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A generalization of monotone comparative statics: Correction

Koji Shirai
Graduate School of Economics, Waseda University

Abstract

We have made a correction to "A generalization of monotone comparative statics", which is published in Economics Bulletin Vol. 3, No. 39. We correct the following three aspects of the original paper: the first and the second are the name and the definition of some fundamental notions, respectively. The third is the proof of the main proposition [Proposition 2.1, pp.5].

I am grateful to Richard Ruble for his detailed comments and suggestions. Needless to say, the remaining errors are my own.
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1 Correction

We have made a correction to "A generalization of monotone comparative statics", which is published in *Economics Bulletin Vol.3, No.39*. In that paper, we generalized Milgrom and Shannon's Theorem (Milgrom and Shannon (1994)) from a partially ordered set to a preordered set. As a result, we showed the following two necessary and sufficient relations. The first is the equivalence of the "*w-quasisupermodularity*" of objective function and the monotonicity of the solution of the constrained optimization problem with respect to "*w-strong set order*". The second is the similar relation of the "*s-quasisupermodularity*" of the function and the monotonicity of solution with respect to "*s-strong set order*".

We correct the following three aspects of the original paper: The first is *the name of a fundamental notion*. In our main proposition, the notion that is called "*prelattice*" in the original paper plays a crucial role. However, it turns out that the term "prelattice" has already been used as the name of a mathematical notion that is different from ours. Hence, we alter the name of ours to "*preordered lattice structure*". The second is *the definition of s-quasisupermodularity*. Thanks to a private communication from Richard Ruble, it has become clear that the original version of s-quasisupermodularity is not the necessary and sufficient condition of the monotonicity of the solution of the optimization problem with respect to the s-strong set order but only a necessary condition of it. Hence, we alter the definition of s-quasisupermodular in such a way that the necessary and sufficient relation is realized. The third is *the proof of [Proposition 2.1, pp.5] in the original paper, specifically, the necessity part of it*. It turns out that the original version is incomplete. Hence, we intend to replace it. For these purposes, we introduce some basic notions as follows.

Definition 1: Let X be a preordered set endowed with a preorder $\preccurlyeq$. We say that $U_{x,y}$ is *the set of upper bounds of* $x, y \in X$ if $x \preccurlyeq u$ and $y \preccurlyeq u$ for all $u \in U_{x,y}$. Similarly, we say that $L_{x,y}$ is *the set of lower bounds of* $x, y \in X$ if $l \preccurlyeq x$ and $l \preccurlyeq y$ for all $l \in L_{x,y}$.

Definition 2: We say that $A_{x,y} \subset U_{x,y}$ is *the set of supremums of* $x, y \in X$ if $a \preccurlyeq u$ for all $a \in A_{x,y}$ and $u \in U_{x,y}$. Similarly, we say that $T_{x,y} \subset L_{x,y}$ is *the set of infimums of* $x, y \in X$ if $l \preccurlyeq t$ for all $t \in T_{x,y}$ and $l \in L_{x,y}$.

The following is our first correction: the name of this notion is altered from the original paper. In Shirai (2008), we called this a "prelattice".

Definition 3: We say that X is *a preordered lattice structure* if $A_{x,y} \neq \emptyset$ and $T_{x,y} \neq \emptyset$ for every $x, y \in X$.

Then, we proceed to the second and the third corrections. We introduce the following notion and lemmas. In particular, the lemmas stated below play fundamental roles in the third correction: that is, in the corrected version of the proof of Proposition 2.1 in the original paper.

Definition 4: We say that x and y are indifferent to each other if we have both $x \preccurlyeq y$ and $y \preccurlyeq x$. We write this as $x \sim y$ or $y \sim x$. We define the set $I_x = \{y \in X \mid x \sim y\}$, which is called *the indifference set of* x in the rest of this paper.

Lemma 1: *If $x' \in I_x$ for some x, then $U_{x,y} = U_{x',y}$ and $L_{x,y} = L_{x',y}$ for every $y \in X$.*

Proof. Let $u \in U_{x,y}$ and $u' \in U_{x',y}$. By the definition, $x \preccurlyeq u$ and $y \preccurlyeq u$. Since $x \sim x'$, by transitivity, we have $x' \preccurlyeq u$ and thus $U_{x,y} \subset U_{x',y}$. By similar arguments, we can prove that $U_{x',y} \subset U_{x,y}$, hence $U_{x,y} = U_{x',y}$. The rest of our claim also follows from almost the same arguments. *[Q.E.D.]*

Lemma 2: *If $b \sim a$ for some $a \in A_{x,y}$, then $b \in A_{x,y}$. Moreover, $A_{x,y} = I_a$ for every $a \in A_{x,y}$. Similarly, if $c \sim t$ for some $t \in T_{x,y}$, then $c \in T_{x,y}$, and we have $T_{x,y} = I_t$ for every $t \in T_{x,y}$.*

