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A note on Condorcet's other paradox

William v. Gehrlein
university of Delaware

Dominique Lepelley
CEMOI, university of La Réunion

Abstract

We report in this note some results on the theoretical likelihood of Condorcet's Other Paradox in three alternative elections. This paradox occurs when we have a voting situation such that no Wheighted Scoring Rule (WSR) will select the Pairwise Majority Rule Winner as the WSR winner. We conclude from our study that actual observances of Condorcet's Other Paradox should be very rare events.

1. Introduction

We consider in this note the problem of a group of n voters having to choose an alternative among a set of three alternatives (or candidates) A, B and C. Individual voter preferences on alternatives are supposed to be both complete and transitive and are expressed as *linear preference rankings*. Fig. 1.1 shows each of the six possible linear preference rankings that each voter might have in a three-alternative election.

$$
\begin{array}{cccccc}
A & A & B & C & B & C \\
B & C & A & A & C & B \\
C & B & C & B & A & A \\
n_1 & n_2 & n_3 & n_4 & n_5 & n_6
\end{array}
$$

Fig. 1.1 The six possible linear preference rankings on three candidates

Here, n_i denotes the number of voters that have the associated linear preference ranking, with $\sum_{i=1}^{6} n_i = n$. A *voting situation* denotes any particular combination of $n_i's$ that sum to n. Voting situations just report the n_i values that are associated with each possible individual preference ranking for a given election, without specifying the preferences of any individual voter. A *voter preference profile*, or *voter profile*, gives a complete list that shows the specific linear preference order that is held by each individual voter. A voting situation can be obtained from a voter profile simply by determining the number of voters within the profile that have each of the possible linear preference rankings. As a result, voters' preferences are not anonymous in the case of a voter profile, but they are in a voting situation.

Let *AMB* denote the event that A defeats B by *Pairwise Majority Rule* (PMR) when only preferences on the pair of candidates A and B are considered in voters' preference rankings. It follows from the notation adopted in Fig. 1.1 that *AMB* if $[n_1 + n_2 + n_4 > n_3 + n_5 + n_6]$. Then, A will be the winner by PMR, or the *Pairwise Majority Rule Winner* (PMRW) when both *AMB* and *AMC*. The PMRW is commonly referred to as the *Condorcet Winner*, since Condorcet (1785) was a very strong advocate of the argument that the PMRW should always be selected as the winner of an election. Condorcet was also the first to demonstrate that a PMRW does not always exist (it can occur that, for instance, *AMB*, *BMC* and *CMA*) and this phenomenon is known as the *Condorcet's Paradox*.

In this note, we are interested in another paradox that was considered by Condorcet (1785) in his seminal work. This paradox is concerned with the general notion of a *Weighted Scoring Rule* (WSR). A WSR gives some number of points to candidates according to their relative position within individual voter's preference rankings. For three-candidate elections, we will assume in the following that a general WSR assigns 1 point to a candidate for each time it is most preferred in a voter's preference ranking, λ points for each time it is ranked as second most preferred, and 0 point for each least preferred ranking. Three well known WSR's in voting theory are the Borda Rule (BR) which takes $\lambda = 1/2$, the Plurality Rule (PR) which takes $\lambda = 0$ and the Negative Plurality Rule (NPR) which takes $\lambda = 1$. We suppose $0 \le \lambda \le 1$ since it would not make sense to award more points to the middle ranked candidate in a voter's preference ranking than to

the most preferred candidate in the ranking, or to award fewer points to the middle ranked candidate than to the least preferred candidate. Consequently, the weighted score for each alternative is

$$Score(A, \lambda) = n_1 + n_2 + \lambda(n_3 + n_4)$$
$$Score(B, \lambda) = n_3 + n_5 + \lambda(n_1 + n_6)$$
$$Score(C, \lambda) = n_4 + n_6 + \lambda(n_2 + n_5).$$

Condorcet (1785) gives the example voting situation in Fig. 1.2 to show a phenomenon that Fishburn (1974) refers to as *Condorcet's Other Paradox*.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A

$$n_1 = 30 \quad n_2 = 1 \quad n_3 = 29 \quad n_4 = 10 \quad n_5 = 10 \quad n_6 = 1.$$

