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Testing for asymmetry in economic time series using bootstrap methods

Claudio Lupi
ISAE − Institute for Studies and Economic Analyses

Patrizia Ordine
University of Calabria, Dept. of Economics Statistics

Abstract

In this paper we show that phase−scrambling bootstrap offers a natural framework for asymmetry testing in economic time series. A comparison with other bootstrap schemes is also sketched. A Monte Carlo analysis is carried out to evaluate the size and power properties of the phase−scrambling bootstrap−based test.

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1. Introduction

The idea that some important macroeconomic variables may feature an asymmetric behavior over the business cycle is an old one. It is already present in Crum (1923) and later it becomes a cornerstone of Burns and Mitchell's (1946) business cycle theory. Asymmetries are also advocated by Keynes (1936) and Hicks (1950) as explanations of the different characteristics of recessions and booms. More recently, according to Blatt (1983) "[...] a pronounced lack of symmetry is the rule".

McQueen and Thorley (1993) classify asymmetries into three categories: *sharpness, steepness,* and *deepness*, respectively. Sharpness asymmetry is characterized by sharp business cycle peaks and rounded troughs and can be associated to the business cycle pattern described by Keynes. Steepness asymmetry is a subject common to many recent papers (see, *e.g.*, De Long and Summers, 1986 and Neftçi, 1984) and features steep downward slopes during recessions and gradual upward slopes in expansions. Finally, deep troughs and low peaks distinguish deepness asymmetry (see, *e.g.*, Sichel, 1991).

In this paper we study the characteristics of a nonparametric test for asymmetry that can be applied to relatively short time series. By not assuming any specific process for the observed series, we bypass the drawbacks of other conventional tests. Also, since we use a simulation-based approach that exploits the features of the observed sample, implicitly we take into account the sample length and do not rely exclusively on asymptotic results.

The paper is organized as follows: the next section is devoted to a brief discussion of the econometric problem. Section three describes the simulation-based test. A comparison with other bootstrap procedures follows. The final section reports some Monte Carlo results and draws some conclusions. Details of the algorithm used in the simulation approach are reported in the appendix.

2. Testing for asymmetry

Our null hypothesis is that the observed time series $\{y_t\}_0^{T-1}$ is a realization of the stationary linear symmetric process

$$y_t = \vartheta(B)\varepsilon_t \qquad \varepsilon_t \text{ uncorrelated and symmetrically distributed with } \mathsf{E}(\varepsilon_t) = 0,\ \mathsf{E}(\varepsilon_t^2) = \sigma_\varepsilon^2 < \infty \tag{1}$$

with invertible $\vartheta(B)$.

Our alternative is that the distribution of the y_t's is skewed. This can happen, for example, when the linear process (1) is driven by shocks which have an asymmetric distribution, so that

$$y_t = \vartheta(B)\varepsilon_t \qquad \varepsilon_t \text{ uncorrelated and asymmetrically distributed with } \mathsf{E}(\varepsilon_t) = 0,\ \mathsf{E}(\varepsilon_t^2) = \sigma_\varepsilon^2 < \infty. \tag{2}$$

Another economically attractive possibility is that the process reacts asymmetrically to positive and negative shocks, as in

$$y_t = \vartheta^+(B)\varepsilon_t^+ + \vartheta^-(B)\varepsilon_t^- \qquad \mathsf{E}(\varepsilon_t) = 0,\ \mathsf{E}(\varepsilon_t^2) = \sigma_\varepsilon^2 < \infty \tag{3}$$

where $\varepsilon_t^+ \equiv \max(\varepsilon_t, 0)$, $\varepsilon_t^- \equiv \min(\varepsilon_t, 0)$, and the lag polynomials $\vartheta^+(B)$ and $\vartheta^-(B)$ are such that $\vartheta^+(B) = \vartheta_0^+ + \vartheta_1^+ B + \ldots$, and $\vartheta^-(B) = \vartheta_0^- + \vartheta_1^- B + \ldots$, with $\vartheta_i^+ \neq \vartheta_i^-$ in general. The process (3) is equivalent to (1) if and only if $\vartheta^+(B) \equiv \vartheta^-(B)$. The asymmetric moving average (3) has been introduced by Wecker (1981) with the aim of finding a process that could be representative of some observed price asymmetries. Indeed, (3) features some interesting properties:

- y_t "reacts" differently according to whether the innovation is positive or negative;
- asymmetry may be not only a matter of "size" (in absolute value) of the reaction (*e.g.,* larger [smaller] reaction in the presence of negative [positive] shocks) but also a matter of timing (*e.g.,* when some of the ϑ_i^+ or ϑ_i^- are zero);
- contrary to the symmetric process (1), (3) has in general a non-zero mean (whose exact value depends on the values of the ϑ_i^+ and ϑ_i^-);
- the autocorrelation function of (3) cannot be distinguished from that of (1), unless $\vartheta_i^+ = -\vartheta_i^-$ $\forall i,$ in which case it cannot be distinguished from that of a white noise process.

It must be stressed that, since the first two sample moments of (1), (2), and (3) are observationally equivalent, if we want to assess if the observed sample $\{y_t\}_0^{T-1}$ has been generated by the symmetric process (1) or by one of the asymmetric processes (2) or (3), we must use higher order moments. In particular we can fruitfully use the properties of the processes in terms of their skewness. A consistent estimator of the coefficient of skewness is $sk = M_3 M_2^{-3/2}$ with $M_r = T^{-1} \sum_{t=0}^{T-1} (y_t - \bar{y})^r$ where $\bar{y}$ is the sample average. If $\{y_t\}_0^{T-1}$ were independent Gaussian, then $\sqrt{T/6} sk \xrightarrow{d} \mathsf{N}(0,1)$ (see *e.g.* Kendall and Stuart, 1969). However, the y_t's are in general autocorrelated. This problem has been tackled by De Long and Summers (1986) assuming a specific ARMA process with Gaussian errors for the observed $\{y_t\}_0^{T-1}$ and simulating a large number of realizations from this process. Since the simulated processes are symmetric by construction, with a given autocorrelation structure, De Long and Summers (1986) derive the finite sample empirical distribution of the coefficient of skewness under the null and build confidence intervals for the estimated coefficient of skewness of the observed series. However, this approach has three main drawbacks. First, the null hypothesis is not just that of symmetry, but also the implicit hypothesis on the parametric form of the DGP (Data Generating Process). If the process for the DGP is incorrectly specified, inference is invalidated. Second, a different ARMA process must be identified and simulated for each series to be analyzed. Third, the procedure relies on a non-pivotal statistic. As far as the series are Gaussian, the last disadvantage can be removed by using a result due to Lomnicki (1961) that proves that the coefficient of skewness of the linear Gaussian process $x_t = \varphi(B)\varepsilon_t$, with $\varepsilon_t \sim \mathsf{N}(0, \sigma_\varepsilon^2)$ uncorrelated, is such that $\sqrt{T/6} sk^* = \sqrt{T/6} \left(\sum_{j=-\infty}^{\infty} \rho_j^3 \right)^{-1/2} sk \xrightarrow{d} \mathsf{N}(0,1)$, with ρ_j the lag-j autocorrelation coefficient. The other two drawbacks can be overcome if we can simulate symmetric time series with the same autocorrelation structure as the observed one, without imposing parametric models. This is what we do in the next section.

3. The phase-scrambling bootstrap

In this paper we propose the use of a nonparametric procedure that, although remaining in the spirit of De Long and Summers (1986), bypasses the difficulties arising from their approach. The idea is that of using the observed series nonparametrically in order to bootstrap linear Gaussian time series that on average have the same correlogram as the original one. The bootstrap replications are then used to derive the distribution of the coefficient of skewness under the null, which is used to carry out the test on the observed series. Being nonparametric, contrary to LM-type tests, this test does not require a parametric alternative (see *e.g.* Luukkonen *et al.*, 1988; Luukkonen and Teräsvirta, 1991). This, of course, may imply that it lacks power as compared to those tests for specific alternatives.

