ECONOMICS BULLETIN

On fuzzy non−discrimination

Dinko Dimitrov
Ruhr−University Bochum

Abstract

We show that the incompatibility between the Pareto principle and the notion of non−discrimination as presented in Xu (2000) continues to hold when the individuals have exact preferences and the social preference relation is allowed to be a reflexive and transitive fuzzy binary relation. Our result can be seen as a strengthening of the result of Xu in two directions: (1) the range of the aggregation rule is enlarged and (2) a weaker condition on non−discrimination is used.

I thank Gergana Dimitrova, Luis G. Gonzalez−Morales, Susanne Maidorn, Yongsheng Xu and an anonymous referee for helpful comments. Financial support by the Deutsche Forschungsgemeinschaft, Graduiertenkolleg "Allokationstheorie, Wirtschaftspolitik und kollektive Entscheidungen" is gratefully acknowledged.
Citation: Dimitrov, Dinko, (2001) "On fuzzy non−discrimination." *Economics Bulletin,* Vol. 4, No. 3 pp. 1−6
Submitted: March 17, 2001. **Accepted:** April 18, 2001.
URL: http://www.economicsbulletin.com/2001/volume4/EB−01D70001A.pdf

1. Introduction

In a recent paper Xu (2000) introduces the notion that “equals should be treated equally“ as a condition of minimal non-discrimination in an Arrow-Sen social choice framework. The condition says that if there are two individuals (j and k) and the society grants the wishes of the first individual then it must also grant the wishes of the second individual. And, by the same principle, if the society denies the wishes of the first individual then the wishes of the second individual must also be socially denied. In his analysis Xu proposes this notion as a requirement on a social welfare function and studies its compatibility with the weak Pareto principle when the domain of the aggregation rule is unrestricted. The result is an impossibility one: there exists no social welfare function satisfying simultaneously these requirements. The purpose of this paper is to examine the above statement in the framework of fuzzy preferences.

In our analysis the individual preferences are exact and the social preference relation is allowed to be fuzzy. In this way the domain of the aggregation rule remains the same as in Xu (2000) but the range is enlarged because every exact (social) preference relation is also a fuzzy preference relation but not vice versa. We introduce also a weaker notion of non-discrimination which allows (1) a mild notion of discrimination to be included without breaking the initial intuition behind the non-discrimination principle and (2) the wishes of the individuals who have opposite preferences in comparison to j and k, respectively, to be not completely denied in a sense described below.

2. Background

Let X denote the finite set of alternatives from which a choice must be made. A *fuzzy preference relation* on X is a function R: X → [0, 1]. For x, y ∈ X the number R(x, y) is usually interpreted as the degree to which the alternative x is at least as good as the alternative y. It is clear that if R : X → {0, 1} then R would be an *exact* preference relation. A fuzzy preference relation is (see Billot (1995, p. 10-15)): *reflexive*, iff for all x ∈ X: R(x, x) ∈ [0, 1] and *transitive* iff for all distinct x, y, z ∈ X: $[R(x, y) \geq R(y, x) \wedge R(y, z) \geq R(z, y)] \rightarrow [R(x, z) \geq R(z, x)]$. The set of all fuzzy preference relations which are reflexive and transitive is denoted by T.

The set of the individuals in the society is N = {1, ..., n}. E stands for the set of all (exact) preference orderings (i.e. reflexive, complete and transitive binary relations) on X, and D is a subset of E. D^n and E^n denote the n-fold Cartesian products of D and E, respectively. A profile of individual preference orderings is denoted by $\{R_i\}$, $\{R_i\} \in D^n$. A *fuzzy aggregation rule* (FAR) is a function g: $D^n \rightarrow T$. In other words, the aggregation rule maps each profile of individual exact preference orderings $\{R_i\}$ into a social fuzzy preference relation R which is reflexive and transitive.

3. Conditions

The first condition we impose on the aggregation rule is that of unrestricted domain, i.e. we allow every logically possible set of individual (exact) preference orderings to be included in this domain.

Unrestricted Domain (UD): The domain of an FAR, D^n, is E^n.

The second condition is the Pareto principle. Its definition, which seems reasonable for the fuzzy case, says that if all individuals in the society exactly prefer x to y then the degree to which the society prefers x to y must be greater than the degree of the social preference of y over x.

Pareto Principle (PP): For all $x, y \in X$, and for all $\{R_i\} \in D^n$:

$$[\forall i \in N: \{R_i(x, y) = 1 \wedge R_i(y, x) = 0\}] \rightarrow [R(x, y) > R(y, x)].$$

Let an individual j exactly prefer x to y and an individual k exactly prefer z to w. Our third condition, the weak minimal non-discrimination, states that if the society (the aggregation rule) grants the wishes of j in the sense that the degree of social preference of x over y is higher than the degree of social preference of y over x then it must also grant the wishes of k in the sense that the degree of social preference of z over w is higher than the degree of social preference of w over z. And, by the same principle, if the society denies the wishes of the first individual in the sense that the degree of social preference of x over y is smaller than the degree of social preference of y over x, then it must also deny the wishes of k in the sense that the degree of social preference of z over w is smaller than the degree of social preference of w over z.

