# WHERE UTILITY FUNCTIONS DO NOT EXIST

## A NOTE ON LEXICOGRAPHIC ORDERS

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## Abstract

There seems to be some amount of confusion in the finance text books regarding the conditions under which an individual's preferences can be represented by a utility function . Fama and Miller, for example, assert that two axioms (comparability and transitivity) are sufficient to establish the existence of a utility function (when the set of alternatives is $R^n$). This is totally false: a real valued utility function need not exist even in the single good case ($R^1$). One might hope that a vector valued utility might exist (with lexicographic ordering of the utility vector); but this is not the case. Indeed we cannot salvage the situation even by allowing the utility to be a vector in $R^\infty$ (i.e. to have an (countably) infinite number of components); only an uncountable number of real components can do the job.

None of these results are new, but they do not seem to be sufficiently well known to researchers in finance. This may be because the original papers are mathematically forbidding or because they are scattered in sources somewhat removed form the mainstream finance literature. If that be so, this note should be of some help; some of our proofs and examples are new and hopefully more elementary (for example we avoid taking recourse to Sierpinski's lemma).

# WHERE UTILITY FUNCTIONS DO NOT EXIST

# A NOTE ON LEXICOGRAPHIC ORDERS

**Jayanth Rama Varma**

## 1. Introduction

There is a considerable body of literature which deals with the conditions under which an individual's preferences can be represented by a utility function which assigns numbers (utilities) to each available alternative in such a way that the individual prefers alternative x to alternative y if and only if x has the higher utility assigned to it. Necessary and sufficient conditions for this are well known (see Debreu, 1964, Fishburn, 1970 or Richter, 1966).

Nevertheless there seems to be some amount of confusion in the finance text books in this regard. Fama and Miller, for example, assert that two axioms (comparability and transitivity) are sufficient to establish the existence of a utility function (when the set of alternatives is $R^n$). This is totally false: a real valued utility function need not exist even in the single good case ($R^1$). One might hope that a vector valued utility might exist (with lexicographic ordering of the utility vector); but this is not the case. Indeed we cannot salvage the situation even by allowing the utility to be a vector in $R^\infty$ (i.e. to have an (countably) infinite number of components); only an uncountable number of real components can do the job.

None of these results are new, but they do not seem to be sufficiently well known to researchers in finance. This may be because the original papers are mathematically forbidding or because they are scattered in sources somewhat removed form the mainstream finance literature. If that be so, this note should be of some help; some of our proofs and examples are new and hopefully more elementary (for example we avoid taking recourse to Sierpinski's lemma).

## 2. Real Valued Utilities

The axioms of comparability and transitivity are:

1. <u>Comparability</u>: For every pair of alternatives x and y, either (1) x is preferred to y, or (2) y is preferred to x, or (3) x and y are indifferent to each other. The individual does not say that x and y are incomparable.

2. <u>Transitivity</u>: If x is preferred to y, and y is preferred to z, then, x is preferred to z. The same holds for indifference.

These two axioms state simply that the set S of alternatives is a pre-ordered set (the ordering being the preference ordering). S is only pre-ordered and not ordered because there may exist distinct elements x and y which are indifferent to each other (this is not allowed in an ordered set). However, we can consider the set S' of equivalence (or indifference) classes of S; this set S' is an ordered set. Without loss of generality, we shall, therefore, assume that S itself is an ordered set.

FM assume that the set S of alternatives is the set $R^n$ of all n- tuples $q = (q_1,...,q_n)$; the alternative q is interpreted as consisting of $q_j$ units of good j, where it is assumed that there are n goods 1,...,n. FM's alleged proposition is thus that any arbitrary order on $R^n$ can be represented by a real valued utility function. This is not true even for n = 1, but we shall first refute it for n = 2.

Consider, therefore, the lexicographic order in $R^2$: (a,b) > (p,q) if and only if either (1) a > p, or (2) a = p and b > q. Assume that a utility function exists in this case. For any real number p, consider the alternatives (p,0) and (p,1); these must be assigned utilities, say U(p) and V(p) respectively with U(p) < V(p). For any other real number q, we will have another pair of utilities U(q) < V(q), and the open intervals ( U(p), V(p) ) and ( U(q), V(q) ) are disjoint. In other words, for every real number, there exists an open interval disjoint from the open interval associated with any other number. But, this is impossible: (1) each open interval must contain a rational number (there is a rational number between any two reals), (2) there are only a countable number of rationals, and (3) there are an uncountable number of reals. Conclusion: the alleged utility function does not exist.

