E C O N O M I C S   B U L L E T I N

Generalized monotonicity and strategy−proofness for non−resolute social choice correspondences

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Abstract

Recently there are several works which analyzed the strategy−proofness of non−resolute social choice rules such as Duggan and Schwartz (2000) and Ching and Zhou (2001). In these analyses it was assumed that individual preferences are linear, that is, they excluded indifference from individual preferences. We present an analysis of the strategy−proofness of non−resolute social choice rules when indifference in individual preferences is allowed. Following to the definition of the strategy−proofness by Ching and Zhou (2001) we shall show that a generalized version of monotonicity and the strategy−proofness are equivalent. It is an extension of the equivalence of monotonicity and the strategy−proofness for resolute social choice rules with linear individual preferences proved by Muller and Satterthwate (1980) to the case of non−resolute social choice rules with general individual preferences.

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1 Introduction

The problem of strategy-proofness for non-resolute (or set-valued, multi-valued) social choice correspondences recently has been analyzed in several works, for example, Duggan and Schwartz (2000) and Ching and Zhou (2001). In these analyses it was assumed that individual preferences over alternatives are linear (or strict, asymmetric), that is, they excluded indifference from individual preferences.

Ching and Zhou (2001) established that when individual preferences are linear, social choice correspondences are single-valued or constant. But if indifference is allowed, their result does not hold.

We present an analysis of strategy-proofness for non-resolute social choice correspondences when indifference in individual preferences is allowed. Following the definition of strategy-proofness by Ching and Zhou (2001) we shall show that a generalized version of monotonicity (generalized monotonicity) and strategy-proofness are equivalent. It is an extension of the equivalence of monotonicity (or strong positive association) and strategy-proofness for resolute social choice rules with linear individual preferences proved by Muller and Satterthwaite (1977) to the case of non-resolute social choice rules with individual preferences which allow indifference.

In the next section we present notation, definitions and preliminary results. In Section 3 we shall show the equivalence of generalized monotonicity and strategy-proofness.

2 Notation, definitions and a preliminary result

There is a society with n individuals, and a social problem with more than two alternatives. n is a finite positive integer which is larger than 1, and the number of the alternatives is a finite positive integer which is larger than 2. The set of individuals is denoted by N, and the set of alternatives is denoted by A. The individuals are represented by individual i, j and so on, and the alternatives are represented by x, y, z and so on. The preference of individual i over the alternatives is represented by a weak order R_i, which is reflexive, complete (connected) and transitive. The asymmetric part (strict preference) and the symmetric part (indifference) of R_i are denoted by P_i and I_i. xP_iy means that individual i prefers x to y, and so on.

A social choice correspondence (or voting rule) is a mapping from an n-tuple of individual preferences to a subset of A. It is non-resolute, that is, it may choose multiple alternatives. We assume unrestricted domain of social choice correspondences. An n-tuple of individual preferences is called a *profile* of individual preferences (or an *individual preference profile*). The profiles are denoted by a, b, c and so on. For example, at a profile a individual i's preference is denoted by R_i^a, P_i^a and I_i^a. Denote the set of alternatives chosen by a social choice correspondence at a profile a by $C(a)$. We call it the *social choice set* at a.

non-imposition and non-constancy We assume that social choice correspondences are *non-imposed* or *onto*, that is, their ranges are A. It means that for any alternative there is an individual preference profile at which the alternative is included in the social choice set. This assumption implies that any alternative may be included in some social choice set. But we do not assume that there is a profile at which any alternative may be chosen by a social choice correspondence as a singleton social choice set.

If social choice sets for all individual preference profiles are identical, the social choice correspondence is said to be constant. Because such a social choice rule is not interesting, we assume that social choice correspondences are not constant. It implies that there is at least one individual preference profile (denoted by a) at which the social choice set does not include all alternatives, that is, $C(a) \neq A$.

Although a social choice set may include multiple alternatives, only one alternative actually realizes. Each individual (represented by i) has a subjective probability measure p and a von Neumann-Morgenstern utility function u_i over A. If we have $u_i(x) > u_i(y)$ when xP_iy and $u_i(x) = u_i(y)$ when xI_iy, it is said that u_i is *consistent* with the preference of individual i.

