimensional distributions. Hence, if the critical value is based on this limiting process, a KS test does not necessarily have correct asymptotic null rejection probability. In fact, we show that it can over-reject the null hypothesis substantially.

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<sup>32</sup>Note that drifting sequences of distributions are of interest because correct asymptotic coverage probabilities under all drifting sequences is necessary, though not sufficient, for correct uniform asymptotic coverage probabilities.

The same phenomenon does not arise with CvM statistics, which are “average” statistics. This is because the averaging smooths out the non-uniform convergence of the normalized mean function.

The results in the first section of this Appendix show that the problem discussed above does not arise with the KS statistic when the critical value employed is a GMS critical value that satisfies Assumption GMS1, see Section 4, or a PA critical value. The validity of these critical values is established using a uniform asymptotic approximation of the distribution of the KS statistic, rather than using asymptotics under sequences of true distributions.

To start, we give a very simple deterministic example to illustrate a situation in which a deterministic KS statistic does not converge to the supremum of the pointwise limit, but an “average” CvM statistic does converge to the average of the pointwise limit. Consider the piecewise linear functions  $f_n : [0, 1] \rightarrow [0, 1]$  defined by

$$f_n(x) = \begin{cases} x/\varepsilon_n & \text{for } x \in [0, \varepsilon_n] \\ 1 - (x - \varepsilon_n)/\varepsilon_n & \text{for } x \in [\varepsilon_n, 2\varepsilon_n] \\ 0 & \text{for } x \in [2\varepsilon_n, 1], \end{cases} \quad (12.16)$$

where  $0 < \varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then, for all  $x \in [0, 1]$ ,

$$f_n(x) \rightarrow f(x) = 0 \text{ as } n \rightarrow \infty. \quad (12.17)$$

The KS statistic does not converge to the supremum of the limit function:

$$\sup_{x \in [0,1]} f_n(x) = 1 \not\rightarrow 0 = \sup_{x \in [0,1]} f(x) \text{ as } n \rightarrow \infty. \quad (12.18)$$

On the other hand, the CvM statistic does converge to the average of the limit function:

$$\int_0^1 f_n(x) dx = \varepsilon_n \rightarrow 0 = \int_0^1 f(x) dx \text{ as } n \rightarrow \infty. \quad (12.19)$$

The convergence result for the KS statistic in (12.18) is potentially problematic because in a testing problem with a KS statistic the critical value might be obtained from the distribution of the supremum of the limit process. If convergence in distribution of the KS statistic to the “sup” of the limit process does not hold, then such a critical value is not necessarily appropriate.

Now we show that the phenomenon illustrated in (12.16)-(12.19) arises in conditional moment inequality models. We consider a particular conditional moment inequality model with a single linear moment inequality, a fixed true value  $\theta_0$ , and a particular drifting sequence of distributions. (Note that CX stands for “counterexample.”)

**Assumption CX.** (a)  $m(W_i, \theta) = Y_i - \theta$  for  $Y_i, \theta \in R$ ,

(b)  $m(W_i, \theta_0) = Y_i = U_i + 1(X_i \in (\varepsilon_n, 1])$ , where the true value  $\theta_0$  equals 0,  $EU_i = 0$ ,  $EU_i^2 = 1$ , the distribution of  $U_i$  does not depend on  $n$ ,  $U_i$  and  $X_i$  are independent, and the constants  $\{\varepsilon_n : n \geq 1\}$  satisfy  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ ,

(c)  $X_i = \varepsilon_n$  with probability 1/2 and  $X_i$  is uniform on  $[0, 1]$  with probability 1/2,

(d)  $\{W_i = (Y_i, X_i)' : i \leq n, n \geq 1\}$  is a row-wise independent and identically distributed triangular array (with the dependence of  $W_i, Y_i$ , and  $X_i$ , on  $n$  suppressed for notational simplicity),

(e)  $S(m, \Sigma) = S(m)$  for  $m \in R$ ,

(f)  $S$  satisfies Assumptions S1 and S2, and

(g)  $\mathcal{G} = \{g_{a,b} : g_{a,b} = 1(x \in (a, b]) \text{ for some } 0 \leq a < b \leq 1\}$ .

The function  $S_1(m) = [m]_-^2$  satisfies Assumptions CX(e)-(f). Assumption CX(e) is made for simplicity. It could be removed and with some changes to the proofs the results given below would hold for  $S = S_2$  as well. The class of functions  $\mathcal{G}$  specified in Assumption CX(g) is the class of one-dimensional boxes, as in Example 1 of Section 3.3.

We write

$$\begin{aligned} n^{1/2}\bar{m}_n(\theta_0, g_{a,b}) &= n^{-1/2} \sum_{i=1}^n Y_i g_{a,b}(X_i) = \nu_n(g_{a,b}) + h_{1,n}(g_{a,b}), \text{ where} \\ \nu_n(g_{a,b}) &= n^{1/2}(\bar{m}_n(\theta_0, g_{a,b}) - E_{F_n}\bar{m}_n(\theta_0, g_{a,b})) \text{ and} \\ h_{1,n}(g_{a,b}) &= n^{1/2}E_{F_n}\bar{m}_n(\theta_0, g_{a,b}). \end{aligned} \tag{12.20}$$

The KS statistic is

$$\sup_{g_{a,b} \in \mathcal{G}} S(n^{1/2}\bar{m}_n(\theta_0, g_{a,b})) = \sup_{g_{a,b} \in \mathcal{G}} S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})).^{33} \tag{12.21}$$

Let  $\nu(\cdot)$  be a mean zero Gaussian process indexed by  $g_{a,b} \in \mathcal{G}$  with covariance kernel  $K(\cdot, \cdot)$  and with sample paths that are uniformly  $\rho$ -continuous, where  $K(\cdot, \cdot)$  and  $\rho(\cdot, \cdot)$  are specified in the proof of Theorem B4 given in the next subsection.

The KS statistic satisfies the following result.



**Theorem B4.** *Suppose Assumption CX holds. Then,*

- (a)  $\nu_n(\cdot) \Rightarrow \nu(\cdot)$  as  $n \rightarrow \infty$ ,
- (b)  $h_{1,n}(g_{a,b}) \rightarrow h_1(g_{a,b}) = \infty$  as  $n \rightarrow \infty$  for all  $g_{a,b} \in \mathcal{G}$ ,
- (c)  $\sup_{g_{a,b} \in \mathcal{G}} |h_{1,n}(g_{a,b}) - h_1(g_{a,b})| \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (d)  $S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) \rightarrow_d S(\nu(g_{a,b}) + h_1(g_{a,b}))$  as  $n \rightarrow \infty$  for all  $g_{a,b} \in \mathcal{G}$ ,
- (e)  $\sup_{g_{a,b} \in \mathcal{G}} S(\nu(g_{a,b}) + h_1(g_{a,b})) = 0$  a.s.,
- (f)  $\sup_{g_{a,b} \in \mathcal{G}} S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) \geq S(\nu_n(g_{0,\varepsilon_n}) + h_{1,n}(g_{0,\varepsilon_n})) \rightarrow_d S(Z^*)$  as  $n \rightarrow \infty$ , where  $Z^* \sim N(0, 1/2)$  and the inequality holds a.s., and
- (g)  $\sup_{g_{a,b} \in \mathcal{G}} S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) \rightarrow_d \sup_{g_{a,b} \in \mathcal{G}} S(\nu(g_{a,b}) + h_1(g_{a,b}))$  as  $n \rightarrow \infty$ .

**Comments. 1.** Theorem B4(g) shows that the KS statistic does not have an asymptotic distribution that equals the supremum over  $g_{a,b} \in \mathcal{G}$  of the pointwise limit given in Theorem B4(d). This is due to the lack of uniform convergence of  $h_{1,n}(g_{a,b})$  shown in Theorem B4(c). (Note that the convergence in part (d) of the Theorem also holds jointly over any finite set of  $g_{a,b} \in \mathcal{G}$ .)

**2.** Let  $c_{\infty,1-\alpha}$  denote the  $1-\alpha$  quantile of  $\sup_{g_{a,b} \in \mathcal{G}} S(\nu(g_{a,b}) + h_1(g_{a,b}))$ . By Theorem B4(e),  $c_{\infty,1-\alpha} = 0$ . Theorem B4(f) and some calculations (given in the proof of Theorem B4 below) yield

$$\liminf_{n \rightarrow \infty} P \left( \sup_{g_{a,b} \in \mathcal{G}} S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) > c_{\infty,1-\alpha} \right) \geq 1/2. \quad (12.22)$$

That is, if one uses  $c_{\infty,1-\alpha}$  as the critical value, the nominal level  $\alpha$  test based on the KS statistic has an asymptotic null rejection probability that is bounded below by  $1/2$ , which indicates substantial over-rejection.

Next, we provide results for a CvM statistic defined by

$$\int S(n^{1/2} \overline{m}_n(\theta_0, g_{a,b})) dQ(g_{a,b}) = \int S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) dQ(g_{a,b}), \quad (12.23)$$

where  $Q$  is a probability measure on  $\mathcal{G}$ . In contrast to the KS statistic, the CvM statistic is well-behaved asymptotically.

**Theorem B5.** *Suppose Assumption CX holds. Then,*

$$\int S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) dQ(g_{a,b}) \rightarrow_d \int S(\nu(g_{a,b}) + h_1(g_{a,b})) dQ(g_{a,b}) \text{ as } n \rightarrow \infty.$$

**Comment.** Theorem B5 is not proved using the continuous mapping theorem due to the non-uniform convergence of  $h_{1,n}(g_{a,b})$ . Rather, it is proved using an almost sure representation argument coupled with the bounded convergence theorem.

## 12.5 Problems with Pointwise Asymptotics

In the case of unconditional moment inequalities, pointwise asymptotics have been shown in Andrews and Guggenberger (2009) to be deficient in the sense that they fail to capture the finite-sample properties of a typical test statistic of interest. This is due to the discontinuity in the asymptotic distribution of the test statistic. In the case of *conditional* moment equalities, the deficiency of pointwise asymptotics is even greater. We show in a simple example that the asymptotic distribution of a test statistic  $T_n(\theta_0)$  under a fixed distribution  $F_0$  often is *pointmass at zero* even when the true parameter  $\theta_0$  is on the boundary of the identified set. This does not reflect the statistic's finite-sample distribution.

Suppose (i)  $W_i = (Y_i, X_i)'$ , (ii) there is one moment inequality function  $m(W_i, \theta) = Y_i - \theta$  and no moment equalities (i.e.,  $p = 1$  and  $v = 0$ ), (iii) the true distribution is  $F_0$  for all  $n \geq 1$ , (iv)  $Y_i = \theta_0 + \mu(X_i) + U_i$ , where  $X_i, U_i \in \mathbb{R}$  and  $\mu(\cdot) = \mu_{F_0}(\cdot)$ , (v)  $\mu(x) \geq 0 \forall x \in \mathbb{R}$ ,  $\mathcal{X}_{zero} = \{x \in \text{Supp}_{F_0}(X_i) : \mu(x) = 0\} \neq \emptyset$ , and  $\mu(\cdot)$  is continuous on  $\mathbb{R}$ , and (vi) under  $F_0$ ,  $(X_i, U_i)$  are i.i.d.,  $X_i$  and  $U_i$  are independent,  $E_{F_0}U_i = 0$ ,  $\text{Var}_{F_0}(U_i) = 1$ ,  $X_i$  is absolutely continuous, and  $\text{Var}_{F_0}(X_i) \in (0, \infty)$ . As defined, the conditional moment inequality is

$$E_{F_0}(m(W_i, \theta_0)|X_i) = \mu(X_i) \geq 0 \text{ a.s.} \quad (12.24)$$

The inequality in (12.24) is strict except when  $X_i \in \mathcal{X}_{zero}$ . Often, the latter occurs with probability zero. For example, this is true if  $\mathcal{X}_{zero}$  is a singleton (or a set with Lebesgue measure zero). In spite of the moment inequality being strict with probability one, the true value  $\theta_0$  is on the boundary of the identified set  $\Theta_{F_0}$ , i.e.,  $\Theta_{F_0} = (-\infty, \theta_0]$ .<sup>34</sup>

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<sup>34</sup>This holds because, for any  $\theta > \theta_0$ , (a)  $E_{F_0}(m(W_i, \theta)|X_i) = \mu(X_i) + \theta_0 - \theta$ , (b)  $\forall \delta > 0$ ,  $P_{F_0}(X_i \in B(\mathcal{X}_{zero}, \delta)) > 0$  by the absolute continuity of  $X_i$ , where  $B(\mathcal{X}_{zero}, \delta)$  denotes the closed set of points that are within  $\delta$  of the set  $\mathcal{X}_{zero}$ , (c) for  $\delta^* > 0$  sufficiently small,  $\mu(x) < \theta - \theta_0 \forall x \in B(\mathcal{X}_{zero}, \delta^*)$  by the continuity of  $\mu(\cdot)$ , and, hence, (d)  $0 < P_{F_0}(X_i \in B(\mathcal{X}_{zero}, \delta^*)) \leq P_{F_0}(E_{F_0}(m(W_i, \theta)|X_i) < 0)$ , which implies that  $\theta \notin \Theta_{F_0}$ .

We consider a test statistic based on  $S(n^{1/2}\overline{m}_n(\theta, g), I)$  with  $S = S_1 = S_2$ :

$$\begin{aligned}
T_n(\theta_0) &= \int [n^{1/2}\overline{m}_n(\theta_0, g)]_-^2 dQ(g) \\
&= \int \left[ n^{1/2} \left( n^{-1} \sum_{i=1}^n (U_i + \mu(X_i))g(X_i) - \Delta(g) \right) + n^{1/2}\Delta(g) \right]_-^2 dQ(g), \text{ where} \\
\overline{m}_n(\theta_0, g) &= n^{-1} \sum_{i=1}^n (Y_i - \theta_0)g(X_i) \text{ and } \Delta(g) = E_{F_0}\mu(X_i)g(X_i). \tag{12.25}
\end{aligned}$$

The first summand in the integrand in (12.25) is  $O_p(1)$  uniformly over  $g \in \mathcal{G}$  by a functional CLT and is identically zero if  $P_{F_0}(g(X_i) = 0) = 1$ . The second summand,  $n^{1/2}\Delta(g)$ , diverges to infinity unless  $\Delta(g) = 0$ . In addition,  $[x_n]_-^2 \rightarrow 0$  as  $x_n \rightarrow \infty$ . Hence, if  $\Delta(g) > 0$ , the integrand converges in probability to zero. In the leading case in which  $\mathcal{X}_{zero}$  is a singleton set (or any set with Lebesgue measure zero),  $\Delta(g) = 0$  only if  $P_{F_0}(g(X_i) = 0) = 1$  (using the absolute continuity of  $X_i$ ). In consequence, if  $\Delta(g) = 0$ , the integrand in (12.25) equals zero a.s. Combining these results shows that the asymptotic distribution of  $T_n(\theta_0)$  under the fixed distribution  $F_0$  is pointmass at zero even though the true parameter is on the boundary of the identified set.<sup>35</sup>

The pointmass asymptotic distribution of  $T_n(\theta_0)$  does not mimic its finite-sample distribution well at all. In finite samples, the distribution of  $T_n(\theta_0)$  is non-degenerate because the quantity  $n^{1/2}\Delta(g)$  is finite and far from infinity for all functions  $g$  for which  $\mu(x)$  is not large for  $x \in \text{Supp}(g)$ . Pointwise asymptotics fail to capture this.

The implication of the discussion above is that to obtain asymptotic results that mimic the finite-sample situation it is necessary to consider uniform asymptotics or, at least, asymptotics under drifting sequences of distributions.

## 12.6 Subsampling Critical Values

### 12.6.1 Definition

Here we define subsampling critical values and CS's. Let  $b$  denote the subsample size when the full sample size is  $n$ . We assume  $b \rightarrow \infty$  and  $b/n \rightarrow 0$  as  $n \rightarrow \infty$ . The number of different subsamples of size  $b$  is  $q_n$ . There are  $q_n = n! / (b!(n-b)!)$  different subsamples of size  $b$ .

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<sup>35</sup>This argument is only heuristic. The result can be proved formally using a combination of an almost sure representation result and the bounded convergence theorem as in the proofs given in Appendix A.

Let  $\{T_{n,b,j}(\theta) : j = 1, \dots, q_n\}$  be subsample statistics where  $T_{n,b,j}(\theta)$  is defined exactly the same as  $T_n(\theta)$  is defined but based on the  $j$ th subsample rather than the full sample. The empirical distribution function and the  $1 - \alpha$  quantile of  $\{T_{n,b,j}(\theta) : j = 1, \dots, q_n\}$  are

$$\begin{aligned} U_{n,b}(\theta, x) &= q_n^{-1} \sum_{j=1}^{q_n} 1(T_{n,b,j}(\theta) \leq x) \text{ for } x \in R \text{ and} \\ c_{n,b}(\theta, 1 - \alpha) &= \inf\{x \in R : U_{n,b}(\theta, x) \geq 1 - \alpha\}, \end{aligned} \quad (12.26)$$

respectively. The subsampling critical value is  $c_{n,b}(\theta_0, 1 - \alpha)$ . The nominal level  $1 - \alpha$  CS is given by (2.5) with  $c_{n,1-\alpha}(\theta) = c_{n,b}(\theta, 1 - \alpha)$ .<sup>36</sup>

### 12.6.2 Asymptotic Coverage Probabilities of Subsampling Confidence Sets

Next, we show that nominal  $1 - \alpha$  subsampling CS's have asymptotic coverage probabilities greater than or equal to  $1 - \alpha$  under drifting sequences of parameters and distributions  $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$ . The sequences that we consider are those in the set  $Seq^b$ , which is defined as follows.

Let  $\mathcal{H}_1, \mathcal{H}_2$ , and  $\mathcal{H}$  be defined as in (5.5). Let  $\mathcal{H}_1^*(h_1) = \{h_1^* \in \mathcal{H}_1 : h_{1,j}^*(g) > 0 \text{ only if } h_{1,j}(g) = \infty \text{ for } j \leq p, \forall g \in \mathcal{G}\}$ .

**Definition  $Seq^b(\mathbf{h}_1^*, \mathbf{h})$ .** For  $h \in \mathcal{H}$  and  $h_1^* \in \mathcal{H}_1^*(h_1)$ , define  $Seq^b(h_1^*, h)$  to be the set of sequences  $\{(\theta_n, F_n) : n \geq 1\}$  such that

- (i)  $(\theta_n, F_n) \in \mathcal{F} \forall n \geq 1$ ,
- (ii)  $\lim_{n \rightarrow \infty} h_{1,n,F_n}(\theta_n, g) = h_1(g) \forall g \in \mathcal{G}$ ,
- (iii)  $\lim_{n \rightarrow \infty} \sup_{g, g^* \in \mathcal{G}} \|D_{F_n}^{-1/2}(\theta_n) \Sigma_{F_n}(\theta_n, g, g^*) D_{F_n}^{-1/2}(\theta_n) - h_2(g, g^*)\| = 0$ , and
- (iv)  $\lim_{n \rightarrow \infty} b^{1/2} D_{F_n}^{-1/2}(\theta_n) E_{F_n} m(W, \theta_n, g) = h_1^*(g) \forall g \in \mathcal{G}$ .

Let

$$Seq^b = \bigcup_{h_1^* \in \mathcal{H}_1^*(h), h \in \mathcal{H}} Seq^b(h_1^*, h). \quad (12.27)$$

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<sup>36</sup>The subsampling critical value defined above is a non-recentered subsampling critical value. One also could consider recentered subsampling critical values, see Andrews and Soares (2010) for the definition. But, there is little reason to do so because tests based on recentered subsampling critical values have the same first-order asymptotic power properties as PA tests and recentered bootstrap tests and worse behavior than the latter two tests in terms of the magnitude of errors in null rejection probabilities asymptotically.

We use the following assumptions.

**Assumption SQ.** For all functions  $h_1 : \mathcal{G} \rightarrow R_{[+\infty]}^p \times \{0\}^v$ ,  $h_2 : \mathcal{G}^2 \rightarrow \mathcal{W}$ , and mean zero Gaussian processes  $\{\nu_{h_2}(g) : g \in \mathcal{G}\}$  with finite-dimensional covariance matrix  $h_2(g, g^*)$  for  $g, g^* \in \mathcal{G}$ , the distribution function of  $\int S(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) dQ(g)$  at  $x \in R$  is

- (a) continuous for  $x > 0$  and
- (b) strictly increasing for  $x > 0$  unless  $v = 0$  and  $h_1(g) = \infty^p$  a.s.  $[Q]$ .

Lemma B3 below shows that Assumption SQ is satisfied by  $S_1$  and  $S_2$ .

**Lemma B3.** *Assumption SQ holds when  $S = S_1$  or  $S_2$ .*

The following Assumption C is needed only to show that subsampling CS's are not asymptotically conservative. For  $(\theta, F) \in \mathcal{F}$ , define  $h_{1,j,F}(\theta, g) = \infty$  if  $E_F m_j(W_i, \theta, g) > 0$  and  $h_{1,j,F}(\theta, g) = 0$  if  $E_F m_j(W_i, \theta, g) = 0$  for  $g \in \mathcal{G}, j = 1, \dots, p$ . Let  $h_{1,F}(\theta, g) = (h_{1,1,F}(\theta, g), \dots, h_{1,p,F}(\theta, g), 0'_v)'$ .

**Assumption C.** For some  $(\theta, F) \in \mathcal{F}$ ,  $\int S(\nu_{h_{2,F}}(\theta, g) + h_{1,F}(\theta, g), h_{2,F}(\theta, g) + \varepsilon I_k) dQ(g)$  is continuous at its  $1 - \alpha$  quantile, where  $\{\nu_{h_{2,F}}(\theta, g) : g \in \mathcal{G}\}$  is a mean zero Gaussian process concentrated on the space of uniformly  $\rho$ -continuous bounded  $R^k$ -valued functionals on  $\mathcal{G}$ , i.e.,  $U_\rho^k(\mathcal{G})$ , with covariance kernel  $h_{2,F}(\theta, g, g^*)$  for  $g, g^* \in \mathcal{G}$ .

Assumption C is not very restrictive.

The exact and asymptotic confidence sizes of a subsampling CS are

$$ExCS_n = \inf_{(\theta, F) \in \mathcal{F}} P_F(T_n(\theta) \leq c_{n,b}(\theta, 1 - \alpha)) \text{ and } AsyCS = \liminf_{n \rightarrow \infty} ExCS_n. \quad (12.28)$$

The next assumption is used to establish *AsyCS* for subsampling CS's. It is a high-level condition that is difficult to verify and hence is not very satisfactory.

**Assumption Sub.** For some subsequence  $\{v_n : n \geq 1\}$  of  $\{n\}$  for which  $\{(\theta_{v_n}, F_{v_n}) \in \mathcal{F} : n \geq 1\}$  satisfies  $\lim_{n \rightarrow \infty} P_{F_{v_n}}(T_n(\theta_{v_n}) \leq c_{n,b}(\theta_{v_n}, 1 - \alpha)) = AsyCS$  (such a subsequence always exists), there is a subsequence  $\{m_n\}$  of  $\{v_n\}$  such that  $\{(\theta_{m_n}, F_{m_n}) \in \mathcal{F} : n \geq 1\}$  belongs to  $Seq^b$ , where  $Seq^b$  is defined with  $m_n$  in place of  $n$  throughout.

Part (a) of the following Theorem shows that subsampling CS's have correct asymptotic coverage probabilities under drifting sequences of parameters and distributions.

**Theorem B6.** *Suppose Assumptions M, S1, S2, and SQ hold. Then, a nominal  $1 - \alpha$*

subsampling confidence set based on  $T_n(\theta)$  satisfies

(a)  $\inf_{\{(\theta_n, F_n): n \geq 1\} \in Seq^b} \liminf_{n \rightarrow \infty} P_{F_n}(T_n(\theta_n) \leq c_{n,b}(\theta_n, 1 - \alpha)) \geq 1 - \alpha,$

(b) if Assumption C also holds, then

$$\inf_{\{(\theta_n, F_n): n \geq 1\} \in Seq^b} \liminf_{n \rightarrow \infty} P_{F_n}(T_n(\theta_n) \leq c_{n,b}(\theta_n, 1 - \alpha)) = 1 - \alpha, \text{ and}$$

(c) if Assumptions Sub and C also hold, then  $AsyCS = 1 - \alpha.$

**Comment.** Theorem B6(c) establishes that subsampling CS's have correct *AsyCS* provided Assumption Sub holds. The latter condition is difficult to verify. Hence, this result is not nearly as useful as the uniformity results given for GMS and PA CS's in Section 5.

## 13 Appendix C

In this Appendix, we prove all the results stated in the main paper except for Theorems 1 and 2(a), which are proved in Appendix A, and Lemma A1, which is proved in Appendix E. The proofs are given in the following order: Lemma 2, Lemma 3, Theorem 2(b), Lemma 4, Theorem 3, Theorem 4, and Lemma 1.

### 13.1 Proofs of Lemmas 2 and 3 and Theorem 2(b)

**Proof of Lemma 2.** We have:  $\theta \notin \Theta_F(\mathcal{G})$  implies that  $E_F m_j(W_i, \theta) g_j(X_i) < 0$  for some  $j \leq p$  or  $E_F m_j(W_i, \theta) g_j(X_i) \neq 0$  for some  $j = p + 1, \dots, k$ . By the law of iterated expectations and  $g_j(x) \geq 0$  for all  $x \in R^{d_x}$  and  $j \leq p$ , this implies that  $P_F(X_i \in \mathcal{X}_F(\theta)) > 0$  and, hence,  $\theta \notin \Theta_F$ .

On the other hand,  $\theta \notin \Theta_F$  implies that  $P_F(X_i \in \mathcal{X}_F(\theta)) > 0$  and the latter implies that  $\theta \notin \Theta_F(\mathcal{G})$  by Assumption CI.  $\square$

The proof of Lemma 3 uses the following Lemma, which is an existence and uniqueness result. The proof of the Lemma utilizes an extended measure result from Billingsley (1995, Thm. 11.3), which delivers the existence part of the Lemma. The proof is given after the proof of Lemma 3.

**Lemma C1.** *Let  $\mathcal{R}$  be a semi-ring of subsets of  $R^{d_x}$ . Let  $\mu$  be a bounded countably additive set function on  $\sigma(\mathcal{R})$  such that  $\mu(\phi) = 0$  and  $\mu(C) \geq 0$  for all  $C \in \mathcal{R} \cup \{R^{d_x}\}$ . If  $R^{d_x}$  can be written as the union of a countable number of disjoint sets in  $\mathcal{R}$ , then  $\mu$  is a measure on  $\sigma(\mathcal{R})$  (and hence  $\mu(C) \geq 0$  for all  $C \in \sigma(\mathcal{R})$ ).<sup>37</sup>*

**Proof of Lemma 3.** First, we establish Assumption CI for  $\mathcal{G} = \mathcal{G}_{box}$  with  $\bar{r} = \infty$ . It suffices to show

$$\begin{aligned} E_F(m_j(W_i, \theta) g_j(X_i)) \geq 0 \quad \forall g \in \mathcal{G} &\Rightarrow E_F(m_j(W_i, \theta) | X_i) \geq 0 \text{ a.s.} \\ &\text{for } j = 1, \dots, p \text{ and} \\ E_F(m_j(W_i, \theta) g_j(X_i)) = 0 \quad \forall g \in \mathcal{G} &\Rightarrow E_F(m_j(W_i, \theta) | X_i) = 0 \text{ a.s.} \\ &\text{for } j = p + 1, \dots, k. \end{aligned} \tag{13.1}$$

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<sup>37</sup>A class of subsets,  $\mathcal{R}$ , of a universal set is called a semi-ring if (a) the empty set  $\phi \in \mathcal{R}$ ; (b)  $A, B \in \mathcal{R}$  implies  $A \cap B \in \mathcal{R}$ ; (c) if  $A, B \in \mathcal{R}$  and  $A \subset B$ , then there exist disjoint sets  $C_1, \dots, C_N \subset \mathcal{R}$  such that  $B - A = \bigcup_{i=1}^N C_i$ , see Billingsley (1995, p.138).

