Economics Bulletin

Volume 29, Issue 3

Arrow's theorem for weak orders

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Abstract

We characterize binary decision rules which are independent and strongly paretian,or independent and almost strongly paretian when the individual preferences and the collective preference are weak orders.

We are grateful to all members of CREME (University of Yaounde I) for discussions and helpful remarks on previous drafts.
Citation: Nicolas Gabriel Andjiga and Joël Moulen, (2009) "Arrow's theorem for weak orders", *Economics Bulletin*, Vol. 29 no.3 pp. 2038-2044.
Submitted: Jun 06 2009. **Published:** August 20, 2009.

1 Introduction

Arrow's Theorem (Arrow (1951)) states that, when the domain and the codomain of a binary decision rule (BDR) are the set of weak orders, and when the BDR is paretian and satisfies the Independence of Irrelevant Alternatives (IIA) property, there exists a player i_0 such that, whenever i_0 strictly prefers an option x to another option y, then the collectivity also strictly prefers x to y. No information is given to the collective preference when x and y are equivalent for i_0 who is called the dictator. The aim of this note is to characterize, as a consequence of the Arrow's theorem, two subfamilies of arrovian BDRs by strengthening the Pareto condition. The result obtained on strong paretian BDR is also obtained by many other authors in different frameworks [see Fishburn (1974), Craven (1992), Hild (2001), Xu (2003)].

2 Notations and definitions

Let N be a the finite set of n voters and A a finite set of m alternatives with $n \geq 2$ and $m \geq 3$. We will denote by $\mathcal{B}$ the set of binary relations on A; $\mathcal{W}$ (resp. $\mathcal{W}^N$) the set of weak orders (transitive and complete binary relations) on A (resp. the set of profiles of weak orders on A). Given a binary relation R on A and a subset $\{x,y\}$ of A , we write $R|_{\{x,y\}} = xy$ if x is strictly preferred to y and $R|_{\{x,y\}} = (xy)$ if the indifference holds between x and y. Moreover, given a subset $\{x,y\}$ of A and two profiles R^N and Q^N, $\pi(x,y,R^N)$ is the set of all voters who strictly prefer x to y; the notation $R^N|_{\{x,y\}} = Q^N|_{\{x,y\}}$ stands for $\pi(x,y,R^N) = \pi(x,y,Q^N)$ and $\pi(y,x,R^N) = \pi(y,x,Q^N)$; and $\left(R^S, Q^{N-S}\right)$ is the profile where preferences of voters in S are given by R^S and preferences of voters in $N-S$ by Q^{N-S}. As usual, 2^N is the set of all non empty subsets of N.

Definition 1 *Let F be a BDR, that is a mapping F from $\mathcal{W}^N$ to $\mathcal{B}$.*

(i) F satisfies ***Independence of Irrelevant Alternatives (IIA)*** *if*

$$\forall R^N, Q^N \in \mathcal{W}^N, \forall \{x,y\} \subset A : R^N|_{\{x,y\}} = Q^N|_{\{x,y\}} \Longrightarrow F(R^N)|_{\{x,y\}} = F(R^N)|_{\{x,y\}}.$$

(ii) F is ***paretian*** *if*

$$\forall R^N \in \mathcal{W}^N, \forall \{x,y\} \subset A : \pi(x,y,R^N) = N \Longrightarrow F(R^N)|_{\{x,y\}} = xy.$$

(iii) F is ***almost paretian*** *if*

$$\forall R^N \in \mathcal{W}^N, \forall \{x,y\} \subset A : \pi(x,y,R^N) = N \Longrightarrow F(R^N)|_{\{x,y\}} \in \{xy, (xy)\}.$$

(iv) F is ***strongly paretian*** *if $\forall R^N \in \mathcal{W}^N, \forall \{x,y\} \subset A$:*

(iv-a) $\left[\pi(x,y,R^N) \neq \emptyset \text{ and } \pi(y,x,R^N) = \emptyset\right] \Longrightarrow F(R^N)|_{\{x,y\}} = xy;$

