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Consistent conjectural variations equilibria in strategic multilateral exchange

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Abstract

In this note, we address the problem of determining which conjectural variations general equilibria are consistent in strategic multilateral exchange. We therefore consider expectations in a simple conjectural general equilibrium model of a pure exchange economy under strategic interactions. Three results are obtained. Firstly, the competitive equilibrium is a locally consistent conjectural general equilibrium. Secondly, the symmetric Cournot oligopoly equilibrium is not a locally consistent conjectural general equilibrium. Thirdly, the symmetric collusive equilibrium is a locally consistent conjectural general equilibrium.

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1. Introduction

In imperfectly competitive economies, each agent when making a decision generally does consider the effect of his action on the market (Bowley (1924), Hicks (1935)). The conjectural approach takes into account the perceptions by individuals of their market environment and intends to study price formation without an auctioneer by attempting a general equilibrium analysis of imperfect competition (Gale (1978), Hahn (1977)). Consistent conjectures have mainly been developed in the context of production economies (Bresnahan (1981), Figuières *et al.* (2004a, b), Dixit (1986), Laitner (1980), Perry (1982) and Ulph (1983)).

In this note, we propose to study the consistency of conjectural variations equilibria in pure exchange economies. The conjectural variations approach has been criticized by Fellner (1949), and later by Friedman (1977)[1]. Both authors put into perspective the feature that *ad hoc* conjectural variations are generally inconsistent with rational behavior, except at the equilibrium. A static definition of consistency was proposed by Bresnahan (1981) for the duopoly case. A consistent conjectural variation is a conjecture that is locally correct: expected change in the relevant decision variable is what would actually occur. This self-fulfilling nature of conjectural variations can be extended to oligopolistic environments (Perry (1982)).

The consistency problem of conjectures is here cast into the strategic multilateral exchange model explored in Julien (2006). We thus refer to the framework initially developed by Gabszewicz and Vial (1972) in exchange economies with production and later pursued by Codognato-Gabszewicz (1991), (1993) in exchange economies. This framework captures the working and the consequences of market power in general equilibrium. Several concepts of oligopoly equilibria can be developed depending on the way strategic behavior is introduced (Busetto *et al.* (2008), Gabszewicz-Michel (1997), Gabszewicz (2002)). In these Nashian perspectives, the oligopoly equilibria can notably coincide with the competitive equilibrium for large economies under a replication procedure or an asymptotic identification.

Throughout an example, we determine the conditions under which rational agents form consistent conjectures in a static environment. It is shown that the conjectural variations are consistent for the competitive and the collusive general equilibria. Additionally, the Cournot general equilibrium is not consistent. The approach we retain is essentially static, even if the working of conjectural variations presupposes some dynamical adjustment process (Figuières *et al.* (2004a)). The intrinsically dynamic nature of the process that governs the working of conjectural variations can be omitted in a first approximation in favor of the multi-markets interactions. Additionally, the properties that prevail in industrial economics under partial equilibrium analysis can be extended to cover a general equilibrium framework.

The paper is organized as follows. The basic economy is described in section 2. In section 3, we define the conjectural general equilibrium and we characterize it. In section 4, we determine the consistent equilibria among the class of symmetric conjectural general equilibria. In section 5, we conclude.

[1] Both authors considered the concept as not being a static one, and that it might consequently be inserted into a dynamic framework in order to capture the sequential process of players' reactions.

2. The economy

Consider a pure exchange economy with two divisible consumption goods, indexed ℓ, $\ell = 1,2$, and *(m+n)* traders, indexed *i*, $i = 1,...,m+n$. The preferences of trader *i* are represented by the following utility function:

$$U_i = x_{i1}^{\alpha} x_{i2}^{1-\alpha} \quad , \; 0 < \alpha < 1 \quad , \; \forall i \, . \tag{1}$$

The structure of the initial endowments is assumed to be the same as in the case of the homogeneous oligopoly developed by Gabszewicz-Michel (1997):

$$\omega_i = \left(\frac{1}{m}, 0 \right) \quad , \; i = 1,...,m \, ,$$

$$\omega_i = \left(0, \frac{1}{n} \right) \quad , \; i = m+1,...,m+n \, . \tag{2}$$

It is assumed that good 2 is taken as the *numéraire*, so p is the price of good 1 as expressed in units of good 2.