The procedure we suggest utilizes an algorithm originally proposed by Theiler *et al.* (1992) and adapted by Braun and Kulperger (1997) and Davison and Hinkley (1997). The procedure is based on the fact that the sample periodogram summarizes the second-order sample moments of the observed series. Given that the periodogram can be expressed as the squared modulus of the Fourier transform of the data, the idea is that of randomizing the phases of the periodogram of the (demeaned) series under investigation, while preserving the moduli, and then recomputing the simulated series via the inverse Fourier transform.[1] Indicating with an asterisk the bootstrap quantities, it can be shown that, not only $\mathrm{cov}^*(y_t^*, y_{t-k}^*) = cov(y_t, y_{t-k})$, but also that the odd joint moments of the simulated series are all zero (see Davison and Hinkley, 1997). Furthermore, Braun and Kulperger (1997) prove that under fairly general conditions the series simulated in this way are Gaussian.[2] Using normality of the bootstrapped series, following Lomnicki (1961), we compute, instead of the coefficient of skewness, the asymptotically pivotal statistic $\sqrt{T/6}sk^*$.

Figure 1 shows the distributions of $\sqrt{T/6}sk$ (dotted line) and of $\sqrt{T/6}sk^*$ (dashed line) computed using 5000 bootstrap replications and compared to that of independent standard normal variates (solid line) for six experiments. In the first column of the figure the driving shocks are standard normal; in the second they are $\chi^2(5)$. The rows correspond respectively to a white noise process, to an ARMA process, and to an asymmetric MA process. In all instances the distribution of the asymptotically pivotal statistic coincides nearly perfectly with the standard normal. Note also that the variance of $\sqrt{T/6}sk$ is larger than that of $\sqrt{T/6}sk^*$ when the series is autocorrelated, as expected.

4. Comparison with other bootstrap schemes

Other bootstrap schemes could in principle be used. Logical possibilities are represented by model-based bootstrap and moving-blocks bootstrap.

The first amounts to fitting appropriate (generally AR or ARMA) models to the data and applying resampled estimated residuals to the model. This procedure is generally fine as far as the model is specified correctly. However, it is not entirely suitable for our purposes. Indeed, if the original series is driven by asymmetric shocks, the residuals from an estimated ARMA will in general be asymmetric and the bootstrap will generate asymmetric series. This is rather clearly illustrated in Figure 2 where the distributions of $\sqrt{T/6}sk$ and $\sqrt{T/6}sk^*$ are plotted under model-based resampling.[3] The presence of asymmetric shocks shifts the distributions with respect to the standard normal. This means that a test based on this bootstrap procedure is likely not to reject the null of symmetry in those cases. Furthermore, it appears that some

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1 A more precise account is given in Appendix A.

2 A sufficient condition for this result to hold is that the original series $\{X_t\}_0^{T-1}$ be a realization of a process such that $\mathsf{E}(X_t X_{t+k}) \underset{k\to\infty}{\longrightarrow} 0$ and $\mathsf{E}(X_t^2 X_{t+k}^2) \underset{k\to\infty}{\longrightarrow} \sigma^4$, thus including, *e.g.*, FARMA processes.