We can now show quite easily that FM's alleged result is not true even for the single good case (n = 1). Every real number x has a unique decimal expansion $x_n x_{n-1}...x_1 x_0.x_{-1}x_{-2}...$ (for terminating decimals, we choose the expansion with an infinite number of zeros). If we put together all the even numbered digits, we get a real number $E(x) = x_n x_{n-2}...x_2 x_0.x_{-2}x_{-4}...$ (if n is even). Putting together all the odd numbered digits, we get another real number $O(x) = x_{n-1}x_{n-3}...x_3 x_1.x_{-1}x_{-3}...$ . This is the classical one-to-one mapping from $R^1$ to $R^2$. We now define an order on $R^1$ by the lexicographic order on the image ( E(x), O(x) )in $R^2$ under this mapping. As we have shown above, this order is not representable by a utility function. This demonstrates that FM's proposition is false even in the case of only one good. We note that the preference order which we defined on $R^1$ is extremely discontinuous: in any interval, howsoever small, the commodity switches from being a good to being a bad and back again infinitely often.

### 3. Vector Valued Utilities

We thus find that real valued utility functions may not exist at all. One way of getting round this suggests itself: relax the requirement that the utility function be real valued and allow it to be a vector. For example if we allow the utility to consist of two numbers (with the usual lexicographic order), we can, quite trivially, represent the lexicographic order on $R^2$ by the identity map U(x) = x. But this does not get us very far. To represent the lexicographic preference on $R^3$, we would need three numbers and so on. This means that, for no finite number n, is a utility function with n components adequate, as we can always consider a lexicographic preference on $R^{n+1}$. Indeed, by the same method as in section 1, we can construct a preference order in $R^1$ that replicates the lexicographic order in $R^{n+1}$. This means that even in the single good case, no finite number of components is adequate. The only hope is to allow utilities with an infinite number of components.

## 4. $R^\infty$ Valued Utilities

To show that even this will not do requires mathematical machinery of a far higher order than we have used so far. We begin with several definitions.

Two ordered sets, A and B are said to be <u>similar</u> if there exists a one-to-one order preserving correspondence between them. To ask whether a utility function exists is equivalent to asking whether the ordered set S is similar to a subset of the reals(R). Two similar sets are said to have the same <u>order type</u>. So the existence of a utility function is equivalent to the ordered set being of the same order type as a subset of the reals.

Let A and B be any ordered sets. We let $A^{B^*}$ denote the power set $A^B$ (the set of all functions from B to A) endowed with the following lexicographic <u>partial</u> order:

Given f and g $\varepsilon$ $A^{B^*}$ define

$$D(f,g) = \{q \,\varepsilon\, B \mid f(q) \neq g(q)\}$$

If D(f,g) has no first element then f and g are incomparable; else let d be the first element of D(f,g) and define

$$f \quad \begin{matrix} > \\ = \\ < \end{matrix} \quad g \text{ according as } f(d) \quad \begin{matrix} > \\ = \\ < \end{matrix} \quad g(d)$$

If the set B is well ordered (i.e. every subset of B has a first element), then D(f,g) always has a first element; $A^{B^*}$ is then <u>totally</u> ordered.

$R^\infty$ is the usual notation for $R^{\omega^*}$ where $\omega$ is the smallest infinite ordinal (the order type of the positive integers); we have also so far used the notation $R^\infty$, but shall henceforth use the more precise notation $R^{\omega^*}$. $R^{\omega^*}$ is lexicographically totally ordered.

To show that an $R^{\omega^*}$ valued utility is not sufficient to represent a preference order on $R^n$ it is sufficient to consider in place of $R^n$, any set of the same cardinality. This was the general principle which allowed us to show that the single good case $R^1$ is no better than $R^2$ or $R^n$ since they all have the same cardinality: the cardinality of the continuum.

We are now ready to state and prove the non existence theorem.