Fig. 1.2 A voting situation showing Condorcet's Other Paradox from Condorcet (1785)

Condorcet notes that *AMB* (41-40) and *AMC* (61-20) in this voting situation, so that Candidate *A* is the PMRW, and then computes $Score(A)$ and $Score(B)$ for the general WSR with weights 1, λ and 0:

$$Score(A, \lambda) = 30 + 1 + \lambda(29 + 10)$$
$$Score(B, \lambda) = 29 + 10 + \lambda(30 + 1).$$

In order for Candidate *A* to be elected by a WSR, we must have:

$$Score(A, \lambda) > Score(B, \lambda)$$
$$31 + 39\lambda > 39 + 31\lambda$$
$$8\lambda > 8$$
$$\lambda > 1.$$

This contradicts our definition of a WSR, so that no WSR can elect the PMRW in this example, which is Condorcet's Other Paradox. Fishburn (1974) generalizes this result for all $m \geq 3$, to show that there is always some voting situation with a PMRW in an m-candidate election, such that every WSR will have at least $m - 2$ candidates with a greater score than the PMRW. Such voting situations are obviously problematic and there is a resulting interest in determining estimates for the likelihood that they occur.

In the present paper, which focuses on three-alternative elections, we consider that Condorcet's Other Paradox occurs when we have a voting situation such that no WSR with $0 \leq \lambda \leq 1$ will select the PMRW as the WSR winner and our aim is to compute the probability of such a phenomenon. Two alternative (and common) probability models will be used, both of which being based on a notion of equiprobability. The first one is known as the *Impartial Culture* (IC) condition and assumes that every *voter profile* is equally likely to occur. The second one is the *Impartial Anonymous Culture* (IAC) condition which supposes that every *voting situation* is equally likely to occur.

2. Probability of Condorcet's Other Paradox

A considerable amount of research effort has been done to develop mathematical representations for the probability that Condorcet's Paradox will be observed (see

Gehrlein, 2006). By contrast, much less attention has been paid to Condorcet's Other Paradox. There has been however some earlier work done to compute the probability that a *stronger* version of this paradox will be observed under the assumption of Impartial Culture (IC) as $n \to \infty$. To describe this earlier work, let **BDA** denote the event that candidate *B* will *dominate* candidate *A* by defeating it for every possible WSR with $0 \le \lambda \le 1$. Similarly, let $BD\{X\}$ denote the event that *B* dominates each candidate in a set *X*. A study by Merlin et al (2002) uses geometric techniques to obtain a representation that can lead to the conditional probability, $P(BD\{A,C\}, \infty, IC / A\ is\ PMRW)$, that candidate *B* will be the overall winner for every WSR with $0 \le \lambda \le 1$, given that *A* is the PMRW as $n \to \infty$ with the assumption of IC. This situation is obviously more restrictive than our definition of Condorcet's Other Paradox.

We start by replicating the basic results from Merlin et al (2002) by using another (standard) approach to the problem, based on the Central Limit Theorem, to obtain a representation for the probability $P(BD\{A,C\} \& A\ is\ PMRW, \infty, IC)$ that candidate *B* dominates both *A* and *C* when *A* is the PMRW. We know from Saari (1992) that, if candidate *B* is the overall winner by both PR and NPR, then *B* dominates both *A* and *C*. Consequently, there are six events that must occur simultaneously in a voting situation for having *B* dominating both *A* and *C* when *A* is the PMRW:

$$AMC \qquad\qquad \left[n_1 + n_2 + n_3 > n_4 + n_5 + n_6\right] \qquad\qquad (1)$$
$$AMB \qquad\qquad \left[n_1 + n_2 + n_4 > n_3 + n_5 + n_6\right] \qquad\qquad (2)$$
$$Score(B,0) > Score(A,0) \quad \left[n_3 + n_5 > n_1 + n_2\right] \qquad\qquad (3)$$
$$Score(B,0) > Score(C,0) \quad \left[n_3 + n_5 > n_4 + n_6\right] \qquad\qquad (4)$$
$$Score(B,1) > Score(A,1) \quad \left[n_5 + n_6 > n_2 + n_4\right] \qquad\qquad (5)$$
$$Score(B,1) > Score(C,1) \quad \left[n_1 + n_3 > n_2 + n_4\right] \qquad\qquad (6)$$

