E C O N O M I C S   B U L L E T I N

Generalized monotonicity and strategy−proofness: A note

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Abstract

In this note we define generalized monotonicity which is a generalized version of monotonicity due to Muller and Satterthwaite (1979) for a social choice function under individual preferences which permit indifference, and shall show that generalized monotonicity and strategy−proofness are equivalent.

I would like to thank Bhaskar Dutta and the anonymous referee for helpful comments.
Citation: Tanaka, Yasuhito, (2001) "Generalized monotonicity and strategy−proofness: A note." *Economics Bulletin,* Vol. 4, No. 11 pp. 1−6
Submitted: June 25, 2001. **Accepted:** October 10, 2001.
URL: http://www.economicsbulletin.com/2001/volume4/EB−01D70005A.pdf

1 Introduction

In this note we define generalized monotonicity which is a generalized version of monotonicity in Muller and Satterthwaite (1979) for a social choice function under individual preferences which permit indifference, and we shall show that generalized monotonicity and strategy-proofness are equivalent.

2 Notation, definitions and preliminary results

There are a set of individuals N, and a set of alternatives A for a social problem. The number of individuals n is a finite positive integer which is larger than or equal to 2. The number of alternatives is also a finite positive integer which is larger than or equal to 3. The individuals are represented by individual i, j and so on, and the alternatives are represented by x, y, z and so on. The preference of individual i about the alternatives is represented by a weak order R_i, which is reflexive, complete (connected) and transitive. The asymmetric part (strict preference) and the symmetric part (indifference) of R_i are denoted by P_i and I_i. We allow indifference in individual preferences.

A social choice function (or voting rule) is a mapping from an n-tuple of reported preferences of the individuals to an alternative. An n-tuple of individual preferences is called a *profile* of individual preferences (or an *individual preference profile*). Each profile is denoted by a, b, c and so on. At a profile a, for example, individual i's preference is denoted by R_i^a, P_i^a and I_i^a. When a social choice function chooses x at a profile a, we denote $C(a) = x$. We call the alternative which is chosen by a social choice function the *winner* of the social choice function. We consider a resolute social choice function, which chooses only one of the alternatives at any profile. Further we assume that social choice functions are *non-imposed* or *onto*. It means that for any alternative and any social choice function there is a profile of individual preferences at which the alternative is chosen by the social choice function.

We define strategic manipulability and strategy-proofness of a social choice function.

Strategic manipulability There are two individual preference profiles a and b such as a social choice function chooses x at a and y at b. Between a and b only the preference of one individual (denoted by i) is different (b is an i-variant of a). If individual i has a preference $xP_i^b y$, the social choice function is strategically manipulable by him at b because he can make the social choice function choose x by reporting falsely his preference R_i^a when his true preference is R_i^b. Similarly, if he has a preference $yP_i^a x$, the social choice function is strategically manipulable by him at a.

Strategy-proofness If a social choice function is not strategically manipulable by any individual at any individual preference profile, it is *strategy-proof*.

Next, we define generalized monotonicity which is a generalized version of monotonicity due to Muller and Satterthwaite (1979).

generalized monotonicity There is a profile of individual preferences a such as for alternatives x and y

(1) individuals in a group V ($V \subset N$): $xP_i^a y$

(2) individuals in a group V' ($V' \subset N,\ V' \cap V = \emptyset$): $xI_i^a y$

(3) others (group V''): $yP_i^a x$

and a social choice function chooses x ($C(a) = x$). We do not assume any specification of individual preferences about alternatives other than x and y. There is another profile b such as

(1) individuals in V: $xP_i^b y$, other preferences are not specified

(2) individuals in V': $xP_i^b y$ or their preferences are the same as those at a

(3) V'': not specified

Then, the social choice function does not choose y at b ($C(b) \neq y$).

Now we show the following lemma.

Lemma 1. *Strategy-proofness implies generalized monotonicity.*

In the following proof we use notation in the above definition of generalized monotonicity.

Proof. Let individuals 1 to m ($0 \leq m \leq n$) belong to V, individuals $m + 1$ to m' ($m \leq m' \leq n$) belong to V', and individuals $m' + 1$ to n belong to V''. Consider a preference profile c other than a and b such as individuals in V and V' have a preference $xP_i^c yP_i^c z$, and individuals in V'' have a preference $yP_i^c xP_i^c z$, where z is an arbitrary alternative other than x and y.