Weak Minimal Non-Discrimination (WMND): There exist two individuals $j, k \in N$ and two distinct pairs $(x, y), (z, w) \in X \times X$ such that for all $\{R_i\} \in D^n$ one and only one of the following three is true:

(1) $[\{R_j(x, y) = 1 \wedge R_j(y, x) = 0\} \wedge \{R_k(z, w) = 1 \wedge R_k(w, z) = 0\}] \rightarrow$
$[R(x, y) > R(y, x) \wedge R(z, w) > R(w, z)]$;

(2) $[\{R_j(x, y) = 1 \wedge R_j(y, x) = 0\} \wedge \{R_k(z, w) = 1 \wedge R_k(w, z) = 0\}] \rightarrow$
$[R(x, y) = R(y, x) \wedge R(z, w) = R(w, z)]$;

(3) $[\{R_j(x, y) = 1 \wedge R_j(y, x) = 0\} \wedge \{R_k(z, w) = 1 \wedge R_k(w, z) = 0\}] \rightarrow$
$[R(x, y) < R(y, x) \wedge R(z, w) < R(w, z)]$.

4. Interpretations

A *first* interpretation concerns the notion of non-discrimination itself. In part (1) of WMND the society does not discriminate the individuals j and k, because they exactly prefer x to y and z to w, respectively, and the degree to which the society prefers x to y *and* z to w is higher than the corresponding degree of social preference of y to x *and* w to z. In part (2) of WMND the exact preferences of x over y for the individual j *and* of z over w for the individual k result in the fact that the aggregation rule equals the degrees of social preference of x over y and y over x *and* of z over w and w over z. In part (3) of WMND the society treats the individuals also equally but in a negative sense: the degree to which it prefers x to y *and* z to w is smaller than the corresponding degree of preference of y to x *and* w to z.

As a *second* remark, it must be noted that a notion of minimal discrimination (in a mild sense) is also included in WMND. For example, in part (1) of WMND the exact preferences of the individuals j and k can result in different degrees of social preference of x over y *and* of z over w. Such a notion of minimal discrimination is also included in (2) of WMND, because the equalities $R(x, y) = R(y, x)$ *and* $R(z, w) = R(w, z)$ can be fixed by the aggregation rule at different levels. This means that the society treats the individuals j and k equally with respect to their corresponding pairs, but at different distances from the exact preferences over these pairs. The analogous statement is also included in part (3) of WMND but this is done in terms of different punishments of the individuals by the society. It follows that the above WMND condition implicitly includes a notion of minimal discrimination of the individuals without breaking the initial intuition behind the non-discrimination principle and can be viewed as a generalization of the corresponding condition in Xu (2000).

A *third* remark concerns the interpretation of a social fuzzy preference relation in the context of this paper. Let us view R(x, y) as reflecting the wishes of other people who are in favour of x over y, apart from individual j´s preference of x over y. In analogy, R(z, w) reflects the wishes of other people who are in favour of z over w, apart from individual k´s preference of z over w. With this interpretation let us look at the exact variant of part (1) in WMND. It would require $[\{R_j(x, y) = 1 \wedge R_j(y, x) = 0\} \wedge \{R_k(z, w) = 1 \wedge R_k(w, z) = 0\}] \rightarrow [\{R(x, y) = 1 \wedge R(y, x) = 0\} \wedge \{R(z, w) = 1 \wedge R(w, z) = 0\}]$, i.e. in this (exact) case the wishes of the individuals j and k are completely granted and there is no discrimination between them. Note that the society also grants the wishes of all individuals who prefer x over y *and* z over w, respectively, and fully denies the wishes of all individuals who have opposite preferences (R(y, x) = 0 and R(w, z) = 0). When the social preference relation is fuzzy and the corresponding individuals exactly prefer x to y *and* w to z, respectively, we allow (i) their wishes to be weakly granted in the sense of R(x, y) > R(y, x) and R(z, w) > R(w, z); (ii) these individuals to be weakly non-discriminated in the sense of $[\{R(x, y) > R(y, x)\} \wedge \{R(z, w) > R(w, z)\}]$; (iii) the wishes of the individuals who prefer y over x and w over z, respectively, to be not completely denied in the sense of not excluding both R(y, x) > 0 and R(w, z) > 0. Note that the latter possibility can not be included in the exact variant of the non-discrimination condition. The analogous interpretation can be given also to parts (2) and (3) in our WMND condition.

5. Result

The proof of our result follows the proof of the impossibility result in Xu (2000).

Proposition 1. There exists no FAR g: $D^n \rightarrow T$ satisfying UD, PP and WMND.

