E C O N O M I C S   B U L L E T I N

On Dynkin's model of economic equilibrium under uncertainty

Guillaume Carlier
Department of Applied Mathematics, University of Bordeaux 1

Igor Evstigneev
School of Economic Studies, University of Manchester

Abstract

The article analyzes the Dynkin (1975) stochastic model of economic equilibrium. We solve a question regarding this model that was open for a long time. We provide arguments yielding a complete proof of Dynkin's existence theorem for equilibrium paths.

Financial support from grants NSF DMS−9705017, INTAS 97−1050, RFBR 00−01−00297, and the Santa Fe Institute (New Mexico, USA) is gratefully acknowledged. The authors are deeply thankful to E.B. Dynkin for his encouragement and help in this work.
Citation: Carlier, Guillaume and Igor Evstigneev, (2001) "On Dynkin's model of economic equilibrium under uncertainty." *Economics Bulletin,* Vol. 3, No. 12 pp. 1−8
Submitted: September 24, 2001. **Accepted:** September 30, 2001.
URL: http://www.economicsbulletin.com/2001/volume3/EB−01C00001A.pdf

1. Introduction

In this article, we consider a stochastic model of dynamic economic equilibrium proposed by E.B. Dynkin (1975,1976). Dynkin's study was aimed at the integration – in a stochastic context – of the Arrow-Debreu general equilibrium theory and the theory of economic growth[1]. As the background for his analysis, he used the work of Polterovich (1973,1978), focusing on some specialized ("fixed income") models.

In his paper (Dynkin 1975), presented at the Congress of Mathematicians in Vancouver in 1974, Dynkin described the model and stated an existence theorem for equilibrium. The results and their proofs were set out in detail in a subsequent publication (Dynkin 1976). To obtain the results, it turned out to be necessary to overcome substantial technical difficulties and to develop new general methods and concepts (regular conditional expectations of correspondences depending on parameters, Dynkin and Evstigneev (1976)).

The argumentation in Dynkin (1976) was quite sophisticated, and it turned out later that one of the stages in the proof of the main result – the existence theorem for equilibrium – contained a gap. The purpose of our note is to fill this gap. We hope that our comments will complete Dynkin's elegant and deep study.

2. The model

Let s_t, $t \in \{1, 2, ..., T+1\}$, be a stochastic process such that, for each t, the random variable s_t takes values in a measurable space $(S_t, \mathcal{F}_t)$. Elements of S_t describe "states of the world", which might influence the economic system at time t. In the economy under consideration, there are m commodities. Vectors $x = (x^1, ..., x^m)$ in the non-negative cone R^m_+ of the Euclidean space R^m represent commodity bundles. There are I producers and J consumers. A producer i at time t is characterized by the *technology set* $\mathcal{T}_{ti}(s^t) \subseteq R^m_+ \times R^m_+$ depending on the history $s^t = (s_1, ..., s_t)$ of the process $\{s_t\}$ up to time t. Elements $(x, y) \in \mathcal{T}_{ti}(s^t)$ are construed as feasible *technological processes* (with *input* x and *output* y). A consumer j at time t is described by a *consumption set* $C_{tj}(s^t) \subseteq R^m_+$, a *utility function* $u_{tj}(s^t, c)$, $c \in C_{tj}(s^t)$, and *income* $w_{tj}(s^t) \geq 0$. For each *price vector* $p \in R^m_+$, we denote by $\phi_{tj}(s^t, p)$ the (possibly empty) set of those $c \in C_{tj}(s^t)$ which maximize the utility function $u_{tj}(s^t, c)$ over *consumption vectors* $c \in C_{tj}(s^t)$ satisfying the budget constraint $pc \leq w_{jt}(s^t)$. The mapping $p \mapsto \phi_{tj}(s^t, p)$ is the *demand correspondence* of consumer j.