3 Here the models used are $AR(p)$ with p selected on the basis of the AIC.

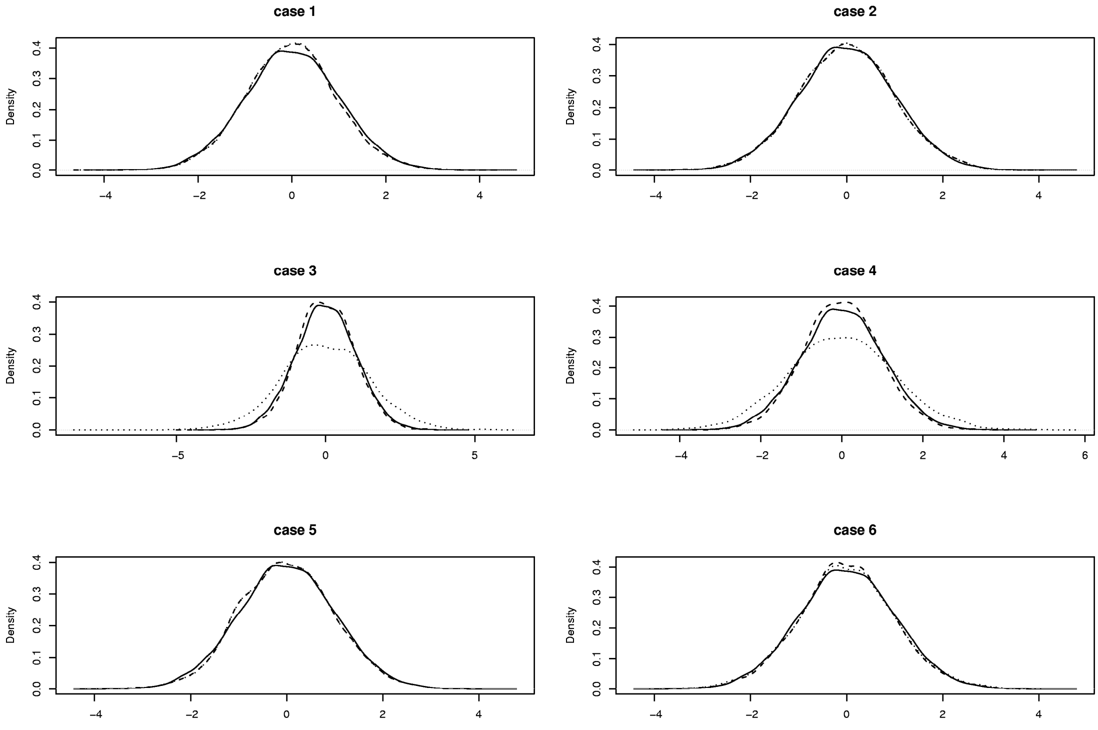

Figure 1: Densities of $\sqrt{T/6}sk$ (dotted line) and $\sqrt{T/6}sk^*$ (dashed line) under phase-scrambling bootstrap (5000 bootstrap replications) as compared to standard normal (solid line). The original $\{y_t\}_0^{T-1}$ series is: case 1, normal IID; case 2, IID $\chi^2(5)$; case 3, $y_t = 0.6y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1}$, $\varepsilon_t \sim \mathsf{N}(0,1)$; case 4 $y_t = 0.6y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1}$, $\varepsilon_t \sim \chi^2(5)$; case 5, $y_t = \varepsilon_t + 0.6\varepsilon_{t-1}^+ + 0.15\varepsilon_{t-1}^-$, $\varepsilon_t \sim \mathsf{N}(0,1)$; case 6 $y_t = \varepsilon_t + 0.6\varepsilon_{t-1}^+ + 0.15\varepsilon_{t-1}^-$, $\varepsilon_t \sim \chi^2(5)$.