Let a^1 be a preference profile such as only the preference of individual 1 has changed from R_1^a to R_1^c, and suppose that at a^1 the social choice function chooses an alternative other than x. Then, individual 1 has an incentive to report falsely his preference R_1^a when his true preference is R_1^c, and so we have $C(a^1) = x$. By the same logic we find that when the preferences of individuals 1 to m' change from R_i^a to R_i^c, the social choice function chooses x ($C(a^{m'}) = x$). Next, let $a^{m'+1}$ be a preference profile such as the preference of individual $m' + 1$, as well as the preferences of the first m' individuals, has changed from $R_{m'+1}^a$ to $R_{m'+1}^c$, and suppose that at $a^{m'+1}$ the social choice function chooses y. Then, individual $m' + 1$ has an incentive to report falsely his preference $R_{m'+1}^c$ when his true preference is $R_{m'+1}^a$ because $yP_{m'+1}^a x$. On the other hand, if the social choice function chooses an alternative other than x and y at $a^{m'+1}$, individual $m' + 1$ has an incentive to report falsely his preference $R_{m'+1}^a$ when his true preference is $R_{m'+1}^c$ because $xP_{m'+1}^c z$. Therefore, we have $C(a^{m'+1}) = x$. By the same logic we find that when the preferences of all individuals have changed from R_i^a to R_i^c, the social choice function must choose x ($C(c) = x$).

Now, suppose that from c to b the individual preferences change one by one from R_i^c to R_i^b. Then, when the preference of some individual changes, the winner of the social choice function can not change directly from x to y. If the social choice function chooses y when the preference of an individual in V or V' (denoted by j) changes from R_j^c to R_j^b, individual j has an incentive to report falsely his preference R_j^c when his true preference is R_j^b because $xP_j^b y$. On the other hand, if the social choice function chooses y when the preference of an individual in V'' (denoted by k) changes from R_k^c to R_k^b, individual k has an incentive to report falsely his preference R_k^b when his true preference is R_k^c because $yP_k^c x$.

It remains the possibility, however, that the winner of the social choice function changes from x through $z(\neq x, y)$ to y. Suppose that when the preferences of some individuals have changed from R_i^c to R_i^b, the winner of the social choice function is $z(\neq x,\ y)$, and further when the preference of individual l has changed from R_l^c to R_l^b, the winner of the social choice function becomes y. Since he prefers y to z at c, he can get y by misrepresenting his preference R_l^b when his true preference is R_l^c. Therefore, if the social choice function is strategy-proof, in the sequence of changes of individual preferences the winner of the social choice function does not change from x through z to y. Hence, we must have $C(b) \neq y$. □

A group V in this lemma may be the set of all individuals, or may be a set consisting of only one individual.

3 Equivalence of generalized monotonicity and strategy-proofness

In this section we shall show the equivalence of generalized monotonicity and strategy-proofness.

Theorem 1. *Generalized monotonicity and strategy-proofness are equivalent.*

Proof. Lemma 1 has shown that strategy-proofness implies generalized monotonicity so that only the converse needs to be proved.

Suppose that at a profile of individual preferences a a social choice function chooses x ($C(a) = x$), and assume that the social choice function which satisfies generalized monotonicity is strategically manipulable. Then, there is a case where, when the preference of one individual (denoted by i) changes from R_i^a to R_i^b (denote such a profile by b), the winner of the social choice function changes from x to y, and individual i has a preference $yP_i^a x$.

Consider another profile of individual preferences c at which individual i has a preference $yP_i^c xP_i^c z$ where z is a arbitrary alternative other than x and y, and the preferences of the other individuals are the same as those at a. If the social choice function chooses y at c, since individual i prefers y to x at a and c, generalized monotonicity implies that the social choice function does not choose x at a. This contradicts with the assumption, and so y is not chosen at c. Comparing a and c about x and z, the preferences of individuals other than individual i have not changed, and individual i has a preference $xP_i^c z$ at c and his preference at a about x and z is not specified. Therefore, from generalized monotonicity z is not chosen at c, and so the social choice function must choose x at c.

On the other hand, comparing b and c about x and y, the preferences of individuals other than individual i have not changed, individual i has a preference $yP_i^c x$ at c, and his preference at b is not specified. Therefore, from generalized monotonicity x is not chosen at c. This contradicts with the above result. Hence, the social choice function must be strategy-proof. □

4 Concluding remarks

The equivalence of strategy-proofness and generalized monotonicity presented in this paper does not require all preference orderings to exist like as the proof of the Gibbard-Satterthwaite theorem by Sen(2000) in the case of linear individual preferences. All that is required is that for all pairs of alternatives x and y there exists an admissible ordering where x is ranked first uniquely and y is ranked second uniquely.

We can show the Gibbard-Satterthwaite theorem (Gibbard(1973) and Satterthwaite(1975)) in the case of individual preferences which permit indifference using generalized monotonicity.

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