E C O N O M I C S   B U L L E T I N

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ON THE ACCURACY OF THE ESTIMATED POLICY FUNCTION USING THE BELLMAN CONTRACTION METHOD

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Abstract. In this paper we show that the approximation error of the optimal policy function in the stochastic dynamic programing problem using the policies defined by the Bellman contraction method is lower than a constant (which depends on the modulus of strong concavity of the one-period return function) times the square root of the value function approximation error. Since the Bellman's method is a contraction it results that we can control the approximation error of the policy function. This method for estimating the approximation error is robust under small numerical errors in the computation of value and policy functions.

Keywords: Stochastic dynamic programing problem, estimation of the policy function

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1. Introduction

One of the classical methods for estimating the optimal policy function in a stochastic dynamic programming problem is the Bellman contraction method (Christiano (1990), Tauchen (1990), Santos and Vigo (1998)). Since this is based on the iterations of a contraction mapping, such a method gives us a fast algorithm for estimating the value function of the problem with high precision, so we can find an estimated value function as close as we want to the exact value function. Unfortunately this method cannot say anything about the precision in the estimated policy function. The asymptotic convergence of the sequence

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of policies defined in each iteration is the general argument for using a high iteration as the approximated optimal policy function.

In Santos and Vigo (1998) it is shown that the optimal policy function of the discretization of the dynamic programming problem is close to the optimal policy function of the original problem. The distance between them depends on the mesh of the discretization. Then the problem is reduced to find accurate estimations of the optimal policy function for the discretization of the problem; again we do not have an estimative of the approximation error in computing the optimal policy function of the discretization of the problem.

There exist other approaches for obtaining approximations of the optimal policy function. The Euler equation grid method (Baxter *et al.* (1990), Coleman (1990,1991)), the parameterized expectations method (Marcet and Marshall (1991)) and projection methods (Judd (1992)). Eventhough all this methods give sequences of approximated policy functions that converge pointwise to the optimal policy function and it is not provided an estimation of the approximation error.

In this paper we show that the strong concavity of the return function is a sufficient condition for obtaining a control of the error in the approximation of the optimal policy function by the policy function of the n-th Bellman operator iteration. Namely, the distance between the optimal policy function and the solution of the $n-th$ iteration of the Bellman equation is lower than a constant (which depends on the modulus of the strong concavity of the return function) times the square root of the distance between the value function and the $n-th$ iteration of the Bellman's operator from any initial function. In particular, if the return function is bounded then the approximation error of the policy function is bounded by a constant which depends on the primitives of the problem and the $n-th$ power of the discount fator.

We also prove that the Bellman equation provides a robust numerical method for obtaining accurate estimatives of the optimal policy function.

This paper is divided in five sections. Section 2 describes the framework and the hypotheses. Section 3 gives the main theorem relating the error bounds of the estimated policy function and the estimated value function when it is using the Bellman contraction method. Section 4 gives an extension of the result in section 3 for alternative hypotheses and shows the robustness of the error control in the policy function when it is used the Bellman method. Conclusions are given in section 5 and the proofs are given in the appendix.

2. The framework

The stochastic dynamic programming problem is defined from the following elements: the set of values for the endogenous state variables $X \subset R^l$ (which is a convex Borel set), the set of values for the exogenous shocks $Z \subset R^k$ (which is a compact set); both are mesurable spaces with their σ-algebras $\mathcal{X}$ and $\mathcal{Z}$ respectively. The evolution of the stochastic shocks is given by the transition function Q defined on $(Z, \mathcal{Z})$ with the Feller property. A (measurable) set $\Omega \subset X \times X \times Z$ describing the feasibility of decisions, *i.e.* if $(x, z) \in X \times Z$ are the current values of the state variable and the shock then $y \in X$ is feasible for the next period if and only if $(x, y, z) \in \Omega$. From this we can define the correspondence $\Gamma : X \times Z \to R^l$ by $\Gamma(x, z) = \{y \in X;\ (x, y, z) \in \Omega\}$. The one-period return function $F : \Omega \to R$ is such that $F(x, y, z)$ is the current return if y is chosen for the next period from (x, z). The discount factor is $\beta \in (0, 1)$. With all these elements, the stochastic dynamic programming problem is to find a sequence of contingent plans $(\hat{x}_t)_{t\geq 1}$ (where for all $t \geq 1$, $\hat{x}_t : Z^t \to X$ is a measurable function) such that it solves the following maximization:

$$v(x_0, z_0) = \text{Max} \sum_{t=0}^{\infty} \int_{Z^t} \beta^t F(x_t, x_{t+1}, z_t)\, Q^t(z_0, dz^t)$$
$$\text{subject to } (x_t, x_{t+1}, z_t) \in \Omega \text{ for all } t \geq 0$$
$$(x_0, z_0) \in X \times Z \text{ given}$$

The following hypotheses will be used in this work.

Hypothesis 1. The correspondence Γ is nonempty, compact-valued, continuous and for all $x, x' \in X$, $z \in Z$ and $\alpha \in [0, 1]$ it satisfies:

$$\alpha\Gamma(x, z) + (1 - \alpha)\Gamma(x', z) \subset \Gamma(\alpha x + (1 - \alpha)x', z).$$

Hypothesis 2. The function F is bounded, continuous and there exists $\eta_1 > 0$ such that $F(x, y, z) + (\eta_1/2)|x|^2$ is a concave function in (x, y).

3. The main result

In this section we will set the main theorem that relates the approximation error in the optimal policy function with the approximation error in the value function. Let T be the Bellman operator on $C(X \times Z)$ (the set of continuous and bounded functions defined in $X \times Z$ with the topology induced by the supremum norm) defined by:

$$TV(x, z) = \underset{\{y\in X;\, (x,y,z)\in\Omega\}}{\text{Max}} F(x, y, z) + \beta \int_Z V(y, z')\, Q(z, dz').$$

It is well known (see Stokey and Lucas (1989)) that under hypotheses 1 and 2 this operator is a contraction with modulus β and fixed point v (the value function). The numerical method based on this contraction is defined as follows. Let $v_0 \in C(X \times Z)$ be a concave function in x and define the sequence $(v_n)_{n\geq 0}$ by:

$$v_{n+1}(x, z) = Tv_n(x, z),\ \forall\, (x, z) \in X \times Z, \forall\, n \geq 0 \tag{1}$$

Lemma 3.1. *With hypotheses 1 and 2, v_n defined by (1), is strongly concave*[1] *in x with modulus η_1 for all $n \geq 1$. In particular the value function v is also strongly concave in x with the same modulus.*

Since for each $n \geq 1$, v_n is strongly concave, we can define:

$$g_n(x, z) = \underset{\{y\in X;\, (x,y,z)\in\Omega\}}{\text{Argmax}} F(x, y, z) + \beta \int_Z v_n(y, z')\, Q(z, dz'), \tag{2}$$

and the policy function is defined by:

$$g(x, z) = \underset{\{y\in X;\, (x,y,z)\in\Omega\}}{\text{Argmax}} F(x, y, z) + \beta \int_Z v(y, z')\, Q(z, dz').$$

[1] A function $f : C \subset R^n \to R$ is strongly concave with modulus $\eta \geq 0$ if $f(x) + (\eta/2)|x|^2$ is a concave function.

The main theorem. *With hypotheses 1 and 2 the sequence* $(v_n, g_n)_{n\geq 1}$ *defined by (1) and (2) satisfies:*

$$\|g - g_n\| \leq (\frac{2}{\eta_1})^{1/2} \|v - v_n\|^{1/2}$$

where $\|.\|$ *is the supremum norm in* $C(X \times Z)$. *In particular, if* $v_0 \equiv 0$ *then:*

$$\|g - g_n\| \leq (\frac{2}{\eta_1})^{1/2} \frac{\|F\|}{1-\beta}^{1/2} \beta^{n/2}$$

4. Extensions and robustness of the approximation method

In this section we will give an alternative to hypothesis 2 which still guarantees the same type of estimation for the optimal policy function. We also consider the case in which the approximated value function is generated by an arbitrary numerical process.