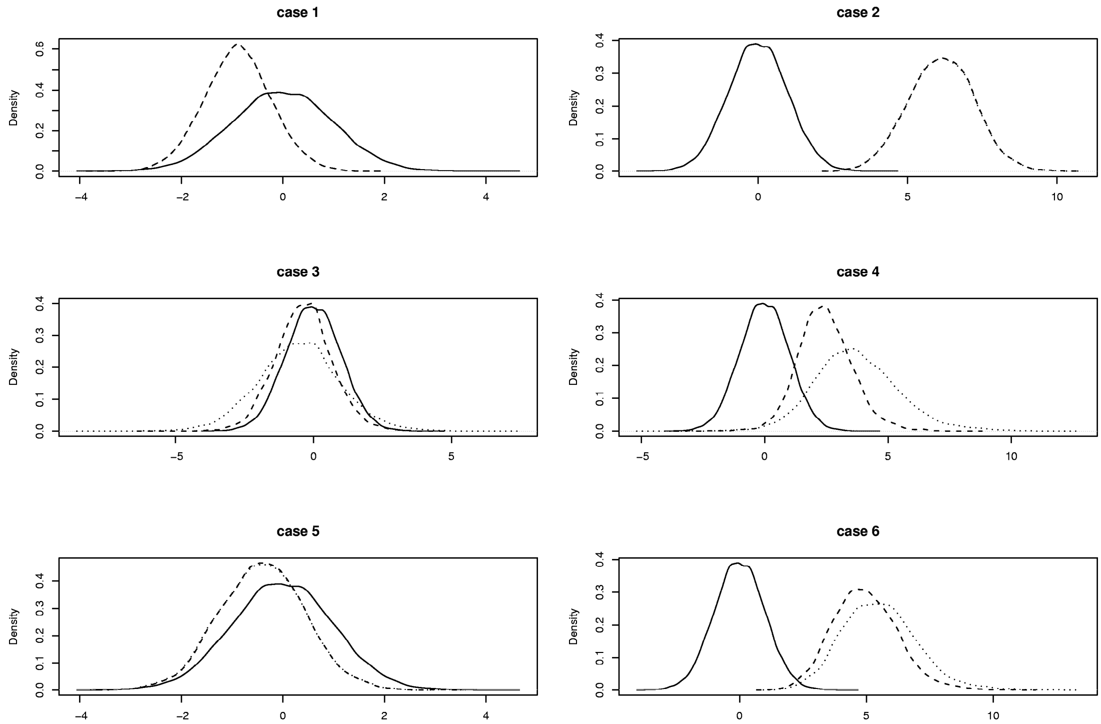

Figure 2: Densities of $\sqrt{T/6}sk$ (dotted line) and $\sqrt{T/6}sk^*$ (dashed line) under model-based bootstrap (5000 bootstrap replications) as compared to standard normal (solid line). The original $\{y_t\}_0^{T-1}$ series is: case 1, normal IID; case 2, IID $\chi^2(5)$; case 3, $y_t = 0.6y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1}$, $\varepsilon_t \sim \mathsf{N}(0,1)$; case 4 $y_t = 0.6y_{t-1}+\varepsilon_t+0.6\varepsilon_{t-1}$, $\varepsilon_t \sim \chi^2(5)$; case 5, $y_t = \varepsilon_t+0.6\varepsilon^+_{t-1}+0.15\varepsilon^-_{t-1}$, $\varepsilon_t \sim \mathsf{N}(0,1)$; case 6 $y_t = \varepsilon_t + 0.6\varepsilon^+_{t-1} + 0.15\varepsilon^-_{t-1}$, $\varepsilon_t \sim \chi^2(5)$.

problems are present also in the asymmetric moving average case (case 5).

Moving-blocks bootstrap is another standard bootstrap technique for time series. In this case, blocks of consecutive observations are resampled. Figure 3 shows that the problems highlighted for model-based bootstrap are present also in this case.

5. Monte Carlo analysis of the bootstrap-based tests

In this section we study the size and power properties of the test for asymmetry based on the phase-scrambling bootstrap. For comparison, we also investigate the properties of the test built upon the asymptotic standard normal distribution. Given the evidence reported in the previous section and the fact that the method is computer-intensive, we will not investigate the relation with other bootstrap schemes any further. Since we are mainly interested in the application with fairly short time series, in all experiments simulated time series of 100 observations are used. The bootstrap tests are based on 1000 bootstrap samples. Given the time required by the bootstrap procedures, the number of Monte Carlo simulations for each parameter configuration had to be limited to 500.