Define $(S^t, \mathcal{F}^t) = (S_1, \mathcal{F}_1) \times ... \times (S_t, \mathcal{F}_t)$ and consider the distribution μ^t on the random element s^t in the space S^t. Denote by $\bar{\mathcal{F}}^t$ the completion of the σ-algebra $\mathcal{F}^t$ with respect to the measure μ^t. A sequence of vector functions $(x_{ti}(s^t), y_{ti}(s^t))$,

[1]Various approaches to this subject are surveyed, e.g., in Radner (1982), Grandmont (1988), and Mas-Colell, Whinston and Green (1995).

$t \in \{1, ..., T\}$, is called a *plan of producer i* if x_{ti} and y_{ti} are measurable with respect to $\bar{\mathcal{F}}^t$ and

$$(x_{ti}(s^t), y_{ti}(s^t)) \in \mathcal{T}_{ti}(s^t) \tag{1}$$

for all t, s^t. A sequence of vector functions $c_{tj}(s^t)$, $t \in \{1, ..., T+1\}$, is called a *plan of consumer j* if c_{tj} is $\bar{\mathcal{F}}^t$ -measurable and

$$c_{tj}(s^t) \in C_{tj}(s^t) \tag{2}$$

for all t, s^t. Conditions (1) and (2) mean that the mappings $(x_{ti}(\cdot), y_{ti}(\cdot))$ and $c_{tj}(\cdot)$ are *selectors* of the correspondences $\mathcal{T}_{ti}(\cdot)$ and $C_{tj}(\cdot)$, respectively.

Let $p_1(s^1), ..., p_{T+1}(s^{T+1})$ be a sequence of functions with values in R^m_+ such that p_t is $\bar{\mathcal{F}}^t$ -measurable. We shall interpret $\{p_t\}$ as a *price system.* For a commodity vector $x \in R^m_+$, the scalar product $p_t(s^t)x$ expresses the *cost* of the commodity bundle x at time t in the random situation s^t. Given the price system $\{p_t\}$, we shall say that a plan (x_{ti}, y_{ti}), $t \in \{1, ..., T\}$, of producer i is *optimal* if it maximizes the *expected profit*

$$\mathbf{E} \sum_{t=1}^{T} (p_{t+1} y_t - p_t x_t) \tag{3}$$

over all plans (x_t, y_t), $t \in \{1, ..., T\}$, of producer i for which the expectation in (3) is well-defined. A plan c_{tj}, $t \in \{1, ..., T+1\}$, of consumer j is said to be *optimal* if $c_t(s^t) \in \phi_t(s^t, p_t(s^t))$ for all t and s^t.

Throughout the paper, we will assume a nonnegative vector $y_0 \in R^m_+$ (the *initial stock*) to be fixed. Let

$$p_t,\ t \in \{1, ..., T+1\}, \tag{4}$$

be a price system and let

$$(x_{ti}, y_{ti}),\ t \in \{1, ..., T\},\ i \in \{1, ..., I\}, \tag{5}$$

$$c_{tj},\ t \in \{1, ..., T+1\},\ j \in \{1, ..., J\}, \tag{6}$$

be optimal programs of producers and consumers. We shall say that the price system (4) and the programs (5), (6) form an *equilibrium* (with initial stock y_0) if, for all t and s^t, we have

$$\Delta_t(s^t) \geq 0 \text{ and } p_t(s^t)\Delta_t(s^t) = 0, \tag{7}$$

where $\Delta_t = \sum_{i=1}^{I} (y_{t-1,i} - x_{t,i}) - \sum_{j=1}^{J} c_{tj}$, $t \in \{2, ..., T+1\}$, and $\Delta_1 = y_0 - \sum_{i=1}^{I} x_{1,i} - \sum_{j=1}^{J} c_{1j}$. Conditions (7) mean that each component of the vector $\Delta_t(s^t)$ (describing *excess supply)* is nonnegative, and it is strictly positive if and only if the corresponding component of the price vector $p_t(s^t)$ is zero.

3. The assumptions and the result

Assume the following.