We consider that every agent behaves strategically. Each agent *i* will manipulate the price by contracting his supply, i.e. the quantity of good 1 or 2 he offers. The strategy set of trader *i* is given by:

$$S_i = \left\{ s_{i1} \in IR_+ \,\middle|\, 0 \le s_i \le \tfrac{1}{m} \right\} \, , i = 1,...,m, \tag{3}$$

$$S_i = \left\{ s_{i2} \in IR_+ \,\middle|\, 0 \le s_i \le \tfrac{1}{n} \right\} \, , i = m+1,...,m+n, \tag{4}$$

where s_{i1} denotes the pure strategy of trader *i*, $i = 1,...,m$, and s_{i2} the pure strategy of trader *i*, $i = m+1,...,m+n$. Each trader *i*, $i = 1,...,m$, obtains in exchange of s_{i1} a quantity ps_{i1} of good 2. Similarly, each trader *i*, $i = m+1,...,m+n$, obtains in exchange of s_{i2} a quantity s_{i2} / p of good 1.

Finally, let us assume the traders form conjectural variations. These conjectures indicate how any trader *i* expects his rivals' supply choices will vary when he modifies the strategic supply of the good he is initially endowed with. Since there are two goods in this economy, we respectively define:

$$\frac{\partial \sum_{-i} s_{-i1}}{\partial s_{i1}} = \nu_1 \quad , \text{where} \, \nu_1 \in [-1, m-1] \, , \tag{5}$$

$$\frac{\partial \sum_{-i} s_{-i2}}{\partial s_{i2}} = \nu_2 \quad , \text{where} \, \nu_2 \in [-1, n-1] \, . \tag{6}$$

We thus assume constant conjectures and that ν_1 and ν_2 to be the same for all traders of each type. The latter assumption precludes heterogeneous conjectures, while the former ensures that (5) and (6) are independent of both the supply and the number of traders. Certain values taken by conjectures in (5) and (6) are of particular interest in the context of production economies (Perry (1982)), and correspondingly in pure exchange economies (Julien (2006)).

Definition 1. *An economy* Ξ *is a collection of agents, endowments, strategy sets and conjectures* $\Xi = \left\{ (U_i, \omega_i, S_i), \nu_\ell \right\}_{i=1,...,m+n}^{\ell=1,2}$.

3. Conjectural general equilibrium: definition and characterization

Definition 2. *A conjectural general equilibrium for* $\Xi = \{(U_i, \omega_i, S_i), \nu_\ell\}_{i=1,...,m+n}^{\ell=1,2}$ *is given by a vector of strategies* $(\widetilde{s}_{11},...,\widetilde{s}_{m1},\widetilde{s}_{m+12},...,\widetilde{s}_{m+n2})$, *with* $\widetilde{s}_{i1} \in [0,1/m]$ *and* $\widetilde{s}_{i2} \in [0,1/n]$, *a vector of conjectural variations* $\nu = (\nu_1, \nu_2)$, *with* $\nu_1 \in [-1, m-1]$ *and* $\nu_2 \in [-1, n-1]$, *and an allocation* $(\widetilde{x}_1,,...,\widetilde{x}_m,\widetilde{x}_{m+1},...,\widetilde{x}_{m+n}) \in IR_+^{2(m+n)}$ *such that:*

(i) $\widetilde{x}_i = x_i(\widetilde{s}_{i1}(\nu), \widetilde{s}_{-i1}(\nu))$ *and* $U_i\left(x_i(\widetilde{s}_{i1}(\nu), \widetilde{s}_{-i1}(\nu))\right) \geq U_i\left(x_i(s_{i1}(\nu), \widetilde{s}_{-i1}(\nu))\right)$ *for* $i = 1,...,m$,

(ii) $\widetilde{x}_i = x_i(\widetilde{s}_{i2}(\nu), \widetilde{s}_{-i2}(\nu))$ *and* $U_i\left(x_i(\widetilde{s}_{i2}(\nu), \widetilde{s}_{-i2}(\nu))\right) \geq U_i\left(x_i(s_{i2}(\nu), \widetilde{s}_{-i2}(\nu))\right)$ *for* $i = m+1,...,m+n$.

