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Population monotonicity, consistency and the random arrival rule

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Abstract

In bankruptcy problems we characterize the random arrival rule by means of CG-consistency and population monotonicity.

1. Introduction

The Talmud rule and the random arrival rule are two well-known rules in bankruptcy problems. Having associated a game to every bankruptcy problem, it is well known that the prenucleolus gives place to the Talmud rule and the Shapley value to the random arrival rule (Aumann & Maschler, 1985; and O'Neill, 1982, respectively). In the axiomatic theory of TU games, we have known that the max-reduced games and the self-reduced games are imposed to characterize the prenucleolus and the Shapley value by Sobolev (1975) and Hart & Mas-Colell (1989), respectively. Corresponding to the max-reduced game, Aumann & Maschler (1985) introduced a definition of a reduced problem in bankruptcy problems and used it to characterize the Talmud rule. Hence, a natural problem raises whether there is an appropriate definition of a reduced problem to characterize the random arrival rule. The aim of this note is to answer this question.

As we have known, the fundamental definition of consistency applies only if the reduction produces an admissible problem. A stronger version is obtained by *adding the requirement that this should indeed be the case.* Albizuri, Leroux and Zarzuelo (2008) introduced a definition of a reduced problem, which corresponds to the self-reduced game, and used *consistency* to characterize the random arrival rule in bankruptcy problems. The axiomatic result in Albizuri, Leroux and Zarzuelo (2008) is true if we apply a stronger version of consistency. However, if we apply the fundamental definition of consistency, their proof is not completely correct because a reduced problem of a bankruptcy problem may be not a bankruptcy problem. We propose an example (see Example 1) to illustrate this fact. To overcome this drawback, we add an axiom, population monotonicity, to a rule in order to make sure the closedness under the reduction operation. Also, by CG-consistency and population monotonicity, we characterize the random arrival rule.

2. Preliminaries and Example

Let U be the universe of agents. Let $N \subseteq U$, the cardinality of N is denoted $|N|$. If $x \in \mathbb{R}^N$ and $S \subseteq N$, write x_S for the restriction of x to S. We denote $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, and $\mathbb{R}_+^N = \{x \in \mathbb{R}^N \mid x_i \geq 0 \text{ for all } i \in N\}$. Also, if $a \in \mathbb{R}$ we denote $a_+ = \max\{a, 0\}$.

A triple (N, E, c) is called a bankruptcy problem, if N is a non-empty finite set of U (the set of claimants), $E \in \mathbb{R}_+$ (the estate), and $c \in \mathbb{R}_+^N$ (the

vector of claims) is such that $\sum_{i\in N} c_i \geq E$. We will denote $\bar{c}_i = \min\{E, c_i\}$. Let β^U denote the set of all bankruptcy problems.

Let $(N, E, c) \in \beta^U$. The feasible set $X(N, E, c)$ is defined to be

$$X(N, E, c) = \{x \in \mathbb{R}_+^N \mid \sum_{i\in N} x_i = E \text{ and } x_i \leq c_i \text{ for all } i \in N\}.$$

Clearly, if $x \in X(N, E, c)$, $x_i \leq \bar{c}_i$ for all $i \in N$.

A rule on β^U is a function σ that assigns to each bankruptcy problem $(N, E, c) \in \beta^U$ a vector $\sigma(N, E, c) \in X(N, E, c)$.

Here we study the random arrival rule (O'Neill, 1982) in bankruptcy problems. To define this rule, imagine claimants arriving one at a time, and compensate them fully until money runs out. The resulting awards vector of course depends on the order in which claimants arrive. To remove the unfairness associated with a particular order, take the arithmetic average over all orders of arrival of the awards vector calculated in this way. For a formal definition of the resulting rule, let Π^N be the class of permutations of N.

Random arrival rule, RA. For each $(N, E, c) \in \beta^U$ and each $i \in N$,

$$RA_i(N, E, c) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \min\left\{\bar{c}_i, \left(E - \sum_{\substack{j\in N \\ \pi(j)<\pi(i)}} \bar{c}_j\right)_+\right\}$$

We refer to the two-claimants version of the random arrival rule as the contested garment rule as follows.

Contested garment rule, CG. For each $(N, E, c) \in \beta^U$ with $|N| = 2$ and each $i, j \in N$, $i \neq j$,

$$CG_i(N, E, c) = (E - c_j)_+ + \frac{E - (E - c_i)_+ - (E - c_j)_+}{2} = \frac{E + \bar{c}_i - \bar{c}_j}{2}$$

Inspired by Hart & Mas-Colell (1989), Albizuri, Leroux and Zarzuelo (2008) defined a reduced problem in bankruptcy problems as follows.