(iv-b) $\pi(y,x,R^N) = \emptyset \Longrightarrow F(R^N)|_{\{x,y\}} \in \{xy, (xy)\}.$

(v) F is ***almost strongly paretian*** *if*

$$\forall R^N \in \mathcal{W}^N, \forall \{x,y\} \subset A : \pi(y,x,R^N) = \emptyset \Longrightarrow F(R^N)|_{\{x,y\}} \in \{xy, (xy)\}.$$

Definition 2 *Let F be a BDR.*
(i) F is ***complete*** *if $\forall R^N \in \mathcal{W}^N$, $F(R^N)$ is complete.*

(ii) F is ***transitive*** *if $\forall R^N \in \mathcal{W}^N$, $F(R^N)$ is transitive.*

(iii) F is ***dictatorial*** *if*
$$\exists i_0 \in N / \ \forall R^N \in \mathcal{W}^N, \forall \{x,y\} \subset A : R^{i_0}|_{\{x,y\}} = xy \Longrightarrow F(R^N)|_{\{x,y\}} = xy$$

(iv) F is ***strongly dictatorial*** *if $\exists i_0 \in N / \ \forall R^N \in \mathcal{W}^N, F(R^N) = R^{i_0}$.*

(v) F is a ***lexicographic dictatorship of order q*** *if there exists q distinct voters $i_1, i_2, ..., i_q$ such that $\forall R^N \in \mathcal{W}^N, \forall \{x,y\} \subset A$, $\forall k \in \{1,2,..,q\}$:*

a) $\left[R^{i_k}|_{\{x,y\}} = xy \text{ and } \forall t < k, R^{i_t}|_{\{x,y\}} = (xy)\right] \Longrightarrow F(R^N)|_{\{x,y\}} = xy$;

b) $R^{i_k}|_{\{x,y\}} = (xy)\, , \forall k < q \Longrightarrow F(R^N)|_{\{x,y\}} = R^q|_{\{x,y\}}$.

For $k \in \{1,2,..,q\}$, i_k will then be called the dictator of order k for F and the decisive dictator for F given $S \in 2^N$ is the dictator for F in S with the smallest order.
(vi) F is a ***lexicographic dictatorship*** *if F is a lexicographic dictatorship of order n.*

(vii) F is ***null*** *if $\forall R^N \in \mathcal{W}^N, \forall \{x,y\} \subset A : F(R^N)|_{\{x,y\}} = (xy)$.*
When F is null, F will be called a lexicographic dictatorship of order $q = 0$.

3 The main result

Let us recall the well-known Arrow's Theorem :

Theorem 1 (Arrow 1951) *If F is paretian, IIA, transitive and complete then F is dictatorial.*

We use theorem 1 to establish the following :

Theorem 2 *F is strongly paretian, IIA, transitive and complete if and only if F is a lexicographic dictatorship.*

Proof. (a) *Sufficiency.* Let F be a lexicographic dictatorship.

(a1) *Completeness.* Let $R^N \in L^N$ and $\{x,y\} \subset A$. Suppose that $\pi(x,y,R^N) \cup \pi(y,x,R^N) = \emptyset$. Then $R^i|_{\{x,y\}} = (xy)$ for all $i \in N$. By definition of a lexicographic dictatorship, $F\left(R^N\right)|_{\{x,y\}} = (xy)$. Now suppose that $\pi(x,y,R^N) \cup \pi(y,x,R^N) \neq \emptyset$ and let i_p be the decisive dictator for F given $\pi(x,y,R^N) \cup \pi(y,x,R^N)$. Then $F|_{\{x,y\}} = R^{i_p}|_{\{x,y\}}$. Since $R^{i_p} \in W$, F is complete.