A conjectural general equilibrium for $\Xi = \{U_i, \omega_i, S_i, \nu_\ell\}_{i=1,...,m+n}^{\ell=1,2}$ is a non cooperative equilibrium of a game where the players are the traders, the strategies are their supply decisions for the quantity of the only good they initially own, and the payoffs are their utility levels. At this equilibrium, each trader *i* determines a strategy in such a way that, according to the expected reactions from rivals as represented by ν, no unilateral deviation from his choice at equilibrium can increase his utility, taking into account the impact of such deviations on the resulting market-clearing price vector[2].

The market clearing condition implies that the price must be $p = \dfrac{\sum_{i=m+1}^{i=m+n} s_{i2}}{\sum_{i=1}^{i=m} s_{i1}} = \dfrac{s_2}{s_1}$.

Since $s_1 = s_{i1} + (m-1)s_{-i1}$ and $s_2 = s_{i2} + (n-1)s_{-i2}$, the payment of trader *i* can be written as $V_i(s_{i1}, (m-1)s_{-i1})$ for $i = 1,2,...,m$ and $V_i(s_{i2}, (n-1)s_{-i2})$ for $i = m+1,...,m+n$. The non-cooperative equilibrium is thus associated with the resolution of the simultaneous strategic programs:

$$\underset{\{\widetilde{s}_{i1} \in S_i\}}{Arg\max}\, V(s_{i1}, (m-1)s_{-i1}) = \left(\frac{1}{m} - s_{i1}\right)^{\alpha} \left(\frac{s_2}{s_{i1} + (m-1)s_{-i1}} s_{i1}\right)^{1-\alpha}, \; i = 1,...,m, \quad (7)$$

$$\underset{\{\widetilde{s}_{i2} \in S_i\}}{Arg\max}\, V_i(s_{i2}, (n-1)s_{-i2}) = \left(\frac{s_1}{s_2} s_{i2}\right)^{\alpha} \left(\frac{1}{n} - s_{i2}\right)^{1-\alpha}, \; i = m+1,...,m+n. \quad (8)$$

The *(m+n)* conditions of optimality $\partial U_i / \partial s_{i1} = 0$ for $i = 1,...,m$, and $\partial U_i / \partial s_{i2} = 0$ for $i = m+1,...,m+n$, lead to the following reaction functions:

$$\left[1 - \left(\frac{1-\alpha}{\alpha}\right)\nu_1\right](s_{i1})^2 + \left[\frac{1}{\alpha}(m-1)s_{-i1} + \frac{1}{m}\left(\frac{1-\alpha}{\alpha}\right)\nu_1\right](s_{i1}) - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{m-1}{m}\right)s_{-i1} = 0 \quad (9)$$

$$\left[\left(\frac{1-\alpha}{\alpha}\right) - \nu_2\right](s_{i2})^2 + \left[\frac{1}{\alpha}(n-1)s_{-i2} + \frac{1}{n}\nu_2\right](s_{i2}) - \left(\frac{n-1}{n}\right)s_{-i2} = 0. \quad (10)$$

[2] This characterizes Nash equilibria conditional on expectations formation.

At a symmetric general equilibrium, one has $\widetilde{s}_{i1} = \widetilde{s}_{-i1}$, $\forall i \in \{1,...,m\}$ and $\widetilde{s}_{i2} = \widetilde{s}_{-i2}$, $\forall i \in \{m+1,...,m+n\}$, which yield the equilibrium strategies:

$$\widetilde{s}_{i1} = \frac{(1-\alpha)[m-(1+\nu_1)]}{m[m-(1-\alpha)(1+\nu_1)]} \quad , \; i = 1,...,m \tag{11}$$

$$\widetilde{s}_{i2} = \frac{\alpha[n-(1+\nu_2)]}{n(n-\alpha(1+\nu_2))] \quad , \; i = m+1,...,m+n \, . \tag{12}$$

The market price $\widetilde{p} = \dfrac{\sum_{i=m+1}^{i=m+n} \widetilde{s}_{i2}}{\sum_{i=1}^{i=m} \widetilde{s}_{i1}}$ may be written:

$$\widetilde{p} = \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{m-(1-\alpha)(1+\nu_1)}{m-(1+\nu_1)} \right) \left(\frac{n-(1+\nu_2)}{n-\alpha(1+\nu_2)} \right). \tag{13}$$

