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Monotone and local potential maximizers in symmetric 3x3 supermodular games

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Abstract

Generalized notions of potential maximizer, monotone potential maximizer (MP-maximizer) and local potential maximizer (LP-maximizer), are studied. It is known that 2x2 coordination games generically have a potential maximizer, while symmetric 4x4 supermodular games may have no MP- or LP-maximizer. This note considers the case inbetween, namely the class of (generic) symmetric 3x3 supermodular coordination games. This class of games are shown to always have a unique MP-maximizer, and its complete characterization is given. A nondegenerate example demonstrates that own-action quasiconcave supermodular games may have more than one LP-maximizers.

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1. Introduction

A game is a potential game (Monderer and Shapley 1996) if it admits a potential function, a real-valued function defined on the set of action profiles such that the change in any player's payoff resulting from switching actions is proportional to the change in the value of this function.[1] The best response correspondence of a potential game thus coincides with that of a pure common interest game in which the payoff function of each player is given by the potential function. A potential maximizer, an action profile that maximizes a potential function, is a pure-strategy Nash equilibrium, and this equilibrium does not depend on a particular choice of a potential function.

Potential maximizers have nice "equilibrium selection" properties: Ui (2001) shows that a unique potential maximizer is robust to incomplete information (Kajii and Morris 1997), while Hofbauer and Sorger (1999, 2002) show that it is a unique equilibrium that is absorbing and globally accessible under perfect foresight dynamics for small frictions (Matsui and Matsuyama 1995). In 2×2 coordination games, potential maximization agrees with risk dominance: that is, 2×2 coordination games are potential games, where the risk-dominant equilibrium maximizes the potential function. Beyond 2×2 games, however, the class of potential games is nongeneric as they are defined by equalities.

In games with certain monotonicity properties, in particular in *supermodular games* (also known as games with strategic complementarities), the nice results have been proved (through versions of "comparison principle") to continue to hold in "generalized" potential games where those equalities are replaced with certain inequalities. More precisely, Morris and Ui (2005) introduce the concepts of *monotone potential maximizer* (*MP-maximizer*) and *local potential maximizer* (*LP-maximizer*) and show that an MP-maximizer of a supermodular game is robust to incomplete information and so is an LP-maximizer if the game has diminishing marginal returns (in which case LP-maximizer coincides with MP-maximizer). By Oyama *et al.* (2008, OTH henceforth), an MP-maximizer of a supermodular game is shown also to satisfy the stability condition under perfect foresight dynamics.[2] On the other hand, except for 2×2 games these generalized potential conditions are typically not easy to inspect. Morris (1999) presents an example of a symmetric 4×4 supermodular game that has no MP- or LP-maximizer.[3]

[1] More precisely, such a game is called a "weighted" potential game, while it is called an "exact" potential game if the payoff change is always exactly equal to the change in the potential. In this note, the term "potential game" refers to the former.

[2] See also Oyama and Tercieux (2009) for further developments. Okada and Tercieux (2008) show that in supermodular games, an LP-maximizer with constant weights is stochastically stable under the log-linear dynamics of Blume (1993).

[3] Beyond two-player games, it has been known that asymmetric three-player unanimity

In this note, we study MP- and LP-maximizers in symmetric 3×3 supermodular coordination games. We first establish a generic existence and a characterization of MP-maximizer for these games. Note that an MP-maximizer, if any, is known to be unique in generic supermodular games (OTH 2008). While those of LP-maximizer for this class of games have been given by Morris (1999) and Frankel *et al.* (2003, FMP henceforth), we also show that there is a non-empty open set of symmetric 3×3 supermodular coordination games that have two LP-maximizers, and these games possibly satisfy own-action quasiconcavity (FMP 2003).

2. Definitions and Known Facts

By $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$, we denote the set of real numbers, the set of nonnegative real numbers, and the set of positive real numbers, respectively.