(a2) *IIA.* Let $R^N, Q^N \in L^N$ and $\{x,y\} \subset A$ such that $R^N|_{\{x,y\}} = Q^N|_{\{x,y\}}$. Suppose that $\pi(x,y,R^N) \cup \pi(y,x,R^N) = \emptyset$. Then $F\left(R^N\right)|_{\{x,y\}} = (xy)$. Since $R^N|_{\{x,y\}} = Q^N|_{\{x,y\}}$, $\pi(x,y,Q^N) \cup \pi(y,x,Q^N) = \emptyset$. Hence $F\left(Q^N\right)|_{\{x,y\}} = (xy)$. Now suppose that $\pi(x,y,R^N) \cup \pi(y,x,R^N) \neq \emptyset$ and let i_p be the decisive dictator for F given $\pi(x,y,R^N) \cup \pi(y,x,R^N)$. Then $F\left(R^N\right)|_{\{x,y\}} = R^{i_p}|_{\{x,y\}}$. Since $R^N|_{\{x,y\}} = Q^N|_{\{x,y\}}$, $\pi(x,y,Q^N) \cup$

$\pi(y,x,Q^N) \neq \emptyset$ and i_p is also the decisive dictator for F given $\pi(x,y,Q^N) \cup \pi(y,x,Q^N)$. Hence $F\left(Q^N\right)|_{\{x,y\}} = Q^{i_p}|_{\{x,y\}}$. Therefore $F\left(R^N\right)|_{\{x,y\}} = F\left(Q^N\right)|_{\{x,y\}}$.

(a3) *Transitivity.* Let $R^N \in L^N$ and $\{x,y,z\} \subset A$.

(a3-1) Suppose that $F\left(R^N\right)|_{\{x,y\}} = (xy)$ and $F\left(R^N\right)|_{\{y,z\}} = (yz)$. Then by the definition of a lexicographic dictatorship, $R^i|_{\{x,y\}} = (xy)$ and $R^i|_{\{y,z\}} = (yz)$ for all $i \in N$. By transitivity of individual preferences, $R^i|_{\{x,z\}} = (xz)$ for all $i \in N$. Hence $F\left(R^N\right)|_{\{x,z\}} = (xz)$.

(a3-2) Suppose that $F\left(R^N\right)|_{\{x,y\}} = (xy)$ and $F\left(R^N\right)|_{\{y,z\}} = yz$. Since F is a lexicographic dictatorship, $R^i|_{\{x,y\}} = (xy)$ for all $i \in N$ and $\pi(y,z,R^N) \neq \emptyset$. By transitivity, $\pi(y,z,R^N) = \pi(x,z,R^N)$ and $\pi(z,y,R^N) = \pi(z,x,R^N)$. Since $F\left(R^N\right)|_{\{y,z\}} = yz$, the decisive dictator for F given $\pi(y,z,R^N) \cup \pi(z,y,R^N)$ belongs to $\pi(y,z,R^N)$. Hence the decisive dictator for F given $\pi(x,z,R^N) \cup \pi(z,x,R^N)$ belongs to $\pi(x,z,R^N)$. Thus $F\left(R^N\right)|_{\{x,z\}} = xz$.

(a3-3) Suppose that $F\left(R^N\right)|_{\{x,y\}} = xy$ and $F\left(R^N\right)|_{\{y,z\}} = (yz)$. Since F is a lexicographic dictatorship, $R^i|_{\{y,z\}} = (yz)$ for all $i \in N$ and $\pi(x,y,R^N) \neq \emptyset$. By transitivity, $\pi(x,z,R^N) = \pi(x,y,R^N)$ and $\pi(z,x,R^N) = \pi(y,x,R^N)$. Since $F\left(R^N\right)|_{\{x,y\}} = xy$, the decisive dictator for F given $\pi(x,y,R^N) \cup \pi(y,x,R^N)$ belongs to $\pi(x,y,R^N)$. Hence the decisive dictator for F given $\pi(x,z,R^N) \cup \pi(z,x,R^N)$ belongs to $\pi(x,z,R^N)$. Thus $F\left(R^N\right)|_{\{x,z\}} = xz$.