The individual allocations are:

$$(\widetilde{x}_{i1}, \widetilde{x}_{i2}) = \left(\frac{\alpha}{m-(1-\alpha)(1+\nu_1)}, \frac{\alpha[n-(1+\nu_2)]}{m[n-\alpha(1+\nu_2)]} \right) \; , \; i = 1,...,m \, , \tag{14}$$

$$(\widetilde{x}_{i1}, \widetilde{x}_{i2}) = \left(\frac{(1-\alpha)[m-(1+\nu_1)]}{n[m-(1-\alpha)(1+\nu_1)]}, \frac{1-\alpha}{n-\alpha(1+\nu_2)} \right) \; , \; i = m+1,...,m+n \, . \tag{15}$$

4. Consistent conjectural general equilibria

Definition 3. *A locally consistent conjectural general equilibrium for* $\Xi = \{(U_i, \omega_i, S_i), \nu_\ell\}_{i=1,...,m+n}^{\ell=1,2}$ *is a conjectural general equilibrium for the vector of conjectures* $\nu = (\nu_1, \nu_2)$ *such that if* $(\widetilde{s}_{11},...,\widetilde{s}_{m1},\widetilde{s}_{m+12},...,\widetilde{s}_{m+n2})$ *is a solution to* $Arg\max V_i(s_{i1},(m-1)s_{-i1}) \quad \forall i = 1,...,m$ *and to* $Arg\max V_i(s_{i2},(n-1)s_{-i2})$ $\forall i = m+1,...,m+n$, *then* $\dfrac{\partial \sum_{-i \neq i} s_{-i1}(\widetilde{s}_{i1}, \nu_1)}{\partial s_{i1}} = \nu_1$ *and* $\dfrac{\partial \sum_{-i \neq i} s_{-i2}(\widetilde{s}_{i2}, \nu_2)}{\partial s_{i2}} = \nu_2$.

Remark 1. The consistency of conjectural variations equilibrium is here defined locally. This means that conjectures and reactions are the same only at equilibrium: it entails the coincidence of slopes of the reaction functions with the defined conjectural variations at the equilibrium (see Bresnahan (1981), Perry (1982)). Thus, each trader's conjectures about other trader's reactions are perfectly correct.

Result 1. *The competitive general equilibrium is a locally consistent conjectural general equilibrium.*

Proof. We show that when $\nu_1 = \nu_2 = -1$, the conjectural general equilibrium coincides with the competitive equilibrium. Then, we verify the local consistency of conjectures at this equilibrium.

Step 1. A *competitive equilibrium* is defined by a relative price p^* and an allocation $(x_1^*,...,x_m^*,x_{m+1}^*,...,x_{m+n}^*) \in IR_+^{2(m+n)}$ such that both markets simultaneously clear, and $Max\, U_i(x_{i1}^*, x_{i2}^*)$ *s.t.* $p^* x_{i1}^* + x_{i2}^* \leq p^*/m$ for $i = 1,...,m$ and

$Max\, U_i(x_{i1}^*, x_{i2}^*)$ *s.t.* $p^* x_{i1}^* + x_{i2}^* \le 1/n$ for $i = m+1,...,m+n$. We first compute the conjectural equilibrium for $\nu_1 = \nu_2 = -1$, and we compare it with the competitive equilibrium supplies and allocations. When $\nu_1 = \nu_2 = -1$, (11)-(15) become respectively $\tilde{p} = \alpha/(1-\alpha)$, $\tilde{s}_{i1} = (1-\alpha)/m$ and $(\tilde{x}_{i1}, \tilde{x}_{i2}) = (\alpha/m, \alpha/m)$ for $i = 1,...,m$, and $\tilde{s}_{i2} = \alpha/n$ and $(\tilde{x}_{i1}, \tilde{x}_{i2}) = ((1-\alpha)/n, (1-\alpha)/n)$ for $i = m+1,...,m+n$. The competitive equilibrium price, supplies and allocations are respectively $p^* = \alpha/(1-\alpha)$, $s_{i1}^* = (1-\alpha)/m$ and $(x_{i1}^*, x_{i2}^*) = (\alpha/m, \alpha/m)$ for $i = 1,...,m$, $s_{i2}^* = \alpha/n$ and $(x_{i1}^*, x_{i2}^*) = ((1-\alpha)/n, (1-\alpha)/n)$ for $i = m+1,...,m+n$. Thus $\tilde{p}_{|\gamma=\varepsilon=-1} = p^*$, $\tilde{s}_{i1|\gamma=\varepsilon=-1} = s_{i1}^*$ and $(\tilde{x}_{i1}, \tilde{x}_{i2})_{|\gamma=\varepsilon=-1} = (x_{i1}^*, x_{i2}^*)$ for $i = 1,...,m$, and $\tilde{s}_{i2|\gamma=\varepsilon=-1} = s_{i2}^*$ and $(\tilde{x}_{i1}, \tilde{x}_{i2})_{|\gamma=\varepsilon=-1} = (x_{i1}^*, x_{i2}^*)$ for $i = m+1,...,m+n$.