We use the following set function:

$$\mu_j(C) = \sigma_{F,j}^{-1}(\theta) E_F m_j(W_i, \theta) 1(X_i \in C) \text{ for } C \in \sigma(\mathcal{C}_{box}) = \mathcal{B}(R^{d_x}), \quad (13.2)$$

where  $\sigma(\mathcal{C}_{box})$  denotes the  $\sigma$ -field generated by  $\mathcal{C}_{box}$ ,  $\mathcal{B}(R^{d_x})$  is the Borel  $\sigma$ -field on  $R^{d_x}$ , and  $\sigma(\mathcal{C}_{box}) = \mathcal{B}(R^{d_x})$  is a well-known result. First we show  $\mu_j(R^{d_x}) \geq 0$ . Let  $I_L = (-L, L]^{d_x}$ . Then,  $I_L \in \mathcal{C}_{box}$  and  $\mu_j(I_L) \geq 0$ . We have

$$\begin{aligned} 0 &\leq \lim_{L \rightarrow \infty} \mu_j(I_L) = \lim_{L \rightarrow \infty} \sigma_{F,j}^{-1}(\theta) E_F m_j(W_i, \theta) 1(X_i \in I_L) \\ &= \sigma_{F,j}^{-1}(\theta) E_F m_j(W_i, \theta) 1(X_i \in R^{d_x}) = \mu_j(R^{d_x}), \end{aligned} \quad (13.3)$$

where the second equality holds by the dominated convergence theorem,  $\sigma_{F,j}^{-1}(\theta) m_j(w, \theta) \times 1(x \in I_L) \rightarrow \sigma_{F,j}^{-1}(\theta) m_j(w, \theta) 1(x \in R^{d_x})$  as  $L \rightarrow \infty$ ,  $|\sigma_{F,j}^{-1}(\theta) m_j(w, \theta) 1(x \in I_L)| \leq \sigma_{F,j}^{-1}(\theta) |m_j(w, \theta)|$  for all  $w$ , and  $\sigma_{F,j}^{-1}(\theta) E_F |m_j(W_i, \theta)| < \infty$ .

Next, we treat the cases  $j \leq p$  and  $j > p$  separately because different techniques are employed. First, we consider  $j = 1, \dots, p$ . Suppose  $E_F m_j(W_i, \theta) g_j(X_i) \geq 0 \forall g \in \mathcal{G}$ . Then,  $\mu_j(C) \geq 0 \forall C \in \mathcal{C}_{box}$ . We want to show that  $E_F m_j(W_i, \theta) 1(X_i \in C) \geq 0 \forall C \in \mathcal{B}(R^{d_x})$  because this implies that  $E_F(m_j(W_i, \theta) | X_i) \geq 0$  a.s. since  $X_i$  is Borel measurable.

By Lemma C1, we have  $\mu_j(C) \geq 0 \forall C \in \sigma(\mathcal{C}_{box})$  if (a)  $\mathcal{C}_{box}$  is a semi-ring of subsets of  $R^{d_x}$ , (b)  $\mu_j$  is bounded, (c)  $\mu_j$  is countably additive, (d)  $\mu_j(\phi) = 0$ , (e)  $\mu_j(R^{d_x}) \geq 0$ , and (f)  $R^{d_x}$  can be written as the union of a countable number of disjoint sets in  $\mathcal{C}_{box}$ . It is a well-known result that (a) holds (provided  $\phi$  is added to  $\mathcal{C}_{box}$ ). By condition (vi) in (2.3), (b) holds. Condition (c) holds by the dominated convergence theorem. Because  $1(X_i \in \phi) = 0$ , condition (d) holds. By (13.3), condition (e) holds. Condition (f) holds because

$$R^{d_x} = \bigcup_{\{i_1, i_2, \dots, i_k\} \in \mathbb{N}^k} \prod_{j=1}^k (i_j, i_j + 1], \quad (13.4)$$

where  $\mathbb{N}$  is the set of all natural numbers. Therefore,  $\mu_j(C) \geq 0 \forall C \in \sigma(\mathcal{C}_{box}) = \mathcal{B}(R^{d_x})$ , i.e.,

$$E_F m_j(W_i, \theta) 1(X_i \in C) \geq 0 \forall C \in \mathcal{B}(R^{d_x}). \quad (13.5)$$

Next, we consider  $j = p + 1, \dots, k$ . Suppose  $E_F m_j(W_i, \theta) g_j(X_i) = 0 \forall g \in \mathcal{G}_{box}$ . Then,  $\mu_j(C) = 0 \forall C \in \mathcal{C}_{box}$ . We want to show that  $E_F m_j(W_i, \theta) 1(X_i \in C) = 0 \forall C \in \mathcal{B}(R^{d_x})$  because this implies that  $E_F(m_j(W_i, \theta) | X_i) = 0$  a.s. because  $X_i$  is Borel measurable. To do so, we show that  $\mathcal{C}_0 = \mathcal{B}(R^{d_x})$ , where  $\mathcal{C}_0 \equiv \{C \in \mathcal{B}(R^{d_x}) : \mu_j(C) = 0\}$ . It suffices to



show  $\mathcal{B}(R^{d_x}) \subset \mathcal{C}_0$ . Because  $\mathcal{C}_{box} \subset \mathcal{C}_0$  and  $\sigma(\mathcal{C}_{box}) = \mathcal{B}(R^{d_x})$ , it suffices to show that  $\mathcal{C}_0$  is a  $\sigma$ -field. The set  $\mathcal{C}_0$  is indeed a  $\sigma$ -field because (a)  $R^{d_x} \in \mathcal{C}_0$  by (13.3), (b) if  $C \in \mathcal{C}_0$ , then  $\mu_j(C^c) = \mu_j(R^{d_x}) - \mu_j(C) = 0$ , i.e.,  $C^c \in \mathcal{C}_0$ , and (c) if  $C_1, C_2, \dots$  are disjoint sets in  $\mathcal{C}_0$ , then  $\mu_j(\bigcup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} \mu_j(C_i) = 0$  because  $\mu_j$  is an additive set function, i.e.,  $\bigcup_{i=1}^{\infty} C_i \in \mathcal{C}_0$ . This completes the proof of Assumption CI for  $\mathcal{G} = \mathcal{G}_{box}$  with  $\bar{r} = \infty$ .

Assumption CI holds for  $\mathcal{G} = \mathcal{G}_{box}$  with  $\bar{r} = \infty$  implies that Assumption CI holds for  $\mathcal{G} = \mathcal{G}_{box}$  when  $\bar{r} \in (0, \infty)$ . The reason is that if some deviation is captured by a big box, it also must be captured by some smaller box contained in the big box (because a big box is a finite disjoint union of smaller boxes).

For  $\mathcal{G} = \mathcal{G}_{c-cube}$ , Assumption CI holds by the same argument as for  $\mathcal{G}_{box}$  but with  $\mathcal{C}_{c-cube}$  in place of  $\mathcal{C}_{box}$  provided (i)  $\mathcal{C}_{c-cube} \cup \{\phi\}$  is a semi-ring of subsets of  $[0, 1]^{d_x}$ , (ii)  $[0, 1]^{d_x}$  can be written as the union of a countable number of disjoint sets in  $\mathcal{C}_{c-cube}$ , and (iii)  $\sigma(\mathcal{C}_{c-cube}) = \mathcal{B}([0, 1]^{d_x})$ . Condition (i) is straightforward to verify. Condition (ii) is verified by using  $\bigcup_{\ell=1}^{2r} ((\ell-1)/(2r), \ell/(2r)] = [0, 1]$  (since the interval  $(0, 1/(2r)]$  is defined specially to include 0) to construct a finite number of  $d_x$ -dimensional boxes whose union is  $[0, 1]^{d_x}$ . Condition (iii) holds because every element of  $\mathcal{C}_{box}$  can be written as a countable union of sets in  $\mathcal{C}_{c-cube}$  and  $\sigma(\mathcal{C}_{box}) = \mathcal{B}([0, 1]^{d_x})$ .

Finally, we establish Assumption M. For  $\mathcal{G} = \mathcal{G}_{box}$ , Assumptions M(a) and M(b) hold by taking  $G(x) = 1 \forall x$  and  $\delta_1 = 4/\delta + 3$ . Assumption M(c) holds because  $\mathcal{C}_{box}$  forms a Vapnik-Cervonenkis class of sets. Assumption M holds for  $\mathcal{G}_{c-cube}$  because  $\mathcal{G}_{c-cube} \subset \mathcal{G}_{box}$ .  $\square$

**Proof of Lemma C1.** Because (i)  $\mu : \sigma(\mathcal{R}) \rightarrow R$  is a bounded countably additive set function, (ii)  $\mu(\phi) = 0$ , and (iii)  $\mu(C) \geq 0 \forall C \in \mathcal{R}$ , Billingsley's (1995) Thm. 11.3 implies that there exist a measure,  $\mu^*$ , on  $\sigma(\mathcal{R})$  that agrees with  $\mu$  on  $\mathcal{R}$ . We want to show that  $\mu^*$  agrees with  $\mu$  on  $\sigma(\mathcal{R})$ . That is, we want to show that  $\mathcal{C}_{eq} = \sigma(\mathcal{R})$ , where

$$\mathcal{C}_{eq} = \{C \in \sigma(\mathcal{R}) : \mu^*(C) = \mu(C)\}. \quad (13.6)$$

It suffices to show that  $\sigma(\mathcal{R}) \subseteq \mathcal{C}_{eq}$  because by definition,  $\sigma(\mathcal{R}) \supseteq \mathcal{C}_{eq}$ . We use Dynkin's  $\pi$ - $\lambda$  theorem, e.g., see Billingsley (1995, p.33), to establish this.

Because  $\mathcal{R}$  is a semi-ring,  $\mathcal{R}$  is a  $\pi$ -system. Now, we show that  $\mathcal{C}_{eq}$  is a  $\lambda$ -system. By definition, the set  $\mathcal{C}_{eq}$  is a  $\lambda$ -system if (a)  $R^{d_x} \in \mathcal{C}_{eq}$ , (b)  $\forall C_1, C_2 \in \mathcal{C}_{eq}$  such that  $C_1 \subset C_2$ ,  $C_2 - C_1 \in \mathcal{C}_{eq}$ , and (c)  $\forall C_1, C_2, \dots \in \mathcal{C}_{eq}$  such that  $C_i \uparrow C$ ,  $C \in \mathcal{C}_{eq}$ . We show (a), (b), and (c) in turn.

(a) By assumption,  $R^{d_x}$  can be written as the union of countable disjoint  $\mathcal{R}$ -sets, say  $C_1, C_2, \dots \in \mathcal{R}$ , where  $R^{d_x} = \bigcup_{i=1}^{\infty} C_i$ . By countable additivity of both  $\mu$  and  $\mu^*$ , we have

$$\mu(R^{d_x}) = \sum_{i=1}^{\infty} \mu(C_i) = \sum_{i=1}^{\infty} \mu^*(C_i) = \mu^*(R^{d_x}), \quad (13.7)$$

where the second equality holds because  $C_1, C_2, \dots \in \mathcal{R}$  and  $\mu^*$  agrees with  $\mu$  on  $\mathcal{R}$ . Thus condition (a) holds.

(b) Suppose  $C_1, C_2 \in \mathcal{C}_{eq}$  and  $C_1 \subset C_2$ , then  $C_2 = (C_2 - C_1) \cup C_1$ . Thus,

$$\mu(C_2 - C_1) = \mu(C_2) - \mu(C_1) = \mu^*(C_2) - \mu^*(C_1) = \mu^*(C_2 - C_1), \quad (13.8)$$

where the first and the third equalities hold by the countable additivity of  $\mu$  and  $\mu^*$  and the second equality holds because  $C_1, C_2 \in \mathcal{C}_{eq}$ . Thus, condition (b) holds.

(c) Suppose  $C_1, C_2, \dots \in \mathcal{C}_{eq}$  and  $C_i \uparrow C$ , then  $C = C_1 \cup (\bigcup_{i=2}^{\infty} (C_i - C_{i-1}))$  and  $C_1, C_2 - C_1, \dots$  are mutually disjoint. By condition (b),  $C_i - C_{i-1} \in \mathcal{C}_{eq}$  for  $i \geq 2$ . Thus,

$$\mu(C) = \mu(C_1) + \sum_{i=2}^{\infty} \mu(C_i - C_{i-1}) = \mu^*(C_1) + \sum_{i=2}^{\infty} \mu^*(C_i - C_{i-1}) = \mu^*(C). \quad (13.9)$$

That is, condition (c) holds.

Therefore,  $\mathcal{C}_{eq}$  is a  $\lambda$ -system. Because  $\mathcal{R} \subset \mathcal{C}_{eq}$  by Dynkin's  $\pi$ - $\lambda$  theorem,  $\sigma(\mathcal{R}) \subseteq \mathcal{C}_{eq}$ . In consequence,  $\sigma(\mathcal{R}) = \mathcal{C}_{eq}$ , i.e.,  $\mu^*$  agrees with  $\mu$  on  $\sigma(\mathcal{R})$ . Because  $\mu^*$  is a measure on  $\sigma(\mathcal{R})$ ,  $\mu$  must be a measure on  $\sigma(\mathcal{R})$ .  $\square$

**Proof of Theorem 2(b).** Consider the parameters  $(\theta_c, F_c)$  that appear in Assumption GMS2. First, we determine the asymptotic behavior of the critical value  $c(\varphi_n(\theta_c), \widehat{h}_{n,2}(\theta_c), 1 - \alpha)$  under  $(\theta_c, F_c)$ . We have

$$\begin{aligned} \xi_n(\theta_c, g) &= \kappa_n^{-1} n^{1/2} \overline{D}_n^{-1/2}(\theta_c, g) \overline{m}_n(\theta_c, g) \\ &= \overline{D}_n^{-1/2}(\theta_c, g) D_{F_c}^{1/2}(\theta_c) \kappa_n^{-1} [\nu_{n, F_c}(\theta_c, g) + h_{1, n, F_c}(\theta_c, g)] \\ &= Dia g^{-1/2} (\overline{h}_{2, n, F_c}(\theta_c, g)) \kappa_n^{-1} [\nu_{n, F_c}(\theta_c, g) + h_{1, n, F_c}(\theta_c, g)]. \end{aligned} \quad (13.10)$$

Note that  $\overline{h}_{2, n, F_c}(\theta_c, g)$  is a function of  $\widehat{h}_{2, n, F_c}(\theta_c, g, g)$  by (5.2). In addition,  $\overline{h}_{2, n}(\theta_c, g)$ , which appears in  $\varphi_n(\theta_c, g)$ , see (4.10), is a function of  $\widehat{h}_{2, n, F_c}(\theta_c, g, g)$  by the argument

in (10.27). Let

$$T_n^{GMS}(\theta_c) = \int S(\nu_{\widehat{h}_{2,n}(\theta_c)}(g) + \varphi_n(\theta_c, g), \widehat{h}_{2,n}(\theta_c, g) + \varepsilon I_k) dQ(g). \quad (13.11)$$

Equations (4.10), (10.27), (13.10), and (13.11) imply that the distribution of  $T_n^{GMS}(\theta_c)$  is determined by the joint distribution of  $\{\nu_{\widehat{h}_{2,n}(\theta_c)}(g) : g \in \mathcal{G}\}$ ,  $\{\widehat{h}_{2,n,F_c}(\theta_c, g) : g \in \mathcal{G}\}$ , and  $\{\kappa_n^{-1}\nu_{n,F_c}(\theta_c, g) : g \in \mathcal{G}\}$ .

We have  $\{(\theta_c, F_c) : n \geq 1\} \in \text{SubSeq}(h_{2,F_c}(\theta_c))$  because  $(\theta_c, F_c) \in \mathcal{F}$ . Hence, by Lemma A1(b),  $d(\widehat{h}_{2,n,F_c}(\theta_c), h_{2,F_c}(\theta_c)) \rightarrow_p 0$  as  $n \rightarrow \infty$ . By the same argument as in (10.27), this yields  $d(\widehat{h}_{2,n}(\theta_c), h_{2,F_c}(\theta_c)) \rightarrow_p 0$ . The latter, the independence of  $\widehat{h}_{2,n,F_c}(\theta_c)$  and  $\{\nu_{h_2}(\cdot) : h_2 \in \mathcal{H}_2\}$ , and an almost sure representation argument imply that the Gaussian processes  $\{\nu_{\widehat{h}_{2,n}(\theta_c)}(\cdot) : n \geq 1\}$  converge weakly to  $\nu_{h_{2,F_c}(\theta_c)}(\cdot)$  as  $n \rightarrow \infty$ . The sequence of random processes  $\{\widehat{h}_{2,n}(\theta_c, \cdot) : n \geq 1\}$  converges in probability uniformly (and hence in distribution) to  $h_{2,F_c}(\theta_c, \cdot)$ , where  $\widehat{h}_{2,n}(\theta_c, g) = \widehat{h}_{2,n}(\theta_c, g, g)$  and  $h_{2,F_c}(\theta_c, g) = h_{2,F_c}(\theta_c, g, g)$ . The sequence  $\{\kappa_n^{-1}\nu_{n,F_c}(\theta_c, \cdot) : n \geq 1\}$  converges in probability to zero uniformly over  $g \in \mathcal{G}$  because  $\kappa_n \rightarrow \infty$  and  $\{\nu_{n,F_c}(\theta_c, \cdot) : n \geq 1\}$  converges to a Gaussian process with sample paths that are bounded a.s. Therefore, we have

$$\begin{pmatrix} \nu_{\widehat{h}_{2,n}(\theta_c)}(\cdot) \\ \widehat{h}_{2,n}(\theta_c, \cdot) \\ \kappa_n^{-1}\nu_{n,F_c}(\theta_c, \cdot) \end{pmatrix} \Rightarrow \begin{pmatrix} \nu_{h_{2,F_c}(\theta_c)}(\cdot) \\ h_{2,F_c}(\theta_c, \cdot) \\ 0_{\mathcal{G}} \end{pmatrix} \text{ as } n \rightarrow \infty, \quad (13.12)$$

where  $\widehat{h}_{2,n}(\theta_c)$  that appears in  $\nu_{\widehat{h}_{2,n}(\theta_c)}(\cdot)$  is a function on  $\mathcal{G} \times \mathcal{G}$  whereas  $\widehat{h}_{2,n}(\theta_c, \cdot)$  is a function on  $\mathcal{G}$ , likewise for  $\nu_{h_{2,F_c}(\theta_c)}(\cdot)$  and  $h_{2,F_c}(\theta_c, \cdot)$ , and  $0_{\mathcal{G}}$  denotes the  $R^k$ -valued function on  $\mathcal{G}$  that is identically  $(0, \dots, 0)' \in R^k$ .

By the almost sure representation theorem, see Pollard (1990, Thm. 9.4), there exist  $\{(\tilde{\nu}_n(g), \tilde{h}_{2,n}(g), \tilde{\nu}_{\kappa,n}(g)) : g \in \mathcal{G}, n \geq 1\}$  and  $\{\tilde{\nu}(g), \tilde{h}_2(g) : g \in \mathcal{G}\}$  such that (i)  $\{(\tilde{\nu}_n(g), \tilde{h}_{2,n}(g), \tilde{\nu}_{\kappa,n}(g)) : g \in \mathcal{G}\}$  has the same distribution as  $\{(\nu_{\widehat{h}_{2,n}(\theta_c)}(g), \widehat{h}_{2,n}(\theta_c, g), \kappa_n^{-1}\nu_{n,F_c}(\theta_c, g)) : g \in \mathcal{G}\}$  for all  $n \geq 1$ , (ii)  $\{(\tilde{\nu}(g), \tilde{h}_2(g)) : g \in \mathcal{G}\}$  has the same distribution as  $\{(\nu_{h_{2,F_c}(\theta_c)}(g), h_{2,F_c}(\theta_c, g)) : g \in \mathcal{G}\}$ , and

$$(iii) \sup_{g \in \mathcal{G}} \left\| \begin{pmatrix} \tilde{\nu}_n(g) \\ \tilde{h}_{2,n}(g) \\ \tilde{\nu}_{\kappa,n}(g) \end{pmatrix} - \begin{pmatrix} \tilde{\nu}(g) \\ \tilde{h}_2(g) \\ 0 \end{pmatrix} \right\| \rightarrow 0 \text{ a.s.} \quad (13.13)$$

Let

$$\tilde{T}_n^{GMS} = \int S(\tilde{\nu}_n(g) + \tilde{\varphi}_n(g), \tilde{h}_{2,n}(g) + \varepsilon I_k) dQ(g), \quad (13.14)$$

where  $\tilde{\varphi}_n(g)$  is defined just as  $\varphi_n(\theta, g)$  is defined in (4.10) but with  $\tilde{h}_{2,n,j}(g) + \varepsilon \tilde{h}_{2,n,j}(1_k)$  in place of  $\bar{h}_{2,n,F_n,j}(\theta, g)$ , where  $\tilde{h}_{2,n,j}(g)$  denotes the  $(j, j)$  element of  $\tilde{h}_{2,n}(g)$ , and  $\tilde{\xi}_n(g)$  in place of  $\xi_n(\theta, g)$ , where

$$\tilde{\xi}_n(g) = \text{Diag}(\tilde{h}_{2,n}(g) + \varepsilon \tilde{h}_{2,n}(1_k))^{-1/2} (\kappa_n^{-1} \tilde{\nu}_{\kappa,n}(g) + \kappa_n^{-1} h_{1,n,F_c}(\theta_c, g)). \quad (13.15)$$

Then,  $\tilde{T}_n^{GMS}$  and  $T_n^{GMS}(\theta_c)$  have the same distribution for all  $n \geq 1$  and the same asymptotic distribution as  $n \rightarrow \infty$ . Let  $\tilde{c}_n(1 - \alpha)$  denote the  $1 - \alpha + \eta$  quantile of  $\tilde{T}_n^{GMS}$  plus  $\eta$ , where  $\eta$  is as in the definition of  $c(h, 1 - \alpha)$ . Then,  $\tilde{c}_n(1 - \alpha)$  has the same distribution as  $c(\varphi_n(\theta_c), \hat{h}_{2,n}(\theta_c), 1 - \alpha)$  for all  $n \geq 1$ .

Let  $\tilde{\Omega}^*$  be the collection of  $\omega \in \Omega$  such that at  $\omega$ ,  $\tilde{\nu}(g)(\omega)$  is bounded and the convergence in (13.13) holds. By (13.13) and the fact that the sample paths of  $\{\tilde{\nu}(g) : g \in \mathcal{G}\}$  are bounded a.s., we have  $P_{F_c}(\tilde{\Omega}^*) = 1$ .

Under  $(\theta_c, F_c)$  for all  $n \geq 1$ ,

$$\kappa_n^{-1} h_{1,n,F_c}(\theta_c, g) = \kappa_n^{-1} n^{1/2} D_{F_c}^{-1/2}(\theta_c) E_{F_c} m(W_i, \theta_c, g) \rightarrow h_{1,\infty,F_c}(\theta_c, g) \quad (13.16)$$

as  $n \rightarrow \infty$  using Assumption GMS2(c). Thus, for fixed  $\omega \in \tilde{\Omega}^*$ ,

$$\begin{aligned} \tilde{\xi}_n(g)(\omega) &= \text{Diag}^{-1/2}(\tilde{h}_2(g) + \varepsilon \tilde{h}_2(1_k) + o(1))(o(1) + \kappa_n^{-1} h_{1,n,F_c}(\theta_c, g)) \rightarrow h_{1,\infty,F_c}(\theta_c, g), \\ (\tilde{h}_{2,n,j}(g)(\omega) + \varepsilon \tilde{h}_{2,n,j}(1_k)(\omega))^{1/2} B_n &= (\tilde{h}_{2,j}(g) + \varepsilon \tilde{h}_{2,j}(1_k) + o(1))^{1/2} B_n \rightarrow \infty \end{aligned} \quad (13.17)$$

as  $n \rightarrow \infty$  for all  $g \in \mathcal{G}$ , where  $\tilde{h}_{2,j}(g)$  denotes the  $(j, j)$  element of  $\tilde{h}_2(g)$ , using (13.13),  $\tilde{h}_2(1_k) = I_k$  (which holds by (5.1) and Definition SubSeq( $h_2$ )),  $\tilde{h}_{2,j}(g) \geq 0$ ,  $\varepsilon > 0$ , and Assumption GMS2(b).

By (13.17), Assumption GMS1(a), and the fact that  $h_{1,\infty,F_c}(\theta_c, g)$  equals either 0 or  $\infty$  by definition, we have

$$\tilde{\varphi}_n(g)(\omega) \rightarrow h_{1,\infty,F_c}(\theta_c, g) \text{ as } n \rightarrow \infty \quad (13.18)$$

for all  $\omega \in \tilde{\Omega}^*$ .

By (13.13), (13.15), (13.18), and Assumption S1(d), we have

$$\begin{aligned} & S(\tilde{\nu}_n(g) + \tilde{\varphi}_n(g), \tilde{h}_{2,n}^*(g) + \varepsilon I_k)(\omega) \\ \rightarrow & S(\tilde{\nu}(g) + h_{1,\infty,F_c}(\theta_c, g), h_{2,F_c}(\theta_c, g) + \varepsilon I_k)(\omega) \end{aligned} \quad (13.19)$$

as  $n \rightarrow \infty \forall \omega \in \tilde{\Omega}^*, \forall g \in \mathcal{G}$ . Now, by the argument given from (10.14) to the end of the proof of Theorem 1, the quantity on the left-hand side of (13.19) is bounded by a finite constant. This, (13.19), and the BCT give

$$\tilde{T}_n^{GMS} \rightarrow \tilde{T}^{GMS} = \int S(\tilde{\nu}(g) + h_{1,\infty,F_c}(\theta_c, g), h_{2,F_c}(\theta_c, g) + \varepsilon I_k) dQ(g) \quad (13.20)$$

as  $n \rightarrow \infty$  a.s.

