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Existence of competitive equilibrium in economies with multi-member households

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Abstract

This paper focuses on the existence of a competitive equilibrium in the general equilibrium model with multi-member households introduced by Haller (2000). Two main results are obtained: the first is based on the assumption of ``budget exhaustion,'' which is also used in the existence theorem by Gersbach and Haller (1999). The second is based on the assumption that every household has at least one member whose preference is nonsatiated with respect to feasible household consumption, and this is applicable to the case in which budget exhaustion does not hold.

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1 Introduction

As an alternative to the traditional general equilibrium model, in which households are treated as if they were single consumers, Haller (2000) introduces a general equilibrium model in which each household may consist of several (heterogeneous) members. In his model, each member of a household has his or her own utility function, which may depend on the other members' consumption as well as his or her own consumption (in other words, consumption externalities may exist between members of the same household). The exchange of goods and services takes place only among households and not among individuals. However, the behavior of a household in markets is collectively decided by its members: taking a price system as given, the household members collectively decide their demand for goods and services under their common budget constraint. Haller (2000) assumes that as a result of such collective decision-making, every household as a whole chooses a consumption plan that is efficient in the sense that there is no other possible choice that satisfies the budget constraint, and makes at least one member better off without making any other members worse off. [1] Then, a competitive equilibrium among households is defined as a state in which every household makes an efficient choice and all markets clear.

Haller (2000) provides several fundamental results for the properties of competitive equilibrium among households. For example, he establishes the first welfare theorem and generalizes it to the core inclusion statement. He also examines conditions under which a competitive equilibrium allocation among households can be realized through competitive exchange among individuals. Further implications of competition among multi-member households are investigated in Gersbach and Haller (2001, 2005), which extend Haller's (2000) model (in which the household structure is given) to the case of a variable household structure.

In this paper, we focus on the existence of a competitive equilibrium among households and provide two main results. Our first result is based on the assumption of "budget exhaustion." Budget exhaustion asserts that to achieve an efficient consumption choice, households need to exhaust their budget. Under this assumption, Gersbach and Haller (1999) prove the existence of a competitive equilibrium among households with free disposal. [2] However, in their proof they implicitly assume that budget exhaustion holds in a truncated economy rather than in an original economy. In contrast, we assume budget exhaustion in the original economy and derive the existence of a competitive equilibrium among households. Our existence result also differs from the result of Gersbach and Haller (1999), in that we discard the free disposal assumption by allowing negative equilibrium prices. [3] Next we consider an alternative to budget exhaustion. As Haller (2000) illustrates, when we take negative intra-household externalities into account, budget exhaustion may not hold even if every consumer has a monotonic preference with respect to his or her individual consumption. However, we prove that even if budget exhaustion fails, the existence of a competitive equilibrium among households is still ensured, provided that each household has at least one member whose utility function is nonsatiated

[1] This behavioral hypothesis is originated by Chiappori (1988, 1992), who intends to develop a collective decision-making model such that testable implications of household behavior can be derived, and household members' individual preferences and the collective decision-making process can be recovered from the observation of the household behavior.

[2] Budget exhaustion is also used in Haller (2000) to establish the first welfare theorem.

[3] With some additional assumptions, we can obtain the existence of a strictly positive equilibrium price.

with respect to the feasible household consumption.

This paper is organized as follows. In Section 2, we introduce the model and definitions. In Section 3, we provide a preliminary result to be used in the proofs of our main existence results. In Section 4, we prove the existence of a competitive equilibrium among households under budget exhaustion. In Section 5, we provide another existence result by replacing budget exhaustion with the nonsatiation assumption. In Section 6, we make concluding remarks. Finally, all the proofs are provided in the Appendix.

2 Model and Definitions

Following Haller (2000), we define a pure exchange economy with multi-member households as follows.

In the economy, there exist ℓ commodities and H households ($\ell, H \in \mathbb{N}$, $1 \leq \ell < \infty$, $1 \leq H < \infty$). [4] Each household $h = 1, \cdots, H$ consists of finitely many members $i = hm$ with $m = 1, \cdots, M(h)$ and $M(h) \geq 1$.

Let I denote the set of all consumers in the economy, that is, $I = \{hm : h = 1, \cdots, H \text{ and } m = 1, \cdots, M(h)\}$. Each consumer $i \in I$ has a consumption set $X_i = \mathbb{R}^\ell_+$. Let $\mathcal{X}_h = \prod_m X_{hm}$ for each h, and let $\mathcal{X} = \prod_i X_i$. We call an element $\mathbf{x}_h = (x_{hm})_m$ of $\mathcal{X}_h$ a household consumption bundle of h, and $\mathbf{x} = (x_i)_i$ of $\mathcal{X}$ an allocation. For a given allocation $\mathbf{x} \in \mathcal{X}$, we denote household h's consumption bundle in $\mathbf{x}$ by $\mathbf{x}_h$.

Let $U_i : \mathcal{X}_h \to \mathbb{R}$ be the utility function of consumer $i = hm$. We allow consumption externalities only between members of the same households (intra-household externalities).

Each household h is endowed with a commodity bundle $\omega_h \in \mathbb{R}^\ell_+$.

A price system p is a ℓ-dimensional vector in $\mathbb{R}^\ell$.

Note that this model coincides with the standard Arrow–Debreu economy if $M(h) = 1$ for all h.

An allocation $\mathbf{x} \in \mathcal{X}$ is *feasible* if $\sum_h \sum_m x_{hm} = \sum_h \omega_h$ (free disposal is not allowed).