(a3-4) Suppose that $F\left(R^N\right)|_{\{x,y\}} = xy$ and $F\left(R^N\right)|_{\{y,z\}} = yz$. Let i_p and i_q be respectively the decisive dictator for F given $\pi(x,y,R^N) \cup \pi(y,x,R^N)$ and given $\pi(y,z,R^N) \cup \pi(z,y,R^N)$. Then $i_p \in \pi(x,y,R^N)$ and $i_q \in \pi(y,z,R^N)$. Consider i_k a dictator for F the order of which is k. Suppose that $k < p$ and $k < q$. Then $R^{i_k}|_{\{x,y\}} = (xy)$ and $R^{i_k}|_{\{y,z\}} = (yz)$. By transitivity $R^{i_k}|_{\{x,z\}} = (xz)$. Now suppose that $p \leq q$. Then $R^{i_p}|_{\{y,z\}} = yz$ for $p = q$ or $R^{i_p}|_{\{y,z\}} = (yz)$ for $p < q$. Hence $R^{i_p}|_{\{x,z\}} = xz$ and i_p is the dictator for F given $\pi(x,z,R^N) \cup \pi(z,x,R^N)$. Similarly if $p > q$ then $R^{i_q}|_{\{y,z\}} = yz$ and i_q is the decisive dictator given $\pi(x,z,R^N) \cup \pi(z,x,R^N)$. In both cases, $F\left(R^N\right)|_{\{x,z\}} = xz$.

Clearly, $F\left(R^N\right)$ is transitive.

(a4) *Strongly Paretian.* Consider a profile R^N and $\{x,y\} \subset A$.

(a4-1) Suppose that there exists $S \in 2^N$ such that $\pi(x,y,R^N) = S$ and $\pi(y,x,R^N) = \emptyset$. Then the decisive dictator for F given $\pi(x,y,R^N) \cup \pi(y,x,R^N)$ belongs to $\pi(x,y,R^N)$. Therefore $F\left(R^N\right)|_{\{y,z\}} = xy$.

(a4-2) Suppose that $F\left(R^N\right)|_{\{x,y\}} = xy$. By the definition of F the decisive dictator for F given $\pi(x,y,R^N) \cup \pi(y,x,R^N)$ belongs to $\pi(x,y,R^N)$. Therefore $\pi(x,y,R^N) \neq \emptyset$.

(b) *Necessity.* Conversely let F be a strongly paretian, IIA, transitive and complete BDR.

(b1) Since F is strongly paretian, F is paretian and thus F is paretian, IIA, transitive and complete. By Theorem 1, F is dictatorial. Hereafter the dictator of F is denoted by i. Let $N_1 = N - \{i_1\}$ and define the BDR F_1 on $\mathcal{W}^{N_1}$ by $F_1\left(R^{N_1}\right) = F\left(I^{\{i_1\}}, R^{N_1}\right)$. There is no difficulty to observe that F_1 is strongly paretian, IIA, transitive and complete.

F_1 is therefore paretian, IIA, transitive and complete. By Theorem 1, F_1 is dictatorial and the dictator for F_2 is denoted i_2.

(b2) Suppose that there exists q voters i_1, i_2, ..., i_q with $q < n$ such that i_1 is the dictator for F and for all $k \in \{2, ..., q\}$, i_k is the dictator for F_{k-1} where for all $k \in \{1, 2, ..., q\}$, F_k is the BDR defined on $\mathcal{W}^{N_k}$ by $F_k\left(R^{N_k}\right) = F\left(I^{\{i_1,i_2,...,i_k\}}, R^{N_k}\right)$ with $N_k = N - \{i_1, i_2, ..., i_k\}$. Since F is strongly paretian, IIA, transitive and complete on $\mathcal{W}^N$, it is easy to observe that F_q is strongly paretian, IIA, transitive and complete on $\mathcal{W}^{N_q}$. By Theorem 1, F_q is dictatorial and the dictator for F_q is denoted by i_{q+1}.

(b3) After n iterations described at **(b1)** and **(b2)**, F is a lexicographic dictatorship.

■

Theorem 3 (Arrow bis) *If F is almost paretian, IIA, transitive and complete then F is either dictatorial or null.*

Proof. Let F be an almost paretian, IIA, transitive and complete BDR.

Case 1 : Suppose that F is paretian. Then by theorem 1 F is dictatorial.