Let $G = (I, (A_i)_{i \in I}, (u_i)_{i \in I})$ be an $N \geq 2$ player supermodular game, where $I = \{1, \ldots, N\}$ is the set of players, $A_i = \{0, 1, \ldots, n_i\}$ the linearly ordered set of actions for player $i \in I$, and $u_i \colon A = \prod_{i \in I} A_i \to \mathbb{R}$ the payoff function for player i satisfying *supermodularity*: for all $i \in I$, all $a_i, a_i' \in A_i$, and all $a_{-i}, a'_{-i} \in A_{-i} = \prod_{j \neq i} A_j$,

$$u_i(a_i', a_{-i}) - u_i(a_i, a_{-i}) \leq u_i(a_i', a'_{-i}) - u_i(a_i, a'_{-i})$$

whenever $a_i < a_i'$ and $a_{-i} \leq a'_{-i}$. For $i \in I$, denote by $\Delta(A_{-i})$ the set of probability distribution over A_{-i}. For a function $f \colon A \to \mathbb{R}$, a probability distribution $\pi_i \in \Delta(A_{-i})$, and a nonempty subset of actions $A_i' \subset A_i$, let

$$br^i_f(\pi_i | A_i') = \arg\max_{a_i \in A_i'} f(a_i, \pi_i),$$

where $f(a_i, \cdot)$ is extended to $\Delta(A_{-i})$ by $f(a_i, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) f(a_i, a_{-i})$. When $A_i' = A_i$, we write $br^i_f(\pi_i) = br^i_f(\pi_i | A_i)$. For $a_i', a_i'' \in A_i$, we denote $[a_i', a_i''] = \{a_i \in A_i \mid a_i' \leq a_i \leq a_i''\}$.

An action profile $a^* \in A$ is a (weighted) *potential maximizer* (Monderer and Shapley 1996) of G if there exists a function $v \colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists $\mu_i \in \mathbb{R}_{++}$ such that

$$\mu_i\big(v(a_i', a_{-i}) - v(a_i, a_{-i})\big) = u_i(a_i', a_{-i}) - u_i(a_i, a_{-i}).$$

Such a function v is called a *potential function* for a^*. Note that in this case, $br^i_v(\pi) = br^i_{u_i}(\pi_i)$ holds for all $i \in I$ and $\pi_i \in \Delta(A_{-i})$. It is easy to see that any game has at most one potential maximizer. We shall be interested in two variants of the notion of potential maximizer due to Morris

games (Carlsson 1989, Morris and Ui 2005, OTH 2008, 2009) and symmetric $3 \times 3 \times 3$ supermodular games (Takahashi 2008) may not have an MP- or LP-maximizer.

and Ui (2005), *monotone potential maximizer* (*MP-maximizer*) and *local potential maximizer* (*LP-maximizer*). We employ the simplified versions and their refinements due to OTH (2008).

MP-maximizer and its refinement, strict MP-maximizer, are defined as follows:

Definition 1. (a) Action profile $a^* \in A$ is an *MP-maximizer* of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br^i_v(\pi_i|[0, a^*_i]) \leq \max br^i_{u_i}(\pi_i|[0, a^*_i]), \tag{1}$$

and

$$\max br^i_v(\pi_i|[a^*_i, n_i]) \geq \min br^i_{u_i}(\pi_i|[a^*_i, n_i]). \tag{2}$$

Such a function v is called a *monotone potential function* for a^*.

(b) Action profile $a^* \in A$ is a *strict MP-maximizer* of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br^i_v(\pi_i|[0, a^*_i]) \leq \min br^i_{u_i}(\pi_i|[0, a^*_i]), \tag{3}$$

and

$$\max br^i_v(\pi_i|[a^*_i, n_i]) \geq \max br^i_{u_i}(\pi_i|[a^*_i, n_i]). \tag{4}$$

Such a function v is called a *strict monotone potential function* for a^*.

Notice that the 'max' ('min', resp.) in the right-had side of (1) ((2), resp.) is replaced by the 'min' ('max', resp.) in the right-had side of (3) ((4), resp.).

A (strict) MP-maximizer is a (strict) Nash equilibrium, and a potential maximizer is a strict MP-maximizer. A strict MP-maximizer is always an MP-maximizer, and for a generic choice of payoffs, an MP-maximizer is a strict MP-maximizer.