Discrete variables, X_t^j for $t = 1,2,3,4,5,6$, are defined for the event that each of these six restrictions will be observed in a randomly selected linear preference ranking for the jth voter. Let p_i denote the probability that a randomly selected voter from the population of voters will have the corresponding linear preference ranking (see Fig. 1.1). The six discrete variables are defined in terms of the p_i probabilities for preference rankings in the following way:

$$X_1^j = +1 : p_1 + p_2 + p_3 \qquad\qquad X_2^j = +1 : p_1 + p_2 + p_4$$
$$-1 : p_4 + p_5 + p_6 \qquad\qquad\qquad -1 : p_3 + p_5 + p_6$$

$$X_3^j = +1 : p_3 + p_5 \qquad\qquad X_4^j = +1 : p_3 + p_5$$
$$-1 : p_1 + p_2 \qquad\qquad\qquad -1 : p_4 + p_6$$
$$0 : p_4 + p_6 \qquad\qquad\qquad 0 : p_1 + p_2$$

$$X_5^j = +1 : p_5 + p_6 \qquad\qquad X_6^j = +1 : p_1 + p_3$$
$$-1 : p_2 + p_4 \qquad\qquad\qquad -1 : p_2 + p_4$$
$$0 : p_1 + p_3 \qquad\qquad\qquad 0 : p_5 + p_6$$

Given that IC implies $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6$, the correlation matrix that results from these definitions is $\boldsymbol{R}^1$, with

$$\boldsymbol{R}^1 = \begin{bmatrix} 1 & \dfrac{1}{3} & -\sqrt{\dfrac{1}{6}} & \sqrt{\dfrac{1}{6}} & -\sqrt{\dfrac{1}{6}} & \sqrt{\dfrac{1}{6}} \\ & 1 & -\sqrt{\dfrac{2}{3}} & -\sqrt{\dfrac{1}{6}} & -\sqrt{\dfrac{2}{3}} & -\sqrt{\dfrac{1}{6}} \\ & & 1 & \dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{4} \\ & & & 1 & \dfrac{1}{4} & \dfrac{1}{2} \\ & & & & 1 & \dfrac{1}{2} \\ & & & & & 1 \end{bmatrix}$$

Then $P(B\mathbf{D}\{A,C\} \& A \text{ is } PMRW, \infty, IC)$ is equivalent to the multivariate normal positive orthant probability $\Phi_6(\boldsymbol{R}^1)$. And the symmetry of IC with respect to candidates requires that the probability that the same candidate, that is not the PMRW, will dominate the two other candidates, including the PMRW, is obtained as $6\Phi_6(\boldsymbol{R}^1)$. Merlin et al (2002) obtain a very complex representation for this probability and use quadrature to obtain a value of .01808. By using the procedure of Naylor et al (1966) to obtain Monte Carlo simulation estimates of $\Phi_6(\boldsymbol{R}^1)$, we obtain a similar probability value. To make this probability conditional on the fact that a PMRW exists, we simply make a modification and use the relationship $6\Phi_6(\mathbf{R}^1)/P_{PMRW}(3,\infty,IC)$, where $P_{PMRW}(3,\infty,IC)$ denotes the probability that a PMRW exists. As it is well known that $P_{PMRW}(3,\infty,IC)=.9123$ (see e.g. Gehrlein, 2006), we obtain a probability value of .01982.

All of this indicates that the probability of observing this phenomenon is quite small. This result, however, could be biased on two accounts.