Proof. By transitivity, we must have $x \preccurlyeq b$, $y \preccurlyeq b$ and $b \preccurlyeq u$ for all $u \in U_{x,y}$. This proves that $b \in A_{x,y}$. Since it is obvious that $a \sim a'$ for all $a, a' \in A_{x,y}$, our claims on $A_{x,y}$ follow. For the proof of the claims on $T_{x,y}$, we use the same logic as above. *[Q.E.D.]*

Lemma 3: *If $x' \in I_x$ for some x, then we have $A_{x,y} = A_{x',y}$ and $T_{x,y} = T_{x',y}$ for every $y \in X$.*

Proof. Let $a \in A_{x,y}$ and $a' \in A_{x',y}$. By the previous lemma, it is sufficient to show that $a' \in I_a$. By the definition, we have $x \preccurlyeq a$ and $a \preccurlyeq u$ for all $u \in U_{x,y}$. By transitivity and Lemma 1, this means $a \preccurlyeq x'$ and $a \preccurlyeq u'$ for all $u' \in U_{x',y}$ and thus, $A_{x,y} \subset A_{x',y}$. The converse relation can be shown similarly, hence $A_{x,y} = A_{x',y}$. The proof of the claim on the set of infimums is similar. *[Q.E.D.]*

We introduce the following notions, which play central roles in our proposition. In particular, as mentioned in the opening sentence, the definition of "*s-quasisupermodularity*" in Definition 6 is altered from the original paper Shirai (2008). This is our second correction.

Definition 5: Let X be a preordered lattice structure and $S, S' \subset X$. We say that S' is higher than S with respect to *w-strong set order* if $A_{x,y} \cap S' \neq \emptyset$ and $T_{x,y} \cap S \neq \emptyset$ for every $x, y \in X$. We write this as $S \leq_{wa} S'$. We say that S' is higher than S with respect to *s-strong set order* if $A_{x,y} \subset S'$ and $T_{x,y} \subset S$. We write this as $S \leq_{sa} S'$.

Definition 6: Let X be a preordered lattice structure. We say that a function $f : X \to \mathbb{R}$ is *w-quasisupermodular* if

$$\forall t \in T_{x,y}; f(x) \geq (>) f(t) \Rightarrow \exists a \in A_{x,y}; f(a) \geq (>) f(y).$$

We say that f is *s-quasisupermodular* of

$$\exists t \in T_{x,y}; f(x) \geq (>) f(t) \Rightarrow \forall a \in A_{x,y}; f(a) \geq (>) f(y).$$

It should be noted that, in the original paper, s-quasisupermodularity is defined as

$$\forall t \in T_{x,y}; f(x) \geq (>) f(t) \Rightarrow \forall a \in A_{x,y}; f(a) \geq (>) f(y).$$

However, it has become clear that this definition is too weak to assure the monotonicity of the solution set of the maximization problem with respect to s-strong set relation. In the following, we write the solution set of the maximization problem: $\max_{x \in S \subset X} f(x)$ as $M(S)$. Our main proposition can now be stated as follows. This is nothing but the corrected version of *Proposition 2.1* in the original paper.

Proposition 1: *(a): Let X be a prelattice, $S, S' \subset X$, and $S \leq_{sa} S'$. Then, $M(S) \leq_{wa} M(S')$ if and only if $f : X \to \mathbb{R}$ satisfies w-quasisupermodularity. (b): $M(S) \leq_{sa} M(S')$ if and only if f satisfies s-quasisupermodularity.*

Proof. (a): For the sufficiency part, see the original paper. The necessity part can be shown as follows. Let $S = I_x \cup T_{x,y}$ and $S' = I_y \cup A_{x,y}$ for some $x, y \in X$. It is obvious that $S \leq_{sa} S'$. Assume $M(S) \leq_{wa} M(S')$. Suppose $x \in M(S)$ and $y \in M(S')$. Since $M(S) \leq_{wa} M(S')$, $A_{x,y} \cap M(S') \neq \emptyset$, which implies $f(a) \geq f(y)$ for some $a \in A_{x,y}$. Suppose $x \notin M(S)$, $f(x) \geq f(t)$ for all $t \in T_{x,y}$ and $y \in M(S')$. Then, there exists some $x' \in I_x$ and $x' \in M(S)$.

