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A simple ultrafilter proof for an impossibility theorem in judgment aggregation

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Abstract

We show how ultrafilters can be used to prove a central impossibility result in judgement aggregation introduced by Nehring and Puppe (2005), namely that for a logically strongly interconnected agenda, an independent and monotonic judgement aggregation rule which satisfies universal domain, collective rationality and sovereignty is necessarily dictatorial.

A similar application of the ultrafilter proof strategy was presented at a workshop on judgment aggregation, Freudenstadt, September 2007.Many thanks to Clemens Puppe, Franz Dietrich, Christian List, Philippe Mongin, Ron Holzman, and two anonymous referees for helpful comments.
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1 Introduction

Although ultrafilters[1] have long been used in the proof of Arrow's theorem (for a survey see e.g. Monjardet 1983), this proof technique has not been exploited in the recent, closely related literature on judgment aggregation[2] with the notable exception of Gaerdenfors 2006, Dietrich and Mongin 2007 (especially for the infinite case), Herzberg 2008 (for the use of ultraproducts) and Daniels 2006 (the latter in the different context of the logical formalization of judgment aggregation). This is allthemore astonishing as the very first application of an ultrafilter proof strategy to Arrow's theorem can be found in an early extension of this result to the aggregation of logically interconnected propositions by Guilbaud (1952) which makes this paper the first contribution to judgment aggregation. (For the reconstruction and historical analysis of Guilbaud's result see Monjardet 2003 and 2005.)

In the following we give a simple ultrafilter proof of a typical impossibility result in judgment aggregation by Nehring and Puppe (2005). The ultrafilter proof approach is not only technically appealing, it also makes transparent how the logical structure of the agenda of a collective decision problem determines the social structure, i.e. the distribution of decisiveness among the individuals.

2 Ultrafilters for the analysis of judgment aggregation problems

The problem of judgment aggregation consists in the derivation of collective judgments over an agenda of logically interconnected propositions from individual judgment sets. Following Dietrich 2007, the agenda is given by a non-empty finite set X of propositions (sentences) from a formal language $\mathbf{L}$ which is closed under negation (i.e. if $p \in \mathbf{L}$, then $\neg p \in \mathbf{L}$) and satisfies the following consistency conditions on sets of propositions:

(C1) For any proposition $p \in \mathbf{L}$, the proposition-negation pair $\{p, \neg p\}$ is inconsistent.

(C2) Subsets of consistent sets $S \subseteq \mathbf{L}$ are consistent.

[1] An ultrafilter on a nonempty set S is a collection $\mathcal{F} \subset 2^S$ of subsets of S such that
(i) $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,
(ii) if $X \in \mathcal{F}$ and $Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$,
(iii) if $X, Y \subset S$, $X \in \mathcal{F}$, and $X \subset Y$, then $Y \in \mathcal{F}$,
(iv) for every $X \subset S$, either $X \in \mathcal{F}$ or $S \backslash X \in \mathcal{F}$.

[2] See List and Puppe 2007 for a survey and the bibliography at:
http://personal.lse.ac.uk/LIST/doctrinalparadox.htm

(C3) The empty set $\emptyset$ is a consistent set, and each consistent set $S \subseteq \mathbf{L}$ can be completed to a consistent superset $T \subseteq \mathbf{L}$ containing a member of each proposition-negation pair $p, \neg p \in \mathbf{L}$.

An individual judgment set is a non-empty subset $A \subset X$ of the agenda, which is assumed to consist only of contingent propositions, i.e. for any proposition $p \in X$, the sets $\{p\}$ and $\{\neg p\}$ are both consistent. Typically, individual judgment sets are assumed to be fully rational in the sense of being not only consistent but also complete (i.e. $p \in A$ or $\neg p \in A$ for every proposition p). For N being a finite set of n individuals (with $n \geq 3$), a profile of judgment sets is an n-tuple $\underline{A} = (A_1, ..., A_n)$. A judgment aggregation rule is a function which assigns to each profile in a set of admissible profiles a collective judgment set. Obviously, the aggregation problem crucially depends on the properties of the agenda, essentially the logical interconnections between the propositions in the agenda.

Following Nehring and Puppe 2002, and Dokow and Holzman 2005, the logical interconnections between the propositions in an agenda X are captured by a binary relation $\vdash^* \subset X \times X$ of conditional entailment between propositions.