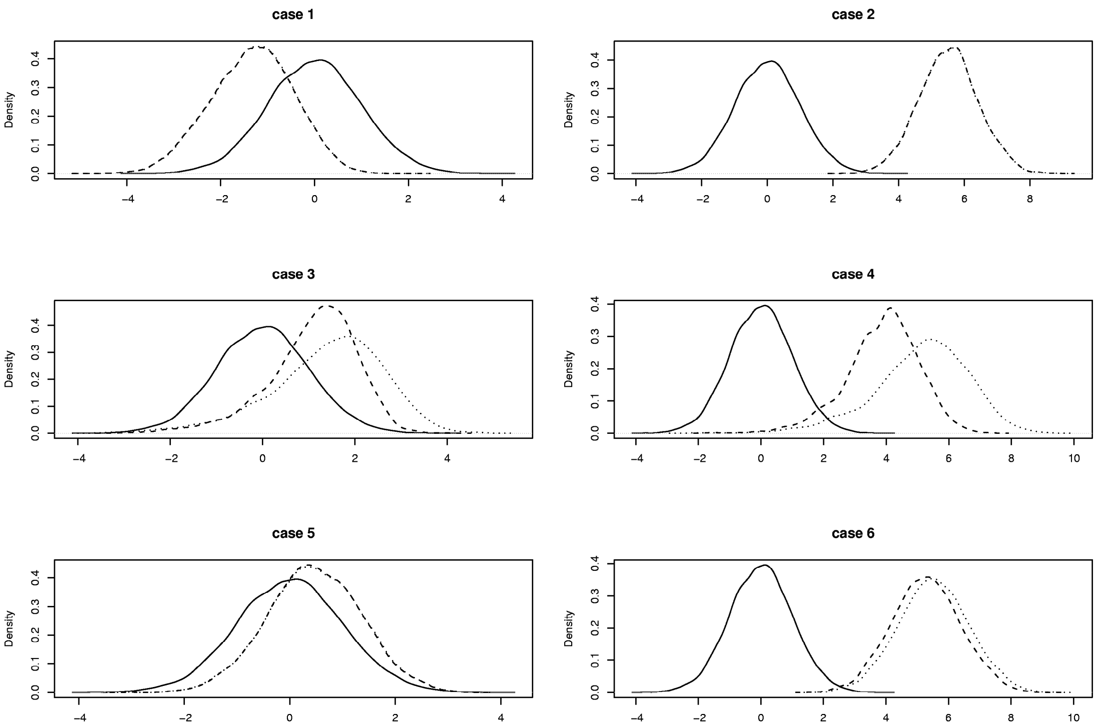

Figure 3: Densities of $\sqrt{T/6}sk$ (dotted line) and $\sqrt{T/6}sk^*$ (dashed line) under moving-blocks bootstrap (5000 bootstrap replications) as compared to standard normal (solid line). The original $\{y_t\}_0^{T-1}$ series is: case 1, normal IID; case 2, IID $\chi^2(5)$; case 3, $y_t = 0.6y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1}$, $\varepsilon_t \sim \mathsf{N}(0,1)$; case 4 $y_t = 0.6y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1}$, $\varepsilon_t \sim \chi^2(5)$; case 5, $y_t = \varepsilon_t + 0.6\varepsilon_{t-1}^+ + 0.15\varepsilon_{t-1}^-$, $\varepsilon_t \sim \mathsf{N}(0,1)$; case 6 $y_t = \varepsilon_t + 0.6\varepsilon_{t-1}^+ + 0.15\varepsilon_{t-1}^-$, $\varepsilon_t \sim \chi^2(5)$.

Table 1: Monte Carlo analysis: Size

		$\sqrt{T/6}sk^*$-based bootstrap Nominal size			$\mathsf{N}(0,1)$		
ϕ	θ	0.010	0.050	0.100	0.010	0.050	0.100
-0.8	-0.8	0.008	0.052	0.098	0.000	0.020	0.048
-0.8	-0.5	0.006	0.040	0.080	0.004	0.014	0.038
-0.8	0.0	0.010	0.048	0.098	0.004	0.024	0.062
-0.8	0.5	0.020	0.066	0.116	0.014	0.056	0.084
-0.4	-0.8	0.012	0.054	0.126	0.010	0.030	0.070
-0.4	-0.5	0.020	0.072	0.114	0.016	0.048	0.086
-0.4	0.0	0.032	0.090	0.130	0.024	0.064	0.102
-0.4	0.5	0.012	0.060	0.130	0.010	0.042	0.092
-0.4	0.8	0.018	0.060	0.118	0.014	0.042	0.088
0.0	-0.8	0.020	0.050	0.114	0.018	0.036	0.080
0.0	0.5	0.014	0.048	0.092	0.012	0.046	0.074
0.0	0.0	0.012	0.042	0.090	0.006	0.038	0.068
0.0	0.5	0.018	0.078	0.128	0.014	0.056	0.098
0.0	0.8	0.014	0.076	0.132	0.006	0.050	0.094
0.4	-0.8	0.018	0.056	0.112	0.016	0.046	0.084
0.4	-0.5	0.022	0.074	0.124	0.022	0.068	0.100
0.4	0.0	0.010	0.064	0.106	0.002	0.036	0.078
0.4	0.5	0.016	0.072	0.136	0.010	0.038	0.076
0.4	0.8	0.020	0.066	0.114	0.012	0.036	0.062
0.8	-0.5	0.012	0.048	0.102	0.010	0.024	0.052
0.8	0.0	0.014	0.058	0.120	0.004	0.018	0.046
0.8	0.5	0.010	0.056	0.124	0.006	0.014	0.040
0.8	0.8	0.016	0.048	0.116	0.002	0.016	0.036

