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Cournot−Walras equilibrium without profit feedback

Leo Kaas
University of Vienna

Abstract

In this note we consider a general equilibrium model with oligopolistic competition between firms who ignore the feedback effect of their dividend payments on demand. The outcome of this competition coincides with the perfectly competitive equilibrium solution, provided that firms have identical production technologies.

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1 Introduction

The concept of a Cournot–Walras equilibrium has been introduced by Gabszewicz and Vial (1972) to describe oligopolistic interaction between firms in a general equilibrium setting. Firms decide production plans, knowing the objective inverse demand curve which captures the competitive demand behavior of their consumers, and which also takes into account the feedback effect of firms' dividend payments on consumers' wealth.

The objective demand approach imposes strong informational assumptions on oligopolists. For this reason, some authors proposed general equilibrium models with imperfect competition assuming that firms do not take into account the feedback effect of their dividend payments on demand (see Marschak and Selten 1974, Silvestre 1977 and Hart 1985). Such an assumption seems reasonable in an economy consisting of a large number of sectors (islands) in which producers and consumers trade on only one island, but in which consumers hold profit shares in firms on other islands. Such an "island economy" has been introduced by Hart (1985) who formulates a general equilibrium model of price competition in which there is no profit feedback. Implicitly, also most of the macroeconomic literature with oligopolistic or monopolistic competition uses the assumption that firms ignore the profit feedback on demand (see e.g. Blanchard and Kiyotaki 1987 and Hart 1982).

This note considers a modification of the Cournot–Walras equilibrium concept of Gabszewicz and Vial (1972) in which firms are ignorant of the profit feedback. Unlike the original Cournot–Walras equilibrium whose outcome depends critically on the choice of the price normalization rule (see Böhm 1994 and Grodal 1996), this equilibrium concept is independent of price normalization. More surprisingly, our equilibrium concept gives rise to a Bertrand–like result: competition between firms with identical technology sets leads to the perfectly competitive solution. Using two examples, we illustrate that our modified Cournot–Walras equilibrium differs from the original Cournot–Walras equilibrium (for several standard price normalization rules). It typically also differs from the competitive equilibrium, when firms have non–identical technologies.

2 The economy

Description of the model

Consider a private–ownership economy

$$\Big(I, J, L, (X_i, u_i, \omega_i)_{i\in I}, (Y_j)_{j\in J}, (\delta_{ij})_{i\in I, j\in J}\Big)$$

with the usual interpretation: I and J are the sets of consumers and firms with typical elements i and j, respectively, and cardinalities of these sets are also denoted I and J. There are L commodities. Consumer $i \in I$ is described by a utility function u_i defined over his consumption set X_i (here assumed to be $\mathbb{R}^L_+$), and his vector of initial endowments $\omega_i \in \mathbb{R}^L$. Firms $j \in J$ choose production plans y_j out of their production sets $Y_j \subset \mathbb{R}^L$. δ_{ij} is i's share of the profits of firm j.

Let $p \in \mathbb{R}^L_{++}$ denote a price vector. If firms choose production plans $(y_j)_{j\in J}$, the profit income of consumer i is denoted $\pi_i = \sum_{j\in J} \delta_{ij} p y_j$. Unlike Gabszewicz and Vial (1972), we assume that firms take the profit incomes of their consumers $\pi = (\pi_i)_{i\in I}$ as given. This assumption can be justified in an economy consisting of a large number of islands in which consumers and producers trade on only one island, but in which consumers hold profit shares in firms on other islands. When the number of islands is large and profit shares are sufficiently distributed, these indirect profit feedbacks are small, and it is therefore reasonable that firms ignore these effects (see Hart 1985). For ease of exposition, we do not formulate such an island economy explicitly, but we simply assume that firms take profit incomes as given when they decide on production plans.