Definition 1 *For any propositions $p, q \in X$ such that $p \neq \neg q$, $(p, q) \in \vdash^*$ if there exists a minimal inconsistent superset S of $\{p, \neg q\}$ (i.e. a set $P \subset X \backslash \{p, \neg q\}$ such that $S = P \cup \{p, \neg q\}$ is inconsistent while every proper subset of S is a consistent set of propositions).*

Thus for any contingent propositions $p, q \in X$, $(p, q) \in \vdash^*$ means that there exists a set of propositions $P \subset X \backslash \{p, \neg q\}$ conditional on which holding proposition p entails holding proposition q.

Example 2 *Consider an agenda $X = \{p, \neg p, q, \neg q, p \vee q, \neg(p \vee q), p \wedge q, \neg(p \wedge q)\}$. Then the set $S = \{\neg p, p \vee q, \neg q\}$ is a minimal inconsistent set which establishes the conditional entailment relation between e.g. $\neg p$ and q.*

Definition 3 *An agenda X will be called **totally blocked** if the relation of conditional entailment is transitively closed, i.e. if the transitive closure $T(\vdash^*) = X \times X$.*

Total blockedness means that any contingent proposition in the agenda is related to any other one by a sequence of conditional entailments.

Example 4 *Verify that the above agenda $X = \{p, \neg p, q, \neg q, p \vee q, \neg(p \vee q), p \wedge q, \neg(p \wedge q)\}$ is totally blocked, while neither the agenda $Y = \{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}$ nor $Z = \{p, \neg p, q, \neg q, p \vee q, \neg(p \vee q)\}$ is totally blocked.*

Observe that in the case of an agenda with more than two non-equivalent proposition-negation pairs (a condition of minimal logical richness that can plausibly be assumed), total blockedness implies that there always exists a minimal inconsistent set of propositions with cardinality strictly larger than two.

The social choice literature provides a wide range of domain conditions for aggregation rules. The following conditions are extensions of classical Arrovian conditions to the area of judgment aggregation.

Definition 5 *A judgment aggregation rule $f : D^n \to D$ satisfies **universal domain** and **collective rationality** if D is the set of fully rational judgment sets.*

Definition 6 *A judgment aggregation rule $f : D^n \to D$ satisfies **sovereignty** if $f(D^n) = D$, i.e. if it is surjective.*

Like social welfare functions, judgment aggregation rules can be analyzed with the help of decisive sets of individuals (decisive coalitions). In the following and for any profile $\underline{A} \in D^n$ and any proposition $p \in X$ denote by $\underline{A}(p) := \{i \in N : p \in A_i\}$ the set of all individuals who hold proposition p. This yields the following definition of decisiveness.

Definition 7 *A coalition $U \subseteq N$ is **decisive** for proposition $p \in X$ if for all profiles $\underline{A} \in D^n$,*
$U = \underline{A}(p) \Rightarrow p \in f(\underline{A})$.
The collection of all decisive coalitions for proposition p will be denoted by $\mathcal{W}_p \subseteq 2^N$.

This allows to define the core property of (non)dictatoriality in terms of decisive coalitions.

Definition 8 *A judgment aggregation rule $f : D^n \to D$ is **dictatorial** if there exists an individual $i \in N$ such that for any proposition $p \in X$ the collection of decisive coalitions $\mathcal{W}_p$ is the set of all supersets of the singleton set $\{i\}$.*

An important class of properties are independence conditions which are - in general - crucial for the derivation of impossibility results in aggregation problems. We use another but equivalent definition of the usual independence condition[3]:

[3] In the standard formulation the judgment aggregation rule f is independent if for any proposition $p \in X$, and any profiles $\underline{A}, \underline{A}' \in D^n$, $[\underline{A}'(p) = \underline{A}(p)] \Rightarrow [p \in f(\underline{A}) \Leftrightarrow p \in f(\underline{A}')]$.