Table 2: Monte Carlo analysis: Power against ARMA processes with chi-squared innovations

		$\sqrt{T/6}sk^*$-based bootstrap Nominal size			$\mathsf{N}(0,1)$		
ϕ	θ	0.010	0.050	0.100	0.010	0.050	0.100
-0.8	-0.8	0.130	0.262	0.336	0.044	0.156	0.248
-0.8	-0.5	0.024	0.070	0.142	0.008	0.042	0.078
-0.8	0.5	0.432	0.642	0.738	0.358	0.588	0.698
-0.4	-0.8	0.066	0.160	0.236	0.042	0.102	0.186
-0.4	-0.5	0.036	0.106	0158	0.014	0.074	0.124
-0.4	0.5	0.646	0.848	0.908	0.612	0.824	0.896
-0.4	0.8	0.516	0.744	0.822	0.466	0.696	0.790
0.0	-0.8	0.064	0.150	0.262	0.042	0.118	0.204
0.0	0.5	0.348	0.560	0.678	0.320	0.514	0.624
0.0	0.5	0.448	0.674	0.766	0.386	0.586	0.718
0.0	0.8	0.288	0.482	0.608	0.234	0.448	0.540
0.4	-0.8	0.382	0.606	0.718	0.338	0.576	0.672
0.4	-0.5	0.634	0.822	0.900	0.588	0.788	0.878
0.4	0.5	0.222	0.410	0.544	0.150	0.300	0.450
0.4	0.8	0.174	0.372	0.502	0.110	0.274	0.406
0.8	-0.5	0.420	0.616	0.728	0.352	0.538	0.628
0.8	0.5	0.052	0.176	0.252	0.020	0.066	0.142
0.8	0.8	0.054	0.138	0.222	0.014	0.070	0.110

The test size is investigated using as the "observed series" simulated stationary and invertible ARMA(1,1) models

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \qquad \varepsilon_t \sim \mathsf{NIID}(0,1)$$

with $\phi \in \{-0.8, -0.4, 0.0, +0.4, +0.8\}$ and $\theta \in \{-0.8, -0.5, 0.0, +0.5, +0.8\}$. Size properties have been investigated for both the asymptotically pivotal bootstrap ($\sqrt{T/6}sk^*$) and the asymptotic test based on the standard normal distribution. From the results reported in Table 1 it appears that the asymptotic test tends to under-reject in the presence of strongly autocorrelated series. On average, the bootstrap test seems preferable, given that, contrary to the asymptotic one, it does not display large deviations from the theoretical size.

Power comparisons are reported in Tables 2 and 3 that show that power is always larger for the bootstrap test as compared to the asymptotic one, irrespective of the specific alternative. In some instances, when the bootstrap test has low power, the power of the test based on the asymptotic distribution of $\sqrt{T/6}sk^*$ is even lower than the nominal size.

As a concluding remark we note that the simulation-based test studied in this paper appears to have better small sample properties than its asymptotic analog. Also, it appears to be better suited than other resampling-based tests, in that the series simulated under phase-scrambling bootstrap are surely symmetric, independently of the nature of the original series.