Hypothesis 3 The function F is bounded, continuous and there exists $\eta_2 > 0$ such that $F(x, y, z) + (\eta_2/2)|y|^2$ is a concave function in (x, y).

Theorem 4.1. *With hypotheses 1 and 3, the sequence* $(v_n, g_n)_{n\geq 0}$ *defined by (1) and (2) satisfies:*

$$\|g - g_n\| \leq (\frac{2\beta}{\eta_2})^{1/2} \|v - v_n\|^{1/2}.$$

The estimation of the error in the approximation of the optimal policy function described in the method above is robust in the following sense: Suppose that a numerical method is used for computing the value function (discretization of the state space, linear or spline approximation of the one-period return function, etc.) and an approximated value function $\tilde{v} \in C(X \times Z)$ is found. Then the approximation error in the policy function obtained from $\tilde{v}$ can be estimated using the approximation error in the estimated value function.

The policy obtained from $\tilde{v}$ is:

$$\tilde{g}(x, z) = \underset{\{y\in X;\, (x,y,z)\in\Omega\}}{\text{Argmax}} \; F(x, y, z) + \beta \int_Z \tilde{v}(y, z') \, Q(z, dz'). \tag{3}$$

Observe that in general $\tilde{g}$ can be a correspondence. Even though, we have the following estimation.

Theorem 4.2. *Take* $\tilde{v} \in C(X \times Z)$ *and let* $\tilde{g}$ *be defined by (3). If hypotheses 1 and 2 hold then for all* $(x, z) \in X \times Z$,

$$d_H(g(x, z), \tilde{g}(x, z)) \leq (\frac{4}{\eta_1})^{1/2} \|v - \tilde{v}\|^{1/2}$$

where d_H *is the Haussdorf distance:* $d_H(y, A) = Sup\{|y - a|; a \in A\}$.

Alternatively, if hypotheses 1 and 3 hold then for all $(x, z) \in X \times Z$,

$$d_H(g(x, z), \tilde{g}(x, z)) \leq (\frac{4\beta}{\eta_2})^{1/2} \|v - \tilde{v}\|^{1/2}.$$

Remark: Let us observe that the estimation above holds even though the exact value of $\tilde{g}(x,z)$ is not known. Suppose that our numerical method for maximizing calculates an approximated maximizer within a tolerance $\tau > 0$. So define $\tilde{g}_\tau(x,z)$ as those $\tilde{y} \in \Gamma(x,z)$ such that

$$F(x,\tilde{y},z) + \beta \int_Z \tilde{v}(\tilde{y},z')\, Q(z,dz') \geq F(x,y,z) + \beta \int_Z \tilde{v}(y,z')\, Q(z,dz') - \tau, \ \ \forall y \in \Gamma(x,z).$$

In this case, the estimation of the theorem 4.2 (with hypotheses 1 and 2) will take the form:

$$d_H(g(x,z), \tilde{g}_\tau(x,z)) \leq (\frac{4}{\eta_1}\|v - \tilde{v}\| + \tau)^{1/2}.$$

5. Conclusions

In this paper we gave a theoretical justification for using the Bellman contraction method in the stochastic dynamic programming problem in order to find accurate estimates for the optimal policy function. In applied models this method is largely used in different ways (making a grid or discretization of the state space, or approximating the one-period return function by linear-quadratic or log-linear quadratic functions). All these techniques are used with success and this paper gives an explanation for it.