Step 2. We now verify the consistency of conjectures. At the conjectural general equilibrium, each trader on the same side of the market forms the same conjectural variations ν_1 and ν_2 . In order to construct the consistent conjectural variations for the competitive equilibrium, we need to characterize the aggregate equilibrium responses of the *(m-1)* (resp. the *(n-1)*) other traders consequent to change of the strategic supply of the *i*-th trader, $i \in \{1,...,m\}$ (resp. $i \in \{m+1,...,m+n\}$). We have to verify that these aggregate equilibrium conditions coincide with any individual response. Consider the reaction function as given in (9) by $s_{i1} = s_{i1}((m-1)s_{-i1}, m, \nu_1)$, which implicitly defines the reaction function of trader *i* as a function of the strategies of all other traders, of the number of traders and of the conjectural variations. Given s_{i1} , let $\sum_{-i\neq 1} s_{-i1}$ be the equilibrium reaction function for the *(m-1)* other traders behaving under ν_1 . Condition (9) can be written (*)

$$\left[1-\left(\frac{1-\alpha}{\alpha}\right)\nu_1\right]\left(\frac{s_{i1}+\sum_{-i} s_{-i1}}{m}\right)^2 + \left[\sum_{-i\neq i} s_{-i1} + \frac{1}{m}\left(\frac{1-\alpha}{\alpha}\right)\nu_1\right]\left(\frac{s_{i1}+\sum_{-i} s_{-i1}}{m}\right) - \left(\frac{1-\alpha}{\alpha}\right)\sum_{-i} s_{-i1} = 0 \quad .$$

The equilibrium response of all other traders -*i* to a unit of change in the supply of the *i*-th trader evaluated at the symmetric equilibrium is given by the differentiation of (*) with respect to s_{i1} . For the conjectural variations to be consistent, it must be equivalent to this local equilibrium response of the other traders at the competitive equilibrium. Consistent conjectural variations are then the fixed points of $\frac{\partial \sum_{-i\neq i} s_{-i1}(s_{i1}^*, m, \nu_1)}{\partial s_{i1}} = \nu_1$. When $\nu_1 = -1$, we must have $\frac{\partial \sum_{-i\neq i} s_{-i1}(s_{i1}^*, m, -1)}{\partial s_{i1}} = -1$. An application of the Implicit Function Theorem to (*) leads to

$$\frac{\partial \sum_{-i\neq i} s_{-i1}}{\partial s_{i1}} = -\frac{\frac{2[\alpha-(1-\alpha)\nu_1]s_{i1}^*}{m} + \frac{(1-\alpha)\sum_{-i\neq i} s_{-i1}^*}{m} + \frac{(1-\alpha)\nu_1}{m}}{\frac{2[\alpha-(1-\alpha)\nu_1]s_{i1}^*}{m} + \frac{2\sum_{-i\neq i} s_{-i1}^*}{m} + \frac{(1-\alpha)\nu_1 + s_{i1}^*}{m} - (1-\alpha)} \quad .$$

At the competitive equilibrium, one has $s_{i1}^* = (1-\alpha)/m$, then $\frac{\partial \sum_{-i\neq i} s_{-i1}}{\partial s_{i1}} = -\frac{(1-\alpha)}{(1-\alpha)} = -1$. The same conclusion holds for (10).

This completes the proof.