By (13.20),

$$P(\tilde{T}_n^{GMS} \leq x) \rightarrow P(\tilde{T}^{GMS} \leq x) \text{ as } n \rightarrow \infty \quad (13.21)$$

for all continuity points  $x$  of the distribution of  $\tilde{T}^{GMS}$ . Let  $\tilde{c}_0(1 - \alpha)$  denote the  $1 - \alpha$  quantile of  $\tilde{T}^{GMS}$ . Let  $\tilde{c}(1 - \alpha) = \tilde{c}_0(1 - \alpha + \eta) + \eta$ , where  $\eta$  is as in the definition of  $c(h, 1 - \alpha)$ . By Assumption GMS2(a), the distribution function of  $\tilde{T}^{GMS}$ , which equals that of  $T(h_{\infty,F_c}(\theta_c))$ , is continuous and strictly increasing at  $x = \tilde{c}(1 - \alpha)$ . Using Lemma 5 of Andrews and Guggenberger (2010), this gives

$$\begin{aligned} & \tilde{c}_n(1 - \alpha) \rightarrow_p \tilde{c}(1 - \alpha) \text{ and} \\ & c(\varphi_n(\theta_c), \hat{h}_{2,n}(\theta_c), 1 - \alpha) \rightarrow_p \tilde{c}(1 - \alpha), \end{aligned} \quad (13.22)$$

where the second convergence result holds because  $\tilde{c}_n(1 - \alpha)$  and  $c(\varphi_n(\theta_c), \hat{h}_{2,n}(\theta_c), 1 - \alpha)$  have the same distribution.

Next, by the same argument as used above to show (13.20), but with  $\nu_{\hat{h}_{2,n}(\theta_c)}(g)$  and  $\varphi_n(\theta_c, g)$  replaced by  $\nu_{n,F_c}(\theta_c, g)$  and  $h_{1,n,F_c}(\theta_c, g)$ , respectively, we have

$$T_n(\theta_c) \rightarrow_d T(h_{\infty,F_c}(\theta_c)) = \int S(\nu_{h_{2,F_c}(\theta_c)}(g) + h_{1,\infty,F_c}(\theta_c, g), h_{2,F_c}(\theta_c, g) + \varepsilon I_k) dQ(g), \quad (13.23)$$

where  $h_{\infty,F_c}(\theta_c) = (h_{1,\infty,F_c}(\theta_c), h_{2,F_c}(\theta_c))$ ,  $h_{1,n,F_c}(\theta_c) \rightarrow h_{1,\infty,F_c}(\theta_c)$  by straightforward calculations, and  $\nu_{n,F_c}(\theta_c, \cdot) \Rightarrow \nu_{h_{2,F_c}(\theta_c)}(\cdot)$  by Lemma A1(a). Note that  $T(h_{\infty,F_c}(\theta_c))$  and  $\tilde{T}^{GMS}$  have the same distribution because  $\nu_{h_{2,F_c}(\theta_c)}(\cdot)$  and  $\tilde{\nu}(\cdot)$  have the same distribution. Thus,  $\tilde{c}(1 - \alpha) (= \tilde{c}_0(1 - \alpha + \eta) + \eta)$  is the  $1 - \alpha + \eta$  quantile of  $T(h_{\infty,F_c}(\theta_c))$  plus  $\eta > 0$ .

By (13.22), (13.23), Assumption GMS2(a), and Lemma 5 of Andrews and Guggenberger (2010), for  $\eta > 0$ , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} P_{F_c}(T_n(\theta_c) \leq c(\varphi_n(\theta_c), \widehat{h}_{2,n}(\theta_c), 1 - \alpha)) \\ &= P(T(h_{\infty, F_c}(\theta_c)) \leq \tilde{c}_0(1 - \alpha + \eta) + \eta). \end{aligned} \quad (13.24)$$

The limit as  $\eta \rightarrow 0$  of the right-hand side equals  $1 - \alpha$  because distribution functions are right-continuous and the df of  $T(h_{\infty, F_c}(\theta_c))$  at its  $1 - \alpha$  quantile equals  $1 - \alpha$  by Assumption GMS2(a).

Combining (13.24) and the result of Theorem 2(a), which holds for all  $\eta > 0$  and hence holds when the limit as  $\eta \rightarrow 0$  is taken, gives Theorem 2(b).  $\square$

## 13.2 Proofs of Results for Fixed Alternatives

**Proof of Lemma 4.** First, we prove part (a). It holds immediately that  $\text{Supp}(Q_a) = \mathcal{G}_{c\text{-cube}}$  because  $\mathcal{G}_{c\text{-cube}}$  is countable and  $Q_a$  has a probability mass function that is positive at each element in  $\mathcal{G}_{c\text{-cube}}$ .

Next, for part (b), consider  $g = g_{x,r} \in \mathcal{G}_{box}$ , where  $g_{x,r}(y) = 1(y \in C_{x,r}) \cdot 1_k$  and  $(x, r) \in [0, 1]^{d_x} \times (0, \bar{r})^{d_x}$ . Let  $\delta > 0$  be given. The idea of the proof is to find a set  $G_{g, \bar{\eta}} \subset \mathcal{B}_{\rho_X}(g, \delta) (\subset \mathcal{G}_{box})$  such that  $Q_b(G_{g, \bar{\eta}}) > 0$ . This implies that  $Q_b(\mathcal{B}_{\rho_X}(g, \delta)) > 0$ , which is the desired result.

The set  $G_{g, \bar{\eta}}$  needs to be defined differently (for reasons stated below) depending on whether  $x_u < 1$  or  $x_u = 1$ , for  $u = 1, \dots, d_x$ , where  $x = (x_1, \dots, x_{d_x})'$ . For  $\bar{\eta} > 0$ , define

$$\begin{aligned} G_{g, \bar{\eta}} &= \{g_{x+\eta_0, r+\eta_1} : (\eta_0, \eta_1) \in \Xi_{g, \bar{\eta}}\}, \text{ where} \\ \Xi_{g, \bar{\eta}} &= \{(\eta_0, \eta_1) \in R^{2d_x} : \text{for } u = 1, \dots, d_x, \text{ if } x_u < 1, \eta_{0,u} \in [\bar{\eta}, 2\bar{\eta}] \ \& \\ & \eta_{1,u} \in [0, \bar{\eta}] \text{ and for } x_u = 1, \eta_{0,u} \in [-\bar{\eta}, 0] \ \& \ \eta_{1,u} \in [-2\bar{\eta}, -\bar{\eta}]\}. \end{aligned} \quad (13.25)$$

We have  $Q_b(G_{g, \bar{\eta}}) = Q_b^*((x, r) + \Xi_{g, \bar{\eta}}) > 0$  for all  $\bar{\eta} > 0$  because  $Q_b^*$  is the uniform distribution on  $[0, 1]^{d_x} \times (0, \bar{r})^{d_x}$ .

Next, we show that  $G_{g, \bar{\eta}} \subset \mathcal{B}_{\rho_X}(g, \delta)$ . Let  $U_{(x_u < 1)} \subset \{1, \dots, d_x\}$  be the set of indices  $u$  such that  $x_u < 1$  and let  $U_{(x_u = 1)} \subset \{1, \dots, d_x\}$  be the set of indices  $u$  such that  $x_u = 1$ . Let  $g_{x+\eta_0, r+\eta_1} \in G_{g, \bar{\eta}}$ . The  $u$ th lower endpoints of the  $C_{x,r}$  and  $C_{x+\eta_0, r+\eta_1}$  boxes are  $x_u - r_u$  and  $x_u + \eta_{0,u} - (r_u + \eta_{1,u})$ , respectively. The lower endpoint of the  $C_{x+\eta_0, r+\eta_1}$  box is larger than that of  $C_{x,r}$  box because  $\eta_{0,u} - \eta_{1,u} \in [0, 2\bar{\eta}]$  (whether  $u \in U_{(x_u < 1)}$  or  $u \in U_{(x_u = 1)}$ ). The

$u$ th upper endpoints of the  $C_{x,r}$  and  $C_{x+\eta_0,r+\eta_1}$  boxes are  $x_u + r_u$  and  $x_u + \eta_{0,u} + r_u + \eta_{1,u}$ , respectively. If  $u \in U_{x_u < 1}$ , the upper endpoint of the  $C_{x+\eta_0,r+\eta_1}$  box is larger than that of  $C_{x,r}$  box because  $\eta_{0,u} + \eta_{1,u} \in [0, 3\bar{\eta}]$ . If  $u \in U_{(x_u=1)}$ , the  $u$ th upper endpoint of the  $C_{x+\eta_0,r+\eta_1}$  box is smaller than that of  $C_{x,r}$  box because  $\eta_{0,u} + \eta_{1,u} \in [-3\bar{\eta}, 0]$ .

Using the results of the previous paragraph, we have

$$\begin{aligned}
& \rho_X^2(g_{x,r}, g_{x+\eta_0,r+\eta_1}) \\
&= E_{F_{X,0}} [1(X_i \in C_{x,r}) - 1(X_i \in C_{x+\eta_0,r+\eta_1})]^2 \\
&\leq \sum_{u=1}^{d_x} P_{F_{X,0}}(X_{i,u} \in (x_u - r_u, x_u + \eta_{0,u} - (r_u + \eta_{1,u}))) \\
&\quad + \sum_{u \in U_{(x_u < 1)}} P_{F_{X,0}}(X_{i,u} \in (x_u + r_u, x_u + \eta_{0,u} + r_u + \eta_{1,u})) \\
&\quad + \sum_{u \in U_{(x_u = 1)}} P_{F_{X,0}}(X_{i,u} \in (1 + \eta_{0,u} + r_u + \eta_{1,u}, 1 + r_u] \cap [0, 1]) \\
&\leq \sum_{u=1}^{d_x} P_{F_{X,0}}(X_{i,u} \in (x_u - r_u, x_u - r_u + 2\bar{\eta})) + \sum_{u \in U_{(x_u < 1)}} P_{F_{X,0}}(X_{i,u} \in (x_u + r_u, x_u + r_u + 3\bar{\eta})) \\
&\quad + \sum_{u \in U_{(x_u = 1)}} P_{F_{X,0}}(X_{i,u} \in (1 + r_u - 3\bar{\eta}, 1 + r_u] \cap [0, 1]), \tag{13.26}
\end{aligned}$$

where the first inequality uses the  $d_x$ -dimensional extension of the one-dimensional result that  $(a, b] \Delta (c, d] \subset (a, c] \cup (b, d]$  when  $a \leq c$  and  $b \leq d$ , where  $\Delta$  denotes the symmetric difference of two sets.

The first and second summands on the rhs of (13.26) tend to zero as  $\bar{\eta} \downarrow 0$  by the right continuity of distribution functions. The third summand on the rhs equals zero when  $\bar{\eta}$  is sufficiently small (i.e., when  $3\bar{\eta} < \min_{u \leq d_x} r_u$ ). Therefore, for  $\bar{\eta} > 0$  sufficiently small,  $\rho_X^2(g_{x,r}, g_{x+\eta_0,r+\eta_1}) < \delta$  and  $G_{g,\bar{\eta}} \subset \mathcal{B}_{\rho_X}(g, \delta)$ . This completes the proof of part (b).

Note that in the proof of part (b) we cannot treat the case where  $u \in U_{(x_u=1)}$  in the same way that we treat the case for  $u \in U_{(x_u < 1)}$  because for  $u \in U_{(x_u < 1)}$  we use the center point  $x_u + \eta_{0,u} > x_u$  which is not in  $[0, 1]$  if  $x_u = 1$  and hence violates the assumption of part (b) that the centers of the  $\mathcal{G}_{box}$  boxes lie in  $[0, 1]^{d_x}$ . Conversely, we cannot treat the case where  $u \in U_{(x_u < 1)}$  in the same way that we treat the case for  $u \in U_{(x_u = 1)}$  because doing so would lead to a term  $P_{F_{X,0}}(X_{i,u} \in (1 + r_u - 3\bar{\eta}, 1 + r_u])$  in (13.26) that does not go to zero as  $\bar{\eta} \downarrow 0$  if  $X_{i,u}$  has positive probability of equaling  $1 + r_u$ .  $\square$

**Proof of Theorem 3.** Part (a) follows from part (b) because

$$c(\varphi_n(\theta_*), \widehat{h}_{2,n}(\theta_*), 1 - \alpha) \leq c(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha), \quad (13.27)$$

which holds because  $\varphi_n(\theta_*, g) \geq 0_k \forall g \in \mathcal{G}$  by Assumption GMS1(a),  $c(h_1, \widehat{h}_{2,n}(\theta_*), 1 - \alpha)$  is non-increasing in the first  $p$  elements of  $h_1$  by Assumption S1(b), and the last  $v$  elements of  $\varphi_n(\theta_*, g)$  equal zero.

Now we prove part (b). By Assumptions FA(a) and CI,  $\beta(g_0) > 0$  for some  $g_0 \in \mathcal{G}$ . By construction,  $e_j = m_j^*(g_0)/\beta(g_0) \in [-1, \infty)$  for  $j = 1, \dots, k$  and  $e_j = -1$  for some  $j \leq p$  or  $|e_j| = 1$  for some  $j = p+1, \dots, k$ . As defined above,  $\mathcal{B}_{\rho_X}(g_0, \tau_2)$  denotes a  $\rho_X$ -ball centered at  $g_0$  with radius  $\tau_2 > 0$ , where  $\rho_X$  is defined in (6.3). First we show that for some  $\tau_2 > 0$ ,

$$\int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} S(m^*(g)/\beta(g_0), h_{2,0}(g) + \varepsilon I_k) dQ(g) > 0, \quad (13.28)$$

where  $m^*(g) = (m_1^*(g), \dots, m_k^*(g))'$  and  $h_{2,0}(g) = h_{2,F_0}(\theta_*, g)$ . We have: for  $j = 1, \dots, k$ ,

$$\begin{aligned} & |m_j^*(g) - m_j^*(g_0)| \\ &= |E_{F_0} m_j(W_i, \theta_*) g_j(X_i) - E_{F_0} m_j(W_i, \theta_*) g_{0,j}(X_i)| / \sigma_{F_0,j}(\theta_*) \\ &\leq (E_{F_0} m_j^2(W_i, \theta_*) )^{1/2} (E_{F_0} (g_j(X_i) - g_{0,j}(X_i))^2)^{1/2} / \sigma_{F_0,j}(\theta_*) \\ &\leq (E_{F_0} \|m(W_i, \theta_*)\|^2)^{1/2} \rho_X(g, g_0) / \sigma_{F_0,j}(\theta_*), \end{aligned} \quad (13.29)$$

where  $g_{0,j}(X_i)$  denotes the  $j$ th element of  $g_0(X_i)$ .

Given  $\tau_1 \in (0, 1)$ , let

$$\tau_2 = \tau_1 \beta(g_0) \sigma_{F_0,j}(\theta_*) / (E_{F_0} \|m(W_i, \theta_*)\|^2)^{1/2}. \quad (13.30)$$

By (13.29), for all  $g \in \mathcal{B}_{\rho_X}(g_0, \tau_2)$ ,

$$|m_j^*(g) - m_j^*(g_0)| \leq \tau_1 \beta(g_0) \text{ for all } j = 1, \dots, k. \quad (13.31)$$

Hence, for all  $g \in \mathcal{B}_{\rho_X}(g_0, \tau_2)$ , there exists  $j \leq k$  such that either

$$\begin{aligned} & j \leq p \text{ and } m_j^*(g)/\beta(g_0) \leq m_j^*(g_0)/\beta(g_0) + \tau_1 = -1 + \tau_1 < 0 \text{ or} \\ & j \in \{p+1, \dots, k\} \text{ and } |m_j^*(g)/\beta(g_0)| \geq |m_j^*(g_0)/\beta(g_0)| - \tau_1 = 1 - \tau_1 > 0 \end{aligned} \quad (13.32)$$



using the triangle inequality.

By (13.32) and Assumption S3,  $S(m^*(g)/\beta(g_0), h_{2,0}(g) + \varepsilon I_k) > 0$  for all  $g \in \mathcal{B}_{\rho_X}(g_0, \tau_2)$ . In addition, by Assumption Q,  $Q(\mathcal{B}_{\rho_X}(g_0, \tau_2)) > 0$ . These properties combine to give (13.28).

We use (13.28) in the following: for all  $\delta > 0$ ,

$$\begin{aligned}
& (n^{1/2}\beta(g_0))^{-\chi} T_n(\theta_*) \\
&= (n^{1/2}\beta(g_0))^{-\chi} \int_{\mathcal{G}} S(\nu_{n,F_0}(\theta_*, g) + h_{1,n,F_0}(\theta_*, g), \bar{h}_{2,n,F_0}(\theta_*, g)) dQ(g) \\
&\geq (n^{1/2}\beta(g_0))^{-\chi} \int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} S(\nu_{n,F_0}(\theta_*, g) + h_{1,n,F_0}(\theta_*, g), \bar{h}_{2,n,F_0}(\theta_*, g)) dQ(g) \\
&= \int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} S((n^{1/2}\beta(g_0))^{-1}\nu_{n,F_0}(\theta_*, g) + m^*(g)/\beta(g_0), \bar{h}_{2,n,F_0}(\theta_*, g)) dQ(g) \\
&\rightarrow_p \int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} S(m^*(g)/\beta(g_0), h_{2,0}(g) + \varepsilon I_k) dQ(g) \\
&> 0,
\end{aligned} \tag{13.33}$$

where  $\chi$  is as in Assumption S4, the first equality holds by (5.4), the first inequality holds by Assumption S1(c), the second equality holds by Assumption S4 and the definition of  $m_j^*(g)$  in (6.2), the last inequality holds by (13.28), and the convergence holds by the argument given in the following paragraph.

By Lemma A1(a) and the continuous mapping theorem,  $\sup_{g \in \mathcal{G}} \|\nu_{n,F_0}(\theta_*, g)\| = O_p(1)$ . (Note that Lemma A1 applies for  $(\theta_{a_n}, F_{a_n}) = (\theta_*, F_0) \notin \mathcal{F}$  for all  $n \geq 1$  because Assumptions FA(b)-(d) imply conditions (ii)-(v) in the definition of  $SubSeq(h_{2,F_0}(\theta_*))$ .) Also,  $(n^{1/2}\beta(g_0))^{-1} = o(1)$ , because Assumptions FA and CI imply that  $\beta(g_0) > 0$  for some  $g_0 \in \mathcal{G}$ . Hence, (i)  $(n^{1/2}\beta(g_0))^{-1}\nu_{n,F_0}(\theta_*, \cdot) \Rightarrow 0_{\mathcal{G}}$ . In addition, (ii)  $\sup_{g \in \mathcal{G}} \|\bar{h}_{2,n,F_0}(\theta_*, g) - h_{2,0}(g) - \varepsilon I_k\| \rightarrow_p 0$ , where  $h_{2,0}(g) = h_{2,F_0}(\theta_*, g, g)$ , by Lemma A1(b), (10.27), and the definition of  $\bar{h}_{2,n,F}(\theta, g)$ . As in previous proofs, by the almost sure representation theorem, there exists a probability space and random quantities defined on it with the same distributions as  $(n^{1/2}\beta(g_0))^{-1}\nu_{n,F_0}(\theta_*, \cdot)$  and  $\bar{h}_{2,n,F_0}(\theta_*, \cdot)$  for  $n \geq 1$  such that the convergence in (i) and (ii) holds almost surely for these random quantities. Furthermore, using Assumptions S1(b) and S1(e), the integrand in the last equality in (13.33) is bounded by  $\sup_{g \in \mathcal{B}_{\rho_X}^{cl}(g_0, \tau_2), \nu \in R^k: \|\nu\| \leq \delta_*} S(\nu + m^*(g)/\beta(g_0), (\varepsilon - \delta_{**})I_k) < \infty$  for all  $g \in \mathcal{B}_{\rho_X}(g_0, \tau_2)$  for some  $\delta_*, \delta_{**} > 0$  for  $n$  sufficiently large, where  $\mathcal{B}_{\rho_X}^{cl}(g_0, \tau_2)$  denotes the closure of  $\mathcal{B}_{\rho_X}(g_0, \tau_2)$ , because a continuous function on a compact set attains its

supremum using Assumption S1(d) and using an argument analogous to that in (10.14) to treat the second argument of the function  $S$ . Thus, by the bounded convergence theorem, the convergence in (13.33) holds a.s. for the newly constructed random quantities. In consequence, it holds in probability for the original random quantities by the equality in distribution of the original and newly constructed random quantities. This completes the proof of the convergence in (13.33).

Next, we show that under  $F_0$ ,

$$c(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha) = O_p(1). \quad (13.34)$$

This and (13.33) give

$$\begin{aligned} & P_{F_0}(T_n(\theta_*) > c(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha)) \\ &= P_{F_0}\left((n^{1/2}\beta(g_0))^{-\chi} T_n(\theta_*) > (n^{1/2}\beta(g_0))^{-\chi} c(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha)\right) \\ &\geq P_{F_0}\left(\int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} S(m^*(g)/\beta(g_0), h_{2,0}(g) + \varepsilon I_k) dQ(g) + o_p(1) > o_p(1)\right), \\ &\rightarrow 1 \end{aligned} \quad (13.35)$$

as  $n \rightarrow \infty$ , which establishes the result of the Theorem.

It remains to show (13.34). Lemma A5, applied with  $h_{2,n} = h_{2,0}$ ,  $\{h_{2,n}^* : n \geq 1\}$  being any sequence of  $k \times k$ -matrix-valued covariance kernels on  $\mathcal{G} \times \mathcal{G}$  such that  $d(h_{2,n}^*, h_{2,0}) \rightarrow 0$ ,  $h_1 = 0_{\mathcal{G}}$ ,  $\eta$  as in the definition of  $c(h, 1 - \alpha)$ ,  $\alpha$  replaced by  $\alpha - \eta > 0$ , and  $\eta_1 = \delta$ , gives:  $\forall \delta > 0$ ,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} [c_0(0_{\mathcal{G}}, h_{2,0}, 1 - \alpha + \eta + \delta) + \delta - c_0(0_{\mathcal{G}}, h_{2,n}^*, 1 - \alpha + \eta)] \geq 0 \text{ and hence} \\ & \limsup_{n \rightarrow \infty} c_0(0_{\mathcal{G}}, h_{2,n}^*, 1 - \alpha + \eta) \leq c_0(0_{\mathcal{G}}, h_{2,0}, 1 - \alpha + \eta + \delta) + \delta < \infty. \end{aligned} \quad (13.36)$$

By Lemma A1(b) and (10.27), we obtain  $d(\widehat{h}_{2,n}(\theta_*), h_{2,0}) \rightarrow_p 0$ . As in previous proofs, by the almost sure representation theorem, there exists a probability space and random quantities  $\tilde{h}_{2,n}(\cdot, \cdot)$  defined on it with the same distributions as  $\widehat{h}_{2,n}(\theta_*, \cdot, \cdot)$  for  $n \geq 1$  such that  $d(\tilde{h}_{2,n}, h_{2,0}) \rightarrow 0$  a.s. This and (13.36) gives  $\limsup_{n \rightarrow \infty} c_0(0_{\mathcal{G}}, \tilde{h}_{2,n}, 1 - \alpha + \eta) < \infty$  a.s., which implies (13.34) because  $\tilde{h}_{2,n}(\cdot, \cdot)$  and  $\widehat{h}_{2,n}(\theta_*, \cdot, \cdot)$  have the same distribution for all  $n \geq 1$  and  $c(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha) = c_0(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_*), 1 - \alpha + \eta) + \eta$ .  $\square$

### 13.3 Proofs of Results for $n^{-1/2}$ -Local Alternatives

**Proof of Theorem 4.** The proof of part (a) uses the following. By element-by-element mean-value expansions about  $\theta_n$  and Assumptions LA1(a), LA1(b), and LA2,

$$\begin{aligned} & D_{F_n}^{-1/2}(\theta_{n,*})E_{F_n}m(W_i, \theta_{n,*}, g) \\ &= D_{F_n}^{-1/2}(\theta_n)E_{F_n}m(W_i, \theta_n, g) + \Pi_{F_n}(\theta_{n,g}, g)(\theta_{n,*} - \theta_n), \text{ and so} \\ & n^{1/2}D_{F_n}^{-1/2}(\theta_{n,*})E_{F_n}m(W_i, \theta_{n,*}, g) \rightarrow h_1(g) + \Pi_0(g)\lambda, \end{aligned} \quad (13.37)$$

where  $\theta_{n,g}$  may differ across rows of  $\Pi_{F_n}(\theta_{n,g}, g)$ ,  $\theta_{n,g}$  lies between  $\theta_{n,*}$  and  $\theta_n$ ,  $\theta_{n,g} \rightarrow \theta_0$ ,  $\Pi_{F_n}(\theta_{n,g}, g) \rightarrow \Pi_0(g)$ , and by definition  $h_1(g) + \Pi_0(g)\lambda = \infty$  if  $h_1(g) = \infty$ .

Now, the proof of part (a) is the same as the proof of Theorem 2(b) with the following changes: (i)  $(\theta_{n,*}, F_n)$  appears in place of  $(\theta_c, F_c)$  whenever  $(\theta_c, F_c)$  is used in an expression with  $n$  finite, (ii)  $(\theta_0, F_0)$  appears in place of  $(\theta_c, F_c)$  whenever  $(\theta_c, F_c)$  is used in an asymptotic expression, (iii)  $\{(\theta_{n,*}, F_n) : n \geq 1\}$  satisfies the conditions to be in  $SubSeq(h_2)$  (where  $h_2 = h_{2,F_0}(\theta_0)$ ) by Assumptions LA1(a) and LA1(c)-(e) and because  $\{W_i : i \geq 1\}$  are i.i.d. under  $F_n$  and Assumption M holds given that  $(\theta_n, F_n) \in \mathcal{F}$  by Assumption LA1, (iv) equation (13.16) is replaced by

$$\kappa_n^{-1}h_{1,n,F_n}(\theta_{n,*}, g)/\bar{\sigma}_{F_n,j}(\theta_{n,*}, g) \rightarrow \pi_1(g) \text{ as } n \rightarrow \infty, \quad (13.38)$$

which holds by Assumption LA4, (13.37) (because  $\kappa_n^{-1}n^{1/2}\Pi_{F_n}(\theta_{n,g}, g)(\theta_{n,*} - \theta_n) \rightarrow 0$ ), and  $\bar{\sigma}_{F_n,j}(\theta_{n,*}, g)/\bar{\sigma}_{F_n,j}(\theta_n, g) \rightarrow 1$  (using Assumption LA1(c)), (v)  $\pi_1(g)$  appears in place of  $h_{1,\infty,F_c}(\theta_c, g)$  in (13.17), (vi)  $\varphi(\pi_1(g))$  appears in place of  $h_{1,\infty,F_c}(\theta_c, g)$  in (13.18)-(13.20), where (13.18) holds for all  $g \in \mathcal{G}_\varphi$  by Assumption LA5(a) and (13.19) holds for all  $g \in \mathcal{G}_\varphi$ , (vii) Assumption LA5(b) is used in place of Assumption GMS2(a) in two places, (viii)  $(h_1 + \Pi_0\lambda, h_2)$  and  $h_1(g)$  appear in place of  $h_{\infty,F_c}(\theta_c)$  and  $h_{1,\infty,F_c}(\theta_c)$ , respectively, in (13.23) and (13.24), and (ix) (13.23) holds using (13.37) in place of  $h_{1,n,F_c}(\theta_c) \rightarrow h_{1,\infty,F_c}(\theta_c)$  and using  $\nu_{n,F_n}(\theta_{n,*}, \cdot) \Rightarrow \nu_{h_2}(\cdot)$  in place of  $\nu_{n,F_c}(\theta_c, \cdot) \Rightarrow \nu_{h_{2,F_c}(\theta_c)}(\cdot)$ . The result  $\nu_{n,F_n}(\theta_{n,*}, \cdot) \Rightarrow \nu_{h_2}(\cdot)$  holds by Lemma A1(a) because  $\{(\theta_{n,*}, F_n) : n \geq 1\} \in SubSeq(h_2)$  by the argument given in (iii) above. The desired result is given by (13.24) with the changes indicated above. This completes the proof of part (a).