(A.1) The correspondences $\mathcal{T}_{ti}(s^t)$ and $C_{tj}(s^t)$ are measurable[2] with respect to $\bar{\mathcal{F}}^t$. Their values are closed convex sets containing the origin. There exists a constant K such that $|z| \le K$ for all $z \in \mathcal{T}_{ti}(s^t)$, t, i and s^t.

Here and in what follows, we write $|z|$ for the sum of the absolute values of the coordinates of the vector z.

(A.2) The functions $u_{tj}(s^t, c)$ are $\bar{\mathcal{F}}^t$-measurable in $s^t \in S^t$ and continuous in $c \in R^m_+$. For each s^t, the function $u_{tj}(s^t, \cdot)$ is concave on $C_{tj}(s^t)$.

(A.3) The real-valued functions $w_{tj}(s^t)$ are strictly positive and $\bar{\mathcal{F}}^t$-measurable. The expectation $L := \mathbf{E} \sum_{t,j} w_{tj}(s^t)$ is finite.

Define $\mathcal{T}_t(s^t) = \sum_i \mathcal{T}_{ti}(s^t)$ (the *aggregate technology set*).

(A.4) There exists a strictly positive non-random vector $\hat{x}$ such that $(\hat{x}, \hat{x}) \in \mathcal{T}_t(s^t)$ for all t and s^t.

Put $\phi_t(s^t, p) = \sum_j \phi_{tj}(s^t, p)$ (*aggregate demand correspondence*).

(A.5) For every s^{T+1}, t and k, either the kth commodity is necessary for consumers at time t, or it is indispensable for the production at a later period $t' \in \{t+1, ..., T+1\}$ of another commodity necessary for consumers at t'.

Here, the expression "the kth commodity is necessary for consumers at time t" means that, for any price vector p, the kth component of every vector $c \in \phi_t(s^t, p)$ is strictly positive. "The kth commodity is indispensable for the production of the lth commodity at time t'" means that the relations

$$x_t^k = 0,\ (x_t, y_t) \in \mathcal{T}_t(s^t);\ x_{t+1} \le y_t,\ (x_{t+1}, y_{t+1}) \in \mathcal{T}_{t+1}(s^{t+1}); ...$$

$$x_{t'} \le y_{t'-1}, ..., (x_{t'}, y_{t'}) \in \mathcal{T}_{t'}(s^{t'})$$

imply $y^l_{t'} = 0$. In view of (A.5), if

$$c_t \in \phi(s^t, p_t),\ (x_t, y_t) \in \mathcal{T}_t,\ x_t + c_t \le y_{t-1},\ t = 1, ..., T,\ \text{and } c_{T+1} \le y_T, \qquad (8)$$

then all the vectors y_t are strictly positive.

Finally, we introduce, following Dynkin (1976), some technical assumptions regarding the underlying stochastic process s_t and the spaces S_t.

(A.6) For every t, the measurable space $(S_t, \mathcal{F}_t)$ is standard (i.e. isomorphic to a Borel subset of a complete separable metric space). The conditional distributions of s_{t+1} given s^t are atomless.

The main result is as follows (see Dynkin 1976, Theorem 4.2).

Theorem 3.1. *For every strictly positive vector $y_0 \in R^m_+$, there exists an equilibrium with initial stock y_0.*

[2] If $A(u)$ is a correspondence assigning a set $A(u)$ in a Euclidean space E to each point u of a measurable space $(U, \mathcal{U})$, then $A(\cdot)$ is said to be *measurable* if $\{u : A(u) \cap M \ne \emptyset\} \in \mathcal{U}$ for any closed set $M \subseteq E$.

We describe the plan of proving this theorem as proposed by Dynkin (1976). First of all, we may assume in what follows that $I = 1$. Indeed, the case of several producers can be reduced to the case of a single producer. To this end, it suffices to replace the system of technology sets $\mathcal{T}_{ti}(s^t)$, $1 \le i \le I$, by the aggregate technology set $\mathcal{T}_t(s^t)$ (see above).