From OTH (2008), we have:[4]

Fact 1 (OTH 2008)**.** *A supermodular game can have at most one strict MP-maximizer.*

LP-maximizer and strict LP-maximizer are defined as follows:

Definition 2. (a) Action profile $a^* \in A$ is an *LP-maximizer* of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all

[4] OTH (2008, Theorems 4.1 and 4.2) show that a strict MP-maximizer of a supermodular game is a unique equilibrium that is absorbing and globally accessible under perfect foresight dynamics when the degree of friction is small. Hence, a supermodular game cannot have more than one strict MP-maximizers.

$i \in I$, there exists a function $\mu_i \colon A_i \setminus \{a_i^*\} \to \mathbb{R}_+$ such that if $a_i < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(a_i)\big(v(a_i+1,a_{-i}) - v(a_i,a_{-i})\big) \le u_i(a_i+1,a_{-i}) - u_i(a_i,a_{-i}), \quad (5)$$

and if $a_i > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(a_i)\big(v(a_i-1,a_{-i}) - v(a_i,a_{-i})\big) \le u_i(a_i-1,a_{-i}) - u_i(a_i,a_{-i}). \quad (6)$$

Such a function v is called a *local potential function* for a^*.

(b) Action profile a^* is a *strict LP-maximizer* of G if there exists a function $v \colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists a function $\mu_i \colon A_i \setminus \{a_i^*\} \to \mathbb{R}_{++}$ such that if $a_i < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(a_i)\big(v(a_i+1,a_{-i}) - v(a_i,a_{-i})\big) \le u_i(a_i+1,a_{-i}) - u_i(a_i,a_{-i}), \quad (7)$$

and if $a_i > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(a_i)\big(v(a_i-1,a_{-i}) - v(a_i,a_{-i})\big) \le u_i(a_i-1,a_{-i}) - u_i(a_i,a_{-i}). \quad (8)$$

Such a function v is called a *strict local potential function* for a^*.

An LP-maximizer is a strict LP-maximizer if one can take strictly positive numbers for the weights μ_i.[5]

The game G is said to satisfy *diminishing marginal returns* (or own-action concavity) if for all $i \in I$, all $a_i \neq 0, n_i$, and all $a_{-i} \in A_{-i}$,

$$u_i(a_i,a_{-i}) - u_i(a_i-1,a_{-i}) \ge u_i(a_i+1,a_{-i}) - u_i(a_i,a_{-i}).$$

For a function $v \colon A \to \mathbb{R}$, let $G_v = (I, (A_i)_{i\in I}, (v)_{i\in I})$ be the game in which all players have the common payoff function v. By Morris and Ui (2005) (see also OTH 2008, Lemma 4.2), we have:

Fact 2 (Morris and Ui 2005)**.** *If the game G has a (strict) LP-maximizer a^* with a (strict) local potential function v and if G or G_v satisfies diminishing marginal returns, then a^* is a (strict) MP-maximizer with the same function v.*

Therefore, from Fact 1 we have:

Fact 3 (OTH 2008)**.** *A supermodular game that satisfies diminishing marginal returns can have at most one strict LP-maximizer.*

[5]Morris (1999) and FMP (2003) give a slightly different definition of LP-maximizer, which is weaker than strict LP-maximizer.

In Section 3, we will show that a supermodular game without diminishing marginal returns may have multiple strict LP-maximizers.

The notion of **p**-dominance (Kajii and Morris 1997), a many-player generalization of risk-dominance, provides a sufficient condition for MP-maximizer.

Definition 3. Let $\mathbf{p} = (p_1, \dots, p_N) \in [0, 1)^N$.

(a) Action profile $a^* \in A$ is a **p**-*dominant equilibrium* of G if for all $i \in I$, $a_i^* \in br^i(\pi_i)$ holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) \geq p_i$.

(b) Action profile a^* is a *strict* **p**-*dominant equilibrium* of G if for all $i \in I$, $\{a_i^*\} = br^i(\pi_i)$ holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) > p_i$.