Part (b) holds by the same argument as for part (a) but with  $\varphi_n(\theta_{n,*}, g)$  replaced by 0, which simplifies the argument considerably. Assumption LA6 is used in place of Assumption LA5(b) in the proof.

Part (c) holds by the following argument:

$$\begin{aligned}
& \beta^{-\chi} T(h_1 + \Pi_0 \lambda_0 \beta, h_2) \\
&= \beta^{-\chi} \int S(\nu_{h_2}(g) + h_1(g) + \Pi_0(g) \lambda_0 \beta, h_2(g) + \varepsilon I_k) dQ(g) \\
&= \int S(\nu_{h_2}(g)/\beta + h_1(g)/\beta + \Pi_0(g) \lambda_0, h_2(g) + \varepsilon I_k) dQ(g) \\
&\rightarrow \int S(\Pi_0(g) \lambda_0, h_2(g) + \varepsilon I_k) dQ(g) > 0
\end{aligned} \tag{13.39}$$

as  $\beta \rightarrow \infty$  a.s., where  $\chi$  is as in Assumption S4, the second equality holds by Assumption S4, the convergence holds a.s. (with respect to the randomness in  $\nu_{h_2}$ ) by the bounded convergence theorem applied for each fixed sample path  $\omega$  because  $\|\nu_{h_2}(g)\|$  has bounded sample paths a.s. and using Assumption LA3' (which guarantees that  $h_{1,j}(g) < \infty$  and hence  $h_{1,j}(g)/\beta \rightarrow 0$  as  $\beta \rightarrow \infty$  for all  $j \leq p$ , for all  $g$  in a set with  $Q$  measure one), and the inequality holds by Assumptions LA3' and S3.

Equation (13.39) implies that  $T(h_1 + \Pi_0 \lambda_0 \beta, h_2) \rightarrow \infty$  a.s. as  $\beta \rightarrow \infty$ . Because  $T(h_1 + \Pi_0 \lambda_0 \beta, h_2) \sim J_{h, \beta \lambda_0}$  and the quantities  $c(\varphi(\pi_1), h_2, 1 - \alpha)$  and  $c(0_{\mathcal{G}}, h_2, 1 - \alpha)$  do not depend on  $\beta$ , the result of part (c) follows.  $\square$

## 13.4 Proofs Concerning the Verification of Assumptions S1-S4

**Proof of Lemma 1.** Assumptions S1(a)-(d) and S3 hold for the functions  $S_1$ ,  $S_2$ , and  $S_3$  by Lemma 1 of Andrews and Guggenberger (2009). Assumptions S1(e) and S4 hold immediately for the functions  $S_1$ ,  $S_2$ , and  $S_3$  with  $\chi = 2$  in Assumption S4.

To verify Assumption S2 for  $S = S_1$ ,  $S_2$ , or  $S_3$ , it suffices to show that

$$\limsup_{n \rightarrow \infty} |S(m_n + \mu_n, \Sigma_n) - S(m_0 + \mu_n, \Sigma_0)| = 0 \tag{13.40}$$

for all sequences  $\{\mu_n \in R_+^p \times \{0\}^v : n \geq 1\}$  and  $\{(m_n, \Sigma_n) : n \geq 1\}$  such that  $(m_n, \Sigma_n) \rightarrow (m_0, \Sigma_0)$ ,  $m_0 \in R^k$ , and  $\Sigma_0 \in \mathcal{W}$ .

For clarity of the proof, we consider a simple case first. We consider the function  $S_1$  and suppose  $\Sigma_n = \Sigma_0$ . In this case, without loss of generality, we can assume that  $\Sigma_0 = I_k$ . Given that  $S_1$  is additive, it suffices to consider the cases where  $(p, v) = (1, 0)$  and  $(0, 1)$ . It is easy to see that Assumption S2 holds in the latter case because  $\mu_n$  does

not appear. For the case where  $(p, v) = (1, 0)$ , we have

$$\begin{aligned}
& |S_1(m_n + \mu_n, I_k) - S_1(m_0 + \mu_n, I_k)| \\
&= |([m_n + \mu_n]_-^2 - [m_0 + \mu_n]_-^2)| \\
&\leq |[m_n + \mu_n]_- - [m_0 + \mu_n]_-| ([m_n + \mu_n]_- + [m_0 + \mu_n]_-) \\
&\leq |m_n - m_0| (|m_n| + |m_0|) \\
&= o(1)O(1),
\end{aligned} \tag{13.41}$$

where the second inequality holds because  $|[a]_- - [b]_-| \leq |a - b|$  and by Assumption S1(b). This completes the verification of Assumption S2 for the simple case.

Next, we verify Assumption S2 for  $S = S_2$ . For any sequence  $\{\mu_n \in R_+^p \times \{0\}^v : n \geq 1\}$ , there exists a subsequence  $\{u_n : n \geq 1\}$  of  $\{n\}$  such that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} |S_2(m_{u_n} + \mu_{u_n}, \Sigma_{u_n}) - S_2(m_0 + \mu_{u_n}, \Sigma_0)| \\
&= \limsup_{n \rightarrow \infty} |S_2(m_n + \mu_n, \Sigma_n) - S_2(m_0 + \mu_n, \Sigma_0)|.
\end{aligned} \tag{13.42}$$

Let  $\{t_{1,u_n}, t_{0,u_n} \in R_+^p \times \{0\}^v : n \geq 1\}$  be sequences such that

$$\begin{aligned}
& (m_{u_n} + \mu_{u_n} - t_{1,u_n})' \Sigma_{u_n}^{-1} (m_{u_n} + \mu_{u_n} - t_{1,u_n}) \leq S_2(m_{u_n} + \mu_{u_n}, \Sigma_{u_n}) + 2^{-u_n} \text{ and} \\
& (m_0 + \mu_{u_n} - t_{0,u_n})' \Sigma_0^{-1} (m_0 + \mu_{u_n} - t_{0,u_n}) \leq S_2(m_0 + \mu_{u_n}, \Sigma_0) + 2^{-u_n}.
\end{aligned} \tag{13.43}$$

Then,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} [S_2(m_{u_n} + \mu_{u_n}, \Sigma_{u_n}) - S_2(m_0 + \mu_{u_n}, \Sigma_0)] \\
&= \lim_{n \rightarrow \infty} [(m_{u_n} + \mu_{u_n} - t_{1,u_n})' \Sigma_{u_n}^{-1} (m_{u_n} + \mu_{u_n} - t_{1,u_n}) - S_2(m_0 + \mu_{u_n}, \Sigma_0)] \\
&\geq \lim_{n \rightarrow \infty} [(m_{u_n} + \mu_{u_n} - t_{1,u_n})' \Sigma_{u_n}^{-1} (m_{u_n} + \mu_{u_n} - t_{1,u_n}) \\
&\quad - (m_0 + \mu_{u_n} - t_{0,u_n})' \Sigma_0^{-1} (m_0 + \mu_{u_n} - t_{0,u_n})] \\
&= \lim_{n \rightarrow \infty} [(m_{u_n} + \mu_{u_n} - t_{1,u_n})' (\Sigma_{u_n}^{-1} - \Sigma_0^{-1}) (m_{u_n} + \mu_{u_n} - t_{1,u_n}) \\
&\quad + (m_{u_n} - m_0)' \Sigma_0^{-1} (m_0 + m_{u_n} + 2\mu_{u_n} - 2t_{1,u_n})] \\
&= 0,
\end{aligned} \tag{13.44}$$

where the last equality holds if  $\mu_{u_n} - t_{1,u_n} = O(1)$  because  $m_{u_n} \rightarrow m_0 < \infty$  and  $\Sigma_{u_n}^{-1} \rightarrow \Sigma_0^{-1}$  as  $n \rightarrow \infty$ .

We now show that  $\mu_{u_n} - t_{1,u_n} = O(1)$ . We have

$$\begin{aligned} m'_{u_n} \Sigma_{u_n}^{-1} m_{u_n} &\geq S_2(m_{u_n} + \mu_{u_n}, \Sigma_{u_n}) \\ &\geq (m_{u_n} + \mu_{u_n} - t_{1,u_n})' \Sigma_{u_n}^{-1} (m_{u_n} + \mu_{u_n} - t_{1,u_n}) - 2^{-u_n}. \end{aligned} \quad (13.45)$$

Thus,

$$\begin{aligned} &\lim_{n \rightarrow \infty} (m_{u_n} + \mu_{u_n} - t_{1,u_n})' \Sigma_{u_n}^{-1} (m_{u_n} + \mu_{u_n} - t_{1,u_n}) \\ &\leq \lim_{n \rightarrow \infty} [m'_{u_n} \Sigma_{u_n}^{-1} m_{u_n} + 2^{-u_n}] = m'_0 \Sigma_0^{-1} m_0 < \infty, \end{aligned} \quad (13.46)$$

which implies that  $m_{u_n} + \mu_{u_n} - t_{1,u_n} = O(1)$ . The latter and  $m_{u_n} \rightarrow m_0 < \infty$  give

$$\mu_{u_n} - t_{1,u_n} = O(1). \quad (13.47)$$

Next, by an analogous argument to (13.44) with  $\geq$  and  $t_{1,u_n}$  replaced by  $\leq$  and  $t_{0,u_n}$ , respectively, we obtain the following upper bound:

$$\begin{aligned} &\lim_{n \rightarrow \infty} [S(m_{u_n} + \mu_{u_n}, \Sigma_{u_n}) - S(m_0 + \mu_{u_n}, \Sigma_0)] \\ &= \lim_{n \rightarrow \infty} [S(m_{u_n} + \mu_{u_n}, \Sigma_{u_n}) - (m_0 + \mu_{u_n} - t_{0,u_n})' \Sigma_0^{-1} (m_0 + \mu_{u_n} - t_{0,u_n})] \\ &\leq 0, \end{aligned} \quad (13.48)$$

where the inequality uses  $\mu_{u_n} - t_{0,u_n} = O(1)$ , which holds by an analogous argument to that given for (13.47). Equations (13.44) and (13.48) imply that the left-hand side of (13.42) equals zero, which completes the verification of Assumption S2 for  $S_2$ .

The verification of Assumption S2 for  $S = S_1$ , where  $\Sigma_n$  need not equal  $\Sigma_0$ , is obtained by replacing  $\Sigma_n$  and  $\Sigma_0$  in the proof above for  $S_2$  by  $Diag\{\Sigma_n\}$  and  $Diag\{\Sigma_0\}$ , respectively, because  $S_1(m, \Sigma) = S_2(m, \Sigma)$  when  $\Sigma$  is diagonal. Assumption S2 holds for the function  $S_3$  when  $(p, v) = (1, 0)$  and  $(0, 1)$  because  $S_3 = S_1 = S_2$  in these cases. It holds for  $S_3$  in the general  $(p, v)$  case because it holds in these two special cases.  $\square$

## 14 Appendix D

In this Appendix, we provide proofs of the results stated in Appendix B. The first sub-section gives proofs for the Kolmogorov-Smirnov and approximate CvM tests and CS's. The second sub-section gives proofs for results concerning  $\mathcal{G}_{B-spline}$  and  $\mathcal{G}_{c/d}$ . The third sub-section gives proofs for results concerning “asymptotic issues with the Kolmogorov-Smirnov statistic.” The fourth sub-section gives proofs for the subsampling results.

### 14.1 Proofs of Kolmogorov-Smirnov and Approximate Cramér von Mises Results

**Proof of Lemma B1.** To verify Assumption S2' for  $S_1$ ,  $S_2$ , and  $S_3$ , it suffices to show that

$$\limsup_{n \rightarrow \infty} |S(m_n + \mu_n, \Sigma_n) - S(m_{n,0} + \mu_n, \Sigma_{n,0})| = 0 \quad (14.1)$$

for all sequences  $\{\mu_n \in R_+^p \times \{0\}^v : n \geq 1\}$ ,  $\{(m_n, \Sigma_n) \in \mathcal{M} \times \mathcal{W}_{bd} : n \geq 1\}$ , and  $\{(m_{n,0}, \Sigma_{n,0}) \in \mathcal{M} \times \mathcal{W}_{bd} : n \geq 1\}$  for which  $(m_n, \Sigma_n) - (m_{n,0}, \Sigma_{n,0}) \rightarrow 0$  as  $n \rightarrow \infty$ .

The verification of (14.1) is an extension of the verification of (13.40) in the proof of Lemma 1. The extension consists of (i) replacing  $m_0$  and  $\Sigma_0$  by  $m_{u_n,0}$  and  $\Sigma_{u_n,0}$  throughout (13.42)-(13.48), (ii) making use of the fact that  $m_{u_n}$ ,  $m_{u_n,0}$ , and  $\Sigma_{u_n}^{-1}$  are bounded by the definitions of  $\mathcal{M}$  and  $\mathcal{W}_{bd}$ , and (iii) making use of the fact that  $\Sigma_{u_n}^{-1} - \Sigma_{u_n,0}^{-1} \rightarrow 0$  given that  $\Sigma_{u_n} - \Sigma_{u_n,0} \rightarrow 0$  and  $\Sigma_{u_n}, \Sigma_{u_n,0} \in \mathcal{W}_{bd}$ .  $\square$

**Proof of Theorem B1.** When  $T_n(\theta)$  is the KS statistic and when  $T_n(\theta)$  is replaced by the approximate statistic  $\bar{T}_{n,s_n}(\theta)$ , the results of Theorem 1 hold under the assumptions of that Theorem plus Assumption S2'. The proof of Theorem 1 goes through with the following changes: (i) the statistics  $\tilde{T}_{a_n}$  and  $\tilde{T}_{a_n,0}$  are changed from integrals with respect to  $Q$  to suprema over  $g \in \mathcal{G}_n$  or weighted averages over  $\{g_1, \dots, g_{s_n}\}$  with weights  $\{w_{Q,n}(\ell) : \ell = 1, \dots, s_n\}$ , (ii) in the proof of (10.7), (10.10) holds uniformly over  $g \in \mathcal{G}$  because Assumption S2 has been strengthened to Assumption S2', and (iii) (10.11) holds with the supremum over  $g \in \mathcal{G}_n$  added or with the average over  $\{g_1, \dots, g_{s_n}\}$  added, because (10.10) holds uniformly over  $g \in \mathcal{G}$  and the weights are non-negative and sum to at most one by Assumption A2. This completes the proof of Theorem 1 for the KS and A-CvM test statistics.

The result of Theorem B1 is the same as that of Theorem 2(a). The proof of Theorem 2(a) follows immediately from Lemmas A2-A4. The proof of Lemma A4 uses Lemma A5. The proofs of Lemmas A2-A5 go through for the KS and A-CvM test statistics with the following minor changes: (i) in the proof of Lemma A2,  $T(h)$  is replaced by  $\bar{T}_{s_n}(h)$  (defined in (4.6)) and the new version of Theorem 1 for the KS and A-CvM statistics is employed, (ii) in the proof of Lemma A3, the form of the test statistic only enters through the first inequality of (10.24), which holds for the supremum and weighted average forms of the test statistic, (iii) in the proof of Lemma A4, no changes are required because the form of the test statistic only enters through Lemma A5, and (iv) in the proof of Lemma A5,  $T(h)$  is replaced by  $\bar{T}_{s_n}(h)$ .  $\square$

**Proof of Theorem B2.** Theorem B2 is proved by adjusting the proof Theorem 3. The proof of Theorem 3 goes through up to (13.32) with the only change being that  $c(\cdot, \cdot, \cdot)$  is replaced by  $c_{s_n}(\cdot, \cdot, \cdot)$  for A-CvM tests in (13.27)—in particular, the integral with respect to  $Q$  in (13.28) is not changed. Equation (13.33) needs to be replaced, see (14.2) and (14.6) below; (13.34) is established with  $c(\cdot, \cdot, \cdot)$  replaced by  $c_{s_n}(\cdot, \cdot, \cdot)$  for A-CvM tests; (13.35) holds, with  $T_n(\theta_*)$  and  $c(\cdot, \cdot, \cdot)$  replaced by  $\bar{T}_{n,s_n}(\theta_*)$  and  $c_{s_n}(\cdot, \cdot, \cdot)$  for A-CvM tests, using the replacements for (13.33) given in (14.2) and (14.6) below; the first equation in (13.36) holds by Lemma A5 with  $c(\cdot, \cdot, \cdot)$  replaced by  $c_{s_n}(\cdot, \cdot, \cdot)$  for A-CvM tests, noting that Lemma A5 is extended to KS and A-CvM critical values in the proof of Theorem B1 above; in the second equation in (13.36) “ $c_0(0_{\mathcal{G}}, h_{2,0}, 1 - \alpha + \eta + \delta) < \infty$ ” holds for the KS critical value because  $c_0(0_{\mathcal{G}}, h_{2,0}, 1 - \alpha + \eta + \delta)$  does not depend on  $n$  and the KS test statistic  $T(0_{\mathcal{G}}, h_{2,0})$  is finite a.s. since the sample paths of  $\nu_{h_{2,0}}(\cdot)$  and  $h_{2,0}(\cdot)$  are bounded a.s.; and in the second equation in (13.36) “ $\sup_{n \geq 1} c_{0,s_n}(0_{\mathcal{G}}, h_{2,0}, 1 - \alpha + \eta + \delta) < \infty$ ” holds for an A-CvM critical value because  $c_{0,s_n}(0_{\mathcal{G}}, h_{2,0}, 1 - \alpha + \eta + \delta)$  is less than equal to the corresponding quantile based on the KS statistic, which does not depend on  $n$  and is finite a.s.



For the KS test, we replace (13.33) with the following:

$$\begin{aligned}
& (n^{1/2}\beta(g_0))^{-\chi} T_n(\theta_*) \cdot Q(\mathcal{B}_{\rho_X}(g_0, \tau_2)) \\
&= (n^{1/2}\beta(g_0))^{-\chi} \sup_{g \in \mathcal{G}_n} S(\nu_{n, F_0}(\theta_*, g) + h_{1, n, F_0}(\theta_*, g), \bar{h}_{2, n, F_0}(\theta_*, g)) \cdot Q(\mathcal{B}_{\rho_X}(g_0, \tau_2)) \\
&\geq (n^{1/2}\beta(g_0))^{-\chi} \int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} 1(g \in \mathcal{G}_n) S(\nu_{n, F_0}(\theta_*, g) + h_{1, n, F_0}(\theta_*, g), \bar{h}_{2, n, F_0}(\theta_*, g)) dQ(g) \\
&= \int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} 1(g \in \mathcal{G}_n) S((n^{1/2}\beta(g_0))^{-1}\nu_{n, F_0}(\theta_*, g) + m^*(g)/\beta(g_0), \bar{h}_{2, n, F_0}(\theta_*, g)) dQ(g) \\
&\rightarrow_p \int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} S(m^*(g)/\beta(g_0), h_{2, 0}(g) + \varepsilon I_k) dQ(g) > 0, \tag{14.2}
\end{aligned}$$

where  $\chi$  is as in Assumption S4,  $m^*(g) = (m_1^*(g), \dots, m_k^*(g))'$ ,  $m_j^*(g)$  is defined in (6.2) for  $j \leq k$ ,  $h_{2, 0} = h_{2, F_0}(\theta_*)$ , and the convergence uses the argument given in the paragraph following (13.33) as well as  $1(g \in \mathcal{G}_n) \rightarrow 1(g \in \mathcal{G}) = 1$  as  $n \rightarrow \infty$  by Assumption KS.

For A-CvM tests, we replace (13.33) with the following results:

$$\begin{aligned}
& (n^{1/2}\beta(g_0))^{-\chi} \bar{T}_{n, s_n}(\theta_*) \\
&= \sum_{\ell=1}^{s_n} w_{Q, n}(\ell) S((n^{1/2}\beta(g_0))^{-1}\nu_{n, F_0}(\theta_*, g_\ell) + m^*(g_\ell)/\beta(g_0), \bar{h}_{2, n, F_0}(\theta_*, g_\ell)), \tag{14.3}
\end{aligned}$$

using Assumption S4. We have

$$\sup_{g \in \mathcal{G}} |m_j^*(g)| \leq (E_{F_0}(m_j^2(W_i, \theta_*)/\sigma_{F_0, j}^2(\theta_*))^{1/2} (E_{F_0} G^2(X_i))^{1/2} < \infty, \tag{14.4}$$

for  $j = 1, \dots, k$ , using the definition of  $m^*(g)$ , Assumption FA (which imposes Assumption M in part FA(e)), and the Cauchy-Schwarz inequality. Next, we have

$$\begin{aligned}
& \sup_{g \in \mathcal{G}} \left| S((n^{1/2}\beta(g_0))^{-1}\nu_{n, F_0}(\theta_*, g) + m^*(g)/\beta(g_0), \bar{h}_{2, n, F_0}(\theta_*, g)) \right. \\
& \quad \left. - S(m^*(g)/\beta(g_0), h_{2, 0}(g) + \varepsilon I_k) \right| = o_p(1) \tag{14.5}
\end{aligned}$$

under  $F_0$ , using the uniform continuity of  $S$  over a compact set, which holds by Assumption S1(d), where attention can be restricted to a compact set by (i) equation (14.4), (ii)  $\sup_{g \in \mathcal{G}} \|n^{-1/2}\nu_{n, F_0}(\theta_*, g)\| = o_p(1)$  by Lemma A1(a), and (iii)  $\sup_{g \in \mathcal{G}} \|\bar{h}_{2, n, F_0}(\theta_*) - h_{2, 0} - \varepsilon I_k\| = o_p(1)$  using Lemma A1(b) and the definition of  $\bar{h}_{2, n, F_0}(\theta_*)$  in (5.2), and

Lemma A1 applies for the reasons given in the paragraph following (13.33).

Equations (14.3) and (14.5) yield

$$\begin{aligned}
& (n^{1/2}\beta(g_0))^{-x}\bar{T}_{n,s_n}(\theta_*) + o_p(1) \\
&= \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) S(m^*(g_\ell)/\beta(g_0), h_{2,0}(g_\ell)) \\
&\rightarrow \int S(m^*(g)/\beta(g_0), h_{2,0}(g)) dQ(g) \\
&\geq \int_{\mathcal{B}_{\rho_X}(g_0, \tau_2)} S(m^*(g)/\beta(g_0), h_{2,0}(g)) dQ(g) > 0, \tag{14.6}
\end{aligned}$$

where the convergence holds for fixed  $\{g_1, g_2, \dots\}$  by Assumptions A1, A2, and S4, the first inequality holds by Assumption S1(c), and the second inequality holds by (13.28). This completes the proof.  $\square$

**Proof of Theorem B3.** Part (a) follows from part (b) because

$$c_{s_n}(\varphi_n(\theta_{n,*}), \widehat{h}_{2,n}(\theta_{n,*}), 1 - \alpha) \leq c_{s_n}(0_G, \widehat{h}_{2,n}(\theta_{n,*}), 1 - \alpha), \tag{14.7}$$

which holds because  $\varphi_n(\theta_*, g) \geq 0_k \forall g \in \mathcal{G}$  by Assumption GMS1(a),  $c(h_1, \widehat{h}_{2,n}(\theta_*), 1 - \alpha)$  is non-increasing in the first  $p$  elements of  $h_1$  by Assumption S1(b), and the last  $v$  elements of  $\varphi_n(\theta_*, g)$  equal zero.

Now, we prove part (b). When  $T_n(\theta)$  is replaced by the A-CvM statistic  $\bar{T}_{n,s_n}(\theta_{n,*})$ , the results of Theorem 1 hold under Assumptions M, S1, and S2' with  $(\theta, F)$  replaced by  $(\theta_{n,*}, F_n)$ ,  $\sup_{(\theta, F) \in \mathcal{F}: h_{2,F}(\theta) \in \mathcal{H}_{2,cpt}}$  deleted,  $T_n(\theta)$ ,  $T(h_{n,F}(\theta))$ , and  $x_{h_{n,F}(\theta)}$  replaced by  $\bar{T}_{n,s_n}(\theta_{n,*})$ ,  $\bar{T}_{s_n}(h_{n,F_n}(\theta_{n,*}))$  (defined in (4.6)), and  $x_{h_{n,F_n}(\theta_{n,*})}$ , respectively, where  $x_{h_{n,F_n}(\theta_{n,*})} \in R$  is a constant that may depend on  $(\theta_{n,*}, F_n)$  and  $n$  through  $h_{n,F_n}(\theta_{n,*})$ . The adjustments needed to the proof of Theorem 1 are quite similar to those stated at the beginning of the proof of Theorem B1. In addition, the proof uses the fact that  $\{(\theta_{n,*}, F_n) : n \geq 1\}$  satisfies the conditions to be in  $SubSeq(h_2)$  (where  $h_2 = h_{2,F_0}(\theta_0)$ ) by Assumptions LA1(a) and LA1(c)-(e) and because  $\{W_i : i \geq 1\}$  are i.i.d. under  $F_n$  and Assumption M holds given that  $(\theta_n, F_n) \in \mathcal{F}$  by Assumption LA1. Because  $\{(\theta_{n,*}, F_n) : n \geq 1\} \in SubSeq(h_2)$ , Lemma A1 applies, which is used in (10.3). Also,  $(h_{1,n,F}(\theta), h_{2,F}(\theta))$  is changed to  $(h_{1,n,F_n}(\theta_{n,*}), h_{2,F_n}(\theta_{n,*}))$  throughout the proof of Theorem 1.

Next, using the mean-value expansion in (13.37) and the definition  $h_{1,n,F}(\theta, g) =$

$n^{1/2}D_F^{-1/2}(\theta)E_F m(W_i, \theta, g)$ , we have:

$$\begin{aligned}
& \sup_{g \in \mathcal{G}} \|h_{1,n,F_n}(\theta_{n,*}, g) - h_{1,n,F_n}(\theta_n, g) - \Pi_0(g)\lambda\| \\
&= \sup_{g \in \mathcal{G}} \|\Pi_{F_n}(\theta_{n,g}, g)n^{1/2}(\theta_{n,*} - \theta_n) - \Pi_0(g)\lambda\| \\
&\leq \sup_{g \in \mathcal{G}} \sup_{\theta \in \Theta: \|\theta - \theta_0\| \leq \delta_n} \|\Pi_{F_n}(\theta, g)\lambda(1 + o(1)) - \Pi_0(g)\lambda\| \\
&\rightarrow 0,
\end{aligned} \tag{14.8}$$

where  $\theta_{n,g}$  may differ across rows of  $\Pi_{F_n}(\theta_{n,g}, g)$ ,  $\theta_{n,g}$  lies between  $\theta_{n,*}$  and  $\theta_n$ ,  $\delta_n = \|\theta_{n,*} - \theta_n\| + \|\theta_n - \theta_0\| \rightarrow 0$ , the inequality holds using Assumption LA1(a), and the convergence to zero uses Assumption LA2'(b). (Note that the  $(1 + o(1))$  term in (14.8) requires the condition in Assumption LA2'(b) that  $\sup_{g \in \mathcal{G}} \|\Pi_0(g)\lambda\| < \infty$ .)