For every $p, q \in R^m_+$ and $s^t \in S^t$, consider the set $\mathcal{T}_t(s^t, p, q)$ of all pairs $(x, y) \in \mathcal{T}_t(s^t)$ for which

$$qy - px = \sup_{x', y' \in \mathcal{T}_t(s^t)} (qy' - px').$$

It follows from the definition that, in order to construct an equilibrium, it is sufficient to find, for every $t \in \{1, 2, ..., T + 1\}$, a collection of $\bar{\mathcal{F}}^t$ -measurable functions x_t,y_t,c_t,p_t,q_t, possessing the following properties:

(A) $(x_t, y_t) \in \mathcal{T}_t(s^t, p_t, q_t)$ almost surely (a.s.) for each $t \in \{1, ..., T\}$; $x_{T+1} = y_{T+1} = 0$.

(B) $c_t \in \phi_t(s^t, p_t)$ for all $s^t \in S^t$, $t \in \{1, ..., T + 1\}$.

(C) $q_t = \mathbf{E}(p_{t+1}|s^t)$ (a.s.) for $t \in \{1, ..., T\}$.

(D) $c_t + x_t \le y_{t-1}$, $p_t(c_t + x_t) = p_t y_{t-1}$ for all $s^t \in S^t$, $t \in \{1, ..., T + 1\}$.

We shall identify sequences $\{x_t, y_t, c_t, p_t, q_t\}$ described above with equilibria and use the same term for referring to them.

Let $\xi = \{\epsilon_1(s^1), ..., \epsilon_T(s^T))$ be a sequence of strictly positive real-valued functions $\epsilon_1(s^1), ..., \epsilon_T(s^T)$ such that ϵ_t is $\bar{\mathcal{F}}^t$-measurable. Define $\mathcal{T}_t^\xi(s^t)$ as the class of all technological processes $(x, y) \in \mathcal{T}_t^\xi(s^t)$ satisfying $y \ge \epsilon_t e$, where $e = (1, 1, ..., 1) \in R^m$. Consider the model in which the technology sets $\mathcal{T}_t(s^t)$ are replaced by $\mathcal{T}_t^\xi(s^t)$. Equilibria in this model will be called ξ-*equilibria.*

Fix a strictly positive vector $y_0 \in R^m_+$. Denote by κ the smallest coordinate of the strictly positive vector $\hat{x}$ described in assumption (A.4) and by ϵ_0 the minimal coordinate of y_0. It can be proved (Dynkin 1976, Section 3) that a ξ-equilibrium exists if the set of functions $\xi = \{\epsilon_1, ..., \epsilon_T\}$ satisfies the following conditions:

$$\epsilon_1(s^1) \le \kappa,\ \epsilon_t(s^t) \le \theta\epsilon_{t-1}(s^{t-1})\ (t \in \{1, ..., T\}), \tag{9}$$

where $\theta = \kappa/2K$ and K is the constant specified in (A.1). Clearly, for each $y_0 > 0$, one can find a ξ with properties (9). Consequently, for every $y_0 > 0$, one can construct a ξ-equilibrium with initial stock y_0.

Remarkably, it turns out that, for every ξ-equilibrium with given initial stock $y_0 > 0$, the production output vectors y_t are bounded away from zero by certain strictly positive random vectors independent of ξ. This makes it possible to deduce the existence of an equilibrium from the existence of a ξ-equilibrium. We will present detailed proofs of these statements in the next section.

4. From ξ-equilibrium to equilibrium

The main goal of this section is to prove the following assertion (see Dynkin 1976, Theorem 4.1).

Theorem 4.1. *For every $\gamma > 0$ there exist functions $\delta_1(s^t), ..., \delta_T(s^T)$ such that δ_t are $\bar{\mathcal{F}}^t$-measurable and the following conditions are fulfilled: (a) for all t, we have $\delta_t > 0$ almost surely; (b) under condition (9), the inequalities $y_t \geq \delta_t e$ (a.s.), $t = 1, ..., T$, are satisfied for every ξ-equilibrium with initial stock $y_0 \geq \gamma e$.*

We provide arguments which fill a gap in the proof of this result in Dynkin (1976). The corrections are concerned with sections 4.3 and 4.4 of the paper cited. We follow the plan of the proof briefly outlined in Evstigneev (2000).