Definition 9 *A judgment aggregation rule* $f : D^n \to D$ *is* ***independent*** *if for any proposition* $p \in X$*, and any profiles* $\underline{A}, \underline{A}' \in D^n$*,*
$p \in f(\underline{A}) \Rightarrow [\underline{A}'(p) = \underline{A}(p) \Rightarrow p \in f(\underline{A}')]$*.*

For any given proposition, independence excludes the use of any information about the individual support for other propositions. Thus, it also seems natural to introduce a monotonicity condition which guarantees that the social acceptance of a proposition is not reduced by additional individual support for it:

Definition 10 *A judgment aggregation rule* $f : D^n \to D$ *is* ***monotonic*** *if for any profiles* $\underline{A}, \underline{A}' \in D^n$ *and any proposition* $p \in X$*,*
$p \in f(\underline{A}) \wedge \underline{A}(p) \subset \underline{A}'(p) \Rightarrow p \in f(\underline{A}')$*.*

Thus for any monotonic judgment aggregation rule $f : D^n \to D$ and any proposition $p \in X$, the collection $\mathcal{W}_p$ of all decisive coalitions for proposition p is closed under supersets (i.e. $U \in \mathcal{W}_p \wedge U \subset V \Rightarrow V \in \mathcal{W}_p$).

It should be clear that monotonicity excludes any form of inverse decisiveness as in particular through an inverse dictator who indirectly determines the collective outcome by forcing the opposite of her position.

From these properties and their combination follow other facts which are of importance for the proof, but also of significance on their own.

(i) Independence can equivalently be characterised in terms of decisiveness:

Remark 11 *A judgment aggregation rule* $f : D^n \to D$ *is* ***independent*** *if for any proposition* $p \in X$ *and any profile* $\underline{A} \in D^n$*,*
$p \in f(\underline{A}) \Rightarrow \underline{A}(p) \in \mathcal{W}_p$*.*

(ii) Together with monotonicity and sovereignty, independence implies the classical Pareto property:

Definition 12 *A judgment aggregation rule* $f : D^n \to D$ *is* ***paretian*** *if for any proposition* $p \in X$*,* $N \in \mathcal{W}_p$*.*

(iii) An independent and monotonic judgment aggregation rule which satisfies universal domain and collective rationality can be characterised by a family $\{\mathcal{W}_p\}$ of collections of decisive coalitions for each proposition (i.e. for any proposition $p \in X$ and any profile $\underline{A} \in D^n$, $p \in f(\underline{A})$ if and only if $\underline{A}(p) \in \mathcal{W}_p$) which satisfies an intersection property introduced by Nehring and Puppe 2007:

Definition 13 *A family* $\{\mathcal{W}_p\}$ *of collections of decisive coalitions satisfies the* ***intersection property*** *if for any minimal inconsistent set of propositions* $S = \{p_1, ..., p_k\} \subset X$ *and every selection* $W_j \in \mathcal{W}_{p_j}$, $\bigcap_{j=1}^{k} W_j \neq \emptyset$.

From preference aggregation it is well known that Arrovian impossibility results are driven by a "contagion" property (as discussed e.g. in Kelly 1988), which propagates the decisiveness of a coalition from some pair of alternatives to any pair. For the case of independent and monotonic judgment aggregation rules which satisfy universal domain and collective rationality, this contagion property can be captured with the help of the following lemma:

Lemma 14 $\mathcal{W}_p \subseteq \mathcal{W}_q$ *for all propostions* p, q *such that* $(p, q) \in \vdash^*$.

Proof. Consider a pair of propositions $(p, q) \in \vdash^*$ and a non-empty set P of propositions such that $P \cup \{p, \neg q\}$ is a minimal inconsistent set of propositions with cardinality strictly larger than two and a profile $\underline{A} \in D^n$ such that
(i) $\underline{A}(p) = U \in \mathcal{W}_p$
(ii) $\underline{A}(q) = U$
(iii) $\underline{A}(P) = N$.
By the decisiveness of U for p, $p \in f(\underline{A})$ and, by the Pareto property, $P \subset f(\underline{A})$. Therefore, $q \in f(\underline{A})$ by minimal inconsistency of $P \cup \{p, \neg q\}$. By independence, for any profile $\underline{A}' \in D^n$ such that $\underline{A}'(q) = \underline{A}(q)$, $q \in f(\underline{A}')$ and hence $U \in \mathcal{W}_q$. ■

Iterated application of this lemma establishes the following neutrality property, which is equivalent to the condition of systematicity (or equal treatment of all propositions) with which the first impossibility result in judgment aggregation was derived (List and Pettit 2002).

Lemma 15 *For a totally blocked agenda,* $\mathcal{W}_p = \mathcal{W}_q$ *for all propositions* $p, q \in X$.

Hence, one and the same collection $\mathcal{W} \subseteq 2^N$ of decisive coalitions determines the collective acceptance of every proposition in a totally blocked agenda.