For $p \in [0, 1)$, we say that a^* is a (strict) p-dominant equilibrium if it is a (strict) $(p, \dots, p)$-dominant equilibrium. In a 2×2 coordination game, the risk-dominant equilibrium is a strict (p_1, p_2)-dominant equilibrium for some (p_1, p_2) such that $p_1 + p_2 < 1$ (Kajii and Morris 1997; see also Morris 1999 or FMP 2003).

As shown by Morris and Ui (2005) (see also OTH 2008, Lemma 4.1), a **p**-dominant equilibrium with low **p** is an MP-maximizer.

Fact 4 (Morris and Ui 2005)**.** *If a^* is a (strict)* **p**-*dominant equilibrium with $\sum_{i \in I} p_i < 1$, then a^* is a (strict) MP-maximizer with the (strict) monotone potential v given by*

$$v(a) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } a = a^*, \\ -\sum_{i \in C(a)} p_i & \text{otherwise,} \end{cases}$$

where $C(a) = \{i \in I \mid a_i = a_i^*\}$.

Note that 2×2 coordination games generically have a (unique) strict MP- and LP-maximizer. On the other hand, Morris (1999) shows that there is a non-empty open set of symmetric 4×4 supermodular games that have no MP- or LP-maximizer.[6]

In the next section, we obtain the generic existence of strict MP- (as well as LP-) maximizer for the case inbetween, namely, the class of symmetric 3×3 supermodular coordination games. We also show that these games may have multiple strict LP-maximizers.

[6]Morris (1999) presents an example of a symmetric 4×4 supermodular game satisfying diminishing marginal returns that has no robust equilibrium in the sense of Kajii and Morris (1997) (and it remains valid with a small perturbation of payoffs). Since, as shown by Morris and Ui (2005), an MP-maximizer of a supermodular game must be a robust equilibrium, it follows that this game has no MP-maximizer. Furthermore, since it satisfies diminishing marginal returns, this game has no LP-maximizer, either.

3. Symmetric 3×3 Supermodular Coordination Games

We consider symmetric 3×3 coordination games, where $I = \{1, 2\}$, $A_1 = A_2 = \{0, 1, 2\}$, $u_1(h, k) = u_2(k, h)$ for all $h, k \in \{0, 1, 2\}$, and all the action profiles on the diagonal are Nash equilibria, i.e., $u_1(h, h) \geq u_1(k, h)$ for all k. We assume strict supermodularity, i.e., $u_1(h', k) - u_1(h, k) < u_1(h', k') - u_1(h, k')$ if $h < h'$ and $k < k'$.

For $h, k \in \{0, 1, 2\}$, let

$$\Delta^{hk}_{h'k'} = \bigl(u_1(h', h) + u_1(h', k)\bigr) - \bigl(u_1(k', h) + u_1(k', k)\bigr).$$

The inequality $\Delta^{hk}_{h'k'} > 0$ means that action h' is better than action k' against the 50-50 mixture of actions h and k. Note that $\Delta^{kh}_{h'k'} = \Delta^{hk}_{h'k'}$ and $\Delta^{hk}_{k'h'} = -\Delta^{hk}_{h'k'}$. Note also that $\Delta^{hk}_{hk} > 0$ if and only if h pairwise risk-dominates k.

3.1. MP-Maximizer

We have the following complete characterization (for generic games) of the strict MP-maximizer. Recall from Fact 1 that a strict MP-maximizer is unique if it exists. (The proofs of the Propositions are provided in the Appendix.)

Proposition 1.
(1) $\Delta^{02}_{01} > 0$ *and* $\Delta^{02}_{02} > 0$*:* $(0, 0)$ *is the strict MP-maximizer.*

(2) $\Delta^{20}_{21} > 0$ *and* $\Delta^{20}_{20} > 0$*:* $(2, 2)$ *is the strict MP-maximizer.*