Theorem 3.1 is a consequence of Theorem 4.1. The former can be derived from the latter rather easily (see Dynkin 1976, Section 4). We do not repeat this derivation here and proceed to the proof of Theorem 4.1, which is based on Lemma 4.1 below.

Consider the random variable $w\left(s^{T+1}\right) = \sum_{tj} w_{tj}\left(s^t\right)$ involved in (A.3) and define $W = \sum_{t=1}^{T+1} \mathbf{E}_t w\left(s^{T+1}\right)$, where $\mathbf{E}_t\left(\cdot\right)$ stands for the conditional expectation $\mathbf{E}\left(\cdot \mid s^t\right)$. By virtue of (A.3), we have $\mathbf{E}W < \infty$. Let us write $\eta(b)$ for the smallest coordinate of the vector b. For any strictly positive vector $y_0 \in R^m_+$, denote by $\Xi\left(y_0\right)$ the class of sequences $\xi = \left(\epsilon_1\left(s^1\right), ..., \epsilon_T\left(s^T\right)\right)$ of functions such that ϵ_t is $\bar{\mathcal{F}}^t$-measurable, $\epsilon_t > 0$, and condition (9) holds with $\epsilon_0 = \eta\left(y_0\right)$ and $\theta = \kappa/2K$. For any real $\delta > 0$, define $W\left(\delta\right) = 2W/[(\theta\delta) \wedge \kappa]$, where $a \wedge b$ means the least of the numbers a, b.

Lemma 4.1. *Let $t \leq T+1$ be a natural number and let $\delta\left(s^{t-1}\right)$ be a strictly positive $\bar{\mathcal{F}}^{t-1}$-measurable function (a constant if $t = 1$). Let $y_0 > 0$ and $\xi = \left(\epsilon_1\left(.\right), ..., \epsilon_T\left(\cdot\right)\right) \in \Xi\left(y_0\right)$. Then for every sequence $\{(x_t, y_t, c_t, p_t, q_t)\}$ forming a ξ-equilibrium with initial stock y_0 and satisfying $y_{t-1} \geq \delta e$ (a.s.), we have*

$$\mathbf{E}_t\left|p_l\right| \leq W\left(\delta\right) \text{ (a.s.)}, \; l = t, ..., T+1. \tag{10}$$

Proof: Consider a sequence $\{(x'_t, y'_t), ..., (x'_T, y'_T)\}$ such that (x'_j, y'_j) is an $\bar{\mathcal{F}}^j$-measurable selector of $\mathcal{T}_j^{\epsilon_j}\left(s^j\right)$. We have $q_j y'_j - p_j x'_j \leq q_j y_j - p_j x_j$, $j = t, ..., T$, and $\mathbf{E}_t q_j y'_j = \mathbf{E}_t p_{j+1} y'_j$, $j = t, ..., T$, since $\mathbf{E}_j p_{j+1} = q_j$, $j = t, ..., T+1$. Consequently, $\mathbf{E}_t\left(p_{j+1} y'_j - p_j x'_j\right) \leq \mathbf{E}_t\left(p_{j+1} y_j - p_j x_j\right)$, $j = t, ..., T$. By summing up these inequalities from $j = t$ to $j = T$ and adding $p_t y_{t-1}$ to both sides, we obtain

$$\mathbf{E}_t \sum_{j=t}^{T+1} p_j\left(y'_{j-1} - x'_j\right) \leq \mathbf{E}_t \sum_{j=t}^{T+1} p_j\left(y_{j-1} - x_j\right),$$

where y'_{t-1} is defined as y_{t-1}, and $x'_{T+1} = x_{T+1} = 0$. According to (D), $p_j(y_{j-1} - x_j) = p_j c_j \leq w\left(s^{T+1}\right)$, and hence

$$\mathbf{E}_t \sum_{j=t}^{T+1} p_j \left(y'_{j-1} - x'_j\right) \leq W. \tag{11}$$