By our assumption, we have $A_{x',y} \cap M(S') \neq \emptyset$ and by Lemma 3, we must have $A_{x,y} \cap M(S') \neq \emptyset$. Hence, we have

$$\forall t \in T_{x,y}; f(x) \geq f(t) \Rightarrow \exists a \in A_{x,y}; f(a) \geq f(y).$$

We have to show the case with strict inequalities. We prove the contraposition of this: assume $f(a) \leq f(y)$ for all $a \in A_{x,y}$. If $x \in M(S)$, we have $T_{x,y} \cap M(S) \neq \emptyset$, hence there exists some $t \in T_{x,y}$ such that $f(t) \geq f(x)$. Suppose $x \notin M(S)$ and $f(x) \geq f(t)$ for all $t \in T_{x,y}$. Then, there exists $x' \in I_x$ such that $x' \in M(S)$. By our assumption, we must have $T_{x',y} \cap M(S) \neq \emptyset$. However, by Lemma 3, this contradicts the fact that $f(x') > f(t)$. Hence there exists some $t \in T_{x,y}$ such that $f(t) \geq f(x)$. This proves that

$$\forall a \in A_{x,y}; f(a) \leq f(y) \Rightarrow \exists t \in T_{x,y}; f(x) \leq f(t),$$

which is equivalent to

$$\forall t \in T_{x,y}; f(x) > f(t) \Rightarrow \exists a \in A_{x,y}; f(a) > f(y).$$

(b): Let $x \in M(S)$ and $y \in M(S')$. By s-quasisupermodularity, we have $\forall a \in A_{x,y}; f(a) \geq f(y)$, which means $A_{x,y} \subset M(S')$. Then, suppose there exists some t' such that $t' \notin M(S)$, that is, $f(t') < f(x)$. In this case, by s-quasisupermodularity, we must have $f(a) > f(y)$ for all $a \in A_{x,y}$, contradiction.

The necessity part can be shown as follows. Let S and S' be the same as the proof of (a). Suppose $f(y) > f(a)$ for some $a \in A_{x,y}$. What we have to show is that $f(t) > f(x)$ for all $t \in T_{x,y}$. Note that $I_x \subset (M(S))^c$. This is shown as follows. Suppose $y \in M(S')$ and some $x' \in I_x$ is contained in $M(S)$. In this case, by Lemma 3, we must have $A_{x',y} = A_{x,y} \subset M(S')$, which contradicts our assumption. Suppose some $a' \in A_{x,y}$ is contained in $M(S')$ and some $x' \in I_x$ is an element of $M(S)$. In this case, by Lemmas 2 and 3, we must have $A_{x',a'} = I_{a'} = A_{x,y} \subset M(S')$, which again contradicts our assumption. Hence, the set $M(S)$ consists of some elements of $T_{x,y}$. Let $t' \in T_{x,y}$ be an element of $M(S)$. Note that, in fact, $A_{x,y} \subset (M(S'))^c$. Indeed, if some $a' \in A_{x,y}$ contained in $M(S')$, then we must have $A_{t',a'} = I_{a'} = A_{x,y} \subset M(S')$, contradiction. Hence, the set $M(S')$ consists of some elements of I_y. Let $y' \in I_y$ be an element of $M(S')$. Based on the above arguments, we can show that, in fact, $T_{x,y} = M(S)$. Indeed, by Lemmas 2 and 3, we have $T_{t',y'} = I_{t'} = T_{x,y} \subset M(S')$. This proves that

$$\exists a \in A_{x,y}; f(y) > f(a) \Rightarrow \forall t \in T_{x,y}; f(t) > f(x),$$

which is equivalent to

$$\exists t \in T_{x,y}; f(x) \geq f(t) \Rightarrow \forall a \in A_{x,y}; f(a) \geq f(y).$$

The case with strict inequalities is shown as follows: suppose $f(x) > f(t)$ for some $t \in T_{x,y}$. In this case, the set $M(S)$ must consist of some elements of I_x. Let x' be an element of $M(S)$. We show that $I_y \subset (M(S'))^c$. Indeed, if some $y' \in I_y$ is contained in $M(S')$, then by Lemma 3, we must have $T_{x',y'} = T_{x,y'} = T_{x,y} \subset M(S)$, which contradicts our assumption. Hence, the set $M(S')$ consists of some elements of $A_{x,y}$. Let $a' \in A_{x,y}$ be an element of $M(S')$. However, by Lemmas 2 and 3, we have $A_{x',a'} = I_{a'} = A_{x,y} \subset M(S')$, which implies $A_{x,y} = M(S')$. This proves that

$$\exists t \in T_{x,y}; f(x) > f(t) \Rightarrow \forall a \in A_{x,y}; f(a) > f(y).$$

This completes our proof. *[Q.E.D.]*

References

[1] Milgrom, P., Shannon, C. (1994): Motone comparative statics. *Econometrica*, 62, 157-180.

[2] Shirai, K. (2008): A generalization of monotone comparative statics. *Economics Bulletin*, Vol. 3, No. 39.