The objective inverse demand

Assume that utility functions are continuous, strictly quasi–concave, strictly monotone, and twice differentiable. Then the solution of consumer i's utility maximization problem is described by the Walrasian demand function $x_i(p, w_i)$ which fulfills $p \cdot x_i(p, w_i) = w_i$ for all price/income–pairs (p, w_i). Consumers' income levels are $w_i = p\omega_i + \pi_i$, so that we can write the consumers' aggregate demand as

$$z(p, \pi) \equiv \sum_{i\in I} \Big(x_i(p, p\omega_i + \pi_i) - \omega_i \Big) \ ,$$

which is differentiable and homogeneous of degree zero in $(p, \pi) \in \mathbb{R}^L_{++} \times \mathbb{R}^I_+$.

Given production plans of firms, $y = (y_j)_{j\in J}$, and the vector of profit incomes π, a market clearing price vector p fulfills

$$z(p, \pi) = \bar{y} \equiv \sum_{j\in J} y_j \ .$$

Let $\mathcal{W}(\bar{y}, \pi)$ denote the set of market clearing price vectors. Under the assumption $Y_j \subset \mathbb{R}^L_+$ which is also imposed by Gabszewicz and Vial (1972), this set is non–empty for all non–zero production plans:

Lemma: If $\sum_{i\in I} \omega_i \gg 0$ and $Y_j \subset \mathbb{R}^L_+$ for all $j \in J$, then $\mathcal{W}(\bar{y}, \pi) \neq \emptyset$ for all $\bar{y} \in \sum_{j\in J} Y_j \setminus \{0\}$ and $\pi \in \mathbb{R}^I_+ \setminus \{0\}$.

Proof: Take any $\bar{y} \geq 0$, $\bar{y} \neq 0$, and any $\pi \geq 0$, $\pi \neq 0$, and consider the associated pure exchange economy with endowments $\tilde{\omega}_i = \omega_i + (\pi_i/\bar{\pi})\bar{y}$ where

$\bar{\pi} \equiv \sum_{i \in I} \pi_i > 0$. Since $\sum_{i \in I} \tilde{\omega}_i \gg 0$, it has a Walrasian equilibrium price vector $p \in \mathbb{R}^L_{++}$ (see Mas–Colell *et al.* 1995, Ch. 17.C). Thus,

$$\sum_{i \in I} \Big(x_i(p, p\omega_i + (\pi_i/\bar{\pi}) p\bar{y}) - \omega_i \Big) = \bar{y} \ .$$

Since this equation is homogenous of degree zero in p and since $\bar{\pi} > 0$ and $\bar{y} \neq 0$, p can be normalized such that $p\bar{y} = \bar{\pi}$. Hence, $z(p, \pi) = \bar{y}$. □

A few remarks are in order. First, the set of market clearing price vectors may be multi–valued, as a pure–exchange economy may have multiple equilibrium price vectors. We denote a selection of this set by $P(\bar{y}, \pi)$ which, in general, need not be a continuous function. $P(., \pi)$ is the "objective" inverse demand function against which firms who ignore the profit feedback play a Cournot game. Second, since the correspondence $\mathcal{W}(\bar{y}, \pi)$ is homogeneous of degree one in $\pi \in \mathbb{R}^I_+$, we assume that also the selection $P(\bar{y}, \pi)$ is homogeneous of degree one in π (which is, of course, only an assumption when there are multiple equilibrium price vectors). Third, we may restrict price selections to those which have discontinuities only at critical points $(\bar{y}, \pi)$ of the projection of the graph of $\mathcal{W}(., .)$ onto $\mathbb{R}^L \times \mathbb{R}^I$ (similar to Dierker and Grodal 1986, p. 170). If $(\bar{y}, \pi)$ is a regular point of this projection, the objective inverse demand P is then a differentiable function in the neighborhood of this point.

3 The equilibrium

Definition: $(y^*, p^*, \pi^*) \in \prod_{j \in J} Y_j \times \mathbb{R}^L_{++} \times \mathbb{R}^I_+$ is a *Cournot–Walras equilibrium without profit feedback* if

(i) $y_j^* \in \text{argmax}_{y_j \in Y_j} P(y_j + \sum_{k \neq j} y_k^*, \pi^*) y_j \quad \forall\, j \in J$,
(ii) $p^* = P(\bar{y}^*, \pi^*)$,
(iii) $\pi_i^* = \sum_{j \in J} \delta_{ij} p^* y_j^* \quad \forall\, i \in I$.