Case 2 : Suppose that F is not paretian. Then there exists $R^N \in L^N$ and $\{x, y\} \subset A$ such that $\pi\left(x, y, R^N\right) = N$ and $F\left(R^N\right)|_{\{x,y\}} \neq xy$. Since F is complete and almost paretian, $F\left(R^N\right)|_{\{x,y\}} = (xy)$. Now given Q^N and $z \in A \backslash \{x\}$, let prove that $F\left(Q^N\right)|_{\{x,z\}} = (xz)$.

(a) Let prove that for all profile Q^N and $z \in A \backslash \{x\}$ such that $\pi\left(x, z, R^N\right) = N$, $F\left(Q^N\right)|_{\{x,y\}} = (xy)$ holds. Suppose that $z = y$. Then $R^N|_{\{x,y\}} = Q^N|_{\{x,y\}}$ and by IIA, $F\left(Q^N\right)|_{\{x,y\}} = (xy)$. Now suppose that $z \in A \backslash \{x, y\}$. Consider a profile R_1^N at which for all voters, x is the most preferred alternative, z is the second best and y is the third best. Then by transitivity of individual preferences, x is strictly preferred to y by all voters. Therefore $R^N|_{\{x,y\}} = R_1^N|_{\{x,y\}}$ and by IIA, $F\left(R_1^N\right)|_{\{x,y\}} = (xy)$. Also observe that F is almost paretian, $\pi\left(x, z, R_1^N\right) = N$ and $\pi\left(z, y, R_1^N\right) = N$. Consequently $F\left(R_1^N\right)|_{\{x,z\}} \neq zx$ and $F\left(R_1^N\right)|_{\{z,y\}} \neq yz$. By transitivity of F, $F\left(R_1^N\right)|_{\{z,y\}} = (yz)$ and $F\left(R_1^N\right)|_{\{x,z\}} = (zx)$. Since $Q^N|_{\{x,z\}} = R_1^N|_{\{x,z\}}$ and F is IIA, then $F\left(Q^N\right)|_{\{x,z\}} = (zx)$.

(b) Since at **(a)** above, $\pi\left(z, y, R_1^N\right) = N$ and $F\left(R_1^N\right)|_{\{z,y\}} = (yz)$, then by IIA, for all profile Q^N and $z \in A \backslash \{x, y\}$ such that $\pi\left(z, y, R^N\right) = N$, $F\left(Q^N\right)|_{\{z,y\}} = (zy)$ holds.

(c) Let prove that for all profile Q^N and $z \in A \backslash \{x, y\}$ such that $\pi\left(y, z, R^N\right) = N$, $F\left(Q^N\right)|_{\{z,y\}} = (zy)$ holds. Consider a profile R_2^N at which for all voters, x is the most preferred alternative and y is the second best alternative. Then $R^N|_{\{x,y\}} = R_2^N|_{\{x,y\}}$ and by IIA, $F\left(R_2^N\right)|_{\{x,y\}} = (xy)$. Also observe that $R_1^N|_{\{x,z\}} = R_2^N|_{\{x,z\}}$. Thus by IIA, $F\left(R_2^N\right)|_{\{z,x\}} = (xz)$. Since F is transitive, $F\left(R_2^N\right)|_{\{z,y\}} = (yz)$. But $Q^N|_{\{y,z\}} = R_2^N|_{\{y,z\}}$. Therefore by IIA, $F\left(Q^N\right)|_{\{y,z\}} = (yz)$.

(d) Let prove that for all profile Q^N and $z \in A \backslash \{x\}$, $F\left(Q^N\right)|_{\{z,x\}} = (zx)$ holds. Just consider a profile R_3^N at which for all voter i : x and z are ranked according to Q^i and are strictly preferred to any other alternative in $A \backslash \{x, z\}$.

(d-1) Suppose that $z \in A \setminus \{x, y\}$. Then note that $\pi\left(x, y, R_3^N\right) = N$ and $\pi\left(z, y, R_3^N\right) = N$. Then by stages **(a)** and **(b)** above, $F\left(R_3^N\right)|_{\{y,x\}} = (yx)$ and $F\left(R_3^N\right)|_{\{z,y\}} = (zy)$. Since F is transitive, $F\left(R_3^N\right)|_{\{x,z\}} = (xz)$. But $Q^N|_{\{x,z\}} = R_3^N|_{\{z,x\}}$. Therefore by IIA, $F\left(Q^N\right)|_{\{z,x\}} = (zx)$.