Equation (14.8) and Assumption LA2'(a) give: for all  $B < \infty$ ,

$$\sup_{g \in \mathcal{G}: h_1(g) \leq B} \|h_{1,n,F_n}(\theta_{n,*}, g) - h_1(g) - \Pi_0(g)\lambda\| \rightarrow 0. \tag{14.9}$$

By Assumption LA1(c),  $d(h_{2,F_n}(\theta_{n,*}), h_{2,F_0}(\theta_0)) \rightarrow 0$ . This implies that  $\nu_{h_{2,F_n}(\theta_{n,*})}(\cdot) \Rightarrow \nu_{h_2}(\cdot)$ , where  $h_2 = h_{2,F_0}(\theta_0)$ . As in previous proofs, by the almost sure representation theorem, there exists a probability space and random quantities  $\tilde{\nu}_n(\cdot)$  and  $\tilde{\nu}(\cdot)$  defined on it with the same distributions as  $\nu_{h_{2,F_n}(\theta_{n,*})}(\cdot)$  and  $\nu_{h_2}(\cdot)$ , respectively, for  $n \geq 1$ , such that  $\sup_{g \in \mathcal{G}} \|\tilde{\nu}_n(g) - \tilde{\nu}(g)\| \rightarrow 0$  a.s. Hence,  $\bar{T}_{s_n}(h_{n,F_n}(\theta_{n,*}))$  and  $\widetilde{T}_{s_n}(h_{n,F_n}(\theta_{n,*}))$  have the same distribution, where the latter is defined to be

$$\widetilde{T}_{s_n}(h_{n,F_n}(\theta_{n,*})) = \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) S(\tilde{\nu}_n(g_\ell) + h_{1,n,F_n}(\theta_{n,*}, g_\ell), h_{2,F_n}(\theta_{n,*}, g_\ell) + \varepsilon I_k). \tag{14.10}$$

For all  $\beta > 0$ ,  $B < \infty$ , and  $\lambda = \lambda_0\beta$ , we have

$$\begin{aligned}
A_{1,n}(\beta, B) &= \sup_{g \in \mathcal{G}: h_1(g) \leq B} |S(\tilde{\nu}_n(g)/\beta + h_{1,n,F_n}(\theta_{n,*}, g)/\beta, h_{2,F_n}(\theta_{n,*}, g) + \varepsilon I_k) \\
&\quad - S(\tilde{\nu}(g)/\beta + h_1(g)/\beta + \Pi_0(g)\lambda_0, h_2(g) + \varepsilon I_k)| \\
&\rightarrow 0 \text{ as } n \rightarrow \infty \text{ a.s.}
\end{aligned} \tag{14.11}$$

using Assumption S2', (14.9),  $\sup_{g \in \mathcal{G}} \|\tilde{\nu}_n(g) - \tilde{\nu}(g)\| \rightarrow 0$  a.s.,  $\sup_{g \in \mathcal{G}} \|\tilde{\nu}(g)\| < \infty$  a.s., and  $d(h_{2,F_n}(\theta_{n,*}), h_2) \rightarrow 0$ , where  $h_2 = h_{2,F_0}(\theta_0)$ .

In addition, for all  $B < \infty$ , we have

$$\begin{aligned}
A_2(\beta, B) &= \sup_{g \in \mathcal{G}: h_1(g) \leq B} |S(\tilde{\nu}(g)/\beta + h_1(g)/\beta + \Pi_0(g)\lambda_0, h_2(g) + \varepsilon I_k) \\
&\quad - S(\Pi_0(g)\lambda_0, h_2(g) + \varepsilon I_k)| \\
&\rightarrow 0 \text{ as } \beta \rightarrow \infty \text{ a.s.}
\end{aligned} \tag{14.12}$$

We use (14.11) and (14.12) to obtain: for all constants  $B_c^* < \infty$  as in Assumption A3,

$$\begin{aligned}
&\beta^{-\chi} \widetilde{T}_{s_n}(h_{n, F_n}(\theta_{n,*})) \\
&\geq \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) 1(h_1(g_\ell) \leq B_c^*) S(\tilde{\nu}_n(g_\ell)/\beta + h_{1,n, F_n}(\theta_{n,*}, g_\ell)/\beta, h_{2, F_n}(\theta_{n,*}, g_\ell) + \varepsilon I_k) \\
&\geq \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) 1(h_1(g_\ell) \leq B_c^*) S(\Pi_0(g_\ell)\lambda_0, h_2(g_\ell) + \varepsilon I_k) - A_{1,n}(\beta, B_c^*) - A_2(\beta, B_c^*) \\
&\xrightarrow{n \rightarrow \infty \text{ a.s.}} \int 1(h_1(g) \leq B_c^*) S(\Pi_0(g)\lambda_0, h_2(g) + \varepsilon I_k) dQ(g) - A_2(\beta, B_c^*) \\
&\xrightarrow{\beta \rightarrow \infty \text{ a.s.}} \int 1(h_1(g) \leq B_c^*) S(\Pi_0(g)\lambda_0, h_2(g) + \varepsilon I_k) dQ(g),
\end{aligned} \tag{14.13}$$

where the first inequality uses Assumptions S1(c) and S4, the second inequality holds by the definitions of  $A_{1,n}(\beta, B_c^*)$  and  $A_2(\beta, B_c^*)$ , the first convergence result holds by (14.11) and Assumption A3, and the second convergence result holds by (14.12).

Let  $c_{\text{sup},0}(0_{\mathcal{G}}, h_2^*, 1 - \alpha)$  denote the  $1 - \alpha$  quantile of  $T_{\text{sup}}(0_{\mathcal{G}}, h_2^*) = \sup_{g \in \mathcal{G}} S(\nu_{h_2}(g), h_2^*(g) + \varepsilon I_k)$ , where  $h_2^*$  is some  $k \times k$ -matrix-valued covariance kernel on  $\mathcal{G} \times \mathcal{G}$ . Let  $0_{\mathcal{G} \times \mathcal{G}}$  denote the  $k \times k$ -matrix-valued covariance kernel on  $\mathcal{G} \times \mathcal{G}$  that equals the  $k \times k$  zero matrix for all  $(g, g^*) \in \mathcal{G} \times \mathcal{G}$ . The A-PA critical value satisfies

$$\begin{aligned}
c_{s_n}(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_{n,*}), 1 - \alpha) &\leq c_{\text{sup},0}(0_{\mathcal{G}}, \widehat{h}_{2,n}(\theta_{n,*}), 1 - \alpha + \eta) + \eta \\
&\leq c_{\text{sup},0}(0_{\mathcal{G}}, 0_{\mathcal{G} \times \mathcal{G}}, 1 - \alpha + \eta) + \eta \\
&< \infty,
\end{aligned} \tag{14.14}$$

where the first inequality holds because a weighted average over  $\{g_1, \dots, g_{s_n}\}$  with non-negative weights that sum to one or less (by Assumption A2) is less than or equal to the corresponding supremum over  $g \in \mathcal{G}$ , which implies that  $\overline{T}_{s_n}(0_{\mathcal{G}}, h_2^*) \leq T_{\text{sup}}(0_{\mathcal{G}}, h_2^*) \forall h_2^*$ , the second inequality holds because  $S(\nu_{h_2}(g), h_2^*(g) + \varepsilon I_k) \leq S(\nu_{h_2}(g), \varepsilon I_k) \forall g \in \mathcal{G}$ ,

for all covariance kernels  $h_2^*$  by Assumption S1(e), which implies that  $T_{\text{sup}}(0_{\mathcal{G}}, h_2^*) \leq T_{\text{sup}}(0_{\mathcal{G}}, 0_{\mathcal{G} \times \mathcal{G}}) \forall h_2^*$ , and the last inequality holds because  $\sup_{g \in \mathcal{G}} S(\nu_{h_2}(g), \varepsilon I_k) < \infty$  a.s., which holds by Assumption S2' and  $\sup_{g \in \mathcal{G}} \|\nu_{h_2}(g)\| < \infty$  a.s.

We now have: for all  $B_c^*$  as in Assumption A3,

$$\begin{aligned}
& \lim_{\beta \rightarrow \infty} \liminf_{n \rightarrow \infty} P_{F_n} \left( \bar{T}_{s_n}(h_{n, F_n}(\theta_{n, *})) > c_{s_n}(0_{\mathcal{G}}, \hat{h}_{2, n}(\theta_{n, *}), 1 - \alpha) \right) \\
& \geq \lim_{\beta \rightarrow \infty} \liminf_{n \rightarrow \infty} P \left( \beta^{-x} \widetilde{\bar{T}}_{s_n}(h_{n, F_n}(\theta_{n, *})) > \beta^{-x} c(0_{\mathcal{G}}, 0_{\mathcal{G} \times \mathcal{G}}, 1 - \alpha + \eta) + \beta^{-x} \eta \right) \\
& \geq \lim_{\beta \rightarrow \infty} P \left( \int 1(h_1(g) \leq B_c^*) S(\Pi_0(g) \lambda_0, h_2(g) + \varepsilon I_k) dQ(g) - A_2(\beta, B_c^*) \right. \\
& \quad \left. > \beta^{-x} c(0_{\mathcal{G}}, h_2, 1 - \alpha + \eta) + \beta^{-x} \eta \right) \\
& = 1 \left( \int 1(h_1(g) \leq B_c^*) S(\Pi_0(g) \lambda_0, h_2(g) + \varepsilon I_k) dQ(g) > 0 \right), \tag{14.15}
\end{aligned}$$

where the first inequality holds by (14.14) and the equality in distribution of  $\widetilde{\bar{T}}_{s_n}(h_{n, F_n}(\theta_{n, *}))$  and  $\bar{T}_{s_n}(h_{n, F_n}(\theta_{n, *}))$ , the second inequality holds by (i) the first two inequalities in (14.13), (ii) the first convergence result in (14.13), and (iii) the BCT, and the last equality holds by the second convergence result of (14.13) and the BCT.

The left-hand side (lhs) in (14.15) does not depend on  $B_c^*$ . Hence, the lhs is greater than or equal to the limit as  $c \rightarrow \infty$  of the right-hand side, which equals

$$1 \left( \int 1(h_1(g) \leq \infty) S(\Pi_0(g) \lambda_0, h_2(g) + \varepsilon I_k) dQ(g) > 0 \right) = 1 \tag{14.16}$$

by the monotone convergence theorem and the assumption that  $B_c^* \rightarrow \infty$  as  $c \rightarrow \infty$ , where the equality holds by Assumptions LA3' and S3.

Lastly, we prove part (c) regarding KS tests and CS's. The proof is essentially the same as that for parts (a) and (b) with  $\bar{T}_{n, s_n}(\theta_{n, *})$ ,  $c_{s_n}(\cdot, \cdot, \cdot)$ ,  $\sum_{\ell=1}^{s_n} w_{Q, n}(\ell) \dots$ , and  $\int \dots dQ(g)$  replaced by the KS quantities  $T_n(\theta_{n, *})$ ,  $c(\cdot, \cdot, \cdot)$ ,  $\sup_{g \in \mathcal{G}}$ , and  $\sup_{g \in \mathcal{G}} \dots$ , respectively (or with  $\mathcal{G}_n$  in place of  $\mathcal{G}$ ).  $\square$

## 14.2 Proof of Lemma B2 Regarding $\mathcal{G}_{B\text{-spline}}$ , $\mathcal{G}_{\text{box,dd}}$ , and $\mathcal{G}_{c/d}$

**Proof of Lemma B2.** First we verify Assumption CI for  $\mathcal{G} = \mathcal{G}_{B\text{-spline}}$ . Let  $m_{j,F}(\theta, x) = E_F(m_j(W_i, \theta) | X_i = x)$ . Write

$$\mathcal{X}_F(\theta) = \left( \bigcup_{j=1}^p \{x \in R^{d_x} : m_{j,F}(\theta, x) < 0\} \right) \cup \left( \bigcup_{j=p+1}^k \{x \in R^{d_x} : m_{j,F}(\theta, x) \neq 0\} \right). \quad (14.17)$$

If  $P_F(X_i \in \mathcal{X}_F(\theta)) > 0$ , then the probability that  $X_i$  lies in one of the  $k$  sets in (14.17) is positive. Suppose (without loss of generality) that  $P_F(X_i \in \{x : m_{1,F}(\theta, x) < 0\}) > 0$ . The set  $\{x : m_{1,F}(\theta, x) < 0\}$  can be written as the union of disjoint non-degenerate hypercubes in  $\mathcal{C}_{B\text{-spline}}$  (i.e., hypercubes with positive Lebesgue volumes) because continuity of  $m_{1,F}(\theta, x)$  implies that if  $m_{1,F}(\theta, x) < 0$  then  $m_{1,F}(\theta, y) < 0$  for all  $y$  in some hypercube that includes  $x$ . The number of such hypercubes is countable (because otherwise their union would have infinite volume). One of these hypercubes, call it  $H$ , must have positive  $X_i$  probability. (Otherwise, the union of these hypercubes would have  $X_i$  probability zero.)

In sum, we have  $H \in \mathcal{C}_{B\text{-spline}}$ ,  $P_F(X_i \in H) > 0$ , and  $m_{1,F}(\theta, x) < 0$  for all  $x \in H$ . In addition, the B-spline whose support is  $H$  is positive on the interior of  $H$ . Thus, if  $P_F(X_i \in \text{int}(H)) > 0$ , we have  $E_F m_1(W_i, \theta) B_H(X_i) < 0$ , which establishes Assumption CI.

On the other hand, if  $P_F(X_i \in \text{int}(H)) = 0$ , then we must have  $P_F(X_i \in H \setminus \text{int}(H)) > 0$ . Because  $m_{1,F}(\theta, x)$  is a continuous function of  $x$ , there exists a finite number of hypercubes in  $\mathcal{C}_{B\text{-spline}}$  whose interiors have union that includes  $H \setminus \text{int}(H)$  and for which  $m_{1,F}(\theta, x) < 0$  for all  $x$  in each hypercube. One of these hypercubes, say  $H_1$ , must have interior with positive probability because  $P_F(X_i \in H \setminus \text{int}(H)) > 0$ . In sum,  $H_1 \in \mathcal{C}_{B\text{-spline}}$ ,  $P_F(X_i \in \text{int}(H_1)) > 0$ ,  $m_{1,F}(\theta, x) < 0$  for all  $x \in H_1$ , and the B-spline  $B_{H_1}(x)$  is positive for  $x \in \text{int}(H_1)$ . Hence,  $E_F m_1(W_i, \theta) B_{H_1}(X_i) < 0$ , which establishes Assumption CI.

Now we establish Assumption CI for  $\mathcal{G}_{\text{box,dd}}$ . The fact that Assumption CI holds for  $\mathcal{G} = \mathcal{G}_{\text{box}}$  for all  $\bar{r} \in (0, \infty]$  by Lemma 3 implies that Assumption CI holds for  $\mathcal{G} = \mathcal{G}_{\text{box,dd}}$  for all  $\bar{r} \in (0, \infty]$ . The reason is as follows. Let  $\mathcal{G}_{\text{box}}(\bar{r})$  and  $\mathcal{G}_{\text{box,dd}}(\bar{r})$  denote  $\mathcal{G}_{\text{box}}$  and  $\mathcal{G}_{\text{box,dd}}$ , respectively, when  $\bar{r}$  is the upper bound on  $r_u$  or  $r_{1,u}$  and  $r_{2,u}$ . For any box  $C_{x_0,r} \in \mathcal{G}_{\text{box}}(\bar{r})$ , if  $C_{x_0,r}$  captures some deviation from the model, i.e.,  $E_F m_j(W_i, \theta) 1(X_i \in C_{x_0,r}) < 0$  for some  $j = 1, \dots, p$  or  $E_F m_j(W_i, \theta) 1(X_i \in C_{x_0,r}) \neq$

0 for some  $j = p + 1, \dots, k$ , then (i)  $C_{x_0, r} \cap \text{Supp}_{F_{X,0}}(X_i) \neq \emptyset$  and (ii)  $C_{x_0+\eta, r+\eta}$  captures the same deviation for  $\eta > 0$  sufficiently small. Result (ii) holds because  $\lim_{\eta \downarrow 0} E_F m_j(W_i, \theta) 1(X_i \in C_{x_0+\eta, r+\eta}) = E_F m_j(W_i, \theta) 1(X_i \in C_{x_0, r})$ . The latter holds by the bounded convergence theorem because  $(C_{x_0+\eta, r+\eta} - C_{x_0, r}) \downarrow \emptyset$  as  $\eta \downarrow 0$ , and hence  $m_j(w, \theta) 1(x \in C_{x_0+\eta, r+\eta}) \rightarrow m_j(w, \theta) 1(x \in C_{x_0, r})$  as  $\eta \downarrow 0$  for every  $w$ , and  $E_F |m_j(W_i, \theta) 1(X_i \in C_{x_0+\eta, r+\eta})| \leq E_F |m_j(W_i, \theta)| < \infty$ . By (i) and  $\eta \in (0, \bar{r}/2]$ ,  $C_{x_0+\eta, r+\eta}$  can be written as a box,  $C_{x, r_1, r_2}$  in  $\mathcal{G}_{\text{box}, dd}(3\bar{r})$  by picking a point  $x \in C_{x_0, r} \cap \text{Supp}_{F_{X,0}}(X_i)$ , which is necessarily in the interior of  $C_{x_0+\eta, r+\eta}$ , and letting  $r_1 = x - x_0 + r$  and  $r_2 = x_0 + r - x + 2\eta$ . We have  $|x - x_0| \leq \bar{r}$ ,  $r_1 \leq 2\bar{r}$ , and  $r_2 \leq 3\bar{r}$ . Because  $C_{x, r_1, r_2} = C_{x_0+\eta, r+\eta}$  and  $C_{x_0+\eta, r+\eta}$  captures a deviation from the model,  $C_{x, r_1, r_2}$  does as well, and the proof is complete.

Note that in the preceding argument, it is necessary to expand  $C_{x_0, r}$  to  $C_{x_0+\eta, r+\eta}$  because  $C_{x_0, r}$  is not necessarily in  $\mathcal{G}_{\text{box}, dd}(3\bar{r})$  if the only elements of  $C_{x_0, r} \cap \text{Supp}_{F_{X,0}}(X_i)$  are on the boundary of  $C_{x_0, r}$ . Also, note that the argument above does not go through if one uses symmetric side lengths (i.e.,  $r_{1,u} = r_{2,u}$ ) in the definition of  $\mathcal{G}_{\text{box}, dd}$ .

Next, we verify Assumption CI for  $\mathcal{G} = \mathcal{G}_{c/d}$ . We write

$$\mathcal{X}_F(\theta) = \cup_{d \in D} \mathcal{X}_{1,F}(\theta, d), \quad \text{where} \quad (14.18)$$

$$\begin{aligned} \mathcal{X}_{1,F}(\theta, d) = \{x_1 \in R^{d_{x,1}} : E_F(m_j(W_i, \theta) | X_{1,i} = x_1, X_{2,i} = d) < 0 \text{ for some } j \leq p \text{ or} \\ E_F(m_j(W_i, \theta) | X_{1,i} = x_1, X_{2,i} = d) \neq 0 \text{ for some } j = p + 1, \dots, k\}, \end{aligned}$$

for  $d \in D$ . We have

$$\begin{aligned} P_F(X_i \in \mathcal{X}_F(\theta)) &= P_F\left((X'_{1,i}, X'_{2,i})' \in \bigcup_{d \in D} \mathcal{X}_{1,F}(\theta, d)\right) \\ &= \sum_{d \in D} P_F(X_{1,i} \in \mathcal{X}_{1,F}(\theta, d) | X_{2,i} = d) P_F(X_{2,i} = d). \end{aligned} \quad (14.19)$$

If  $P_F(X_i \in \mathcal{X}_F(\theta)) > 0$ , then there exists some  $d^* \in D$  such that  $P_F(X_{2,i} = d^*) > 0$  and

$$P_F((X_{1,i} \in \mathcal{X}_{1,F}(\theta, d^*) | X_{2,i} = d^*) > 0. \quad (14.20)$$

Given the inequality in (14.20), we use the same argument to verify Assumption CI as given for  $\mathcal{G}_{c\text{-cube}}$ ,  $\mathcal{G}_{\text{box}}$ ,  $\mathcal{G}_{B\text{-spline}}$ , or  $\mathcal{G}_{\text{box}, dd}$  with  $d_x$  replaced by  $d_{x,1}$ , but with  $E_F(\cdot)$  replaced by  $E_F(\cdot | X_{2,i} = d^*)$  throughout, and using the fact that  $\{g : g = g_1 1_{\{d^*\}}\}$ ,

$g_1 \in \mathcal{G}_1\} \subset \mathcal{G}_{c/d}$  for  $\mathcal{G}_1 = \mathcal{G}_{c-cube}, \mathcal{G}_{box}, \mathcal{G}_{B-spline}$ , or  $\mathcal{G}_{box,dd}$ .

Next, we verify Assumption M. Assumptions M(a) and M(b) hold for  $\mathcal{G}_{B-spline}$  by taking  $G(x) = 2/3 \forall x$  and  $\delta_1 = 4/\delta + 3$ . Assumption M(c) holds for  $\mathcal{G}_{B-spline}$  because each element of  $\mathcal{G}_{B-spline}$  can be written as the sum of four functions each of which is the product of an indicator function of a box and a polynomial of order four. Manageability of polynomials and indicator functions of boxes hold because they have finite pseudodimension (as defined in Pollard (1990, Sec. 4)). Manageability of finite linear combinations of these functions holds by the stability properties of cover numbers under addition and pointwise multiplication, see Pollard (1990, Sec. 5).

Assumption M holds for  $\mathcal{G}_{box,dd}$  because it holds for  $\mathcal{G}_{box}$  by Lemma 3 and  $\mathcal{G}_{box,dd} \subset \mathcal{G}_{box}$ .

The verification of Assumption M for  $\mathcal{G} = \mathcal{G}_{c/d}$  is the same as in the proof of Lemma 3 when  $\mathcal{G}_1$  is  $\mathcal{G}_{c-cube}, \mathcal{G}_{box}$ , or  $\mathcal{G}_{box,dd}$  because  $\mathcal{C}_{box} \times \{\{d\} : d \in D\}$  is a Vapnik-Cervonenkis class of sets. The verification of Assumption M for  $\mathcal{G} = \mathcal{G}_{c/d}$  when  $\mathcal{G}_1$  is  $\mathcal{G}_{B-spline}$  is essentially the same as the proof above for  $\mathcal{G}_{B-spline}$ . The functions in  $\mathcal{G}_{c/d}$  in this case still can be written as the sum of four functions each of which is the product of an indicator function of a box—in this case, the box is of the form  $B \times \{d\}$ , where  $B$  is a box in  $R^{d_{x,1}}$  and  $d \in D$ —and a polynomial of order four.

Assumption FA(e) holds for  $\mathcal{G}_{B-spline}, \mathcal{G}_{box,dd}$ , and  $\mathcal{G}_{c/d}$  by the same arguments as given above for Assumption M.

This completes the proofs of parts (a)-(d) of the Lemma.

Part (e) of the Lemma holds, i.e.,  $Supp(Q_c) = \mathcal{G}_{B-spline}$ , because  $\mathcal{G}_{B-spline}$  is countable and  $Q_c$  has a probability mass function that is positive at each element in  $\mathcal{G}_{B-spline}$ .

Now, we prove part (f) using a similar argument to that for part (b) of Lemma 4. Consider  $g = g_{x,r_1,r_2} \in \mathcal{G}_{box,dd}$ , where  $g_{x,r_1,r_2}(y) = 1(y \in C_{x,r_1,r_2}) \cdot 1_k$  and  $(x, r_1, r_2) \in Supp(X_i) \times (\prod_{u=1}^{d_x} (0, \sigma_{X,u}\bar{r}))^2$ . Let  $\delta > 0$  be given. Let  $\eta_0 = (\eta_{0,1}, \dots, \eta_{0,d_x})'$  and likewise for  $\eta_1$  and  $\eta_2$ . Define

$$G_{g,\bar{\eta}} = \{g_{x+\eta_0,r_1-\eta_1,r_2+\eta_2} : -\bar{\eta} \leq \eta_{0,u} \leq \bar{\eta}, \bar{\eta} \leq \eta_{1,u}, \eta_{2,u} \leq 2\bar{\eta} \forall u \leq d_x\}. \quad (14.21)$$

By the same sort of argument as for (13.26), for  $g^* = g_{x+\eta_0,r_1-\eta_1,r_2+\eta_2} \in G_{g,\bar{\eta}}$ , we



have

$$\begin{aligned}
\rho_X^2(g, g^*) &= E_{F_{X,0}}[1(X_i \in C_{x,r_1,r_2}) - 1(X_i \in C_{x+\eta_0,r_1-\eta_1,r_2+\eta_2})]^2 \\
&\leq \sum_{u=1}^{d_x} [P_{F_{X,0}}(X_{i,u} \in (x_u - r_{1,u}, x_u + \eta_{0,u} - (r_{1,u} - \eta_{1,u}))) \\
&\quad + P_{F_{X,0}}(X_{i,u} \in (x_u + r_{2,u}, x_u + \eta_{0,u} + r_{2,u} + \eta_{2,u}))] \\
&\leq \sum_{u=1}^{d_x} [F_{X_u,0}(x_u - r_{1,u} + 3\bar{\eta}) - F_{X_u,0}(x_u - r_{1,u})] \\
&\quad + \sum_{u=1}^{d_x} [F_{X_u,0}(x_u + r_{2,u} + 3\bar{\eta}) - F_{X_u,0}(x_u + r_{2,u})], \tag{14.22}
\end{aligned}$$

where  $F_{X_u,0}(\cdot)$  denotes the distribution function of  $X_{i,u}$  and the first inequality holds because  $\eta_{0,u} + \eta_{1,u} \geq 0$  and  $\eta_{0,u} + \eta_{2,u} \geq 0$ . Because distribution functions are right continuous, the rhs of (14.22) converges to zero as  $\bar{\eta} \downarrow 0$ . Thus,  $\rho_X^2(g, g^*)$  converges to zero uniformly over  $G_{g,\bar{\eta}}$  as  $\bar{\eta} \downarrow 0$  and there exists an  $\bar{\eta} > 0$  sufficiently small that  $G_{g,\bar{\eta}} \subset \mathcal{B}_{\rho_X}(g, \delta)$ .

Next, we have  $Q_c(G_{g,\bar{\eta}})$  equals

$$Q_{F_{X,0}}^* \left( \prod_{u=1}^{d_x} [x_u - \bar{\eta}, x_u + \bar{\eta}] \times \prod_{u=1}^{d_x} [r_{1,u} - 2\bar{\eta}, r_{1,u} - \bar{\eta}] \times \prod_{u=1}^{d_x} [r_{2,u} + \bar{\eta}, r_{2,u} + 2\bar{\eta}] \right) > 0, \tag{14.23}$$

where  $Q_{F_{X,0}}^* = F_{X,0} \times Unif((\prod_{u=1}^{d_x} (0, \sigma_{X,u}\bar{r}))^2)$  and the inequality holds because  $x \in Supp(X_i)$  and  $\bar{\eta} > 0$ . This completes the proof of part (f).