Note that a Cournot–Walras equilibrium without profit feedback is independent of price (and profit) normalization: If (y^*, p^*, π^*) is an equilibrium, then $(y^*, \lambda p^*, \lambda \pi^*)$ is also an equilibrium for any $\lambda > 0$, since $P(\bar{y}, .)$ is linearly homogeneous. The determination of absolute prices plays no role in our equilibrium concept, as it is the case in a competitive equilibrium. It contrasts however to Gabszewicz and Vial's concept of a Cournot–Walras equilibrium (with profit feedback) where the determination of absolute prices (the normalization rule) affects the equilibrium allocation. In fact, Grodal (1996) shows that any arbitrary production plan can generally be obtained as an equilibrium by a suitable choice of price normalization. This dependence on price normalization is sometimes attributed to the hypothesis of profit maximization which needs not be in the interest of firms' shareholders. In contrast, in the interpretation of our economy as an island economy, profit maximization would be in the interest of shareholders (who trade only with firms on other islands). We return to the normalization issue in the examples below.

From the definition of the inverse demand function P and the budget constraints of consumers follows that the sum of firms' profits is independent of production plans and equals the aggregate profit income of consumers. Indeed, for all $\pi \in \mathbb{R}^I_+$ and all $\bar{y} \in \sum_{j \in J} Y_j$ we have:

$$\begin{aligned} P(\bar{y},\pi)\bar{y} &= P(\bar{y},\pi) z(P(\bar{y},\pi),\pi) \\ &= P(\bar{y},\pi)\sum_{i\in I}\Big(x_i(P(\bar{y},\pi),P(\bar{y},\pi)\omega_i+\pi_i)-\omega_i\Big) = \sum_{i\in I}\pi_i = \bar{\pi}\ . \end{aligned} \tag{1}$$

The case of a single monopolist ($J = 1$) turns out to be completely indeterminate: For any $y \in Y$, $(y, P(y,(\delta_i)_{i\in I}), (\delta_i)_{i\in I})$ is an equilibrium since (1) implies that the monopolist's profit is constant, whenever the inverse demand function is defined. If there are more firms, the situation is different however. (1) then implies that maximization of any firm's profit is equivalent to the minimization of the cumulative profits of all other firms. A competition with this feature turns out to be efficient if all firms have identical technologies. The following theorem shows that whenever firms have identical technologies, a symmetric Cournot–Walras equilibrium without profit feedback is a competitive equilibrium, and vice versa if profit functions are strictly quasi–concave. A *competitive equilibrium* is defined as a vector (y^*, p^*, π^*) which fulfills (ii) and (iii) in the above definition, but satisfies instead of (i) the profit maximization condition under price–taking behaviour:

$$y_j^* \in \underset{y_j \in Y_j}{\operatorname{argmax}}\, p^* y_j \quad \forall\ j \in J\ .$$

Theorem: Let $J > 1$, suppose $Y_j = Y$ for all $j \in J$ where $Y \subset \mathbb{R}^L$ is convex. Let $(y^*, p^*, \pi^*) \in Y^J \times \mathbb{R}^L_{++} \times \mathbb{R}^I_+$ where $y_j^* = \hat{y}^*$ for all $j \in J$. Assume that $P(., \pi^*)$ is continuously differentiable at $J\hat{y}^*$. Then it follows:

(i) If (y^*, p^*, π^*) is a Cournot–Walras equilibrium without profit feedback, then it is a competitive equilibrium.

(ii) If (y^*, p^*, π^*) is a competitive equilibrium and if the profit function $\Pi(y) = P(y+(J-1)\hat{y}^*, \pi^*)y$ is strictly quasi–concave, then it is a Cournot–Walras equilibrium without profit feedback.