Consider now the $\bar{\mathcal{F}}^{t-1}$-measurable random vector $v = \hat{x} \cdot (1 \wedge \theta\kappa^{-1}\eta(y_{t-1}))$. Choose some $l = t+1, ..., T$ and set $(x'_t, y'_t) = ... = (x'_{l-1}, y'_{l-1}) = (v, v)$, $(x'_l, y'_l) = ... = (x'_T, y'_T) = \frac{1}{2}(v, v)$. We have $(x'_j, y'_j) \in \mathcal{T}_j(s^j)$ for any j. Let us show that $y'_j \geq \epsilon_j e$ for each $j = t, ..., T$. To this end it is sufficient to check the inequality $v \geq \epsilon_t e$. Indeed, we then have

$$y'_j = v \geq \epsilon_t e \geq \epsilon_j e \text{ for } t \leq j < l, \;\; y'_j = \frac{1}{2}v \geq \frac{1}{2}\epsilon_t e \geq \theta\epsilon_t e \geq \epsilon_j e \text{ for } j \geq l.$$

To verify that $v \geq \epsilon_t e$, we observe $v \geq [\kappa \wedge \theta\eta(y_{t-1})]e$, and we consider two cases: $t > 1$ and $t = 1$. In the former case, $\theta\eta(y_{t-1}) = \kappa\eta(y_{t-1})/2K \leq \kappa$ by virtue of (A), and so $v \geq \theta\eta(y_{t-1})e \geq \theta\epsilon_{t-1}e \geq \epsilon_t e$. In the latter case, $v \geq [\kappa \wedge \theta\eta(y_0)]e = [\kappa \wedge \theta\epsilon_0]e \geq \epsilon_1$ in view of (9). Thus, $y'_j \geq \epsilon_j e$ and hence (x'_j, y'_j) is a selector of $\mathcal{T}_j^{\epsilon_j}(s^j)$ for all $j = 1, ..., T$. By substituting (x'_j, y'_j) into (11), we get

$$\mathbf{E}_t[p_t(y_{t-1} - v) + \frac{1}{2}p_l v + \frac{1}{2}p_{T+1}v] \leq W. \tag{12}$$

Since $|v| \leq |\hat{x}|\,\eta(y_{t-1})\theta\kappa^{-1} = (|\hat{x}|\,/2K)\,\eta(y_{t-1}) \leq \eta(y_{t-1})/2$, we find $y_{t-1} - v \geq \eta(y_{t-1})e/2$, and so $y_{t-1} - v \geq v$. Thus, (12) implies the inequalities $\mathbf{E}_t p_j v \leq 2W$ for $j = t, ..., T+1$. But $v \geq \kappa \wedge \theta\eta(y_{t-1})e \geq \kappa \wedge (\theta\delta)\, e$. Therefore (10) holds. □

To obtain Theorem 4.1 it suffices to prove the following lemma.

Lemma 4.2. *For any $t = 0, ..., T$ and $\gamma > 0$, there exists a constant $\delta_0 > 0$ and functions $\delta_1(s^1) > 0$,...,$\delta_t(s^t) > 0$ such that δ_j is $\bar{\mathcal{F}}^j$-measurable and the inequalities $y_j \geq \delta_j e$, $j = 0, ..., t$, hold for every ξ-equilibrium with $y_0 \geq \gamma e$, $\xi \in \Xi(y_0)$.*

For $t = 0$, the above assertion is trivial (put $\delta_0 = \gamma$). If this assertion is established for $t = T$, we immediately obtain Theorem 4.1.