Let σ be a rule, $(N, E, c) \in \beta^U$, and $S \subseteq N$, $S \neq \emptyset$. The reduced problem with respect to S and σ is the problem $(S, E^{S,\sigma}, c^{S,\sigma})$ where [1]

$$\begin{cases} E^{S,\sigma} = E - \sum_{k\in N\setminus S} \sigma_k(N, E, c), \\ c_i^{S,\sigma} = \bar{c}_i - \sum_{k\in N\setminus S} (\sigma_k(N, E, \bar{c}) - \sigma_k(N \setminus \{i\}, E - \bar{c}_i, \bar{c}_{N\setminus\{i\}})) \text{ for all } i \in S. \end{cases}$$

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[1]For more details, see Albizuri, Leroux and Zarzuelo (2008).

The following example illustrates that a reduced problem of a bankruptcy problem may be not a bankruptcy problem.

Example 1 *Let $N = \{1, 2, 3\}$, $E = 5$, and $c = (3, 2, 2)$. Let σ be a rule with*

$$\begin{cases} \sigma(\{1, 2, 3\}, 5, (3, 2, 2)) = (1, 2, 2), \\ \sigma(\{1, 2\}, 3, (3, 2)) = (1, 2), \\ \sigma(\{1, 3\}, 3, (3, 2)) = (2, 1), \\ \sigma(\{2, 3\}, 2, (2, 2)) = (2, 0). \end{cases}$$

It is easy to see that $E^{\{1,2\},\sigma} = 3$, $c_1^{\{1,2\},\sigma} = 1$, and $c_2^{\{1,2\},\sigma} = 1$. Since $c_1^{\{1,2\},\sigma} + c_2^{\{1,2\},\sigma} < E^{\{1,2\},\sigma}$, $(\{1, 2\}, E^{\{1,2\},\sigma}, c^{\{1,2\},\sigma}) = (\{1, 2\}, 3, (1, 1))$ is not a bankruptcy problem.

3. Axiomatization

In this section, we characterize the random arrival rule by CG-consistency and population monotonicity. The two axioms are defined as follows.

CG-consistency: For each $(N, E, c) \in \beta^U$ and each $S \subseteq N$ with $|S| = 2$, if $(S, E^{S,\sigma}, c^{S,\sigma}) \in \beta^U$, then

$$\sigma_S(N, E, c) = CG(S, E^{S,\sigma}, c^{S,\sigma}).$$

Remark 1 *As we stated in Introduction, CG-consistency applies only if the reduction produces an admissible problem. A stronger version is obtained by adding the requirement that this should indeed be the case. To so strengthen CG-consistency, replace*

"if $(S, E^{S,\sigma}, c^{S,\sigma}) \in \beta^U$, then $\sigma_S(N, E, c) = CG(S, E^{S,\sigma}, c^{S,\sigma})$,"

by

"then $(S, E^{S,\sigma}, c^{S,\sigma}) \in \beta^U$, and $\sigma_S(N, E, c) = CG(S, E^{S,\sigma}, c^{S,\sigma})$."

Population monotonicity(PMON): For each $(N, E, c) \in \beta^U$ and each $j \in N$, $\sigma_i(N, E, c) \geq \sigma_i(N \setminus \{j\}, E - \bar{c}_j, \bar{c}_{N \setminus \{j\}})$ for each $i \in N \setminus \{j\}$.

PMON says that if all the members of N agree that a claimant j will get his claim (the maximum payment he can get), then all the members of $N \setminus \{j\}$ should be worse off after the right of the claimant j has been recognized.[2]

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[2]Many rules satisfy PMON, for example, the random arrival rule, the Talmud rule, the proportional rule, and so on.

Remark 2 *Let σ be a rule, $(N, E, c) \in \beta^U$, and $S \subseteq N$, $S \neq \emptyset$. If the reduced problem $(S, E^{S,\sigma}, c^{S,\sigma})$ with (1') $c_i^{S,\sigma} \geq 0$ for each $i \in S$ and (2') $\sum_{i \in S} c_i^{S,\sigma} \geq E^{S,\sigma}$, then the reduced problem $(S, E^{S,\sigma}, c^{S,\sigma})$ is a bankruptcy problem. In particular, if $|S| = 2$, the previous conditions (1') and (2') can be rewritten to the following conditions (1) and (2), respectively.*

$$\sigma_i(N, E, c) - \sigma_i(N \setminus \{j\}, E - \bar{c}_j, \bar{c}_{N \setminus \{j\}}) \geq -\sigma_j(N, E, c) \text{ for each } i \in S, \text{ where } j = S \setminus \{i\}. \tag{1}$$

$$\sum_{i \in S} \left(\sigma_i(N, E, c) - \sigma_i(N \setminus \{j\}, E - \bar{c}_j, \bar{c}_{N \setminus \{j\}}) \right) \geq 0, \text{ where } j = S \setminus \{i\}. \tag{2}$$