Proof: Define $A := d_y P(J\hat{y}^*, \pi^*) \in \mathrm{I\!R}^{L\times L}$. Then differentiation of equation (1) with respect to y (for fixed π^*) at $y = \hat{y}^*$ implies

$$0 = \frac{d}{dy}\Big(P(Jy, \pi^*)Jy\Big)\Big|_{y=\hat{y}^*} = J(A \cdot J\hat{y}^* + p^*) \quad ,$$

where $p^* = P(J\hat{y}^*, \pi^*)$. With $\Pi(y) = P(y + (J-1)\hat{y}^*, \pi^*)y$, this yields

$$d\Pi(\hat{y}^*) = A \cdot \hat{y}^* + p^* = \frac{J-1}{J}p^* \quad . \tag{2}$$

If (y^*, p^*, π^*) is a Cournot–Walras equilibrium without profit feedback, then $\hat{y}^* \in \mathrm{argmax}_{y\in Y}\,\Pi(y)$ and convexity of Y imply $d\Pi(\hat{y}^*)(y - \hat{y}^*) \leq 0$ for all $y \in Y$. Using (2) gives $p^*\hat{y}^* \geq p^*y$ for all $y \in Y$, and claim (i) follows.

If (y^*, p^*, π^*) is a competitive equilibrium, it follows again from (2) and $p^*\hat{y}^* \geq p^*y$ that $d\Pi(\hat{y}^*)(y - \hat{y}^*) \leq 0$ for all $y \in Y$. But now strict quasi–concavity of $\Pi(.)$ and convexity of Y imply $\Pi(\hat{y}^*) \geq \Pi(y)$ for all $y \in Y$, and therefore (ii) follows. □

Part (ii) in this theorem imposes the strong assumption of strict quasi–concavity of profit functions. Such an assumption cannot be derived from hypotheses on fundamentals, but is standard in the literature on general equilibrium with imperfect competition to guarantee existence of equilibrium (see e.g. Hart 1985). Note however, that part (i) does not need this assumption. That is, whenever a Cournot–Walras equilibrium exists it must be a competitive equilibrium (provided that firms have identical technologies).

4 Two examples

We now provide two examples that illustrate the theorem and compare our equilibrium concept to the one of Gabszewicz and Vial. The first is a simple "Robinson Crusoe" economy in which firms have identical technologies. The second example shows that the theorem does not extend to competition between firms with different technologies.

Example 1: Consider an economy with two commodities (output good and labor), 2 firms and one consumer. Firms have identical technology sets

$$Y_j = \Big\{(y_j, -\ell_j) \mid 0 \leq y_j = \ell_j \leq 1\Big\} \quad , \; j = 1, 2 \; .$$

The consumer has an endowment of $\bar{\ell} > 2$ units of labor and zero endowment of the output good, and his utility function is $\ln(y) + \zeta\ell$, $\zeta < 1/2$, where (y, ℓ) denotes consumption of the output good and leisure. Utility maximization implies that the real wage is $w/p = \zeta y$, and the goods market clears when

$y = y_1 + y_2$ (the labor market clears by Walras's law). The assumption $\zeta < 1/2$ implies that the competitive equilibrium has firms producing at full capacity, $y_1 = y_2 = 1$, with $w/p = 2\zeta < 1$. Now consider the Cournot–Walras equilibrium without profit feedback. From $w/p = \zeta y$ and the consumer's budget constraint $py = wy + \pi$ ($\pi = \pi_1 + \pi_2$) follow the inverse demand functions (for all $\pi > 0$ and $y = \ell \in (0, 2]$):

$$p((y,\ell),\pi) = \frac{\pi}{y(1-\zeta y)} \quad \text{and} \quad w((y,\ell),\pi) = \frac{\pi\zeta}{1-\zeta y} \ .$$