Next, we define (strategic) manipulability and strategy-proofness of a social choice correspondence following the definition by Ching and Zhou (2001)[1)]. Let a and b be two profiles of individual preferences between which only the preference of individual i is different, and let $C(a)$ and $C(b)$ be the social choice sets at a and b. Denote the set of alternatives which are included in $C(a)$ but not included in $C(b)$ by $C(a) \setminus C(b)$, and the set of alternatives which are included in $C(b)$ but not included in $C(a)$ by $C(b) \setminus C(a)$, and denote the value of individual i's subjective probability measure on x by $p(x)$. Then, his expected utility conditional on $C(a)$

[1)] The definition of manipulability by Duggan and Schwartz (2000) is different from that by Ching and Zhou (2001). The former requires that misrepresentation of an individual's preference makes him better off for *every* prior subjective probabilities. On the other hand, the latter requires that misrepresentation makes him better off for *some* prior. We think that the definition by Duggan and Schwartz (2000) is too strong.

and that conditional on $C(b)$ evaluated by his utility function which is consistent with his preference at a are written as follows,

$$E_i^a(a) = \frac{1}{\sum_{x \in C(a)} p(x)} \sum_{x \in C(a)} p(x) u_i^a(x)$$

and

$$E_i^a(b) = \frac{1}{\sum_{x \in C(b)} p(x)} \sum_{x \in C(b)} p(x) u_i^a(x)$$

If $C(a) = C(b)$, we have $E_i^a(a) = E_i^a(b)$. If $C(a)$ and $C(b)$ are different, we obtain

$$E_i^a(a) = \frac{\sum_{x \in C(a) \cap C(b)} p(x) u_i^a(x) + \sum_{y \in C(a) \setminus C(b)} p(y) u_i^a(y)}{\sum_{x \in C(a) \cap C(b)} p(x) + \sum_{y \in C(a) \setminus C(b)} p(y)} \tag{1}$$

and

$$E_i^a(b) = \frac{\sum_{x \in C(a) \cap C(b)} p(x) u_i^a(x) + \sum_{y \in C(b) \setminus C(a)} p(y) u_i^a(y)}{\sum_{x \in C(a) \cap C(b)} p(x) + \sum_{y \in C(b) \setminus C(a)} p(y)} \tag{2}$$

If for all probability measures and utility functions which are consistent with his preference at a we have $E_i^a(a) \geq E_i^a(b)$, the social choice correspondence is not manipulable. Conversely, if for some probability measure and some utility function we have $E_i^a(a) < E_i^a(b)$, individual i has an incentive to reveal R_i^b (his preference at b) when his true preference is R_i^a, and the social choice correspondence is manipulable by individual i at a. For example, assume that $p(x) = 0.8 - \varepsilon$, $p(y) = 0.2$, $u_i^a(x) = 0$, $u_i^a(y) = 1$ and $u_i^a(z) = 2$ for the alternative z which is the most preferred alternative of individual i in $C(a)$. $\varepsilon (0 < \varepsilon < 1)$ is the sum of the probabilities of alternatives other than x and y.

From (1) and (2), if for some $x \in C(a)$ and some $y \in C(b) \setminus C(a)$ the preference of individual i is $y P_i^a x$, we obtain

$$E_i^a(b) \geq 0.2$$

and

$$E_i^a(a) < \frac{2\varepsilon}{0.8 - \varepsilon}$$

Let ε be sufficiently small (such that $\varepsilon \leq 0.07$), then we obtain $E_i^a(a) < E_i^a(b)$. Similarly, if for some $x \in C(a) \setminus C(b)$ and some $y \in C(b)$ the preference of individual i is $y P_i^a x$, we obtain

$$E_i^a(b) \geq \frac{0.2}{0.2 + \varepsilon}$$

and

$$E_i^a(a) < \frac{2\varepsilon + 0.2}{0.8 - \varepsilon}$$

Let ε be sufficiently small (such that $\varepsilon \le 0.1$), then we obtain $E_i^a(a) < E_i^a(b)$. Summarizing the results,

Lemma 1. *If for some $x \in C(a)$ and some $y \in C(b) \setminus C(a)$ the preference of individual i is $yP_i^a x$, or for some $x \in C(a) \setminus C(b)$ and some $y \in C(b)$ the preference of individual i is $yP_i^a x$, the social choice correspondence is manipulable by individual i at a preference profile a by R_i^b.*

Conversely, if for all $x \in C(a)$ and $y \in C(b) \setminus C(a)$, and for all $x \in C(a) \setminus C(b)$ and $y \in C(b)$ the preference of individual i is $xR_i^a y$, the social choice correspondence is not manipulable.

strategy-proofness If a social choice correspondence is not manipulable by any individual at any individual preference profile, it is *strategy-proof*.