We can also use the estimations given in the main theorem and theorem 4.1 for numerical purposes. Suppose that we are in the case where hypotheses 1 and 2 are satisfied and we would like to compute an estimated policy function with error $\epsilon > 0$. Let $v_0 \in C(X \times Z)$ be an initial function and define the sequence $(v_n)_{n\geq 1}$ by $v_{n+1} = Tv_n$. Then for $N > 0$ such that $\|v_{N+1} - v_N\| \leq \frac{(1-\beta)\eta_1}{2}\epsilon^2$ the policy function g_N associated with v_N will have the desired property.

Finally, it is important to observe that the Bellman contraction method (and therefore, our estimation) can only be use when the economic model may be written as a dynamic programming problem; in other cases the Euler equation approaches must be used.

APPENDIX

To prove lemma 3.1, let us prove the following

Lemma A. *$F(x,y,z) + (\eta/2)|x|^2$ is a concave function in (x,y) if and only if for all $(x_i, y_i, z) \in \Omega$, $i = 1,2$ and for all $\alpha \in [0,1]$ it holds:*

$$F(x^\alpha, y^\alpha, z) \geq \alpha F(x_1,y_1,z) + (1-\alpha)F(x_2,y_2,z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2,$$

where $x^\alpha = \alpha x_1 + (1-\alpha)x_2$ and $y^\alpha = \alpha y_1 + (1-\alpha)y_2$.

Proof. ($\Rightarrow$) By hypothesis:

$$F(x^\alpha, y^\alpha, z) + \frac{\eta}{2}|x^\alpha|^2 \geq \alpha[F(x_1,y_1,z) + \frac{\eta}{2}|x_1|^2] + (1-\alpha)[F(x_2,y_2,z) + \frac{\eta}{2}|x_2|^2],$$

expanding the square of the left side and simplifying it results:

$$F(x^\alpha, y^\alpha, z) \geq \alpha F(x_1,y_1,z) + (1-\alpha)F(x_2,y_2,z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2.$$

($\Leftarrow$) Completely analogous.

Proof of lemma 3.1. It will be sufficient to prove that if $V(.,z)$ is a concave function then $TV(.,z)$ is a strongly concave function with modulus η_1. Let $x_1, x_2 \in X$, $\alpha \in [0,1]$, $x^\alpha = \alpha x_1 + (1-\alpha)x_2$ and for $i = 1,2$ let $y_i \in \Gamma(x_i, z)$ such that:

$$TV(x_i, z) = F(x_i, y_i, z) + \beta \int_Z V(y_i, z') \, Q(z, dz').$$

Then, using hypotheses 1, 2 and lemma A we have that:

$$\begin{aligned} TV(x^\alpha) \geq & F(x^\alpha, \alpha y_1 + (1-\alpha)y_2, z) + \beta \int_Z V(\alpha y_1 + (1-\alpha)y_2, z') \, Q(z, dz') \\ \geq & \alpha F(x_1, y_1, z) + (1-\alpha)F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2 + \\ & \beta \int_Z [\alpha V(y_1, z') + (1-\alpha)V(y_2, z')] \, Q(z, dz') \\ = & \alpha TV(x_1, z) + (1-\alpha)TV(x_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2. \end{aligned}$$

Since the set of strongly concave functions is a closed set with the topology induced by the sup norm then the fixed point of T is strongly concave.

lemma B. *$f : C \subset R^n \to R$ (C is a convex set) is strongly concave with modulus η if and only if for all $x_1, x_2 \in C$ it holds:*

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2) + (\eta/2)\alpha(1-\alpha)|x_1 - x_2|^2.$$

Proof.
It is analogous to the proof of lemma A.

Lemma C. *Let $f : C \subset R^n \to R$ (C is a convex set) be a strongly concave function with modulus η. If $x^* = Argmax_{x \in C} f(x)$ then*

$$f(x) \leq f(x^*) - \frac{\eta}{2}|x - x^*|^2, \quad \forall x \in C.$$

Proof. Let $x \in C$ and $\alpha \in (0,1)$. By definition of x^* and the characterization of strong concavity given in lemma B we have:

$$f(x^*) \geq f(\alpha x^* + (1-\alpha)x) \geq \alpha f(x^*) + (1-\alpha)f(x) + \frac{\eta}{2}\alpha(1-\alpha)|x - x^*|^2$$

$$\Rightarrow f(x^*) \geq f(x) + \frac{\eta}{2}\alpha|x - x^*|^2,$$

making $\alpha \to 1$ we have the result.