Theorem: There exists an ordered set of the cardinality of the continuum which is not similar to a subset of $R^{\infty}$ (i.e. $R^{\omega^*}$).

Proof: Let Q be the rationals; since Q is countable, $R^Q$ like $R^{\infty}$, has the same cardinality as the continuum. As above, $R^Q$ is lexicographically partially ordered.

Consider the element $\mathbf{0} \in R^Q$ defined by $\mathbf{0}(q) \equiv 0$. Define the subset S of $R^Q$ defined as follows:

$$S = \{ f \in R^Q \mid D(f,\mathbf{0}) \text{ is well ordered}\}$$

We observe that if $f,g \in S$ then $D(f,\mathbf{0}) \cup D(g,\mathbf{0})$ is the union of two well ordered sets and therefore well ordered; $D(f,g)$ being a subset of this well ordered set is itself well ordered. This means that S is totally ordered.

Now consider a fixed infinite sequence of rationals:

$$0 = q_0 < q_1 < ... < q_\omega = 1$$

and define

$$T = \{f \in R^Q \mid f(q) = 0 , q \neq q_j, j \in \omega+1\}$$

where $\omega+1 = \{0,1,...,\omega\} = \{0,1,...\} \cup \{\omega\}$ is the infinite ordinal immediately following $\omega$.

T is contained in S because if $f \in T$, then $D(f,\mathbf{0})$ is a subset of $\{q_j, j\in\omega+1\}$ which is well ordered. Thus T is a totally ordered set of the cardinality of the continuum.

It is obvious that T is similar to $R^{(\omega+1)^*}$; we wish to show that it is not similar to a subset of $R^{\omega^*}$. Since on the one hand $J = [0,1]$ is contained in (and therefore similar to a subset of) R, and on the other hand R is similar to the subset (0,1) of J, it is clearly sufficient to show that $J^{(\omega+1)^*}$ is not similar to a subset of $J^{\omega^*}$. (If $R^{(\omega+1)^*}$ were similar to a subset of $R^{\omega^*}$, then the subset $J^{(\omega+1)^*}$ of $R^{(\omega+1)^*}$ would be similar to a subset of $R^{\omega^*}$ which in turn is similar to a subset of $J^{\omega^*}$).

Our proof closely follows Hausdorff's proof that $J^m$ is not similar to a subset of $J^n$ where $n < m$.

Assume to the contrary that there is a similarity map. Let the images of $(x_1,0,0...,0)$ and $(x_1,1,1...,1)$ be $(y_1,y_2,...)$ and $(z_1,z_2,...)$ respectively. Clearly $y_1 \leq z_1$; but it is not possible for $y_1$ and $z_1$ to be distinct for all $x_1$. For, if this were so, the open intervals $(y_1,z_1)$ would be disjoint, and the set of these intervals would, like the set of the $x_1$, have the cardinality of the continuum; whereas, since each such interval must contain a rational, there are only a countable number of them. Hence for some value of $x_1$ say $a_1$ we have $y_1 = z_1 =$ say $b_1$. This means that if the first coordinate of x is $a_1$ then the first coordinate of its image is $b_1$ regardless of the remaining coordinates.

We now assume that a sequence of numbers $a_1, a_2, ..., a_k$ has been found such that if the first k coordinates of x are $a_1, a_2, ..., a_k$ respectively, then the first k coordinates of its image are $b_1, b_2, ..., b_k$ regardless of what the remaining coordinates are. We now consider the subset $J^{(\omega+1)^*}(a_1,a_2,...,a_k)$ of $J^{(\omega+1)^*}$ whose first k coordinates are $a_1, a_2, ..., a_k$ and the subset $J^{\omega^*}(b_1,b_2,...,b_k)$ of $J^{\omega^*}$ whose first k coordinates are $b_1, b_2, ..., b_k$. Repeating our earlier argument, we can now find $a_{k+1}$ and $b_{k+1}$ such that if the (k+1)'th coordinate of $x \in J^{(\omega+1)^*}(a_1,a_2,...,a_k)$ is $a_{k+1}$, then the (k+1)'th coordinate of its image $y \in J^{\omega^*}(b_1,b_2,...,b_k)$ is $b_{k+1}$ regardless of what the remaining coordinates are. By induction therefore we find the sequences $a_j$ and $b_j$, $j \in \omega$.