Proof. Let there exist two different individuals j, k $\in$ N and two distinct pairs (x, y), (z, w) $\in X \times X$ such that $\{R_j(x, y) = 1 \wedge R_j(y, x) = 0\}$ and $\{R_k(z, w) = 1 \wedge R_k(w, z) = 0\}$. For x, y, z, w $\in$ X there are two cases: (1) (x, y) and (z, w) have one element in common, say w = x, or (2) x, y, z, w are distinct.

(1) In this case we have $\{R_j(x, y) = 1 \wedge R_j(y, x) = 0\}$ and $\{R_k(z, x) = 1 \wedge R_k(x, z) = 0\}$. Consider the social preference over x and y, for which one and only one of the following three possibilities can happen: (1.1) R(x, y) > R(y, x); (1.2) R(y, x) > R(x, y); (1.3) R(y, x) = R(x, y).

(1.1) Consider the following preferences of all i $\in$ N: $\{R_i(y, z) = 1 \wedge R_i(z, y) = 0\}$. By WMND we have R(z, x) > R(x, z) and by PP R(y, z) > R(z, y). From R(x, y) > R(y, x) and R(y, z) > R(z, y), and given that social preference is transitive, follows R(x, z) ≥ R(z, x), which is a contradiction to R(z, x) > R(x, z).

(1.2) Consider the following preferences of all i $\in$ N: $\{R_i(z, y) = 1 \wedge R_i(y, z) = 0\}$. By WMND we have R(x, z) > R(z, x) and by PP R(z, y) > R(y, z). From R(x, z) > R(z, x) and R(z, y) > R(y, z), and given that social preference is transitive, follows R(x, y) ≥ R(y, x), which contradicts R(y, x) > R(x, y).

(1.3) Consider the following preferences of all i $\in$ N: $\{R_i(y, z) = 1 \wedge R_i(z, y) = 0\}$. By WMND we have R(z, x) = R(x, z) and R(x, y) = R(y, x). By PP follows R(y, z) > R(z, y). From R(z, x) = R(x, z) and R(x, y) = R(y, x), and given that social preference is transitive, follows R(z, y) ≥ R(y, z), which is a contradiction to R(y, z) > R(z, y).

Therefore, in case (1) there is no fuzzy aggregation rule satisfying PP and WMND.

(2) In this case we have $\{R_j(x, y) = 1 \wedge R_j(y, x) = 0\}$ and $\{R_k(z, w) = 1 \wedge R_k(w, z) = 0\}$. Consider the social preference over x and y, for which again one and only one of the following two possibilities can happen: (2.1) $R(x, y) > R(y, x)$; (2.2) $R(y, x) > R(x, y)$; (2.3) $R(y, x) = R(x, y)$.

(2.1) Consider the following preferences of all $i \in N$: $[\{R_i(y, z) = 1 \wedge R_i(z, y) = 0\} \wedge \{R_i(w, x) = 1 \wedge R_i(x, w) = 0\}]$. By WMND we have $R(z, w) > R(w, z)$ and by PP $R(y, z) > R(z, y)$ and $R(w, x) > R(x, w)$. From $R(x, y) > R(y, x)$, $R(y, z) > R(z, y)$ and $R(z, w) > R(w, z)$, and given that social preference is transitive, follows $R(x, w) \geq R(w, x)$, which contradicts $R(w, x) > R(x, w)$.

(2.2) Consider the following preferences of all $i \in N$: $[\{R_i(z, y) = 1 \wedge R_i(y, z) = 0\} \wedge \{R_i(x, w) = 1 \wedge R_i(w, x) = 0\}]$. By WMND we have $R(w, z) > R(z, w)$ and by PP $R(z, y) > R(y, z)$ and $R(x, w) > R(w, x)$. From $R(x, w) > R(w, x)$, $R(w, z) > R(z, w)$ and $R(z, y) > R(y, z)$, and given that social preference is transitive, follows $R(x, y) \geq R(y, x)$, which is a contradiction to $R(y, x) > R(x, y)$.

(2.3) Consider the following preferences of all $i \in N$: $[\{R_i(y, z) = 1 \wedge R_i(z, y) = 0\} \wedge \{R_i(w, x) = 1 \wedge R_i(x, w) = 0\}]$. By WMND we have $R(z, w) = R(w, z)$ and by PP $R(y, z) > R(z, y)$ and $R(w, x) > R(x, w)$. From $R(x, y) = R(y, x)$, $R(y, z) > R(z, y)$ and $R(z, w) = R(w, z)$, and given that social preference is transitive, follows $R(x, w) \geq R(w, x)$, which contradicts $R(w, x) > R(x, w)$.

Therefore, in case (2) there is no fuzzy aggregation rule satisfying PP and WMND. Combining (1) and (2), proposition 1 is proved. ♦

References

1. Billot, A. (1995) *Economic Theory of Fuzzy Equlibria: An Axiomatic Analysis*, Springer-Verlag.
2. Xu, Y. (2000) „Non-Discrimination and the Pareto Principle" *The Economic Review* **51**, 54-60.