Remark 2. In order to interpret the consistency of conjectures in such a case, consider that the price of a good initially owned by a trader can be interpreted as an

opportunity cost, so each trader *i*, who has endowment must incur constant marginal costs $p_\ell s_{i\ell}$, $\ell = 1,2$. If a trader decides to increase his supply by one unit, and if all other traders were expected to behave competitively, these increases would shift each aggregate demand facing all the other traders inward by one unit. This would lead to exactly a one-unit contraction along each horizontal supply curves implied by constant marginal costs and perfectly competitive behaviors[3]. The effectiveness of this self-fulfilling mechanism presupposes that traders are perfectly informed about the demand functions which are addressed to them.

Result 2. *The symmetric Cournot oligopoly equilibrium is not consistent.*

Proof. When $\nu_1 = \nu_2 = 0$, the strategic supplies and the allocation become $\tilde{s}_{i1} = \frac{(1-\alpha)(m-1)}{m[m-(1-\alpha)]}$ and $(\tilde{x}_{i1}, \tilde{x}_{i2}) = \left(\frac{\alpha}{m-(1-\alpha)}, \frac{\alpha(n-1)}{m(n-\alpha)} \right)$ for $i = 1,...,m$, and $\tilde{s}_{i2} = \frac{\alpha(n-1)}{n(n-\alpha)}$ and $(\tilde{x}_{i1}, \tilde{x}_{i2}) = \left(\frac{(1-\alpha)(m-1)}{n[m-(1-\alpha)]}, \frac{1-\alpha}{n-\alpha} \right)$ for $i = m+1,...,m+n$. A *symmetric oligopoly equilibrium* is a $(m+n)$ -tuple of strategies $(\hat{s}_{11},...,\hat{s}_{m1},\hat{s}_{m+12},...,\hat{s}_{m+n2})$, with $\hat{s}_{i1} \in [0,1/m]$ for $i = 1,...,m$ and $\hat{s}_{i2} \in [0,1/n]$ for $i = m+1,...,m+n$, and an allocation $(\hat{x}_1,,...,\hat{x}_m,\hat{x}_{m+1},...,\hat{x}_{m+n}) \in IR_+^{2(m+n)}$ such that (i) $\hat{x}_i = x_i(\hat{s}_{i1},\hat{s}_{-i1})$ and $U_i\left(x_i(\hat{s}_{i1},\hat{s}_{-i1})\right) \geq U_i\left(x_i(s_{i1},\hat{s}_{-i1})\right)$ for $i = 1,...,m$ and (ii) $\tilde{x}_i = x_i(\tilde{s}_{i2},\tilde{s}_{-i2})$ and $U_i\left(x_i(\hat{s}_{i2},\hat{s}_{-i2})\right) \geq U_i\left(x_i(s_{i2},\hat{s}_{-i2})\right)$ for $i = m+1,...,m+n$. When each trader plays *à la* Cournot, the reaction functions write $(s_{i1})^2 + \frac{1}{\alpha}(m-1)s_{-i1}(s_{i1}) - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{m-1}{m}\right)s_{-i1} = 0$ for $i = 1,...,m$ and $\left(\frac{1-\alpha}{\alpha}\right)(s_{i1})^2 + \frac{1}{\alpha}(n-1)s_{-i2}(s_{i2}) - \left(\frac{n-1}{n}\right)s_{-i2} = 0$ otherwise. This leads to $\hat{s}_{i1} = \frac{(1-\alpha)(m-1)}{m[m-(1-\alpha)]}$ and $(\hat{x}_{i1}, \hat{x}_{i2}) = \left(\frac{\alpha}{m-(1-\alpha)}, \frac{\alpha(n-1)}{m(n-\alpha)} \right)$ for $i = 1,...,m$, and $\hat{s}_{i2} = \frac{\alpha(n-1)}{n(n-\alpha)}$ and $(\hat{x}_{i1}, \hat{x}_{i2}) = \left(\frac{(1-\alpha)(m-1)}{n[m-(1-\alpha)]}, \frac{1-\alpha}{n-\alpha} \right)$, $i = m+1,...,m+n$. Moreover, when $\nu_1 = 0$ and $\hat{s}_{i1} = \frac{(1-\alpha)(m-1)}{m[m-(1-\alpha)]}$, we easily verify that $\frac{\partial \sum_{-i\neq i} s_{-i1}}{\partial s_{i1}} = -\frac{\frac{2\alpha s_{i1}}{m} + \frac{(1-\alpha)\sum_{-i\neq i} s_{-i1}}{m}}{\frac{2\alpha s_{i1}}{m} + \frac{2\sum_{-i\neq i} s_{-i1}}{m} + \frac{s_{i1}}{m} - (1-\alpha)} \neq 0$. The same conclusion holds for $\nu_2 = 0$. *QED.*

Remark 3. The preceding result has a counterpart in industrial economics: with a linear demand function and constant marginal costs, the conjectural variations associated with the Cournot conjectures are not consistent (Bresnahan (1981))[4].