In this vein, the conditions on the aggregation problem translate into properties of the collection of decisive coalitions, yielding the following dictatorship result.

Theorem 16 *(Nehring and Puppe 2005) If and only if the agenda is totally blocked, an independent and monotonic judgment aggregation rule which satisfies universal domain, collective rationality and sovereignty is dictatorial.*

Our proof of this theorem exploits the relation between ultrafilters and collections of decisive sets of individuals known as simple games (Neumann and Morgenstern 1944, for a recent reference see Taylor and Zwicker 1999).

Definition 17 *A* ***simple game*** *on the set N of individuals is a collection $\mathcal{W} \subseteq 2^N$ of subsets of N which satisfies closure under supersets (i.e. $U \in \mathcal{W} \wedge U \subset V \Rightarrow V \in \mathcal{W}$).*
A simple game is ***strong*** *if for any $U \subseteq N$, $U \notin \mathcal{W} \Rightarrow N \backslash U \in \mathcal{W}$.*

Simple games stand in a close relation to ultrafilters which, explicitly following Guilbaud 1952, was established by Monjardet 1978 with the help of an intersection property (for a similar characterisation of ultrafilters in terms of simple games and their Nakamura number[4] see Monjardet 2003):

Lemma 18 *A strong simple game is an ultrafilter if for any $U, V, W \in \mathcal{W}$, $U \cap V \cap W \neq \emptyset$.*

This yields a simple proof of the above theorem.
Proof. (if part) From Lemma 15 we know that the collection $\mathcal{W} \subseteq 2^N$ of decisive coalitions is the same for all propositions in a totally blocked agenda. The proof of the theorem now proceeds by establishing that any such collection is
(i) a strong simple game, which is
(ii) an ultrafilter.
The dictatorship result then immediately follows from the well known fact that an ultrafilter on a finite set of individuals is a collection $\mathcal{W} = \{U \in N : i \in U\}$ of all supersets of some singleton, - the dictator.
(i) To see that any collection $\mathcal{W} \subseteq 2^{|N|}$ of decisive coalitions is a simple game keep in mind that monotonicity of the judgment aggregation rule implies the closure of $\mathcal{W}$ under supersets. To see that the simple game $\mathcal{W}$ is strong consider that for any profile $\underline{A} \in D^n$, any non-decisive set of individuals $U \notin \mathcal{W}$ and any proposition $p \in X$, $\underline{A}(p) = U$ implies $p \notin f(\underline{A})$ and hence by completeness of the agenda, $\neg p \in f(\underline{A})$, which in turn implies $\underline{A}(\neg p) = N \backslash U \in \mathcal{W}$ (by indenpendence).
(ii) To see that the simple game $\mathcal{W}$ is an ultrafilter, verify that for any decisive sets of individuals $U, V, W \in \mathcal{W}$, it must be the case that $U \cap V \cap W \neq \emptyset$. Otherwise, a profile $\underline{A} \in D^n$ can be constructed such that for a pair of

[4]The Nakamura number of a simple game is the minimal number of decisive coalitions with empty intersection (see Nakamura 1979). Observe that for a totally blocked agenda the intersection property of Nehring and Puppe 2007 implies a Nakamura number strictly larger than three and thus the intersection property of Lemma 18.

propositions $(p, q) \in \vdash^*$ and a minimal inconsistent superset S of $\{p, \neg q\}$ with cardinality strictly larger than two, $\underline{A}(p) = U$, $\underline{A}(S \backslash \{p, \neg q\}) = V$, and $\underline{A}(\neg q) = W$ which contradicts the collective rationality of the aggregation rule.

Necessity is shown by constructing an independent and monotonic judgment aggregation rule which satisfies universal domain, collective rationality and sovereignty such that for the corresponding family $\{\mathcal{W}_p\}$ of decisive coalitions, $\mathcal{W}_p \neq \mathcal{W}_q$ for some pair of propositions $p, q \in X$ (for the details of this construction see Nehring and Puppe 2005). Obviously, while for every totally blocked subset S of the agenda with cardinality strictly larger than two, $\mathcal{W}_p = \mathcal{W}_{p'}$ is an ultrafilter for any $p, p' \in S$, this local dictatorship on subagendas does not carry over to the full agenda if the latter is not totally blocked. ■

Hence the ultrafilter proof strategy shows that it essentially is the logical structure of the agenda of the collective decision problem which determines the distribution of decisiveness and thus drives the dictatorship result.

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