Now consider the points $(a_1, a_2, ..., 0)$ and $(a_1, a_2, ..., 1)$. By induction, we see that both of these points must be mapped into $(b_1, b_2, ...)$. This proves that $J^{(\omega+1)^*}$ cannot be mapped into a subset of $J^{\omega^*}$ in an order preserving way. ■

If it were only desired to prove the bare substance of the theorem, it would have been sufficient to directly introduce the $R^{(\omega+1)^*}$ instead of introducing $R^Q$ and exhibiting $R^{(\omega+1)^*}$ as a subset thereof. That would, however, have left open the possibility that perhaps $R^{(\omega+1)^*}$ or some larger space is sufficient while $R^{\omega^*}$ is not. Our method of proof can be used to show that any attempt to represent all orderings of the continuum by a lexicogrpahically ordered space is doomed to failure if we use only a countable number of real coordinates.

## 5. Higher Countable Ordinals

We wish to show that it is not possible to represent all possible orderings of the continuum with utility functions having only a countable number of real components. This is a consequence of the following considerations:

1. Our proof that $R^{(\omega+1)^*}$ is not similar to a subset of $R^{\omega^*}$ can be extended to show that $R^{(\sigma+1)^*}$ is not similar to a subset of $R^{\sigma^*}$ for any infinite ordinal $\sigma$. All that we have to do is to replace ordinary induction by transfinite induction. The case of finite $\sigma$ is covered by Hausdorff's theorem cited above.

2. Given any countable ordinal $\sigma$, there exists a subset of the rationals whose order type (under the natural order) is the same as $\sigma$. In fact, the rationals contain not merely all the countable ordinals, but all the countable order types (Hausdorff p 60).

3. Consequently, the set S constructed in the proof of Theorem I, contains a subset similar to $R^{\sigma^*}$ for any countable ordinal $\sigma$.

4. If $\sigma$ is any countable ordinal, $\sigma+1$ is also a countable ordinal (adding one element to an infinite set cannot change its cardinality).

We are led to the following theorem:

Theorem: There exists a set of the cardinality of the continuum which is not similar to a subset of $R^{\sigma^*}$ for any countable ordinal σ.

Proof: The set S constructed in Section 4 contains a subset similar to $R^{(\sigma+1)^*}$ which is not similar to a subset of $R^{\sigma^*}$.■

Since $R^{S^*}$ is totally ordered only if S is well ordered, and any well ordered set is similar to an ordinal, this theorem justifies the statement made earlier that to represent all possible orderings of the continuum by a lexicographic order, a countable number of real coordinates is not sufficient. We now proceed to show that utilities with an uncountable number of real components can do the job.

## 6. Utilities with Uncountable Number of Real Components

Theorem: Let A be any ordered set and σ, its cardinality. Then, A is similar to a subset of the lexicographically ordered set $\{0,1\}^{\sigma^*}$ where σ is identified with any well ordered set of cardinality σ.

Proof: Let V be a one-to-one (not necessarily order preserving) map from σ to A (which exists because the two sets have the same cardinality). Define the similarity map U from A to $\{0,1\}^{\sigma^*}$ as follows:

U(a) = f where f(x) = 1 if V(x) < a

= 0 otherwise.

It is clear that if f=U(a), g=U(b), y = $V^{-1}$(a) and a < b, then

f(x) ≤ g(x) for all x; and

0 = f(y) < g(y) = 1

Consequently, U(a) < U(b) and U is order preserving.■

Corollary 1: If A is countable, it is similar to a subset of the reals.

Proof: The unit interval is identical to $\{0,1\}^{\omega^*}$ as every real number in (0,1) has a binary expansion of 0s and 1s, and the natural order on (0,1) is the same as the lexicographic order on these sequences of 0s and 1s.

Remark: As we have stated earlier, if A is countable, it is similar to a subset of the rationals; our corollary, though useful and often used, is not the strongest possible result.

Corollary 2: Any preference ordering on R or $R^n$ or even $R^\infty$ can be represented by a utility function taking values in $\{0,1\}^{\mu^*}$ (or a fortiori in $R^{\mu^*}$) where μ is the smallest ordinal with the cardinality of the continuum.

Proof: Immediate. ■

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