Lastly, we prove part (g). By parts (e) and (f) and parts (a) and (b) of Lemma 4, we have  $\mathcal{G}_1 \subset Supp(Q_1)$ . Because  $Supp(Q_D) = D$  and  $Q_e = Q_1 \times Q_D$ , we have  $\mathcal{G}_{c/d} \subset Supp(Q_e)$ .  $\square$

### 14.3 Proofs of Theorems B4 and B5 Regarding Uniformity Issues

**Proof of Theorem B4.** Part (a) holds by an empirical process central limit theorem because the intervals  $\{(a, b] : 0 \leq a < b \leq 1\}$  form a Vapnik-Cervonenkis class of sets, e.g., see the proof of Lemma A1(a). The covariance kernel of  $\nu(\cdot)$  and the pseudo-metric  $\rho_*$  are specified below.

Let  $c \vee d = \max\{c, d\}$  and  $c \wedge d = \min\{c, d\}$ .

To prove part (b), we write

$$\begin{aligned} Y_i g_{a,b}(X_i) &= (U_i + 1(X_i \in (\varepsilon_n, 1]) \cdot 1(X_i \in (a, b]) \\ &= U_i 1(X_i \in (a, b]) + 1(X_i \in (a \vee \varepsilon_n, b]) \end{aligned} \quad (14.24)$$

and

$$\begin{aligned} E_{F_n} Y_i g_{a,b}(X_i) &= E_{F_n} U_i 1(X_i \in (a, b]) + P_{F_n}(X_i \in (a \vee \varepsilon_n, b]) \\ &= P_{F_n}(X_i \in (a \vee \varepsilon_n, b]) \\ &\rightarrow (b - a)/2, \end{aligned} \quad (14.25)$$

where the second equality uses Assumption CX(b) and the convergence uses Assumption CX(c) and holds by slightly different arguments when  $a = 0$  and  $a > 0$ . Equation (14.25) and  $b - a > 0$  imply that  $h_{1,n}(g_{a,b}) = n^{1/2} E_{F_n} Y_i g_{a,b}(X_i) \rightarrow \infty = h_1(g_{a,b})$  as  $n \rightarrow \infty$  for all  $g_{a,b} \in \mathcal{G}$ , which proves part (b).

Part (c) holds because  $h_1(g_{a,b}) = \infty$  for all  $g_{a,b} \in \mathcal{G}$  and

$$\begin{aligned} \inf_{g_{a,b} \in \mathcal{G}} h_{1,n}(g_{a,b}) &= \inf_{g_{a,b} \in \mathcal{G}} n^{1/2} P_{F_n}(X_i \in (a \vee \varepsilon_n, b]) \\ &= \inf_{a,b: \varepsilon_n \leq a < b \leq 1} n^{1/2} P_{F_n}(X_i \in (a, b]) = 0 \end{aligned} \quad (14.26)$$

for all  $n$ , where the first equality holds by (14.25) and the last equality holds by Assumption CX(c).

Part (d) holds because  $\nu_n(g_{a,b}) + h_{1,n}(g_{a,b}) = O_p(1) + n^{1/2}(b - a)/2 \rightarrow_p \infty$  by part (a) and (14.25) for all  $g_{a,b} \in \mathcal{G}$ . This, combined with Assumption CX(f) (in particular, Assumption S1(d)), proves part (d).

Part (e) holds by part (b) and Assumption CX(f) (in particular, Assumption S2) because  $S(\nu(g_{a,b}) + h_1(g_{a,b})) = S(\infty) = 0$  for all  $g_{a,b} \in \mathcal{G}$ .

To show part (f), we define

$$g_n^*(x) = 1(x \in (0, \varepsilon_n]). \quad (14.27)$$

Then,

$$h_{1,n}(g_n^*) = n^{1/2} E_{F_n} Y_i g_n^*(X_i) = P_{F_n}(X_i \in (0 \vee \varepsilon_n, \varepsilon_n]) = 0 \quad (14.28)$$

for all  $n$ , where the second equality holds by (14.25) with  $a = 0$  and  $b = \varepsilon_n$ .

Next, we have

$$\sup_{g_{a,b} \in \mathcal{G}} S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) \geq S(\nu_n(g_n^*) + h_{1,n}(g_n^*)) = S(\nu_n(g_n^*)), \quad (14.29)$$

where the equality holds by (14.28). The asymptotic distribution of  $S(\nu_n(g_n^*))$  is established as follows:

$$\begin{aligned} \nu_n(g_n^*) &= n^{-1/2} \sum_{i=1}^n [Y_i 1(X_i \in (0, \varepsilon_n]) - E_{F_n} Y_i 1(X_i \in (0, \varepsilon_n])] \\ &= n^{-1/2} \sum_{i=1}^n [U_i 1(X_i = \varepsilon_n) + U_i 1(X_i \in (0, \varepsilon_n)) \\ &\quad + 1(X_i \in (\varepsilon_n, 1]) 1(X_i \in (0, \varepsilon_n]) - E_{F_n} 1(X_i \in (\varepsilon_n, 1]) 1(X_i \in (0, \varepsilon_n))] \\ &= n^{-1/2} \sum_{i=1}^n U_i 1(X_i = \varepsilon_n) + n^{-1/2} \sum_{i=1}^n U_i 1(X_i \in (0, \varepsilon_n)) \\ &\rightarrow_d Z^* \sim N(0, 1/2), \end{aligned} \quad (14.30)$$

where the second equality uses  $E_{F_n} U_i = 0$  and  $U_i$  and  $X_i$  are independent. The convergence in distribution in (14.30) holds by a triangular array CLT for the first summand on the second last line because  $U_i 1(X_i = \varepsilon_n)$  has mean zero and variance  $E_{F_n} U_i^2 1(X_i = \varepsilon_n) = 1 \cdot P_{F_n}(X_i = \varepsilon_n) = 1/2$  for all  $n$  using Assumption CX(b). The second summand on the second last line of (14.30) is  $o_p(1)$  because its mean is zero and its variance is

$$\begin{aligned} \text{Var} \left( n^{-1/2} \sum_{i=1}^n U_i 1(X_i \in (0, \varepsilon_n)) \right) &= \text{Var}(U_i 1(X_i \in (0, \varepsilon_n))) \\ &= E_{F_n} U_i^2 1(X_i \in (0, \varepsilon_n)) = 1 \cdot P_{F_n}(X_i \in (0, \varepsilon_n)) = \varepsilon_n/2, \end{aligned} \quad (14.31)$$

where the first equality holds by Assumption CX(d), the second and third equalities hold by Assumption CX(b), and the last equality holds by Assumption CX(c).

Equations (14.29) and (14.30), Assumption S1(d), and the continuous mapping theorem combine to prove part (f).

Part (g) holds if

$$\sup_{g_{a,b} \in \mathcal{G}} S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) \rightarrow_p 0 \quad (14.32)$$

using part (e). By part (f), for all  $\delta \geq 0$ ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} P \left( \sup_{g_{a,b} \in \mathcal{G}} S(\nu_n(g_{a,b}) + h_{1,n}(g_{a,b})) > \delta \right) &\geq \liminf_{n \rightarrow \infty} P(S(\nu_n(g_n^*)) > \delta) \\ &= P(S(Z^*) > \delta). \end{aligned} \quad (14.33)$$

Now, by the dominated convergence theorem, as  $\delta \rightarrow 0$ ,

$$P(S(Z^*) > \delta) \rightarrow P(S(Z^*) > 0) = 1/2, \quad (14.34)$$

where the equality holds because  $S(m) > 0$  iff  $m < 0$  by Assumption S2 and  $P(Z^* < 0) = 1/2$ . Hence, the right-hand side in (14.33) is arbitrarily close to  $1/2$  for  $\delta > 0$  sufficiently small, which implies that (14.32) holds and part (g) is established.

Lastly, we compute the covariance kernel  $K(g_{a_1, b_1}, g_{a_2, b_2})$  of the Gaussian process  $\nu(\cdot)$ . We have

$$\begin{aligned} &E_{F_n} Y_i^2 g_{a_1, b_1}(X_i) g_{a_2, b_2}(X_i) \\ &= E_{F_n} (U_i + 1(X_i \in (\varepsilon_n, 1]))^2 \cdot 1(X_i \in (a_1 \vee a_2, b_1 \wedge b_2]) \\ &= E_{F_n} U_i^2 1(X_i \in (a_1 \vee a_2, b_1 \wedge b_2]) \\ &\quad + E_{F_n} (2U_i + 1) 1(X_i \in (a_1 \vee a_2 \vee \varepsilon_n, b_1 \wedge b_2]) \\ &= P_{F_n}(X_i \in (a_1 \vee a_2, b_1 \wedge b_2]) + P_{F_n}(X_i \in (a_1 \vee a_2 \vee \varepsilon_n, b_1 \wedge b_2]) \\ &\rightarrow (1/2) 1(a_1 = a_2 = 0) + \max\{(b_1 \wedge b_2) - (a_1 \vee a_2), 0\} \\ &= K_1(g_{a_1, b_1}, g_{a_2, b_2}), \end{aligned} \quad (14.35)$$

where the third equality uses Assumption CX(b) and the convergence uses Assumption CX(c).

In addition, we have

$$\lim_{n \rightarrow \infty} E_{F_n} Y_i g_{a,b}(X_i) = (b - a)/2 = K_2(g_{a,b}), \quad (14.36)$$

where the first equality holds by (14.25). Putting the results of (14.35) and (14.36)

together yields

$$\begin{aligned}
& K(g_{a_1, b_1}, g_{a_2, b_2}) \\
&= \lim_{n \rightarrow \infty} \left( E_{F_n} Y_i^2 g_{a_1, b_1}(X_i) g_{a_2, b_2}(X_i) - E_{F_n} Y_i g_{a_1, b_1}(X_i) \cdot E_{F_n} Y_i g_{a_2, b_2}(X_i) \right) \\
&= K_1(g_{a_1, b_1}, g_{a_2, b_2}) - K_2(g_{a_1, b_1}) K_2(g_{a_2, b_2}). \tag{14.37}
\end{aligned}$$

The square of the pseudo-metric  $\rho_*$  on  $\mathcal{G}$  is

$$\begin{aligned}
& \rho_*^2(g_{a_1, b_1}, g_{a_2, b_2}) \tag{14.38} \\
&= \lim_{n \rightarrow \infty} E_{F_n} \left( Y_i g_{a_1, b_1}(X_i) - Y_i g_{a_2, b_2}(X_i) - E_{F_n} Y_i g_{a_1, b_1}(X_i) + E_{F_n} Y_i g_{a_2, b_2}(X_i) \right)^2.
\end{aligned}$$

The limit in (14.38) exists and can be computed via calculations analogous to those in (14.25) and (14.35).  $\square$

**Proof of Theorem B5.** For notational convenience, we let  $g$  denote  $g_{a, b}$ . By Theorem B4(a),  $\nu_n(\cdot) \Rightarrow \nu(\cdot)$  as  $n \rightarrow \infty$ . As in the proof of Theorem 1(a), by an almost sure representation argument, e.g., see Thm. 9.4 of Pollard (1990), there exist processes  $\tilde{\nu}_n(\cdot)$  and  $\tilde{\nu}(\cdot)$  on  $\mathcal{G}$  that have the same distributions as  $\nu_n(\cdot)$  and  $\nu(\cdot)$ , respectively, for which

$$\sup_{g \in \mathcal{G}} |\tilde{\nu}_n(g) - \tilde{\nu}(g)| \rightarrow 0 \text{ a.s.} \tag{14.39}$$

Let  $\tilde{\Omega}$  denote the sample paths for which the convergence in (14.39) holds. By (14.39),  $P(\tilde{\Omega}) = 1$ .

For each  $\omega \in \tilde{\Omega}$ , we apply the bounded convergence theorem (BCT) to obtain

$$\lim_{n \rightarrow \infty} \int S(\tilde{\nu}_n(g)(\omega) + h_{1, n}(g)) dQ(g) = \int S(\tilde{\nu}(g)(\omega) + h_1(g)) dQ(g), \tag{14.40}$$

which yields the result of the Theorem. Now we check the conditions for the BCT. For all  $g \in \mathcal{G}$ , pointwise convergence holds:

$$S(\tilde{\nu}_n(g)(\omega) + h_{1, n}(g)) \rightarrow S(\tilde{\nu}(g)(\omega) + h_1(g)) \text{ as } n \rightarrow \infty$$

by (14.39), Theorem B4(b), and Assumption S1(d). A bound on  $S(\tilde{\nu}_n(g)(\omega) + h_{1, n}(g))$  over  $g \in \mathcal{G}$  and  $n$  sufficiently large is given by  $S(\inf_{g^* \in \mathcal{G}} \tilde{\nu}(g^*)(\omega) - \varepsilon)$  for some  $\varepsilon > 0$ .

This follows because for all  $\varepsilon > 0$  and  $g \in \mathcal{G}$ , we have

$$\begin{aligned} 0 &\leq S(\tilde{\nu}_n(g)(\omega) + h_{1,n}(g)) \leq S(\tilde{\nu}_n(g)(\omega)) \\ &\leq S(\inf_{g^* \in \mathcal{G}} \tilde{\nu}_n(g^*)(\omega)) \leq S(\inf_{g^* \in \mathcal{G}} \tilde{\nu}(g^*)(\omega) - \varepsilon) < \infty, \end{aligned} \quad (14.41)$$

where the first inequality holds by Assumption S1(c), the second inequality holds by Assumption S1(b) and  $h_{1,n}(g) \geq 0$  for all  $g \in \mathcal{G}$  by (14.25), the third inequality holds by Assumption S1(b), the fourth inequality holds for all  $n$  sufficiently large by (14.39) and Assumption S1(b), and the last inequality holds because  $\inf_{g^* \in \mathcal{G}} \tilde{\nu}(g^*)(\omega) > -\infty$  because the sample paths of  $\tilde{\nu}(\cdot)$  are bounded a.s. (which follows from  $|m(W_i, \theta_0)g(X_i)| \leq |m(W_i, \theta_0)| \leq |U_i| + 1 < \infty$  a.s. and (14.39)). This completes the proof of (14.40) and the Theorem is proved.  $\square$

## 14.4 Proofs of Subsampling Results

**Proof of Lemma B3.** For  $S_1$ , Assumption SQ(a) holds because (i) if  $v \geq 1$ , the summand  $\sum_{j=p+1}^k (\nu_{h_2,j}^2(g)/(h_{2,j,j}(g) + \varepsilon))$  is absolutely continuous for all  $g \in \mathcal{G}$ , where  $\nu_{h_2}(g) = (\nu_{h_2,1}(g), \dots, \nu_{h_2,k}(g))'$  and  $h_{2,j,j}(g)$  denotes the  $j$ th diagonal element of  $h_2(g)$ , (ii) if  $v = 0$  and  $h_1(g) \neq \infty^p$ , the summands  $[\nu_{h_2,j}(g) + h_{1,j}(g)]_-^2 / (h_{2,j,j}(g) + \varepsilon)$  are absolutely continuous for  $x > 0$  and all  $j \leq p$  such that  $h_{1,j}(g) < \infty$ , (iii) if  $v = 0$  and  $h_1(g) = \infty^p$ ,  $S_1(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) = 0$  and its distribution function equals one for all  $x > 0$ , and (iv) if  $S_1(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k)$  is absolutely continuous for all  $g \in \mathcal{G}$ , then  $\int S_1(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) dQ(g)$  is absolutely continuous.

Assumption SQ(b) holds for  $S_1$  because (i) if  $v \geq 1$ , the summand  $\int \sum_{j=p+1}^k (\nu_{h_2,j}^2(g)/(h_{2,j,j}(g) + \varepsilon)) dQ(g)$  has positive density on  $R_+$ , and (ii) if  $v = 0$  and  $h_1(g) \neq \infty^p$  on some  $G \subset \mathcal{G}$  such that  $Q(G) > 0$ , each summand  $\int [\nu_{h_2,j}(g) + h_{1,j}(g)]_-^2 / (h_{2,j,j}(g) + \varepsilon) dQ(g)$  for which  $h_{1,j}(g) < \infty$  on some  $G \subset \mathcal{G}$  such that  $Q(G) > 0$  has positive density on  $R_+$  and so does the sum over  $\sum_{j=1}^p$ .

For  $S_2$ , if  $v = 0$  and  $h_1(g) = \infty^p$  a.s.  $[Q]$ , then  $S_2(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) = 0$  a.s.  $[Q]$ ,  $J_{(h_1, h_2)}(x) = 1$  for all  $x > 0$ , Assumption SQ(a) holds, and Assumption SQ(b) does not impose any restriction. Otherwise,  $v \geq 1$  or  $h_1(g) < \infty^p$  on a subset  $G \subset \mathcal{G}$  such that  $Q(G) > 0$ . In this case, the random variable  $\int S_2(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) dQ(g)$  has support  $R_+$  and is absolutely continuous. Hence, Assumptions SQ(a)-(b) hold.  $\square$

The proof of Theorem B6 uses the following Lemma.

**Lemma D1.** *Suppose Assumptions M and S1 hold. Then, for all  $h \in \mathcal{H}$ , under any sequence  $\{(\theta_n, F_n) : n \geq 1\} \in Seq^b(h_1^*, h)$ ,*

$$T_n(\theta_n) \rightarrow_d \int S(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) dQ(g) \sim J_{(h_1, h_2)} \text{ as } n \rightarrow \infty.$$

**Comment.** Condition (iv) of  $Seq^b(h_1^*, h)$  is not needed for the result of Lemma D1 to hold.

**Proof of Theorem B6.** First, we prove part(a). Suppose  $\{(\theta_n, F_n) : n \geq 1\} \in Seq^b$ . Then, there exist  $h \in \mathcal{H}$  and  $h_1^* \in \mathcal{H}_1^*(h)$  such that  $\{(\theta_n, F_n) : n \geq 1\} \in Seq^b(h_1^*, h)$ . We need to show that under  $\{(\theta_n, F_n) : n \geq 1\}$ ,  $\limsup_{n \rightarrow \infty} P_{F_n}(T_n(\theta_n) \leq c_{n,b}(\theta_n, 1 - \alpha)) \geq 1 - \alpha$ . The asymptotic distribution of  $T_n(\theta_n)$  is given by Lemma D1. We now determine the probability limit of  $c_{n,b}(\theta_n, 1 - \alpha)$ .

Let  $J_{(h_1, h_2)}(x)$  for  $x \in R$  denote the distribution function of  $J_{(h_1, h_2)}$ . By Lemma 5 in Andrews and Guggenberger (2010), if (i)  $U_{n,b}(\theta_n, x) \rightarrow_p J_{(h_1^*, h_2)}(x)$  for all  $x \in C(J_{(h_1^*, h_2)})$ , where  $C(J_{(h_1^*, h_2)})$  denotes the continuity points of  $J_{(h_1^*, h_2)}$ , and (ii) for all  $\xi > 0$ ,  $J_{(h_1^*, h_2)}(c_\infty + \xi) > 1 - \alpha$ , where  $c_\infty$  is the  $1 - \alpha$  quantile of  $J_{(h_1^*, h_2)}$ , then

$$c_{n,b}(\theta_n, 1 - \alpha) \rightarrow_p c_\infty. \quad (14.42)$$

Condition (i) holds by the properties of U-statistics of degree  $b$  and  $T_{n,b,j}(\theta_n) \rightarrow_d J_{(h_1^*, h_2)}$  (see Thm. 2.1(i) in Politis and Romano (1994)). The latter holds by Lemma D1 because subsample  $j$  is an i.i.d. sample of size  $b$  from the population distribution.

By Assumption S1(c),  $J_{(h_1, h_2)}(x) = 0 \forall x < 0$  for  $h \in \mathcal{H}$ . Thus,  $c_\infty \geq 0$ . If  $v = 0$  and  $h_1(g) = \infty^p$  a.s.  $[Q]$ , then  $J_{(h_1^*, h_2)}(0) = 1$ ,  $c_\infty = 0$ ,  $J_{(h_1^*, h_2)}(c_\infty + \xi) = 1 > 1 - \alpha$ . In all other cases, Assumption SQ(b) applies,  $J_{(h_1^*, h_2)}(0) < 1$ , and  $J_{(h_1^*, h_2)}(c_\infty + \xi) > J_{(h_1^*, h_2)}(c_\infty) \geq 1 - \alpha$ . Thus, condition (ii) holds and (14.42) is established.

If  $c_\infty > 0$ ,  $c_\infty \in C(J_{(h_1, h_2)})$  by Assumption SQ(a). Thus,

$$\liminf_{n \rightarrow \infty} P_{F_n}(T_n(\theta_n) \leq c_{n,b}(\theta_n, 1 - \alpha)) = J_{(h_1, h_2)}(c_\infty) \geq J_{(h_1^*, h_2)}(c_\infty) = 1 - \alpha, \quad (14.43)$$

where the first equality holds by (14.42) and Lemma D1, the inequality holds by Assumption S1(b) and  $h_1^* \leq h_1$ , and the second equality holds by Assumption SQ(a) and the definition of  $c_\infty$ .

If  $c_\infty = 0$ , for some set  $G \subset \mathcal{G}$  with  $Q(G) = 1$ , we have

$$\begin{aligned}
& P_{F_n}(T_n(\theta_n) \leq c_{n,b}(\theta_n, 1 - \alpha)) \\
& \geq P_{F_n}(T_n(\theta_n) \leq 0) \\
& = P_{F_n} \left( \frac{n^{1/2} \bar{m}_{n,j}(\theta_n, g)}{\bar{\sigma}_{n,j}(\theta_n, g)} \geq 0 \ \forall j \leq p \ \& \ \frac{\bar{m}_{n,j}(\theta_n, g)}{\bar{\sigma}_{n,j}(\theta_n, g)} = 0 \ \forall j = p+1, \dots, k, \ \forall g \in G \right) \\
& \rightarrow P \left( \frac{\nu_{h,j}(g) + h_{1,j}(g)}{h_{2,j,j}(g) + \varepsilon} \geq 0 \ \forall j \leq p \ \& \ \frac{\nu_{h,j}(g)}{h_{2,j,j}(g) + \varepsilon} = 0 \ \forall j = p+1, \dots, k, \ \forall g \in G \right) \\
& = P(S(\nu_h(g) + h_1(g), h_2(g) + \varepsilon I_k) = 0 \ \forall g \in G) \\
& = J_{(h_1, h_2)}(0) \geq J_{(h_1^*, h_2)}(0) \geq 1 - \alpha, \tag{14.44}
\end{aligned}$$

where  $\bar{\sigma}_{n,j}(\theta, g)$  and  $h_{2,j,j}(g)$  denote the  $j$ th diagonal elements of  $\bar{\Sigma}_n(\theta, g)$  and  $h_2(g)$ , respectively. In (14.44), the first inequality holds because  $c_{n,b}(\theta_n, 1 - \alpha)$  is the  $1 - \alpha$  sample quantile of the subsample test statistics and the test statistics are non-negative (by Assumption S1(a)), the first and second equalities hold by Assumption S2, the convergence holds by Lemma A1(a)-(b), the third equality holds by the definition of  $J_{(h_1, h_2)}$ , and the last inequality holds because 0 is the  $1 - \alpha$  quantile of  $J_{(h_1^*, h_2)}$ .

Next, we prove part (b). Let  $(\theta_n^*, F_n^*) = (\theta, F)$  for  $n \geq 1$ , where  $(\theta, F)$  is specified in Assumption C. Then,  $\{(\theta_n^*, F_n^*) : n \geq 1\} \in Seq^b(h_1^*, h)$ , where  $h_1^* = h_{1,F}(\theta)$  and  $h = (h_{1,F}(\theta), h_{2,F}(\theta))$ . Thus,

$$\liminf_{n \rightarrow \infty} P_{F_n^*}(T_n(\theta_n^*) \leq c_{n,b}(\theta_n^*, 1 - \alpha)) = J_{(h_1, h_2)}(c_\infty) = J_{(h_1^*, h_2)}(c_\infty) = 1 - \alpha. \tag{14.45}$$

This and the result of Theorem B6(a) establish part (b).

Lastly, we prove part (c). Suppose Assumption Sub holds and  $\{(\theta_{m_n}, F_{m_n}) : n \geq 1\}$  belongs to  $Seq^b$  (where  $Seq^b$  is defined with  $m_n$  in place of  $n$ ). Then,

$$\begin{aligned}
AsyCS & = \lim_{n \rightarrow \infty} P_{F_{m_n}}(T_n(\theta_{m_n}) \leq c_{n,b}(\theta_{m_n}, 1 - \alpha)) \\
& \geq \inf_{\{(\theta_n, F_n) : n \geq 1\} \in Seq^b} \liminf_{n \rightarrow \infty} P_{F_n}(T_n(\theta_n) \leq c_{n,b}(\theta_n, 1 - \alpha)) \\
& = 1 - \alpha \tag{14.46}
\end{aligned}$$



using Theorem B6(b). On the other hand,

$$\begin{aligned}
AsyCS &= \liminf_{n \rightarrow \infty} \inf_{(\theta, F) \in \mathcal{F}} P_F(T_n(\theta) \leq c_{n,b}(\theta, 1 - \alpha)) \\
&\leq \inf_{\{(\theta_n, F_n) : n \geq 1\} \in Seq^b} \liminf_{n \rightarrow \infty} P_{F_n}(T_n(\theta_n) \leq c_{n,b}(\theta_n, 1 - \alpha)) \\
&= 1 - \alpha.
\end{aligned} \tag{14.47}$$

Thus, we have  $AsyCS = 1 - \alpha$ .  $\square$

**Proof of Lemma D1.** By the same argument as used above to show (13.20), but with  $\nu_{\widehat{h}_{2,n}(\theta_c)}(g)$  and  $\varphi_n(\theta_c, g)$  replaced by  $\nu_{n, F_n}(\theta_n, g)$  and  $h_{1, n, F_n}(\theta_n, g)$ , respectively, we have

$$T_n(\theta_n) \rightarrow_d T(h) = \int S(\nu_{h_2}(g) + h_1(g), h_2(g) + \varepsilon I_k) dQ(g), \tag{14.48}$$

where  $\nu_{n, F_n}(\theta_n, \cdot) \Rightarrow \nu_{h_2}(\cdot)$  by Lemma A1(a),  $h_{1, n, F_n}(\theta_n, g) \rightarrow h_1(g) \forall g \in \mathcal{G}$  by Definition  $Seq^b(h_1^*, h)$ (ii), and  $d(\widehat{h}_{2,n}(\theta_n), h_2) \rightarrow 0$  by Lemma A1(b) and (10.27). Note that the assumption that  $\{(\theta_n, F_n) : n \geq 1\}$  satisfies Definition  $Seq^b(h_1^*, h)$  and Assumption M implies that  $\{(\theta_n, F_n) : n \geq 1\}$  satisfies Definition  $SubSeq(h_2)$  and hence the conditions of Lemma A1 hold.  $\square$

# 15 Appendix E

This Appendix proves Lemma A1.