(3) $\Delta^{02}_{10} \geq 0$ *and* $\Delta^{02}_{12} \geq 0$*:*

- (a) $(0, 0)$ *is the strict MP-maximizer if* $\Delta^{01}_{01} > 0$ *and either* (i) $\Delta^{12}_{12} \geq 0$ *or* (ii) $\Delta^{21}_{21} > 0$ *and* $\Delta^{02}_{10}/\Delta^{01}_{01} < \Delta^{02}_{12}/\Delta^{21}_{21}$.
- (b) $(1, 1)$ *is the strict MP-maximizer if* $\Delta^{10}_{10} > 0$ *and* $\Delta^{12}_{12} > 0$.
- (c) $(2, 2)$ *is the strict MP-maximizer if* $\Delta^{21}_{21} > 0$ *and either* (i) $\Delta^{10}_{10} \geq 0$ *or* (ii) $\Delta^{01}_{01} > 0$ *and* $\Delta^{02}_{10}/\Delta^{01}_{01} > \Delta^{02}_{12}/\Delta^{21}_{21}$.

Thus, we have:

Result 1. *Every generic symmetric* 3×3 *supermodular coordination game has a strict MP-maximizer.*

In the cases (1), (2), and (3–b), the strict MP-maximizer is also a strict p-dominant equilibrium for some $p < 1/2$, while in the other cases, there is no p-dominant equilibrium with $p < 1/2$.

3.2. LP-Maximizer

The generic existence of LP-maximizer has been shown by Morris (1999) (see also FMP 2003).

Fact 5 (Morris 1999, FMP 2003)**.** *Every generic symmetric* 3×3 *supermodular coordination game has at least one strict LP-maximizer.*

This follows from the following classification reported in Morris (1999) and FMP (2003), which they prove only for two cases. In the Appendix, we provide a proof that covers all the cases.

Proposition 2.
(a) $(0,0)$ *is a strict LP-maximizer if* $\Delta_{01}^{01} > 0$ *and either* (i) $\Delta_{12}^{12} \geq 0$ *or* (ii) $\Delta_{21}^{21} > 0$ *and* $\Delta_{10}^{02}/\Delta_{01}^{01} < \Delta_{12}^{02}/\Delta_{21}^{21}$.
(b) $(1,1)$ *is a strict LP-maximizer if* $\Delta_{10}^{10} > 0$ *and* $\Delta_{12}^{12} > 0$.
(c) $(2,2)$ *is a strict LP-maximizer if* $\Delta_{21}^{21} > 0$ *and either* (i) $\Delta_{10}^{10} \geq 0$ *or* (ii) $\Delta_{01}^{01} > 0$ *and* $\Delta_{10}^{02}/\Delta_{01}^{01} > \Delta_{12}^{02}/\Delta_{21}^{21}$.

However, we show that this is only a *partial* characterization, in that there is another strict LP-maximizer than the one described above in some cases. More precisely:

Proposition 3.
(1) *If* $\Delta_{01}^{02} > 0$*, then* $(0,0)$ *is a strict LP-maximizer.*
(2) *If* $\Delta_{21}^{20} > 0$*, then* $(2,2)$ *is a strict LP-maximizer.*

These conditions (1) and (2) can be simultaneously satisfied by a non-empty open set of symmetric 3×3 supermodular coordination games. Thus, we have:

Result 2. *There is a non-empty open set of symmetric* 3×3 *supermodular coordination games that have two strict LP-maximizers.*

Observe that by supermodularity, $\Delta_{01}^{02} > 0$ and $\Delta_{21}^{20} > 0$ imply $\Delta_{01}^{01} > 0$ and $\Delta_{21}^{21} > 0$, respectively. Therefore, multiplicity occurs in subcases of the cases (a–ii) and (c–ii) in Proposition 2. In these cases, the game cannot have diminishing marginal returns, i.e., the inequalities, $\Delta_{01}^{02} > 0$, $\Delta_{21}^{20} > 0$, $u_1(1,0) - u_1(0,0) \geq u_1(2,0) - u_1(1,0)$, and $u_1(1,2) - u_1(0,2) \geq u_1(2,2) - u_1(1,2)$, cannot be simultaneously satisfied. On the other hand, the game may satisfy the weaker property of own-action *quasi*-concavity (FMP 2003): the game is said to be own-action quasiconcave if for all $i \in I$, the set $\{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq c\}$ is convex (i.e., it is written as $[a_i', a_i'']$ for some $a_i', a_i'' \in A_i$) for all $a_{-i} \in A_{-i}$ and all constants c.