Table 3: Monte Carlo analysis: Power against asymmetric MA processes

		$\sqrt{T/6}sk^*$-based bootstrap			$\mathsf{N}(0,1)$		
		Nominal size			Nominal size		
ϕ	θ	0.010	0.050	0.100	0.010	0.050	0.100
-0.8	-0.8	0.036	0.124	0.218	0.010	0.052	0.106
-0.8	-0.5	0.014	0.052	0.120	0.006	0.026	0.050
-0.8	0.5	0.182	0.376	0.482	0.154	0.314	0.440
-0.8	0.8	0.322	0.588	0.712	0.286	0.534	0.664
-0.4	-0.8	0.020	0.090	0.174	0.006	0.050	0.126
-0.4	-0.5	0.028	0.070	0.120	0.022	0.054	0.092
-0.4	0.5	0.312	0.574	0.698	0.280	0.514	0.646
-0.4	0.8	0.178	0.394	0.534	0.144	0.334	0.462
0.0	-0.8	0.022	0.078	0.152	0.014	0.050	0.116
0.0	0.5	0.162	0.350	0.474	0.128	0.282	0.416
0.0	0.5	0.172	0.352	0.482	0.142	0.304	0.422
0.0	0.8	0.118	.264	0.398	0.088	0.206	0.316
0.4	-0.8	0.136	0.340	0.480	0.108	0.300	0.438
0.4	-0.5	0.310	0.550	0.666	0.264	0.502	0.630
0.4	0.5	0.074	0.180	0.288	0.052	0.120	0.196
0.4	0.8	0.066	0.168	0.264	0.050	0.106	0.176
0.8	-0.8	0.294	0.540	0.702	0.242	0.484	0.634
0.8	-0.5	0.150	0.344	0.486	0.100	0.254	0.384
0.8	0.5	0.016	0.080	0.148	0.006	0.028	0.058
0.8	0.8	0.022	0.082	0.126	0.004	0.034	0.064

Appendix A. The phase-scrambling bootstrap algorithm

Let $z_p \equiv \sum_{t=0}^{T-1} y_t e^{it\omega_p}$ define the finite Fourier transform of $\{y_t\}_0^{T-1}$, when $p = 0, 1, \ldots, (T-1)$ and $\omega_p = 2\pi p/T$. The first simulation step is that of computing $z'_p = z_p e^{i\phi_p}$ with ϕ_p uniformly distributed between 0 and 2π.[4] Since we want to obtain the Fourier transform of a real time series, we need to make z' symmetric around frequency π. Therefore, for each $0 \leq p \leq (T-1)$, we compute

$$\mathrm{Re}\left\{z''_p\right\} = 2^{-1/2}\,\mathrm{Re}\left\{z'_p + z'_{T-p}\right\} \tag{4}$$

and

$$\mathrm{Im}\left\{z''_p\right\} = 2^{-1/2}\,\mathrm{Im}\left\{z'_p - z'_{T-p}\right\}. \tag{5}$$

The simulated series is then derived as the inverse Fourier transform of z''_p. Noting that the periodogram of the simulated series is $I^*(\omega_p) = \left|z''_p\right|^2$ and using the fact that $z_{T-p} = \bar{z}_p$ (where the bar indicates complex conjugation), it is possible to derive that

$$\begin{aligned} I^*(\omega_p) = \left|z''_p\right|^2 &= |z_p|^2 + 2^{-1}\left[e^{i(\phi_p+\phi_{T-p})} + e^{i(\phi_p+\phi_{T-p})}\right] \\ &= |z_p|^2 + |z_p|^2\cos\left(\phi_p + \phi_{T-p}\right) \\ &= I(\omega_p)\left[1 + \cos\left(\phi_p + \phi_{T-p}\right)\right] \end{aligned}$$

which implies that $\mathsf{E}^*\left[I^*(\omega_p)\right] = I(\omega_p)$, as desired, with $\mathsf{E}^*\left[\cdot\right]$ denoting the bootstrap average.

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[4] In this way we maintain that the phases of a harmonic process are uniformly distributed on $[0, 2\pi)$.

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