Result 3. *The symmetric collusive oligopoly equilibrium is consistent.*

[3] This interpretation is usual in industrial organization (Perry (1982)).
[4] See also the interpretation of Daughety (1985).

Proof. When $\nu_1 = m-1$ and $\nu_2 = n-1$, the strategic supplies and the allocations become $\tilde{s}_{i1} = 0$ and $(\tilde{x}_{i1}, \tilde{x}_{i2}) = (1/m, 0)$ for $i = 1,...,m$ and $\tilde{s}_{i1} = 0$ and $(\tilde{x}_{i1}, \tilde{x}_{i2}) = (0, 1/n)$ for $i = m+1,...,m+n$. A *symmetric collusive equilibrium* is a $(m+n)$-tuple of strategies $(\bar{s}_{11},...,\bar{s}_{m1},\bar{s}_{m+12},...,\bar{s}_{m+n2})$, with $\bar{s}_{i1} \in [0,1/m]$ for $i = 1,...,m$ and $\bar{s}_{i2} \in [0,1/n]$ for $i = m+1,...,m+n$, and an allocation $(\bar{x}_1,,...,\bar{x}_m,\bar{x}_{m+1},...,\bar{x}_{m+n}) \in IR_+^{2(m+n)}$ such that (i) $\bar{x}_i = x_i(\bar{s}_{i1}, \bar{s}_{-i1})$ and $U_i\left(x_i(\bar{s}_{i1}, \bar{s}_{-i1})\right) \geq U_i\left(x_i(s_{i1}, \bar{s}_{-i1})\right)$ for $i = 1,...,m$ and (ii) $\bar{x}_i = x_i(\vec{s}_{i2}, \bar{s}_{-i2})$ and $U_i\left(x_i(\bar{s}_{i2}, \bar{s}_{-i2})\right) \geq U_i\left(x_i(s_{i2}, \bar{s}_{-i2})\right)$ for $i = m+1,...,m+n$. Consider now the equilibrium is collusive. The joint maximization program for trader i writes $Arg\max V_i(s_{i1},(m-1)s_{-i1}) + \sum_{-i\neq i} V_{-i}(s_{-i1}, s_{i1},(m-2)s_{j1})$, $j \neq i,-i$ for $i = 1,...,m$ and $Arg\max V_i(s_{i2},(n-1)s_{-i2}) + \sum_{-i\neq i} V_{-i}(s_{-i2}, s_{i1},(n-2)s_{j2})$, $j \neq i,-i$ for $i = m+1,...,m+n$. Then (9) and (10) become

$$\left[1-\left(\frac{1-\alpha}{\alpha}\right)(m-1)\right](s_{i1})^2 + \left[\frac{1}{\alpha}(m-1)s_{-i1} + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{m-1}{m}\right)\right](s_{i1}) - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{m-1}{m}\right)s_{-i1} = 0$$

for $i = 1,...,m$, while for $i = m+1,...,m+n$; it writes

$$\left[1-\left(\frac{1-\alpha}{\alpha}\right)(m-1)\right](s_{i1})^2 + \left[\frac{1}{\alpha}(m-1)s_{-i1} + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{m-1}{m}\right)\right](s_{i1}) - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{m-1}{m}\right)s_{-i1} = 0 .$$

At the symmetric equilibrium, one has $\bar{s}_{i1} = \bar{s}_{-i1}$ and $\bar{s}_{i2} = \bar{s}_{-i2}$. This leads to $\bar{s}_{i1} = 0$ and $(\bar{x}_{i1}, \bar{x}_{i2}) = (1/m, 0)$ for $i = 1,...,m$ and $\bar{s}_{i1} = 0$ and $(\bar{x}_{i1}, \bar{x}_{i2}) = (0, 1/n)$ for $i = m+1,...,m+n$. Therefore, the conjectural equilibrium coincides with the collusive equilibrium. Finally, $\dfrac{\partial \sum_{-i\neq i} s_{-i1}(\bar{s}_{i1})}{\partial s_{i1}} = -\dfrac{\dfrac{(1-\alpha)(m-1)}{m}}{\dfrac{(1-\alpha)(m-1)}{m} - (1-\alpha)} = m-1$ for $i = 1,...,m$ and $\dfrac{\partial \sum_{-i\neq i} s_{-i2}(\bar{s}_{i2})}{\partial s_{i2}} = -\dfrac{\dfrac{\alpha(n-1)}{n}}{\dfrac{\alpha(n-1)}{n} - \alpha} = n-1$ for $i = m+1,...,m+n$. *QED.*