## 15.1 Preliminary Lemmas E1-E3

Before we prove Lemma A1, we review a few concepts from Pollard (1990) and state several lemmas that are used in the proof.

**Definition E1 (Pollard, 1990, Definition 3.3).** *The packing number  $D(\xi, \rho, G)$  for a subset  $G$  of a metric space  $(\mathcal{G}, \rho)$  is defined as the largest  $b$  for which there exist points  $g^{(1)}, \dots, g^{(b)}$  in  $G$  such that  $\rho(g^{(s)}, g^{(s')}) > \xi$  for all  $s \neq s'$ . The covering number  $N(\xi, \rho, G)$  is defined to be the smallest number of closed balls with  $\rho$ -radius  $\xi$  whose union covers  $G$ .*

It is easy to see that  $N(\xi, \rho, G) \leq D(\xi, \rho, G) \leq N(\xi/2, \rho, G)$ .

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be the underlying probability space equipped with probability distribution  $\mathbf{P}$ . Let  $\{f_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  be a triangular array of random processes. Let

$$\mathcal{F}_{n,\omega} = \{(f_{n,1}(\omega, g), \dots, f_{n,n}(\omega, g))' : g \in \mathcal{G}\}. \quad (15.1)$$

Because  $\mathcal{F}_{n,\omega} \subset R^n$ , we use the Euclidean metric  $\|\cdot\|$  on this space. For simplicity, we omit the metric argument in the packing number function, i.e., we write  $D(\xi, G)$  in place of  $D(\xi, \|\cdot\|, G)$  when  $G \subset \mathcal{F}_{n,\omega}$ .

Let  $\odot$  denote the element-by-element product. For example for  $a, b \in R^n$ ,  $a \odot b = (a_1 b_1, \dots, a_n b_n)'$ . Let **envelope functions** of a triangular array of processes  $\{f_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  be an array of functions  $\{F_n(\omega) = (F_{n,1}(\omega), \dots, F_{n,n}(\omega))' : n \geq 1\}$  such that  $|f_{n,i}(\omega, g)| \leq F_{n,i}(\omega) \forall i \leq n, n \geq 1, g \in \mathcal{G}, \omega \in \Omega$ .

**Definition E2 (Pollard, 1990, Definition 7.9).** *A triangular array of processes  $\{f_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is said to be *manageable with respect to envelopes*  $\{F_n(\omega) : n \geq 1\}$  if there exists a deterministic real function  $\lambda$  on  $(0, 1]$  for which (i)  $\int_0^1 \sqrt{\log \lambda(\xi)} d\xi < \infty$  and (ii)  $D(\xi \|\alpha \odot F_n(\omega)\|, \alpha \odot \mathcal{F}_{n,\omega}) \leq \lambda(\xi)$  for  $0 < \xi \leq 1$ , all  $\omega \in \Omega$ , all  $n$ -vectors  $\alpha$  of nonnegative weights, and all  $n \geq 1$ .*

**Lemma E1.** *If a row-wise i.i.d. triangular array of random processes  $\{\phi_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is manageable with respect to the envelopes  $\{F_n(\omega) : n \geq 1\}$  and  $c_n(\omega) = (c_{n,1}(\omega), \dots, c_{n,n}(\omega))'$  is an  $R^n$ -valued function on the underlying probability*

space, then

(a)  $\{\phi_{n,i}(\omega, g)c_{n,i}(\omega) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is manageable with respect to the envelopes

$$F_n(\omega) = (F_{n,1}(\omega)|c_{n,1}(\omega)|, \dots, F_{n,n}(\omega)|c_{n,n}(\omega)|)' \text{ for } n \geq 1, \quad (15.2)$$

(b)  $\{E\phi_{n,i}(\cdot, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is manageable with respect to the envelopes  $\{EF_n : n \geq 1\}$  provided  $EF_{n,1} < \infty$  for all  $n \geq 1$ , and

(c) if another triangular array of random processes  $\{\phi_{n,i}^*(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is manageable with respect to the envelopes  $\{F_n^*(\omega) : n \geq 1\}$ , then  $\{\phi_{n,i}^*(\omega, g) + \phi_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is manageable with respect to the envelopes  $\{F_n(\omega) + F_n^*(\omega) : n \geq 1\}$ .

**Lemma E2.** If the triangular array of processes  $\{f_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is manageable with respect to the envelopes  $\{F_n(\omega) = (F_{n,1}(\omega), \dots, F_{n,n}(\omega))' : n \geq 1\}$ , and there exist  $0 < \eta < 1$  and  $0 < B^* < \infty$  such that  $n^{-1} \sum_{i \leq n} EF_{n,i}^{1+\eta} \leq B^*$  for all  $n \geq 1$ , then

$$\sup_{g \in \mathcal{G}} \left| n^{-1} \sum_{i=1}^n (f_{n,i}(\omega, g) - Ef_{n,i}(\cdot, g)) \right| \rightarrow_p 0. \quad (15.3)$$

Lemma E1(b)-(c) imply that if  $\{f_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  is manageable, then the triangular array of recentered processes  $\{f_{n,i}(\omega, g) - Ef_{n,i}(\cdot, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  also is manageable with respect to their corresponding envelopes. Lemma E2 is a uniform weak law of large numbers for triangular arrays of row-wise independent random processes. Lemma E2 is a complement to Thm. 8.2 in Pollard (1990) which is a uniform weak law of large numbers for independent sequences of random processes.

Lemma A1(a) is a functional central limit theorem result for multi-dimensional empirical processes. We prove it using a functional central limit theorem for real-valued empirical processes given in Pollard (1990, Thm. 10.3) and the Cramér-Wold device.

For  $a \in R^k / \{0_k\}$ , let

$$f_{n,i}(\omega, g) = a'D_{F_n}^{-1/2}(\theta_n)n^{-1/2}[m(W_{n,i}(\omega), \theta_n, g) - E_{F_n}m(W_{n,i}(\cdot), \theta_n, g)],$$

$$\text{for } \omega \in \Omega, g \in \mathcal{G}, \quad (15.4)$$

where  $W_{n,i}(\cdot) = W_i$ , and the index  $n$  in  $W_{n,i}$  signifies the fact that the distribution of  $W_i$  is changing with  $n$ . The random variable  $f_{n,i}(\omega, g)$  depends on  $a$ , but for notational

simplicity,  $a$  does not appear explicitly in  $f_{n,i}(\omega, g)$ . By definition, we have

$$a' \nu_{n, F_n}(\theta_n, g) = \sum_{i=1}^n f_{n,i}(\omega, g). \quad (15.5)$$

Let

$$\rho_{n,a}(g, g^*) = (nE|f_{n,i}(\cdot, g) - f_{n,i}(\cdot, g^*)|^2)^{1/2} \text{ for } g, g^* \in \mathcal{G}. \quad (15.6)$$

We show in the proof of Lemma E3 below that under the assumptions, the sequence  $\{\rho_{n,a}(g, g^*) : n \geq 1\}$  converges for each pair  $g, g^* \in \mathcal{G}$ . In consequence, the pointwise limit of  $\rho_{n,a}(\cdot, \cdot)$  is an appropriate choice for the pseudo-metric on  $\mathcal{G}$ . Denote the limit by  $\rho_a(\cdot, \cdot)$ , i.e.,

$$\rho_a(g, g^*) = \lim_{n \rightarrow \infty} \rho_{n,a}(g, g^*). \quad (15.7)$$

**Lemma E3.** *For all  $a \in R^k / \{0\}$  and any subsequence  $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\} \in \text{SubSeq}(h_2)$ , for some  $k \times k$ -matrix-valued covariance kernel  $h_2$  on  $\mathcal{G} \times \mathcal{G}$ ,*

- (a)  $\mathcal{G}$  is totally bounded under the pseudo-metric  $\rho_a$ ,
- (b) the finite dimensional distributions of  $a' \nu_{a_n, F_{a_n}}(\theta_{a_n}, g)$  have Gaussian limits with zero means and covariances given by  $a' h_2(g, g^*) a$ ,  $\forall g, g^* \in \mathcal{G}$ , which uniquely determine a Gaussian distribution  $\nu_a$  concentrated on the space of uniformly  $\rho_a(\cdot, \cdot)$ -continuous bounded functionals on  $\mathcal{G}$ ,  $U_{\rho_a}(\mathcal{G})$ , and
- (c)  $a' \nu_{a_n, F_{a_n}}(\theta_{a_n}, \cdot)$  converges in distribution to  $\nu_a$ .

The proofs of Lemmas E1-E3 are given below. The proof of Lemma E2 uses the maximal inequality in (7.10) of Pollard (1990). The proof of Lemma E3 uses the real-valued empirical process result of Thm. 10.6 in Pollard (1990).

## 15.2 Proof of Lemma A1(a)

Lemma A1 is stated in terms of subsequences  $\{a_n\}$ . For notational simplicity, we prove it for the sequence  $\{n\}$ . All of the arguments in this subsection and the next go through with  $\{a_n\}$  in place of  $\{n\}$ .

The following three conditions are sufficient for weak convergence: (a)  $(\mathcal{G}, \rho)$  is a totally bounded pseudo-metric space, (b) finite dimensional convergence holds:  $\forall \{g^{(1)}, \dots, g^{(L)}\} \subset \mathcal{G}$ ,  $(\nu_{n, F_n}(\theta_n, g^{(1)})', \dots, \nu_{n, F_n}(\theta_n, g^{(L)})')'$  converges in distribution, and

(c)  $\{\nu_{n,F_n}(\theta_n, \cdot) : n \geq 1\}$  is stochastically equicontinuous. (For example, see Thm. 10.2 of Pollard (1990).)

First, we establish the total boundedness of the pseudo-metric space  $(\mathcal{G}, \rho)$ , i.e.,  $N(\xi, \rho, \mathcal{G}) < \infty$  for all  $\xi > 0$ . This is done by constructing a finite collection of closed balls that covers  $(\mathcal{G}, \rho)$ .

Consider  $\xi > 0$ . Let  $B_\rho(g, \xi)$  denote a closed ball centered at  $g$  with  $\rho$ -radius  $\xi$ . Let  $\#G$  denote the number of elements in  $G$  when  $G$  is a finite set. (Throughout this proof  $G$  denotes a subset of  $\mathcal{G}$ , not the envelope function that appears in Assumption M.) For  $j = 1, \dots, k$ , let  $e_j$  be a  $k$ -dimensional vector with the  $j$ th coordinate equal to one and all other coordinates equal to zero. Then,  $e_j \in R^k/\{0\}$  and by Lemma E3(a), the pseudo-metric spaces  $(\mathcal{G}, \rho_{e_j})$  are totally bounded. Consequently, for all  $G \subset \mathcal{G}$ ,  $(G, \rho_{e_j})$  is totally bounded. Our construction of the collection of closed balls is based on the following relationship between  $\{\rho_{e_j} : j \leq k\}$  and  $\rho$ :  $\forall g, g^* \in \mathcal{G}$ ,

$$\begin{aligned} \rho^2(g, g^*) &= \text{tr}(h_2(g, g) - h_2(g, g^*) - h_2(g^*, g) + h_2(g^*, g^*)) \\ &= \lim_{n \rightarrow \infty} E_{F_n} \|D_{F_n}^{-1/2}(\theta_n)[\tilde{m}(W_i, \theta_n, g) - \tilde{m}(W_i, \theta_n, g^*)]\|^2 \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^k \rho_{n, e_j}^2(g, g^*) = \sum_{j=1}^k \rho_{e_j}^2(g, g^*), \end{aligned} \quad (15.8)$$

where the second equality holds by (15.7), which is proved in (15.40)-(15.41).

We start with  $j = 1$ . Because  $(\mathcal{G}, \rho_{e_1})$  is totally bounded, we can find a set  $G_1 \subset \mathcal{G}$  such that

$$\#G_1 = N(\xi_k, \rho_{e_1}, \mathcal{G}) \text{ and } \sup_{g \in G_1} \min_{g^* \in G_1} \rho_{e_1}(g, g^*) \leq \xi_k, \quad (15.9)$$

where  $\xi_k = \xi/(2\sqrt{k})$ . For all  $g \in G_1$ , let  $B_{\rho_{e_1}}^1(g, \xi_k) = B_{\rho_{e_1}}(g, \xi_k) \cap \mathcal{G}$ . Then,  $\bigcup_{g \in G_1} B_{\rho_{e_1}}^1(g, \xi_k)$  covers  $\mathcal{G}$ .

Because  $B_{\rho_{e_1}}^1(g, \xi_k) \subset \mathcal{G}$ ,  $(B_{\rho_{e_1}}^1(g, \xi_k), \rho_{e_2})$  is totally bounded. We are then able to choose a set  $G_{2,g}$  such that

$$\#G_{2,g} = N(\xi_k, \rho_{e_2}, B_{\rho_{e_1}}^1(g, \xi_k)) \text{ and } \sup_{g' \in B_{\rho_{e_1}}^1(g, \xi_k)} \min_{g^* \in G_{2,g}} \rho_{e_2}(g', g^*) \leq \xi_k. \quad (15.10)$$

Let  $G_2 = \bigcup_{g \in G_1} G_{2,g}$ . We have  $\#G_2 = \sum_{g \in G_1} \#G_{2,g} < \infty$ . For all  $g \in G_1$  and  $g' \in G_{2,g}$ , let

$$B_{\rho_{e_2}}^2(g', \xi_k) = B_{\rho_{e_2}}(g', \xi_k) \cap B_{\rho_{e_1}}^1(g, \xi_k). \quad (15.11)$$

By construction,  $\bigcup_{g' \in G_{2,g}} B_{\rho_{e_2}}^2(g', \xi_k)$  covers  $B_{\rho_{e_1}}^1(g, \xi_k)$ . Because  $\bigcup_{g \in G_1} B_{\rho_{e_1}}^1(g, \xi_k)$  covers  $\mathcal{G}$ ,  $\bigcup_{g' \in G_2} B_{\rho_{e_2}}^2(g', \xi_k)$  covers  $\mathcal{G}$ .

Repeat the previous steps to obtain in turn  $G_3$ ,  $\{B_{\rho_{e_3}}^3(g, \xi_k) : g \in G_3\}$ , ...,  $G_k$ ,  $\{B_{\rho_{e_k}}^k(g, \xi_k) : g \in G_k\}$ . One can induce that (i)  $\#G_k < \infty$ , (ii)  $\bigcup_{g' \in G_k} B_{\rho_{e_k}}^k(g', \xi_k)$  covers  $\mathcal{G}$ , and (iii)  $\forall g \in \mathcal{G}$ , there exists  $(g^{(k)}, g^{(k-1)}, \dots, g^{(1)}) \in G_k \times G_{k-1} \times \dots \times G_1$  such that

$$g \in B_{\rho_{e_k}}^k(g^{(k)}, \xi_k) \subset B_{\rho_{e_{k-1}}}^{k-1}(g^{(k-1)}, \xi_k) \subset \dots \subset B_{\rho_{e_1}}^1(g^{(1)}, \xi_k). \quad (15.12)$$

Thus,

$$\rho(g, g^{(k)}) = \left( \sum_{j=1}^k \rho_{e_j}^2(g, g^{(k)}) \right)^{1/2} \leq \left( \frac{\xi^2}{4k} + \frac{4\xi^2}{4k} + \dots + \frac{4\xi^2}{4k} \right)^{1/2} < \xi. \quad (15.13)$$

Equation (15.13) implies that  $\bigcup_{g \in G_k} B_{\rho}^k(g, \xi)$  covers  $\mathcal{G}$ ,  $G_k$  is the desired finite collection we set out to construct,  $N(\xi, \rho, \mathcal{G}) \leq \#G_k < \infty$ , and  $(\mathcal{G}, \rho)$  is totally bounded.

Second, we show that finite dimensional convergence holds. By Lemma E3, the finite dimensional random vector  $(a'\nu_{n, F_n}(\theta_n, g^{(1)}), \dots, a'\nu_{n, F_n}(\theta_n, g^{(L)}))'$  converges in distribution:

$$\begin{pmatrix} a'\nu_{n, F_n}(\theta_n, g^{(1)}) \\ \vdots \\ a'\nu_{n, F_n}(\theta_n, g^{(L)}) \end{pmatrix} \rightarrow_d N \left( 0, \begin{pmatrix} a'h_2(g^{(1)}, g^{(1)})a & \dots & a'h_2(g^{(1)}, g^{(L)})a \\ \vdots & \dots & \vdots \\ a'h_2(g^{(L)}, g^{(1)})a & \dots & a'h_2(g^{(L)}, g^{(L)})a \end{pmatrix} \right) \quad (15.14)$$

for all  $a \in R^k$ . Thus, by the Cramér-Wold device, for all  $g^{(1)}, g^{(2)}, \dots, g^{(L)} \in \mathcal{G}$ ,

$$\begin{pmatrix} \nu_{n, F_n}(\theta_n, g^{(1)}) \\ \vdots \\ \nu_{n, F_n}(\theta_n, g^{(L)}) \end{pmatrix} \rightarrow_d N \left( 0, \begin{pmatrix} h_2(g^{(1)}, g^{(1)}) & \dots & h_2(g^{(1)}, g^{(L)}) \\ \vdots & \dots & \vdots \\ h_2(g^{(L)}, g^{(1)}) & \dots & h_2(g^{(L)}, g^{(L)}) \end{pmatrix} \right). \quad (15.15)$$

Lastly, we show that  $\{\nu_{n, F_n}(\theta_n, \cdot) : n \geq 1\}$  is stochastically equicontinuous with respect to  $\rho$ . By Lemma E3,  $\{e'_j \nu_{n, F_n}(\theta_n, \cdot) : n \geq 1\}$  is stochastically equicontinuous with respect to  $\rho_{e_j}$  for all  $j \leq k$ . (Weak convergence implies stochastic equicontinuity.) Because  $\rho(g, g^*) \geq \rho_{e_j}(g, g^*)$  for all  $g, g^* \in \mathcal{G}$ ,  $\{e'_j \nu_{n, F_n}(\theta_n, \cdot) : n \geq 1\}$  is stochastically equicontinuous with respect to  $\rho$  for all  $j \leq k$ . Note that  $e'_j \nu_{n, F_n}(\theta_n, \cdot)$  is the  $j$ th coordinate of  $\nu_{n, F_n}(\theta_n, \cdot)$ . Therefore,  $\{\nu_{n, F_n}(\theta_n, \cdot) : n \geq 1\}$  is stochastically equicontinuous

with respect to  $\rho$ .  $\square$

### 15.3 Proof of Lemma A1(b)

It suffices to show that each element of  $D_F^{-1/2}(\theta)\widehat{\Sigma}_n(\theta, g, g^*)D_F^{-1/2}(\theta)$  converges in probability uniformly over  $g, g^* \in \mathcal{G}$ . Suppose  $1 \leq j, j' \leq k$ . The  $(j, j')$ th element of  $D_{F_n}^{-1/2}(\theta_n)\widehat{\Sigma}_n(\theta_n, g, g^*)D_{F_n}^{-1/2}(\theta_n)$  can be decomposed into two parts:

$$\begin{aligned} & n^{-1} \sum_{i=1}^n \sigma_{F_n, j}^{-1}(\theta_n) m_j(W_i, \theta_n) m_{j'}(W_i, \theta_n) \sigma_{F_n, j'}^{-1}(\theta_n) g_j(X_i) g_{j'}^*(X_i) \\ & - \sigma_{F_n, j}^{-1}(\theta_n) \overline{m}_{n, j}(\theta_n, g) \overline{m}_{n, j'}(\theta_n, g^*) \sigma_{F_n, j'}^{-1}(\theta_n) \\ \equiv & n^{-1} \sum_{i=1}^n f_{n, i, j, j'}^{mm}(\omega, g, g^*) - n^{-1} \sum_{i=1}^n f_{n, i, j}^m(\omega, g) \left( n^{-1} \sum_{i=1}^n f_{n, i, j'}^m(\omega, g^*) \right), \end{aligned} \quad (15.16)$$

where

$$\begin{aligned} f_{n, i, j}^m(\omega, g) &= \sigma_{F_n, j}^{-1}(\theta_n) m_j(W_i, \theta_n) g_j(X_i), \text{ and} \\ f_{n, i, j, j'}^{mm}(\omega, g, g^*) &= f_{n, i, j}^m(\omega, g) f_{n, i, j'}^m(\omega, g^*). \end{aligned} \quad (15.17)$$

Note that  $\{f_{n, i, j, j'}^{mm}(\omega, g, g^*) : g, g^* \in \mathcal{G}, i \leq n, n \geq 1\}$  and  $\{f_{n, i, j}^m(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  are triangular arrays of row-wise i.i.d. random processes. We show the uniform convergence of their sample means using Lemma E2.

We first study  $f_{n, i, j}^m(\omega, g)$ . Let

$$\mathcal{F}_{n, \omega, j}^m = \{(f_{n, 1, j}^m(\omega, g), \dots, f_{n, n, j}^m(\omega, g))' : g \in \mathcal{G}\}. \quad (15.18)$$

By Assumption M(c) and Lemma E1,  $\{f_{n, i, j}^m(\omega, g) : i \leq n, g \in \mathcal{G}\}$  are manageable with respect to the envelopes

$$\begin{aligned} F_{n, \cdot, j}^m(\omega) &= (F_{n, 1, j}^m(\omega), \dots, F_{n, n, j}^m(\omega))', \text{ where} \\ F_{n, i, j}^m(\omega) &= G(X_i) \sigma_{F_n, j}^{-1}(\theta_n) |m_j(W_i, \theta_n)|. \end{aligned} \quad (15.19)$$

In consequence, there exist functions  $\lambda_j : (0, 1] \rightarrow R_+$  for  $j \leq k$  such that

$$D(\xi | \alpha \odot F_{n, \cdot, j}^m, \alpha \odot \mathcal{F}_{n, \omega, j}^m) \leq \lambda_j(\xi) \quad (15.20)$$

for all  $\alpha \in R_+^n$ ,  $\omega \in \Omega$ , and  $n \geq 1$  and  $\sqrt{\log \lambda_j(\xi)}$  is integrable over  $(0, 1]$ .

Because the data are i.i.d., we have for all  $0 < \eta \leq 1$  and all  $n$ ,

$$\begin{aligned} n^{-1} \sum_{i=1}^n E(F_{n,i,j}^m)^{1+\eta} &= E(F_{n,1,j}^m)^{1+\eta} \\ &\leq (E_{F_n} G^{\delta_1}(X_i))^{(1+\eta)/\delta_1} \left( E_{F_n} \left| \frac{m_j(W_1, \theta_n)}{\sigma_{F_{n,h,j}}(\theta_n)} \right|^{\delta_2} \right)^{(1+\eta)/\delta_2} < \infty, \end{aligned} \quad (15.21)$$

where  $\delta_2 = (1+\eta)\delta_1/(\delta_1 - 1 - \eta)$ . The first inequality above holds by Hölder's inequality and the second holds by Assumption M(b),  $\delta_2 \leq 2+4/(\delta_1 - 1 - \eta) \leq 2+4/(4\delta^{-1} + 1 - \eta) \leq 2 + \delta$ , and condition (vi) of (2.3). Therefore, by Lemma E2,

$$\sup_{g \in \mathcal{G}} \left| n^{-1} \sum_{i=1}^n f_{n,i,j}^m(\omega, g) - E f_{n,1,j}^m(\cdot, g) \right| \rightarrow_p 0. \quad (15.22)$$

Now we study  $f_{n,i,j,j'}^{mm}(\omega, g, g^*)$ . For all  $n \geq 1$  and  $\omega \in \Omega$ , let

$$\mathcal{F}_{n,\omega,j,j'}^{mm} = \{(f_{n,1,j,j'}^{mm}(\omega, g, g^*), \dots, f_{n,n,j,j'}^{mm}(\omega, g, g^*))' : g, g^* \in \mathcal{G}\}. \quad (15.23)$$

Then,  $\mathcal{F}_{n,\omega,j,j'}^{mm} = \mathcal{F}_{n,\omega,j}^m \odot \mathcal{F}_{n,\omega,j'}^m$ . Let  $F_{n,\cdot,j,j'}^{mm}(\omega) = F_{n,\cdot,j}^m(\omega) \odot F_{n,\cdot,j'}^m(\omega)$ . We have: for all  $\alpha \in R_+^n$ ,  $\omega \in \Omega$ , and  $n \geq 1$ ,

$$\begin{aligned} &D(\xi |\alpha \odot F_{n,\cdot,j,j'}^{mm}(\omega)|, \alpha \odot \mathcal{F}_{n,\omega,j,j'}^{mm}) \\ &= D(\xi |\alpha \odot F_{n,\cdot,j,j'}^{mm}(\omega)|, \alpha \odot \mathcal{F}_{n,\omega,j}^m \odot \mathcal{F}_{n,\omega,j'}^m) \\ &\leq D\left(\frac{\xi}{4} |\alpha \odot F_{n,\cdot,j'}^m(\omega) \odot F_{n,\cdot,j}^m(\omega)|, \alpha \odot F_{n,\cdot,j'}^m(\omega) \odot \mathcal{F}_{n,\omega,j}^m\right) \\ &\quad \cdot D\left(\frac{\xi}{4} |\alpha \odot F_{n,\cdot,j}^m(\omega) \odot F_{n,\cdot,j'}^m(\omega)|, \alpha \odot F_{n,\cdot,j}^m(\omega) \odot \mathcal{F}_{n,\omega,j'}^m\right) \\ &\leq \lambda_j(\xi/4) \lambda_{j'}(\xi/4), \end{aligned} \quad (15.24)$$

where the first inequality holds by equation (5.2) in Pollard (1990) and the second



inequality holds by (15.20). We have

$$\begin{aligned} & \int_0^1 \sqrt{\log(\lambda_j(\xi/4)\lambda_{j'}(\xi/4))} d\xi = \int_0^1 \sqrt{\log \lambda_j(\xi/4) + \log \lambda_{j'}(\xi/4)} d\xi \\ & \leq 4 \int_0^{1/4} \left( \sqrt{\log \lambda_j(\xi)} + \sqrt{\log \lambda_{j'}(\xi)} \right) d\xi < \infty, \end{aligned} \quad (15.25)$$

where the first inequality holds by  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ . Therefore,  $\{f_{n,i,j,j'}^{mm}(\omega, g, g^*) : g, g^* \in \mathcal{G}, i \leq n, n \geq 1\}$  are manageable with respect to the envelopes  $\{F_{n,i,j,j'}^{mm}(\omega) : n \geq 1\}$ .

Let  $\eta$  be a small positive number. We have

$$\begin{aligned} & n^{-1} \sum_{i \leq n} E(F_{n,i,j,j'}^{mm}(\cdot))^{1+\eta} = E(F_{n,j,j'}^{mm}(\cdot))^{1+\eta} \\ & \leq [E_{F_n} G^{\delta_3}(X_1)]^{2(1+\eta)/\delta_3} \left[ E_{F_n} \left| \frac{m_j(W_1, \theta_n)}{\sigma_{F_n,j}(\theta_n)} \right|^{2+\delta} \right]^{(1+\eta)/(2+\delta)} \\ & \quad \cdot \left[ E_{F_n} \left| \frac{m_{j'}(W_1, \theta_n)}{\sigma_{F_n,j'}(\theta_n)} \right|^{2+\delta} \right]^{(1+\eta)/(2+\delta)} \\ & < \infty, \end{aligned} \quad (15.26)$$

where  $\delta_3 = 2(1+\eta)(2+\delta)/(\delta-2\eta)$ , the first inequality holds by Hölder's inequality, and the second holds for sufficiently small  $\eta > 0$  by Assumption M(b) and condition (vi) of (2.3).