• First, it could be the result of the IC assumption. The impact of this assumption can be tested by doing the same analysis with the assumption of IAC. Under IAC, the desired probability can be obtained by dividing the number of voting situations described by Eqs. 1 to 6 (multiplied by two, to take into account the fact that C can be the overall winner for every WSR instead of B) by the number of voting situations such that A is the PMRW. An algorithm based on Ehrhart polynomial theory was used to compute these numbers with IAC as a function of n, following a procedure developed in Lepelley et al (2008). We obtain that, with the assumption of IAC, the probability of having B or C the overall winner for every WSR given that A is the PMRW is given as:

$$\frac{n(n-12)(38n^3 - 429n^2 - 468n + 864)}{3240(n+2)^2(n+4)^2}$$

for n=24, 48, 72, 96... The resulting conditional probability as $n \to \infty$ is then reduced to 38/3240= 19/1620 = .01173, so the small increase in voter dependence that is suggested by IAC (see e.g. Berg and Lepelley, 1994) makes the already small IC probability significantly smaller.

• Second, if Condorcet's Other Paradox, as we have defined it, is to be perceived as a real potential threat to elections, then the restrictions that we have just considered might be creating a significant understatement of the paradox.

3. A More Relaxed Condition

The restrictions that are used by Merlin et al (2002) can be relaxed by just considering the possibility that some other given candidate always dominates the PMRW, but where this given candidate does not necessarily dominate the remaining candidate that is not the PMRW. This would be obtained for one particular such occurrence with the probability $P(B\mathbf{D}A \& A\,is\,PMRW, \infty, IC)$ in which candidate B dominates the PMRW candidate A. The representation for this probability will follow directly from the discussion above, since the conditions that lead to its occurrence in a voting situation follow from the restrictions in Eqs. 1 through 6 above, with the conditions of Eqs. 4 and 6 being removed. It follows directly that we can obtain a representation for $P(B\mathbf{D}A \& A\,is\,PMRW, \infty, IC)$ as a multivariate normal positive orthant probability, $\Phi_4(\mathbf{R}^2)$,with a correlation matrix $\mathbf{R}^2$ that is obtained from $\mathbf{R}^1$ by removing the terms that are associated with variables X_4^j and X_6^j, with

$$
\mathbf{R}^2 = \begin{bmatrix} 1 & \dfrac{1}{3} & -\sqrt{\dfrac{1}{6}} & -\sqrt{\dfrac{1}{6}} \\ & 1 & -\sqrt{\dfrac{2}{3}} & -\sqrt{\dfrac{2}{3}} \\ & & 1 & \dfrac{1}{2} \\ & & & 1 \end{bmatrix}.
$$

The form of $\mathbf{R}^2$ does not lead to a simple representation for $\Phi_4(\mathbf{R}^2)$, so we use a procedure from Gehrlein (1979) to evaluate it by quadrature, with $\Phi_4(\mathbf{R}^2) \approx .003234$. It is not possible to have a voting situation in which both B and C dominate A when A is the PMRW, since this would require the PMRW to be ranked last by BR, which is impossible. The symmetry of IC with respect to candidates therefore leads to the conclusion that the conditional probability that some given candidate dominates the PMRW, given that a PMRW exists is given by $6\Phi_4(\mathbf{R}^2)/P_{PMRW}(3, \infty, IC)$. This value is given by .02127.

The use of this less restrictive condition does not result in a significant increase in the conditional probability of observing the outcome that is described by Merlin et al (2002). When the probability of having BDA, given that A is PMRW, is calculated for IAC as a function of n, we obtain (using the same approach as above):

$$\frac{(n-8)(n^3 - 17n^2 + 24n + 352)}{80(n+2)^2 (n+4)^2}$$

for n=8, 16, 24, 32… Thus, as $n \to \infty$, the resulting probability is $1/80 = .0125$. Again, the slight degree of dependence that is suggested by IAC significantly decreases the already small probability that is obtained with the assumption of IC.