Hence, firms' profit functions are $\pi_j = (p-w)y_j = \pi y_j/(y_1+y_2)$, $j = 1, 2$, which are strictly increasing in y_j, so that the unique Nash equilibrium coincides with the competitive equilibrium. On the other hand, the original Cournot–Walras equilibrium (with profit feedback) depends decisively on the normalization rule. To give an example, suppose first that the wage is the numéraire, $w = 1$, so that the Cournot–Walras equilibrium describes a duopoly. The inverse goods demand is then $p = 1/(\zeta y)$, profit functions are $\pi_j = (1/(\zeta(y_1 + y_2)) - 1)y_j$, $j = 1, 2$, so that the unique Nash equilibrium has $y_1 = y_2 = \min(1, 1/(4\zeta))$ which differs from the competitive equilibrium if $\zeta > 1/4$. On the other hand, suppose we fix the price, $p = 1$, so that the Cournot–Walras equilibrium describes a duopsony. The inverse labor supply is $w = \zeta y$, profit functions are $\pi_j = (1 - \zeta(y_1 + y_2))y_j$, $j = 1, 2$, and the unique Nash equilibrium is $y_1 = y_2 = \min(1, 1/(3\zeta))$ which now differs from both the competitive equilibrium and from the duopoly Cournot-Walras equilibrium if $\zeta > 1/3$. $\diamond$

Example 2: Let $I = 1$, $J = 2$, $L = 2$ and assume that the single consumer has no endowment and that his preferences are represented by $u(x_1, x_2) = v(x_1) + v(x_2)$, where $v : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, and fulfills $v(x) = x - x^2/2$ for all $0 \le x \le 3/4$. Technology sets of the two firms differ:

$$Y_j = \Big\{(y_{j1}, y_{j2}) \mid 0 \le y_{jj} \le 3/4,\ y_{ji} = 0 \text{ if } i \neq j\Big\} \quad , \ j = 1, 2 \ .$$

With abuse of notation, write y_j instead of y_{jj}. The inverse demand functions are (for profit income $\pi > 0$ and production plans $y_j \in (0, 3/4]$, $j = 1, 2$)

$$P_1((y_1,y_2),\pi) = \frac{\pi v'(y_1)}{y_1 v'(y_1) + y_2 v'(y_2)} \quad , \quad P_2((y_1,y_2),\pi) = \frac{\pi v'(y_2)}{y_1 v'(y_1) + y_2 v'(y_2)} \quad .$$

Thus, maximization of $P_1((y_1, y_2), \pi)y_1$ over $y_1 \in [0, 3/4]$ and given $y_2 > 0$ is equivalent to the maximization of $y_1 v'(y_1) = y_1(1 - y_1)$ which has the solution $y_1 = 1/2$. By symmetry, the best response of firm 2 is $y_2 = 1/2$ and equilibrium prices are $p_1 = p_2 = \pi$. Thus, the unique Cournot–Walras equilibrium without profit feedback differs from the unique competitive equilibrium with production plans $y_1 = y_2 = 3/4$ and prices $p_1 = p_2 = 2\pi/3$. Now consider the original Cournot–Walras equilibrium (with profit feedback). If good 1 is the numéraire ($p_1 = 1$), firm 1 produces at full capacity, $y_1 = 3/4$, whereas the inverse demand for good 2 is $p_2 = (1-y_2)/(1-y_1)$. Hence, profit maximization of firm 2 yields $y_2 = 1/2$. Conversely, choosing good 2 as numéraire yields the Cournot–Walras equilibrium $y_1 = 1/2$ and $y_2 = 3/4$. Finally, simplex normalization ($p_1+p_2 = 1$) yields inverse demand functions $p_i = (1-y_i)/(2-y_1-y_2)$, $i = 1, 2$. It turns out that the Cournot–Walras equilibrium is now symmetric and is given by $y_1 = y_2 = 2/3$. This example shows that the Cournot–Walras equilibrium without profit feedback has an activity level which is below the competitive equilibrium and which is also below the activity levels in the Cournot–Walras equilibrium for three standard normalization rules. $\diamond$

5 Conclusion

We considered a general equilibrium model with oligopolistic competition between firms who know the objective competitive demand behavior of their consumers, but who are ignorant of the profit feedback on demand. The outcome of this competition between firms with identical production technologies is the competitive equilibrium. Thus, firms do not only take the profit incomes of their consumers as given, but they effectively behave as if they were price takers.

The number of oligopolists plays no role for this result (as it does not in the classic Bertrand paradox). However, behind our assumption that firms ignore the profit feedback is the idea of a large number of islands (or sectors). Thus it seems that it is the large number of sectors which is responsible for our result.

6 References

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