Let us prove Lemma 4.2 for some $t \in \{1, ..., T\}$ assuming that it is already proven for $t - 1$. We will construct a strictly positive $\bar{\mathcal{F}}^t$-measurable function $\delta_t(s^t)$ such that $y_t \geq \delta_t e$ for every ξ-equilibrium with $y_0 \geq \gamma e$, $\xi \in \Xi(y_0)$.

It is well-known (see, for example, Neveu 1965, Proposition II.4.1) that from any class $\mathcal{H}$ of non-negative measurable functions on a probability space $(\Omega, \mathcal{F}, P)$, it is possible to select a sequence of functions h_n with the property

$$h \geq \inf h_n \text{ a.s. for every } h \in \mathcal{H}. \tag{13}$$

Let us apply this proposition to the space $(S^t, \bar{\mathcal{F}}^t, \mu_t)$ and the class $\mathcal{H}$ defined as follows. Let us write $(y_0, \zeta) \in \mathcal{C}$ if ζ is a ξ-equilibrium with the initial stock $y_0 \geq \gamma e$ and $\xi \in \Xi(y_0)$. Let $h \in \mathcal{H}$ if there exists $(y_0, \zeta) \in \mathcal{C}$ such that $h(s^t) = \eta(y_t(s^t))$.

Choose a sequence h_n with property (13) and denote by δ_t an $\bar{\mathcal{F}}^t$-measurable function for which $\delta_t = \inf h_n$ (a.s.). Condition (b) is satisfied by virtue of (13). It remains to prove that condition (a) holds.

Consider elements (y_0^n, ζ^n) of $\mathcal{C}$ corresponding to h_n. Here, $\zeta^n = \{(x_t^n, y_t^n, c_t^n, p_t^n, q_t^n)\}$ is a ξ-equilibrium. Define $\Lambda^n\left(s^{T+1}\right) = \sum_1^{T+1} \left|p_j^n(s^j)\right|$ and $\lambda^n(s^t) = \int \pi(d\sigma \mid s^t)\Lambda^n(s^t, \sigma)$, where $\pi(d\sigma \mid s^t)$ is the conditional distribution of the collection of random parameters $\sigma = (s_{t+1}, ..., s_{T+1})$ given s^t (this conditional distribution exists since the spaces S_t are standard). We have $\lambda^n(s^t) = \sum_1^t \left|p_j^n(s^j)\right| + \mathbf{E}_t \sum_{t+1}^{T+1} \left|p_j^n(s^j)\right|$, and, by virtue of Lemma 4.1,

$$\sup \lambda^n\left(s^t\right) \leq (T+1)\left[W(\delta_0) + W(\delta_1) + ... + W(\delta_{t-1})\right] < \infty \text{ a.s.}$$

Therefore the set $\Gamma = \{s^t : \sup \lambda^n(s^t) < \infty, \ \inf h_n = \delta_t\}$ has measure 1, and it is sufficient to show that $\delta_t > 0$ on Γ.

Fix $\bar{s}^t = (\bar{s}_0, ..., \bar{s}_t) \in \Gamma$. By passing to a subsequence, we may assume without loss of generality that

$$\sup \lambda^n\left(\bar{s}^t\right) < \infty, \tag{14}$$

$$\eta\left(y_t^n\left(\bar{s}^t\right)\right) = h_n\left(\bar{s}^t\right) \to \inf h_n\left(\bar{s}^t\right) = \delta_t\left(\bar{s}^t\right). \tag{15}$$

By Fatou's lemma, (14) implies $\liminf \Lambda^n(\bar{s}^t, \sigma) < \infty$ for $\pi(d\sigma \mid \bar{s}^t)$-almost all $\sigma = (s_{t+1}, ..., s_{T+1})$. Consequently, there exist $\bar{\sigma} = (\bar{s}_{t+1}, ..., \bar{s}_{T+1})$ and $\{n_k\}$ such that the sequence $\Lambda^{n_k}\left(\bar{s}^{T+1}\right) = \Lambda^{n_k}(\bar{s}^t, \bar{\sigma})$ is bounded. This means that the sets of vectors $p_j^{n_k}(\bar{s}^j)$, $j = 1, ..., T+1$, are bounded.