With the manageability of  $\{f_{n,i,j,j'}^{mm}(\omega, g, g^*) : g, g^* \in \mathcal{G}, i \leq n, n \geq 1\}$  and (15.26), Lemma E2 gives

$$\sup_{g, g^* \in \mathcal{G}} \left| n^{-1} \sum_{i=1}^n f_{n,i,j,j'}^{mm}(\omega, g, g^*) - E f_{n,1,j,j'}^{mm}(\cdot, g, g^*) \right| \rightarrow_p 0. \quad (15.27)$$

By (15.16), (15.22), (15.27), as well as  $|E f_{n,1,j}^{mm}(\cdot, g)| \leq E(F_{n,1,j}^m)^{1+\eta} < \infty$ , we conclude that the difference between the  $(j, j')$ th element of  $D_{F_n}^{-1/2}(\theta_n) \widehat{\Sigma}_n(\theta_n, g, g^*) D_{F_n}^{-1/2}(\theta_n)$  and  $E f_{n,1,j,j'}^{mm}(\cdot, g, g^*) - E f_{n,1,j}^m(\cdot, g) E f_{n,1,j'}^m(\cdot, g^*)$  converges to zero uniformly over  $(g, g^*) \in \mathcal{G}^2$ .

By definition,

$$\begin{aligned}
& E f_{n,1,j,j'}^{mm}(\cdot, g, g^*) - E f_{n,1,j}^m(\cdot, g) E f_{n,1,j'}^m(\cdot, g^*) \\
&= E_{F_n} [\sigma_{F_n,j}^{-1}(\theta_n) \sigma_{F_n,j'}^{-1}(\theta_n) m_j(W_1, \theta_n) g_j(X_1) m_{j'}(W_1, \theta_n) g_{j'}^*(X_1)] \\
&\quad - E_{F_n} [\sigma_{F_n,j}^{-1}(\theta_n) m_j(W_1, \theta_n) g_j(X_1)] E_{F_n} [\sigma_{F_n,j'}^{-1}(\theta_n) m_{j'}(W_1, \theta_n) g_{j'}^*(X_1)] \\
&= \sigma_{F_n,j}^{-1}(\theta_n) \sigma_{F_n,j'}^{-1}(\theta_n) [\Sigma_{F_n}(\theta_n, g, g^*)]_{j,j'} \\
&\rightarrow [h_2(g, g^*)]_{j,j'}, \tag{15.28}
\end{aligned}$$

where the convergence holds uniformly over  $(g, g^*) \in \mathcal{G}^2$  by conditions (i) and (iv) in Definition *SubSeq*( $h_2$ ). This completes the proof of Lemma A1(b).  $\square$

## 15.4 Proof of Lemma E1

Part (a) is proved by a similar, but simpler, argument to that given in (15.24)-(15.25).

Next, we prove part (b). Because  $E F_{n,i} < \infty$  and the processes  $\{\phi_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  are row-wise i.i.d.,  $E \mathcal{F}_n \equiv \{E \phi_{n,i}(\cdot, g) \cdot 1_n : g \in \mathcal{G}\}$  is a subset of a one dimensional affine subspace of  $R^n$  with diameter no greater than  $2E F_{n,i}$ . Thus,  $\alpha \odot E \mathcal{F}_n$  is a subset of a one dimensional affine subspace of  $R^n$  with diameter no greater than  $2\|\alpha\| E F_{n,i}$ . By Lem. 4.1 in Pollard (1990), we have: for all  $n \geq 1$ ,

$$D(\xi \|\alpha \odot E F_n\|, \alpha \odot E \mathcal{F}_n) \leq 6\|\alpha\| E F_{n,i} / (\xi \|\alpha \odot E F_n\|) = 6/\xi. \tag{15.29}$$

Because  $\int_0^1 \sqrt{\log(6/\xi)} d\xi = 3\sqrt{\pi} < \infty$ , part (b) holds.

Finally, we prove part (c). Let  $\lambda_\phi^*(\xi) : (0, 1] \rightarrow R^+$  be the square-root-log integrable function such that

$$D(\xi \|\alpha \odot F_n^*(\omega)\|, \alpha \odot \mathcal{F}_{n,\omega}^*) \leq \lambda_\phi^*(\xi) \text{ for } 0 < \xi \leq 1, \tag{15.30}$$

for all  $\alpha \in R_+^n$ ,  $\omega \in \Omega$ , and  $n \geq 1$ . Let

$$\begin{aligned}
\mathcal{F}_{n,\omega}^* &= \{\phi_n^*(\omega, g) : g \in \mathcal{G}\}, \\
\mathcal{F}_{n,\omega}^{sum} &= \{\phi_n(\omega, g) + \phi_n^*(\omega, g) : g \in \mathcal{G}\}, \text{ and} \\
\mathcal{F}_{n,\omega}^+ &= \mathcal{F}_{n,\omega}^* \oplus \mathcal{F}_{n,\omega} \equiv \{a + b \in R^n : a \in \mathcal{F}_{n,\omega}^*, b \in \mathcal{F}_{n,\omega}\}, \tag{15.31}
\end{aligned}$$

where  $\phi_n(\omega, g) = (\phi_{n,1}(\omega, g), \dots, \phi_{n,n}(\omega, g))'$ . Let

$$F_n^{sum}(\omega) = F_n(\omega) + F_n^*(\omega). \quad (15.32)$$

Then, for  $0 < \xi \leq 1$  and  $\alpha \in R_+^n$ ,

$$\begin{aligned} & D(\xi \|\alpha \odot F_n^{sum}(\omega)\|, \alpha \odot \mathcal{F}_{n,\omega}^{sum}) \\ & \leq D(\xi \|\alpha \odot F_n^{sum}(\omega)\|, \alpha \odot \mathcal{F}_{n,\omega}^+) \\ & \leq D\left(\xi(\|\alpha \odot F_n(\omega)\| + \|\alpha \odot F_n^*(\omega)\|)/\sqrt{2}, \alpha \odot \mathcal{F}_{n,\omega}^+\right) \\ & \leq D(\xi \|\alpha \odot F_n(\omega)\|/(2\sqrt{2}), \alpha \odot \mathcal{F}_{n,\omega}) \\ & \quad \cdot D(\xi \|\alpha \odot F_n^*(\omega)\|/(2\sqrt{2}), \alpha \odot \mathcal{F}_{n,\omega}^*) \\ & \leq \lambda_\phi(\xi/(2\sqrt{2}))\lambda_\phi^*(\xi/(2\sqrt{2})), \end{aligned} \quad (15.33)$$

where  $\lambda_\phi(\xi)$  denotes the packing number bounding function given in Definition E2 for the processes  $\{\phi_n(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$ , the first inequality holds because  $\mathcal{F}_{n,\omega}^{sum} \subset \mathcal{F}_{n,\omega}^+$ , the second inequality holds because  $D(x, G)$  is decreasing in  $x$  and  $\|a + b\| \geq (\|a\| + \|b\|)/\sqrt{2}$  for  $a, b \in R_+^n$ , the third inequality holds by a stability result for packing numbers (see Pollard (1990, p. 22)), and the last inequality holds by the manageability of  $\{\phi_n(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$  and (15.30).

The function  $\lambda_\phi(\xi/(2\sqrt{2}))\lambda_\phi^*(\xi/(2\sqrt{2}))$  is square-root-log integrable by (15.25), which completes the proof of part (c).  $\square$

## 15.5 Proof of Lemma E2

We prove convergence in probability by showing convergence in  $L^1$ . We have

$$\begin{aligned} & E \sup_{g \in \mathcal{G}} \left| n^{-1} \sum_{i=1}^n [f_{n,i}(\cdot, g) - E f_{n,i}(\cdot, g)] \right| \leq n^{-1} K E \left( \sum_{i=1}^n F_{n,i}^2 \right)^{1/2} \\ & \leq n^{-1} K E \left( \sum_{i=1}^n F_{n,i}^{1+\eta} \right)^{1/(1+\eta)} \leq n^{-1} K \left( E \sum_{i=1}^n F_{n,i}^{1+\eta} \right)^{1/(1+\eta)} \\ & \leq n^{-\eta/(1+\eta)} K (B^*)^{1/(1+\eta)} \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned} \quad (15.34)$$

where the first inequality holds for some constant  $K < \infty$  by manageability and the maximal inequality (7.10) in Pollard (1990), the second inequality holds using  $0 < \eta < 1$

by applying the inequality  $\sum_{i=1}^n x_i^s \leq (\sum_{i=1}^n x_i)^s$ , which holds for  $s \geq 1$  and  $x_i \geq 0$  for  $i = 1, \dots, n$ , with  $x_i = F_{n,i}^{1+\eta}$  and  $s = 2/(1+\eta) > 0$ , the third inequality holds by the concavity of the function  $f(x) = x^{1/(1+\eta)}$  when  $\eta > 0$ , and the last inequality holds because  $n^{-1} \sum_{i=1}^n EF_{n,i}^{1+\eta} \leq B^*$  for all  $n \geq 1$ .  $\square$

## 15.6 Proof of Lemma E3

For notational simplicity, we prove Lemma E3 for the sequence  $\{n\}$ , rather than the subsequence  $\{a_n\}$ . All of the arguments in this subsection go through with  $\{a_n\}$  in place of  $\{n\}$ .

The conclusions of Lemma E3 are implied by the result of Thm. 10.6 of Pollard (1990), which relies on the following five conditions:

- (i) the  $\{f_{ni}(\omega, g) : g \in \mathcal{G}\}$  defined in (15.4) are manageable with respect to some envelope  $F_{a,n}(\omega) = (F_{a,n,1}(\omega), \dots, F_{a,n,n}(\omega))'$ ,
- (ii)  $\lim_{n \rightarrow \infty} Ea' \nu_{n, F_n}(\theta_n, g) \nu_{n, F_n}(\theta_n, g^*)' a = a' h_2(g, g^*) a$  for all  $g, g^* \in \mathcal{G}$ ,
- (iii)  $\limsup_{n \rightarrow \infty} \sum_{i=1}^n EF_{a,n,i}^2 < \infty$ ,
- (iv)  $\sum_{i=1}^n EF_{a,n,i}^2 \{F_{a,n,i} > \xi\} \rightarrow 0$  as  $n \rightarrow \infty$  for each  $\xi > 0$ , and
- (v) the limit  $\rho_a(\cdot, \cdot)$  is well defined by (15.7), and for all deterministic sequences  $\{g_{(n)}\}$  and  $\{g_{(n)}^*\}$ , if  $\rho_a(g_{(n)}, g_{(n)}^*) \rightarrow 0$ , then  $\rho_{n,a}(g_{(n)}, g_{(n)}^*) \rightarrow 0$  as  $n \rightarrow \infty$ .

Now we verify the five conditions.

- (i) By (15.4), we have

$$f_{n,i}(\omega, g) = \sum_{j=1}^k a_j \sigma_{F_{n,j}}^{-1}(\theta_n) n^{-1/2} [m_j(W_{n,i}(\omega), \theta_n) g_j(X_{n,i}(\omega)) - E_{F_n} m_j(W_i, \theta_n) g_j(X_i)], \quad (15.35)$$

where  $a_j$  denotes the  $j$ th element of  $a$ . By Assumption M(c),  $\{g_j(X_{n,i}(\omega)) : i \leq n\}$  are manageable with respect to envelopes  $G(X_{n,i}(\omega))$ . Therefore, by Lemma E1(a)-(c),  $\{f_{n,i}(\omega, g) : i \leq n\}$  is manageable with respect to envelopes  $F_{a,n} = (F_{a,n,1}, \dots, F_{a,n,n})'$  defined by

$$F_{a,n,i}(\omega) = n^{-1/2} \sum_{j=1}^k a_j \sigma_{F_{n,j}}^{-1}(\theta_n) [|m_j(W_{n,i}(\omega), \theta_n)| G(X_{ni}(\omega)) + E_{F_n} |m_j(W_i, \theta_n)| G(X_i)]. \quad (15.36)$$

(ii) By (15.5), we have

$$\begin{aligned}
& Ea' \nu_{n, F_n}(\theta_n, g) \nu'_{n, F_n}(\theta_n, g^*) a \\
&= E \left( \sum_{i=1}^n f_{n,i}(\cdot, g) \right) \left( \sum_{i=1}^n f_{n,i}(\cdot, g^*) \right)' = n E f_{n,1}(\cdot, g) f_{n,1}(\cdot, g^*)' \\
&= n^{-1} a' D_{F_n}^{-1/2}(\theta_n) \cdot Cov_{F_n}(m(W_1, \theta_n, g), m(W_1, \theta_n, g^*)) \cdot D_{F_n}^{-1/2}(\theta_n) a \\
&= n^{-1} a' D_{F_n}^{-1/2}(\theta_n) \Sigma_{F_n}(\theta_n, g, g^*) D_{F_n}^{-1/2}(\theta_n) a, \tag{15.37}
\end{aligned}$$

where the second equality holds because the data are i.i.d., the third inequality holds by (15.4). Condition (i) in Definition *SubSeq*( $h_2$ ) completes the verification of condition (ii) above.

(iii) Next, we verify  $\limsup_{n \rightarrow \infty} \sum_{i=1}^n E F_{a,n,i}^2 < \infty$ . By the linear structure of  $F_{a,n,i}$ , it suffices to show that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} E_{F_n} \sigma_{F_{n,j}}^{-2}(\theta_n) |m_j(W_i, \theta_n)|^2 G^2(X_i) < \infty \text{ and} \\
& \limsup_{n \rightarrow \infty} E_{F_n} \sigma_{F_{n,j}}^{-1}(\theta_n) |m_j(W_i, \theta_n)| G(X_i) < \infty. \tag{15.38}
\end{aligned}$$

The latter is implied by the former and the former holds by the same argument as in (15.21) with  $\eta = 1$ .

(iv) For  $B$  as in condition (vi) of (2.3),  $\xi > 0$ , and  $\eta > 0$  sufficiently small,

$$\begin{aligned}
& \sum_{i=1}^n E F_{a,n,i}^2 \{F_{a,n,i} > \xi\} = n E F_{a,n,i}^2 \{F_{a,n,i} > \xi\} \leq n E F_{a,n,i}^{2+\eta} / \xi^\eta \\
& \leq \frac{2(2k)^{2+\eta}}{n^{\eta/2} \xi^\eta} \sum_{j=1}^k |a_j|^{2+\eta} E_{F_n} G^{2+\eta}(X_i) \sigma_{F_{n,j}}^{-2-\eta}(\theta_n) |m_j(W_i, \theta_n)|^{2+\eta} \\
& \leq \frac{2(2k)^{2+\eta}}{n^{\eta/2} \xi^\eta} \sum_{j=1}^k |a_j|^{2+\eta} [E_{F_n} G^{\delta_4}(X_1)]^{(2+\eta)/\delta_4} B^{(2+\eta)/(2+\delta)} \\
& \leq \frac{2(2k)^{2+\eta} B^{(2+\eta)/(2+\delta)} C^{(2+\eta)/\delta_1}}{n^{\eta/2} \xi^\eta} \sum_{j=1}^k |a_j|^{2+\eta} \rightarrow 0, \tag{15.39}
\end{aligned}$$

where the first equality holds because the data are identically distributed, the second inequality holds by Jensen's inequality using the convexity of  $\psi(x) = x^{2+\eta}$ , i.e.,  $((2k)^{-1} \sum_{j=1}^k (|X_j| + E|X_j|))^{2+\eta} \leq (2k)^{-1} \sum_{j=1}^k (|X_j|^{2+\eta} + (E|X_j|)^{2+\eta})$  and  $(E|X_j|)^{2+\eta} \leq$

$E|X_j|^{2+\eta}$ , the third inequality holds with  $\delta_4 = (2 + \eta)(2 + \delta)/(\delta - \eta)$  by the same arguments as in (15.26), and the fourth inequality holds by Assumption M(b) and  $\delta_4 \leq \delta_1$  for sufficiently small  $\eta$ .

(v) First we show that the limit  $\rho_a(\cdot, \cdot)$  is well defined by (15.7). For any  $g, g^* \in \mathcal{G}$ ,

$$\begin{aligned} \rho_{n,a}^2(g, g^*) &= nE(f_{n,i}(\cdot, g) - f_{n,i}(\cdot, g^*))^2 \\ &= a'D_{F_n}^{-1/2}(\theta_n)Var_{F_n}(m(W_i, \theta_n, g) - m(W_i, \theta_n, g^*))D_{F_n}^{-1/2}(\theta_n)a \\ &\rightarrow a'h_2(g, g)a + a'h_2(g^*, g^*)a - a'h_2(g, g^*)a - a'h_2(g^*, g)a, \end{aligned} \quad (15.40)$$

where the convergence hold uniformly over  $\mathcal{G}^2$  by condition (i) in Definition *SubSeq*( $h_2$ ). Thus,  $\rho_a(g, g^*) = \lim_{n \rightarrow \infty} \rho_{n,a}(g, g^*)$  is well defined, and

$$\lim_{n \rightarrow \infty} \sup_{g, g^* \in \mathcal{G}} |\rho_{n,a}(g, g^*) - \rho_a(g, g^*)| = 0. \quad (15.41)$$

Lastly, we show the second property of condition (v). Let  $\xi > 0$  be arbitrary. Suppose  $\rho_a(g_{(n)}, g_{(n)}^*) \rightarrow 0$ . Then, there exists an  $N_0 < \infty$  such that for  $n \geq N_0$ ,

$$\rho_a(g_{(n)}, g_{(n)}^*) \leq \xi/2. \quad (15.42)$$

By (15.41), we have

$$\lim_{m \rightarrow \infty} \sup_{n \geq 1} |\rho_{m,a}(g_{(n)}, g_{(n)}^*) - \rho_a(g_{(n)}, g_{(n)}^*)| = 0. \quad (15.43)$$

Thus, there exists an  $N_1 < \infty$  such that for all  $m \geq N_1$ ,

$$\sup_{n \geq 1} |\rho_{m,a}(g_{(n)}, g_{(n)}^*) - \rho_a(g_{(n)}, g_{(n)}^*)| \leq \xi/2. \quad (15.44)$$

Take  $N = \max\{N_0, N_1\}$ , then we have for  $n \geq N$ ,

$$\rho_{n,a}(g_{(n)}, g_{(n)}^*) \leq \xi. \quad (15.45)$$

Thus,  $\rho_a(g_{(n)}, g_{(n)}^*) \rightarrow 0$  implies  $\rho_{n,a}(g_{(n)}, g_{(n)}^*) \rightarrow 0$ .  $\square$

## 16 Appendix F

This Appendix provides some additional material concerning the Monte Carlo simulations.

### 16.1 Quantile Selection Model

Table S-I provides coverage probability (CP) and false coverage probability (FCP) results for the upper endpoint of the identified interval in the quantile selection model.<sup>38</sup> (Table I of AS provides analogous results for the lower endpoint.) Table S-I provides a comparison of CS's based on the CvM/Sum, CvM/QLR, CvM/Max, KS/Sum, KS/QLR, and KS/Max statistics, coupled with the PA/Asy and GMS/Asy critical values. The relative attributes of the different CS's are quite similar to those reported in Table I of AS for the lower endpoint. None of the CS's under-cover. So, the relative attributes of the CS's are determined by their FCP's. The CvM-based CS's have lower FCP's than the KS-based CS's. The CS's that use the GMS/Asy critical values have lower FCP's than those based on the PA/Asy critical values. The FCP's do not depend on whether the Sum, QLR, or Max version of the statistic is employed. Hence, the best CS of those considered is the CvM/Max-GMS/Asy CS, or this CS with Max replaced by Sum or QLR.

Table S-II reports CP's and FCP's for the lower endpoint with the kinky bound DGP. (Table III of AS reports analogous results for the lower endpoint with the flat bound.) The results are similar to those in Table III of AS. There is relatively little sensitivity to the sample size, the number of cubes  $g$ , and the choice of  $\varepsilon$ . There is relatively little sensitivity of the CP's to the choice of  $(\kappa_n, B_n)$ , but some sensitivity of the FCP's with the basecase choice being superior to values of  $(\kappa_n, B_n)$  that are twice or half as large. The CS with  $\alpha = .5$  is half-median unbiased and avoids the well-known problem of inward-bias. But, it is farther from being median-unbiased than in the flat bound case.

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<sup>38</sup>For the upper endpoint with the flat bound and the upper endpoint with the kinky bound, the FCP's are computed at the points  $\underline{\theta}(1) + 0.40 \times \text{sqr}(250/n)$  and  $\underline{\theta}(1) + 0.75 \times \text{sqr}(250/n)$ , respectively. These points are chosen to yield similar values for the FCP's across the different cases considered.

Table S-I. Quantile Selection Model, Upper Endpoint: Basecase Test Statistic Comparisons

(a) Coverage Probabilities							
DGP	Statistic: Crit Val	CvM/Sum	CvM/QLR	CvM/Max	KS/Sum	KS/QLR	KS/Max
Flat Bound	PA/Asy	.994	.994	.993	.984	.984	.982
	GMS/Asy	.971	.971	.970	.974	.974	.972
Kinky Bound	PA/Asy	.996	.996	.996	.989	.989	.988
	GMS/Asy	.974	.974	.972	.976	.976	.975
(b) False Coverage Probabilities (coverage probability corrected)							
Flat Bound	PA/Asy	.73	.72	.71	.70	.70	.69
	GMS/Asy	.42	.42	.42	.55	.55	.55
Kinky Bound	PA/Asy	.73	.73	.72	.74	.74	.73
	GMS/Asy	.41	.41	.41	.52	.52	.52

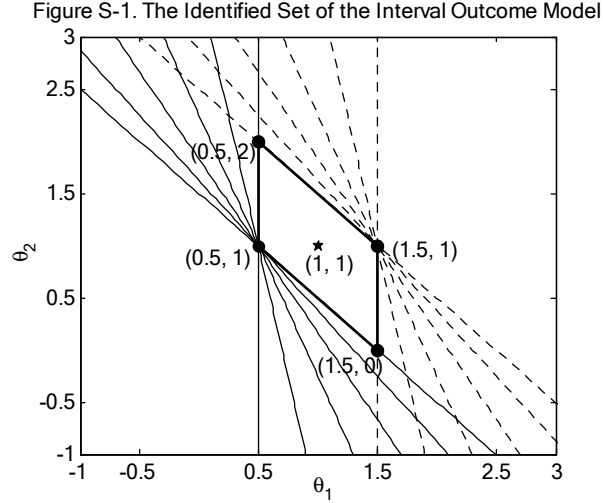


Table S-II. Quantile Selection Model, Kinky Bound, and Lower Endpoint: Variations on the Basecase

Case	Statistic: Crit Val:	(a) Coverage Probabilities		(b) False Cov Probs (CPcor)	
		CvM/Max	KS/Max	CvM/Max	KS/Max
		GMS/Asy	GMS/Asy	GMS/Asy	GMS/Asy
Basecase ( $n = 250, r_1 = 7, \varepsilon = 5/100$ )		.983	.984	.34	.52
$n = 100$		.981	.985	.34	.55
$n = 500$		.984	.984	.39	.54
$n = 1000$		.984	.980	.41	.54
$r_1 = 5$		.981	.981	.34	.49
$r_1 = 9$		.983	.986	.35	.55
$r_1 = 11$		.984	.987	.36	.60
$(\kappa_n, B_n) = 1/2(\kappa_{n,bc}, B_{n,bc})$		.984	.997	.39	.51
$(\kappa_n, B_n) = 2(\kappa_{n,bc}, B_{n,bc})$		.990	.991	.38	.59
$\varepsilon = 1/100$		.981	.981	.34	.56
$\alpha = .5$		.721	.710	.03	.06
$\alpha = .5 \ \& \ n = 500$		.741	.734	.04	.08

## 16.2 Interval Outcome Regression Model

Figure S-I shows the identified set for the interval outcome regression model. The diamond shaped area enclosed by thick solid lines is the identified set of  $(\theta_1, \theta_2)$ . The point  $(1, 1)$  is the true parameter. The thin solid lines are the lower bounds defined by the first moment inequality and the dashed lines are the upper bounds defined by the second moment inequality.



## 16.3 Entry Game Model

In the entry game model, the probit log likelihood function for  $\tau = (\tau_1, \tau_2)$  given  $\theta = (\theta_1, \theta_2)$  is

$$\begin{aligned}
 & \sum_{i=1}^n 1(Y_i = (0, 0)) \ln(\Phi(-X'_{i,1}\tau_1)\Phi(-X'_{i,2}\tau_2)) \\
 & + \sum_{i=1}^n 1(Y_i = (1, 1)) \ln(\Phi(X'_{i,1}\tau_1 - \theta_1)\Phi(X'_{i,2}\tau_2 - \theta_2)) \\
 & + \sum_{i=1}^n 1(Y_i = (1, 0) \text{ or } Y_i = (0, 1)) \ln(g_i(\tau, \theta)), \text{ where} \tag{16.1} \\
 & \quad g_i(\tau, \theta) = 1 - \Phi(-X'_{i,1}\tau_1)\Phi(-X'_{i,2}\tau_2) - \Phi(X'_{i,1}\tau_1 - \theta_1)\Phi(X'_{i,2}\tau_2 - \theta_2)
 \end{aligned}$$

over  $\tau \in R^8$  for fixed  $\theta$ . The estimator  $\hat{\tau}_n(\theta)$  maximizes this function over  $\tau \in R^8$  given  $\theta$ .

The gradient of the probit log likelihood for  $\tau$  given  $\theta$  is

$$\begin{aligned}
& - \sum_{i=1}^n 1(Y_i = (0, 0)) \begin{pmatrix} \psi(-X'_{i,1}\tau_1)X_{i,1} \\ \psi(-X'_{i,2}\tau_2)X_{i,2} \end{pmatrix} \\
& + \sum_{i=1}^n 1(Y_i = (1, 1)) \begin{pmatrix} \psi(X'_{i,1}\tau_1 - \theta_1)X_{i,1} \\ \psi(X'_{i,2}\tau_2 - \theta_2)X_{i,2} \end{pmatrix} \\
& + \sum_{i=1}^n 1(Y_i = (1, 0) \text{ or } Y_i = (0, 1)) \frac{1}{g_i(\tau, \theta)} \\
& \times \begin{pmatrix} \phi(-X'_{i,1}\tau_1)\Phi(-X'_{i,2}\tau_2)X_{i,1} - \phi(X'_{i,1}\tau_1 - \theta_1)\Phi(X'_{i,2}\tau_2 - \theta_2)X_{i,1} \\ \Phi(-X'_{i,1}\tau_1)\phi(-X'_{i,2}\tau_2)X_{i,2} - \Phi(X'_{i,1}\tau_1 - \theta_1)\phi(X'_{i,2}\tau_2 - \theta_2)X_{i,2} \end{pmatrix}, \tag{16.2}
\end{aligned}$$

where  $\psi(x) = \phi(x)/\Phi(x)$ .

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