Example 1. Consider the following supermodular coordination game, which satisfies own-action quasiconcavity, but not diminishing marginal returns:

	0	1	2
0	$7,7$	$0,4$	$0,0$
1	$4,0$	$1,1$	$2,0$
2	$0,0$	$0,2$	$8,8$

Observe that $\Delta_{01}^{01} = 2$, $\Delta_{21}^{21} = 5$, $\Delta_{10}^{02} = -1$, $\Delta_{12}^{02} = -2$, so that $\Delta_{10}^{02}/\Delta_{01}^{01} < \Delta_{12}^{02}/\Delta_{21}^{21}$, and $\Delta_{20}^{20} = 1$. Thus, this game falls in Case (a–ii) in Proposition 2 (and in Case (2) in Proposition 1), and Proposition 3 applies.

This game has two strict LP-maximizers, $(0,0)$ and $(2,2)$, while $(2,2)$ is the unique strict MP-maximizer (and 7/15-dominant). Local potential functions are given by the following (the function (b) is also a monotone potential function for $(2,2)$):

	0	1	2
0	13	4	0
1	4	7	6
2	0	6	12

(a) Local potential function for $(0,0)$

	0	1	2
0	7	4	0
1	4	3	2
2	0	2	8

(b) Local potential function for $(2,2)$

When the game satisfies diminishing marginal returns (in which case a strict LP-maximizer is unique by Fact 3), the unique strict LP-maximizer is given by the classification of Proposition 2. This follows by verifying the strict LP-maximizer given in the classification is also a strict MP-maximizer as given in Proposition 1.

Remark 1. Example 1 is also a counterexample to Theorem 4 in FMP (2003), which claims that an LP-maximizer of a supermodular game that satisfies own-action *quasiconcavity* is noise-independent selection in global games: the game in Example 1 shows that this claim contradicts the limit uniqueness result of FMP (2003, Theorem 1). A correct statement should read that a (strict) LP-maximizer of a supermodular game that satisfies diminishing marginal returns (or own-action *concavity*) is noise-independent selection. More generally, one can show that a strict MP-maximizer of a supermodular game is noise-independent selection in global games.

Appendix

We denote $u_1(h,k) = w_{hk}$; thus $\Delta^{hk}_{h'k'} = w_{h'h} + w_{h'k} - w_{k'h} - w_{k'k}$.

Proof of Proposition 1. Case (1): $(0,0)$ is a strict p-dominant equilibrium with $p < 1/2$, so that Fact 4 applies.

Case (2): Symmetric with Case (1).

Case (3–a–i): A monotone potential function v for $(0,0)$ is the following:

	0	1	2
0	$\varepsilon\Delta^{01}_{01}$	$\varepsilon(w_{01} - w_{11})$	$\varepsilon(w_{02} - w_{12}) + (w_{21} - w_{11})$
1	$\varepsilon(w_{01} - w_{11})$	0	$w_{21} - w_{11}$
2	$\varepsilon(w_{02} - w_{12}) + (w_{21} - w_{11})$	$w_{21} - w_{11}$	0

where $\varepsilon > 0$ is sufficiently small. All entries but $v(0,0)$ are less than or equal to zero (recall $w_{02} - w_{12} < w_{01} - w_{11} \leq 0$ and $w_{21} - w_{11} \leq 0$ by assumption). By verifying that

$$v(0,k) - v(1,k) = \varepsilon\big(u_1(0,k) - u_1(1,k)\big),$$
$$v(1,k) - v(2,k) \leq u_1(1,k) - u_1(2,k),$$
$$v(0,k) - v(2,k) \leq u_1(1,k) - u_1(2,k)$$

for all k (let ε be sufficiently small, and use $w_{20} - w_{10} < w_{21} - w_{11}$ and $\Delta^{12}_{12} \geq 0$), one can show that the conditions in Definition 1(b) (with $a^* = (0,0)$) are satisfied.