Let $\bar{x}_{tk}, \bar{y}_{tk}, \bar{c}_{tk}, \bar{p}_{tk}$ be the values of the functions $x_t^{n_k}, y_t^{n_k}, c_t^{n_k}, p_t^{n_k}$ at the point $\bar{s}^t$. By (B) and (D),

$$(\bar{x}_{tk}, \bar{y}_{tk}) \in \mathcal{T}_t\left(\bar{s}^t\right) \text{ for } t = 1, ..., T, \tag{16}$$

$$\bar{c}_{tk} = \phi_t\left(s^t, \bar{p}_{tk}\right), \quad \bar{x}_{tk} + \bar{c}_{tk} \leq \bar{y}_{t-1,k} \text{ for } t = 1, ..., T+1. \tag{17}$$

The sets of vectors $\bar{x}_{tk}$, $\bar{y}_{tk}$, and $\bar{c}_{tk}$ $(t = 1, ..., T;\ k = 1, 2, ...)$ are bounded by virtue of (16), (A) and (17). The boundedness of $\bar{p}_{tk}$ was established above. Therefore for some sequence k_i, there exist limits $\lim_{i\to\infty}(\bar{x}_{tk_i}, \bar{y}_{tk_i}, \bar{c}_{tk_i}, \bar{p}_{tk_i}) = (\bar{x}_t, \bar{y}_t, \bar{c}_t, \bar{p}_t)$ for $t = 1, ..., T$. It follows from (16) and (17) that these limits satisfy conditions (B) and (8). Hence $\eta(\bar{y}_t) > 0$. But, by virtue of (15), $\delta_t(\bar{s}^t) = \eta(\bar{y}_t)$. □

References

Dynkin, E.B. (1975) "Stochastic dynamic models of economic equilibrium" in International Congress of Mathematicians, Vancouver 1974, Proceedings, Vol. 2, Canad. Math. Congress: Montreal, 509-515.

Dynkin, E.B. (1976) "Economic equilibrium under uncertainty" in Computing Equilibria: How and Why by J. Los and M. W. Los, Eds., North-Holland: Amsterdam, 41-60. Reprinted in Selected Papers of E.B. Dynkin with Commentary, 2000, A.A. Yushkevich, G.M. Seitz, and A.L. Onishchik, Eds., American Mathematical Society: Providence, RI, 529-546.

Dynkin, E.B., and I.V. Evstigneev (1976) "Regular conditional expectations of correspondences" Theory of Probab. and Appl. 21, 333-347.

Evstigneev, I.V. (2000) "Comments on Dynkin's work in mathematical economics" in Selected Papers of E.B. Dynkin with Commentary, A.A. Yushkevich, G.M. Seitz, and A.L. Onishchik, Eds., American Mathematical Society: Providence, RI, 547-548 and 793-796.

Grandmont, J.-M., Ed. (1988) Temporary Equilibrium. Selected Readings, Academic Press: London.

Mas-Colell, A., M.D. Whinston, and J.R. Green (1995) Microeconomic Theory, Oxford University Press: Oxford.

Neveu, J. (1965) Mathematical Foundations of the Calculus of Probability Theory, Holden Day: San Francisco.

Polterovich, V.M. (1973) "Economic equilibrium and optimum" Ekon. i Matem. Metody 9, 835-845. (In Russian.)

Polterovich, V.M. (1978) "Equilibrium trajectories of economic growth" in Methods of Functional Analysis in Mathematical Economics by B. A. Efimov, Ed., Nauka: Moscow, 56-97 (in Russian). English translation in Econometrica 51 (1983), 693-729.

Radner, R. (1982) "Equilibrium under uncertainty" in Handbook of Mathematical Economics, K.J. Arrow and M.D. Intrilligator, Eds., North-Holland: Amsterdam, 923-1006.