Therefore, if a rule satisfies PMON, by conditions (1) and (2), then a two-claimant reduced problem of a bankruptcy problem is still a bankruptcy problem.

Theorem 1 *The RA rule is the unique rule on β^U satisfying CG-consistency and PMON.*

Proof. It only needs to verify a two-claimant reduced problem of a bankruptcy problem is still a bankruptcy problem. Then the remaining proof follows that of Theorem 2 (Albizuri, Leroux and Zarzuelo, 2008). Since the RA rule satisfies PMON, by Remark 2, a two-claimant reduced problem is still a bankruptcy problem. This completes the proof. ∎

The following examples show that CG-consistency and PMON in Theorem 1 are logically independent.

Example 2 *Let M be a rule defined on β^U by for each $(N, E, c) \in \beta^U$ and for each $i \in N$,*

$$M_i(N, E, c) = c_i \min\{1, \frac{\lambda}{|T_i|}\},$$

where $T_i = \{k \in N \mid c_k = c_i\}$ and λ is chosen so that $\sum c_i \min\{1, \frac{\lambda}{|T_i|}\} = E$. It is not difficult to verify that the rule M does not satisfy PMON and there exists a two-claimant reduced problem (with respect to M) of a bankruptcy problem is not a bankruptcy problem.[3] Next let σ be a rule defined on β^U by

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[3] We consider a bankruptcy problem (N, E, c) with $N = \{1, 2, 3\}$, $E = 6$ and $c = (3, 2, 2)$. It is easy to see that $M(N, E, c) = (3, 1.5, 1.5)$, $M(N \setminus \{2\}, E - c_2, c_{N \setminus \{2\}}) = M(N \setminus \{3\}, E - c_3, c_{N \setminus \{3\}}) = (\frac{12}{5}, \frac{8}{5})$ and $c_2^{\{2,3\},M} = c_3^{\{2,3\},M} = \frac{7}{5}$. Since $M_3(N, E, c) = 1.5 < \frac{8}{5} = M_3(N \setminus \{2\}, E - c_2, c_{N \setminus \{2\}})$ and $c_2^{\{2,3\},M} + c_3^{\{2,3\},M} = \frac{14}{5} < 3 = E^{\{2,3\},M}$, the rule M is not PMON and the reduced problem $(\{2, 3\}, E^{\{2,3\},M}, c^{\{2,3\},M}) \notin \beta^U$.

for each $(N, E, c) \in \beta^U$,

$$
\sigma(N, E, c) = \begin{cases} RA(N, E, c) & , \textit{if } (S, E^{S,M}, c^{S,M}) \in \beta^U \textit{ for all } S \subseteq N \textit{ with } |S| = 2 \\ M(N, E, c) & , \textit{otherwise.} \end{cases}
$$

Then σ satisfies CG-consistent but it violates PMON.

Example 3 *The proportional rule[4] satisfies PMON but it violates CG-consistent.*

References

[1] Albizuri, M.J. and J. Leroux and J. Zarzuelo (2008) "Updating Claims in Bankruptcy Problems" Cahiers de recherche from HEC Montreal, Institut d'economie appliquee working paper 08-08.

[2] Aumann, R.J. and M. Maschler (1985) "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud" *Journal of Economic Theory* **36**, 195-213.

[3] Hart, S. and A. Mas-Colell (1989) "Potential, Value and Consistency" *Econometrica* **57**, 589-614.

[4] O'Neill, B. (1982) "A Problem of Rights Arbitration from the Talmud" *Mathematical Social Sciences* **2**, 345-371.

[5] Sobolev, A.I. 1975) "The Characterization of Optimality Principles in Cooperative Games by Functional Equations" *Mathematical Methods in Social Sciences* **6**, 150-165 (in Russian).

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[4]P is the proportional rule if for each $(N, E, c) \in \beta^U$, $P(N, E, c) = \lambda c$, where λ is chosen with $\sum_{i \in N} \lambda c_i = E$.