Ching and Zhou (2001) showed that when individual preferences are linear, social choice correspondences are single-valued or constant. But if indifference is allowed, their result does not hold. There is a simple example. Define the social choice set for each profile as the maximal set of individual 1 for that profile, and assume that his maximal set is not constant and may be multi-valued for some profiles. Then, the social choice correspondence is strategy-proof, neither constant nor single-valued.

Let us consider an example of a manipulable voting rule.

An example There are three individuals 1, 2 and 3, and three alternatives, x, y and z. Suppose the following two preference profiles, *a* and *b*.

(1) $xP_1^a yP_1^a zP_1^a w,\ yP_2^a zP_2^a xP_2^a w,\ zP_3^a xP_3^a yP_3^a w$

(2) $xP_1^b yP_1^b zP_1^b w,\ yP_2^b zP_2^b xP_2^b w,\ zP_3^b wP_3^b yP_3^a x$

Between *a* and *b* only individual 3's preference is different. Consider a so-called Borda rule. Each individual assigns 3 points to his most preferred alternative, 2 points to the second, 1 point to the third and 0 to the last, and the social choice set consists of the alternatives which get the largest total points. Then, we obtain $C(a) = \{x, y, z\}$ and $C(b) = \{y, z\}$. Let p_x, p_y and p_z be the subjective probabilities of individual 3 on x, y and z, and let u_x, u_y and u_y be the values of his utility of x,

y and z at the profile a. Then, if the following relations holds, this voting rule is manipulable by him at a by R_i^b.

$$p_z(u_z - u_y) > p_y(u_x - u_y)$$

Next, we define *generalized monotonicity*[2)].

generalized monotonicity Suppose that at a profile of individual preferences a such that for a pair of alternatives (x, y)

(1) individuals in a group V ($V \subset N$): $xP_i^a y$

(2) individuals in a group V' ($V' \subset N,\ V' \cap V = \emptyset$): $xI_i^a y$

(3) others (group V''): $yP_i^a x$

a social choice correspondence chooses x and does not choose y ($x \in C(a)$ and $y \notin C(a)$). We do not assume any specification of individual preferences about alternatives other than x and y. There is another profile b such that

(1) individuals in V: $xP_i^b y$, other preferences are not specified

(2) individuals in V': $xP_i^b y$ or their preferences are completely identical to those at a

(3) V'': not specified

Then, the social choice correspondence does not choose y at b ($y \notin C(b)$).

First we show the following lemma.

Lemma 2. *Strategy-proofness implies generalized monotonicity.*

In the following proof we use notation in the above definition of generalized monotonicity.

Proof. Let individuals 1 to m ($0 \le m \le n$) belong to V, individuals $m+1$ to m' ($m \le m' \le n$) belong to V', and individuals $m'+1$ to n belong to V''. Consider a preference profile c other than a and b such that individuals in V and V' have a

[2)]Our generalized monotonicity does not imply the so-called Maskin monotonicity, and the latter does not imply the former. The Maskin monotonicity requires the following condition. There is a preference profile a at which x is included in the social choice set. There is another profile b such that between a and b the preference of only one individual (denoted by i) is different, and $xP_i^b y$ for $y \neq x$ if $xP_i^a y$. Then, x is included in the social choice set at b. See Maskin (1999) preference $xP_i^c yP_i^c z$, and individuals in V'' have a preference $yP_i^c xP_i^c z$, where z is an arbitrary alternative other than x and y.

Let a^1 be a preference profile such that only the preference of individual 1 changes from R_1^a to R_1^c, and suppose that at a^1 the social choice correspondence chooses y. Then, individual 1 has an incentive to reveal a false preference R_1^a when his true preference is R_1^c because he prefers x to y at a^1 and y is not chosen at a. Thus, we have $y \notin C(a^1)$. Next, suppose that at a^1 the social choice correspondence does not choose x. Then, individual 1 has an incentive to reveal a false preference R_1^a when his true preference is R_1^c because he prefers x to all other alternatives at a^1 and x is chosen at a. Thus, we have $x \in C(a^1)$. By the same logic we find that when the preferences of individuals 1 to m' change from R_i^a to R_i^c, the social choice correspondence chooses x and does not choose y ($x \in C(a^{m'})$ and $y \notin C(a^{m'})$).