Case (3–a–ii): A monotone potential function v for $(0,0)$ is the following:

	0	1	2
0	ε	$\varepsilon + \lambda_1(w_{10} - w_{00})$	$\lambda_1(w_{02} - w_{12}) + \lambda_2(w_{12} - w_{22})$
1	$\varepsilon + \lambda_1(w_{10} - w_{00})$	$-\lambda_2\Delta^{21}_{21}$	$\lambda_2(w_{12} - w_{22})$
2	$\lambda_1(w_{02} - w_{12}) + \lambda_2(w_{12} - w_{22})$	$\lambda_2(w_{12} - w_{22})$	0

where $\varepsilon > 0$ is sufficiently small, and $\lambda_1 > 0$ and $\lambda_2 > 0$ are such that

$$\frac{\Delta^{21}_{21}}{\Delta^{01}_{01}} < \frac{\lambda_1}{\lambda_2} < \frac{\Delta^{02}_{12}}{\Delta^{02}_{10}}$$

(when $\Delta^{02}_{10} = 0$, in which case $\Delta^{02}_{12} > 0$, let $\Delta^{02}_{12}/\Delta^{02}_{10} = \infty$). This is the local potential function given in Morris (1999). Verify that

$$\begin{aligned}
v(0,k) - v(1,k) &\leq \lambda_1\bigl(u_1(0,k) - u_1(1,k)\bigr),\\
v(1,k) - v(2,k) &\leq \lambda_2\bigl(u_1(1,k) - u_1(2,k)\bigr),\\
v(0,k) - v(2,k) &\leq (\lambda_2 + \lambda_3)\bigl(u_1(1,k) - u_1(2,k)\bigr)
\end{aligned}$$

for all k, where $\lambda_3 > 0$ is such that

$$\frac{w_{22} - w_{12}}{w_{12} - w_{02}} < \frac{\lambda_1}{\lambda_3} < \frac{w_{10} - w_{20}}{w_{12} - w_{02}}.$$

Case (3–b): $(1,1)$ is a strict p-dominant equilibrium with $p < 1/2$.

Case (3–c): Symmetric with Case (3–a). ▮

Proof of Proposition 2. Case (a–i): The monotone potential function constructed in Case (3–a–i) in the proof of Proposition 1 works also as a local potential function for $(0,0)$.

Case (a–ii–1): $\Delta^{01}_{01} > 0$, $\Delta^{21}_{21} > 0$, $\Delta^{02}_{10}/\Delta^{01}_{01} < \Delta^{02}_{12}/\Delta^{21}_{21}$, and $\Delta^{02}_{10} \geq 0$. The monotone potential function constructed in Case (3–a–ii) in the proof of Proposition 1 works also as a local potential function for $(0,0)$.

Case (a–ii–2): $\Delta^{01}_{01} > 0$, $\Delta^{21}_{21} > 0$, $\Delta^{02}_{10}/\Delta^{01}_{01} < \Delta^{02}_{12}/\Delta^{21}_{21}$, and $\Delta^{02}_{01} > 0$. Follows from Proposition 3(1).

Case (b): $(1,1)$ is a strict p-dominant equilibrium with $p < 1/2$. The monotone potential function given in Fact 4 works also as a local potential function for $(1,1)$.

Case (c): Symmetric with Case (a). ▮

Proof of Proposition 3. Case (1): A local potential function v for $(0,0)$ is the following:

	0	1	2
0	Δ^{02}_{01}	$w_{02} - w_{12}$	$w_{02} - w_{12}$
1	$w_{02} - w_{12}$	$\varepsilon(w_{11} - w_{21})$	0
2	$w_{02} - w_{12}$	0	$\varepsilon(w_{22} - w_{12})$

where $\varepsilon > 0$ is sufficiently small. Verify that

$$\begin{aligned}
v(0,k) - v(1,k) &\leq u_1(0,k) - u_1(1,k),\\
v(1,k) - v(2,k) &\leq \varepsilon\bigl(u_1(1,k) - u_1(2,k)\bigr)
\end{aligned}$$

for all k.

Case (2): Symmetric with Case (1). ▮

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