4. Another Relaxation Condition

We can obtain a representation for the probability that Condorcet's Other Paradox is observed by considering another relaxation of the conditions that were given by Merlin et al (2002). Suppose that candidate A is the PMRW and that it is not dominated by either B or C, but A is still never selected as the WSR for any $0 \le \lambda \le 1$. Such an outcome can occur in a voting situation in which

AMC	$[n_1 + n_2 + n_3 > n_4 + n_5 + n_6]$	(7)
AMB	$[n_1 + n_2 + n_4 > n_3 + n_5 + n_6]$	(8)
$Score(B,0) > Score(A,0)$	$[n_3 + n_5 > n_1 + n_2]$	(9)
$Score(A,0) > Score(C,0)$	$[n_1 + n_2 > n_4 + n_6]$	(10)
$Score(C,1) > Score(A,1)$	$[n_5 + n_6 > n_1 + n_3]$	(11)
$Score(A,1) > Score(B,1)$	$[n_2 + n_4 > n_5 + n_6]$	(12)
$Score(C,\lambda^*) > Score(A,\lambda^*)$	$[n_4 + n_6 + \lambda^*(n_2 + n_5) > n_1 + n_2 + \lambda^*(n_3 + n_4)]$	(13)

Candidate C does not dominate A because A beats C under PR due to the restriction in Eq. 10. Similarly, candidate B does not dominate A since A beats B under NPR in Eq. 12. It follows directly from the linearity of $Score(X,\lambda)$ as λ increases for each $X \in \{A,B,C\}$, that A must always be beaten by either B or C if it is also true that a value of λ^* exists for some $0 < \lambda^* < 1$ with $Score(C,\lambda^*) = Score(B,\lambda^*)$ and in addition we have $Score(C,\lambda^*) > Score(A,\lambda^*)$.

In order to have $Score(C,\lambda^*) = Score(B,\lambda^*)$, we need

$$n_4 + n_6 + \lambda^*(n_2 + n_5) = n_3 + n_5 + \lambda^*(n_1 + n_6),$$

so that

$$\lambda^* = \frac{n_3 + n_5 - n_4 - n_6}{n_2 + n_5 - n_1 - n_6}.$$

For $Score(C,\lambda^*) > Score(A,\lambda^*)$,

$$n_4 + n_6 + \lambda^*(n_2 + n_5) > n_1 + n_2 + \lambda^*(n_3 + n_4),$$

so that

$$n_4 + n_6 + \frac{n_3 + n_5 - n_4 - n_6}{n_2 + n_5 - n_1 - n_6}(n_2 + n_5) > n_1 + n_2 + \frac{n_3 + n_5 - n_4 - n_6}{n_2 + n_5 - n_1 - n_6}(n_3 + n_4). \quad (14)$$

If we sum Eqs. 9 through 12, and reduce the results, we obtain

$$n_2 + n_5 > n_1 + n_6.$$

So, the result that is given above to require $Score(C, \lambda^*) > Score(A, \lambda^*)$ in Eq. 14 becomes

$$(n_4 + n_6)(n_2 + n_5 - n_1 - n_6) + (n_3 + n_5 - n_4 - n_6)(n_2 + n_5) >$$
$$(n_1 + n_2)(n_2 + n_5 - n_1 - n_6) + (n_3 + n_5 - n_4 - n_6)(n_3 + n_4).$$

This can be reduced to make the restriction in Eq. (14) equivalent to

$$n_1^2 + n_4^2 + n_5^2 + n_2 n_3 + n_2 n_6 + n_3 n_6 > n_2^2 + n_3^2 + n_6^2 + n_1 n_4 + n_1 n_5 + n_4 n_5. \qquad (15)$$

The nonlinear nature of the restriction in Eq. 15 makes it very difficult to obtain an estimate of this probability with the assumption of IC. However, Monte-Carlo simulation was used to obtain an estimate of this probability with the assumption of IAC as $n \rightarrow \infty$, following a procedure developed in Tovey (1997). The conditional probability that a profile exists for which the PMRW is not dominated by either of the other candidate, and still is never selected as the winner for any WSR, given that a PMRW exists, is approximated as .00017. This is an extremely small probability. Adding this to the IAC probability from the previous section, we obtain the probability that Condorcet's Other Paradox, as we have defined it, will be observed. The resulting probability for IAC as $n \rightarrow \infty$ is only .01267, to indicate that actual observances of Condorcet's Other Paradox should be rare events. If we couple this observation with the knowledge of the fact that IC and IAC are expected to give inflated estimates of the probability that voting paradoxes will be observed, it can be concluded that actual observances of Condorcet's Other Paradox should be *very* rare events.

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