Remark 4. The general oligopoly equilibrium price is indeterminate in the presence of collusion in both sectors, while such a price is virtually infinite when there is collusion in sector 1, whatever happens in sector 2.

5. Conclusion

The role of conjectural variations can be transposed in general equilibrium for pure exchange economies. The competitive and the collusive equilibria are consistent equilibria. These results rest on the Cobb-Douglas specification for the utility function, which entails hyperbolic demand functions and constant marginal costs.

Further investigations could consider the strategic foundations of self-fulfilling conjectures in a dynamic game setting.

References

Bowley, A. (1924), *The mathematical groundwork of economics*, Kelley: New-York (1965).

Bresnahan, T. (1981), "Duopoly models with consistent conjectures", *American Economic Review* **71**, 934-945.

Busetto, F., G. Codognato, S. Ghosal (2008), "Cournot-Walras equilibrium as a subgame perfect equilibrium", *International Journal of Game Theory* **37**, 371-386.

Codognato, G., J.J. Gabszewicz (1993), "Cournot-Walras equilibria with a continuum of traders", *Economic Theory* **3**, 453-464.

Codognato, G., Gabszewicz, J.J. (1991), "Equilibres de Cournot-Walras dans une économie d'échange pure", *Revue Economique* **42**, 1013-1026.

Daughety, A.F. (1985), "Reconsidering Cournot: the Cournot equilibrium is consistent", *Rand Journal of Economics* **16**, 368-379.

Dixit, A. (1986), "Comparative statics for oligopoly", *International Economic Review* **27**, 107-122.

Dockner, E.J. (1992), "A dynamic theory of conjectural variations", *Journal of Industrial Economics* **40**, 377-395.

Fellner, W. (1949), *Competition among the few*, Knopf, New-York.

Figuières, C., A. Jean-Marie, N. Quérou, M. Tidball (2004a), *Theory of conjectural variations*, World Scientific Publishing, London.

Figuières, C., M. Tidball, A. Jean-Marie (2004b), "On the effects of conjectures in a symmetric strategic setting", *Research in Economics* **58**, 75-102.

Friedman, J. (1977), *Oligopoly theory and the theory of games*, North-Holland, Amsterdam.

Gabszewicz, J.J. (2002), *Strategic multilateral exchange. General equilibrium with imperfect competition*, Edward-Elgar, Cheltenham.

Gabszewicz, J.J., P. Michel (1997), "Oligopoly equilibria in exchange economies", in B.C. Eaton and R.G. Harris (Eds), *Trade, Technology and Economics. Essays in honour of R.G. Lipsey*, Edward-Elgar, Cheltenham, 217-240.

Gabszewicz, J.J., J.P. Vial (1972), "Oligopoly 'à la Cournot' in general equilibrium analysis", *Journal of Economic Theory* **4**, 381-400.

Gale, D. (1978), "A note on conjectural equilibria", *Review of Economic Studies* **45**, 33-38.

Hahn, F. (1977), "Exercise on conjectural equilibria", *Scandinavian Journal of Economics* **79**, 210-226.

Hicks, J.R. (1935), "The theory of monopoly", *Econometrica* **3**, 1-20.

Julien, L.A. (2006), "Conjectural variations and general oligopoly equilibrium in pure exchange economies, *Economics Bulletin* **4**, 1-6.

Laitner, J. (1980), "'Rational' duopoly equilibria", *Quarterly Journal of Economics* **94**, 641-662.

Perry, M. (1982), "Oligopoly and consistent conjectural variations", *Bell Journal of Economics* **13**, 197-205.

Ulph, D. (1983), "Rational conjectures in the theory of oligopoly", *International Journal of Industrial Organization* **1**, 131-154.