(d-2) Suppose that $z = y$ and consider $t \in A \setminus \{x, y\}$. Then observe that $\pi\left(x, t, R_3^N\right) = N$ and $\pi\left(y, t, R_3^N\right) = N$. Then by stages **(a)** and **(c)** in the present proof, $F\left(R_3^N\right)|_{\{t,x\}} = (tx)$ and $F\left(R_3^N\right)|_{\{t,y\}} = (ty)$. Since F is transitive, $F\left(R_3^N\right)|_{\{x,z\}} = (xz)$. But $Q^N|_{\{x,z\}} = R_3^N|_{\{z,x\}}$. Therefore by IIA, $F\left(Q^N\right)|_{\{z,x\}} = (zx)$.

To conclude, consider a profile Q^N and $\{a, b\} \subset A$. First suppose that $x \in \{a, b\}$. Then by stage **(d)**, $F\left(Q^N\right)|_{\{a,b\}} = (ab)$. Now suppose that $x \notin \{a, b\}$. Then by stage **(d)**, $F\left(Q^N\right)|_{\{a,x\}} = (ax)$ and $F\left(Q^N\right)|_{\{b,x\}} = (bx)$. By transitivity of F, $F\left(Q^N\right)|_{\{b,a\}} = (ba)$.

As conslusion in **case 2**, F is null. ■

Theorem 4 *F is almost strongly paretian, IIA, transitive and complete if and only if F is either null or there exists $q \leq n$ such that F is a lexicographic dictatorship of order q.*

Proof. (i) *Sufficiency.* It is obvious that lexicographic dictatorships of any order q are almost strongly paretian, IIA, transitive and complete BDRs.

(ii) *Necessity.* Conversely let F be an almost strongly paretian, IIA, transitive and complete BDR.

(ii1) Since F is almost strongly paretian , F is almost paretian and thus F is almost paretian, IIA, transitive and complete. By the Theorem 3, F is either null or F is dictatorial.

If F is null, the proof ends and F is lexicographic dictatorship of order $q = 0$. Otherwise F is dictatorial and its dictator is denoted by i_1. Let $N_1 = N - \{i_1\}$ and define the BDR F_1 on W^{N_1} by $F_1\left(R^{N_1}\right) = F\left(I^{\{i_1\}}, R^{N_1}\right)$. It is easy to prove that F_1 is almost strongly paretian, IIA, transitive and complete. F_1 is therefore almost paretian, IIA, transitive and complete. By Theorem 3, F_1 is either null or F_1 is dictatorial and its dictator is denoted i_2.

If F_1 is null, F is lexicographic dictatorship of order 1 and the proof ends. Otherwise F_2 is dictatorial and its dictator is denoted by i_2.

(ii2) Suppose that there exists q voters i_1, i_2, ..., i_q with $q < n$ such that i_1 is the dictator of F and for all $k \in \{2, ..., q\}$, i_k is the dictator of F_{k-1} where for all $k \in \{1, 2, ..., q\}$, F_k is the BDR defined on $\mathcal{W}^{N_k}$ by $F_k\left(R^{N_k}\right) = F\left(I^{\{i_1,i_2,...,i_k\}}, R^{N_k}\right)$ with $N_k = N - \{i_1, i_2, ..., i_k\}$. Since F is almost strongly paretian, IIA, transitive and complete on $\mathcal{W}^N$, it is easy to observe that F_q is almost strongly paretian, IIA, transitive and complete on $\mathcal{W}^{N_q}$. By Theorem 3, F_q is either null or dictatorial.

If F_q is null, F is a lexicographic dictatorship of order q and the proof ends. Otherwise F_q is dictatorial and its dictator is denoted by i_{q+1}.

(ii3) By at most n iterations described at **(ii1)** and **(ii2)**, F is null or is a lexicographic dictatorship of some order q with $0 \leq q \leq n$. ■

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