Next, let $a^{m'+1}$ be a preference profile such that the preference of individual $m'+1$, as well as the preferences of the first m' individuals, changes from $R_{m'+1}^a$ to $R_{m'+1}^c$, and suppose that at $a^{m'+1}$ the social choice correspondence chooses y. Then, individual $m'+1$ has an incentive to reveal a false preference $R_{m'+1}^c$ when his true preference is $R_{m'+1}^a$ because $yP_{m'+1}^a x$. On the other hand, if the social choice correspondence does not choose x and chooses an alternative $z(\neq x, y)$ at $a^{m'+1}$. Then, individual $m'+1$ has an incentive to reveal a false preference $R_{m'+1}^a$ when his true preference is $R_{m'+1}^c$ because $xP_{m'+1}^c z$ for all $z(\neq x, y)$. Therefore, we have $x \in C(a^{m'+1})$ and $y \notin C(a^{m'+1})$. By the same logic we find that when the preferences of all individuals change from R_i^a to R_i^c, the social choice correspondence chooses x and does not choose y ($x \in C(c)$ and $y \notin C(c)$).

Now, suppose that from c to b the individual preferences change one by one from R_i^c to R_i^b. Then, when the preference of some individual changes, the social choice set can not change directly from a set which includes x and does not include y to a set which includes y. If the social choice correspondence chooses y when the preference of an individual in V or V' (denoted by j) changes from R_j^c to R_j^b, individual j has an incentive to reveal a false preference R_j^c when his true preference is R_j^b because $xP_j^b y$. On the other hand, if the social choice correspondence chooses y when the preference of an individual in V'' (denoted by k) changes from R_k^c to R_k^b, individual k has an incentive to reveal a false preference R_k^b when his true preference is R_k^c because $yP_k^c x$.

It remains the possibility, however, that the social choice set changes from a set which includes x and does not include y through a set which includes $z(\neq x, y)$ and does not include x and y to a set which includes y. Suppose that when the preferences of some individuals change from R_i^c to R_i^b, the social choice correspondence chooses $z(\neq x, y)$ and does not choose x and y, and further when the preference of individual l changes from R_l^c to R_l^b, the social choice correspondence chooses y. Then, individual l has an incentive to reveal a false preference R_l^b when his true preference is R_l^c because $yP_l^c z$. Therefore, we must have $y \notin C(b)$. □

A group V in this lemma may be the set of all individuals, or may be a set consisting of only one individual.

3 Equivalence of generalized monotonicity and strategy-proofness

In this section we show the equivalence of generalized monotonicity and strategy-proofness.

Theorem 1. *Generalized monotonicity implies strategy-proofness. Therefore, with Lemma 2, generalized monotonicity and strategy-proofness are equivalent.*

Proof. Denote the social choice sets at preference profiles a and b by C_a and C_b. Between a and b only the preference of individual i is different. Assume that a social choice correspondence which satisfies generalized monotonicity is manipulable. Then, there is a case where, either of the following (1) or (2) holds.

(1) For some $x \in C_a$ and $y \in C_b \setminus C_a$ individual i's preference is $yP_i^a x$.

(2) For some $x \in C_a \setminus C_b$ and $y \in C_b$ individual i's preference is $yP_i^a x$.

First consider (1). Comparing a and b, individual i has a preference $yP_i^a x$ at a and the preferences of other individuals are the same. Thus, those who prefer x to y at a prefer x to y also at b, and the preferences of individuals who are indifferent between x and y at a do not change. From generalized monotonicity, if y is not included in C_a, it is not included in C_b. Therefore, there is not a case where (1) holds.

Next consider (2). Comparing b and a, individual i has a preference $yP_i^a x$ at a and the preferences of other individuals are the same. Thus, those who prefer y to x at b prefer y to x also at a, and the preferences of individuals who are indifferent between x and y at b do not change or they prefer y to x at a (when individual i is indifferent between x and y at b). From generalized monotonicity, if x is not included in C_b, it is not included in C_a. Therefore, there is not a case where (2) holds. □

We have a conjecture that the Gibbard-Satterthwaite theorem (Gibbard (1973) and Satterthwaite (1975) can be extended to the case of non-resolute social choice correspondences using generalized monotonicity.

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