Proof of the main theorem.

Let us fix some notations. Let $v_n \in C(X \times Z)$ be the n-th iteration ($n \geq 1$) of the Bellman's operator from some initial concave function $v_0 \in C(X \times Z)$. Let

$$g_n(x,z) = \text{Argmax}_{\{y \in X;\, (x,y,z) \in \Omega\}} F(x,y,z) + \beta \int_Z v_n(y,z')\, Q(z,dz'),$$

$$\phi_n(x,y,z) = F(x,y,z) + \beta \int_Z v_n(y,z',)\, Q(z,dz'),$$

$$\phi(x,y,z) = F(x,y,z) + \beta \int_Z v(y,z')\, Q(z,dz').$$

From lemma 3.1 the functions $\phi(x,.,z)$ and $\phi_n(x,.,z)$ are strongly concave with modulus $\beta\eta_1$. Then from lemma C we have that:

$$\phi(x,g(x,z),z) \geq \phi(x,g_n(x,z),z) + \frac{\beta\eta_1}{2}|g(x,z) - g_n(x,z)|^2,$$

$$\phi_n(x,g_n(x,z),z) \geq \phi_n(x,g(x,z),z) + \frac{\beta\eta_1}{2}|g(x,z) - g_n(x,z)|^2.$$

Summing up the above inequalities it results:

$$\beta\{\int_Z [(v_n - v)(g_n(x,z),z') + (v - v_n)(g(x,z),z')]\, Q(z,dz')\} \geq \beta\eta_1|g(x,z) - g_n(x,z)|^2$$

$$\Rightarrow 2\|v - v_n\| \geq \eta_1|g(x,z) - g_n(x,z)|^2$$

this inequality holds for all $(x,z) \in X \times Z$, so we conclude:

$$\|g - g_n\| \leq [\frac{2}{\eta_1}\|v - v_n\|]^{1/2}.$$

Proof of theorem 4.1.

Under hypothesis 3, $\phi_n(x,.,z)$ and $\phi(x,.,z)$ are strongly concave with modulus η_2. Then from lemma C we have that:

$$\phi(x,g(x,z),z) \geq \phi(x,g_n(x,z),z) + \frac{\eta_2}{2}|g(x,z) - g_n(x,z)|^2,$$

$$\phi_n(x,g_n(x,z),z) \geq \phi_n(x,g(x,z),z) + \frac{\eta_2}{2}|g(x,z) - g_n(x,z)|^2.$$

Using the same reasoning as in the proof of the main theorem we conclude that:

$$\|g - g_n\| \leq [\frac{2\beta}{\eta_2}\|v - v_n\|]^{1/2}.$$

Proof of theorem 4.2.

Let

$$\tilde{\phi}(x,y,z) = F(x,y,z) + \beta \int_Z \tilde{v}(y,z')\, Q(z,dz').$$

Take $\tilde{y} \in \tilde{g}(x,z)$. Then

$$\tilde{\phi}(x,\tilde{y},z) \geq \tilde{\phi}(x,g(x,z),z).$$

With hypotheses 1 and 2 we have:

$$\phi(x,g(x,z),z) \geq \phi(x,\tilde{y},z) + \frac{\beta\eta_1}{2}|g(x,z) - \tilde{y}|^2.$$

Adding up these inequalities and using the reasoning above we will obtain the desired result.

With hypotheses 1 and 3 we have

$$\phi(x,g(x,z),z) \geq \phi(x,\tilde{y},z) + \frac{\eta_2}{2}|g(x,z) - \tilde{y}|^2,$$

and the conclusion follows again.

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