For each h, a household consumption bundle $\mathbf{x}_h \in \mathcal{X}_h$ is *feasible for* h if there exists $(\mathbf{x}_k)_{k \neq h} \in \prod_{k \neq h} \mathcal{X}_k$ such that $\mathbf{x} = (\mathbf{x}_h, (\mathbf{x}_k)_{k \neq h}) \in \mathcal{X}$ is feasible. Let $\mathcal{F}_h$ denote the set of household consumption bundles of h that are feasible for h. Since $X_i = \mathbb{R}^\ell_+$ for all $i \in I$, it is easy to check that

$$\mathcal{F}_h = \{\mathbf{x}_h \in \mathcal{X}_h : \textstyle\sum_m x_{hm} \leq \sum_k \omega_k\} \quad \text{for all} \quad h.$$

Note that $\mathcal{F}_h$ is compact for all h, and if $\mathbf{x} = (\mathbf{x}_h)_h \in \mathcal{X}$ is a feasible allocation, then $\mathbf{x}_h \in \mathcal{F}_h$ for all h.

Along with the standard continuity and concavity, we use the following properties of consumers' utility functions.

Strict Monotonicity. *U_{hm} is strictly monotonic if $U_{hm}(x_{hm}, (x_{hn})_{n \neq m})$ is strictly increasing in x_{hm}.* [5]

[4] We use the following mathematical notations. Let $\mathbb{N}$ denote the set of natural numbers. For $k \in \mathbb{N}$, let $\mathbb{R}^k$ denote the k-dimensional Euclidean space. For $x, y \in \mathbb{R}^k$, by $x \geq y$, we mean $x_j \geq y_j$ for all $j = 1, \cdots, k$, and by $x \gg y$, we mean $x_j > y_j$ for all j. Further, by $x > y$, we mean $x \geq y$ and $x \neq y$. Let $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x \geq 0\}$. For $x, y \in \mathbb{R}^k$, we denote by $x \cdot y = \sum_{j=1}^k x_j y_j$ the inner product, by $\|x\| = \sqrt{x \cdot x}$ the Euclidean norm, and by $B(x_0, r) = \{x \in \mathbb{R}^k : \|x - x_0\| < r\}$ the open ball centered at x_0 with radius r. For a set $A \subset \mathbb{R}^k$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{co} A$ the interior of A, closure of A and convex hull of A, respectively.

[5] For $X \subset \mathbb{R}^\ell$, a function $f : X \to \mathbb{R}$ is strictly increasing if $f(y) > f(x)$ for all $x, y \in X$ with $y > x$.

Nonsatiation. *U_{hm} is nonsatiated on $\mathcal{F}_h$ if for all $\mathbf{x}_h \in \mathcal{F}_h$, there exists $\mathbf{y}_h \in \mathcal{X}_h$ such that $U_{hm}(\mathbf{y}_h) > U_{hm}(\mathbf{x}_h)$.*

Local Nonsatiation. *U_{hm} is locally nonsatiated on $\mathcal{F}_h$ if for all $\mathbf{x}_h \in \mathcal{F}_h$ and $r > 0$, there exists $\mathbf{y}_h \in B(\mathbf{x}_h, r) \cap \mathcal{X}_h$ such that $U_{hm}(\mathbf{y}_h) > U_{hm}(\mathbf{x}_h)$.*

Non-Negative Externalities. *U_{hm} exhibits non-negative externalities if U_{hm} is nondecreasing in $(x_{hn})_{n \neq m}$.* [6]

It is clear that the strict-monotonicity of U_{hm} implies the local nonsatiation. Note also that under the concavity of U_{hm}, the nonsatiation implies the local nonsatiation.

For $\mathbf{x}_h \in \mathcal{X}_h$ and $p \in \mathbb{R}^\ell$, we denote

$$p * \mathbf{x}_h = p \cdot \sum_m x_{hm}.$$

Then, we define h's budget set $B_h(p)$ by

$$B_h(p) = \{\mathbf{x}_h \in \mathcal{X}_h : p * \mathbf{x}_h \leq p \cdot \omega_h\}.$$

Further, we define h's efficient budget set $EB_h(p)$ as follows.

Definition 1. *Household h's consumption bundle $\mathbf{x}_h \in \mathcal{X}_h$ belongs to $EB_h(p)$ if and only if $\mathbf{x}_h \in B_h(p)$ and there is no $\mathbf{y}_h \in B_h(p)$ such that*

$$\begin{aligned} U_{hm}(\mathbf{y}_h) &\geq U_{hm}(\mathbf{x}_h) \quad \text{for all } m, \text{ and} \\ U_{hm}(\mathbf{y}_h) &> U_{hm}(\mathbf{x}_h) \quad \text{for some } m. \end{aligned}$$

We call an element $\mathbf{x}_h$ of $EB_h(p)$ an efficient choice or efficient consumption bundle of h. Note that Definition 1 does not specify the exact decision mechanism used in a household to achieve its efficient choices.

Definition 2. *An element $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ is a competitive equilibrium among households if*
(i) $\mathbf{x}_h^* \in EB_h(p^*)$ for all h, and
(ii) $\sum_h \sum_m x_{hm}^* = \sum_h \omega_h$.

In a competitive equilibrium, each household makes an efficient choice under its budget constraint and markets clear. Note that we allow negative equilibrium prices.

3 Preliminary Result

In this paper, we prove the existence of a competitive equilibrium among households under two different sets of assumptions.

In the existence analysis, we suppose that a specific decision mechanism is used in every household to achieve its efficient choices: every household maximizes a weighted sum of its members' utility functions subject to the budget constraint.

[6]For $X \subset \mathbb{R}^\ell$, a function $f : X \to \mathbb{R}$ is nondecreasing if $f(y) \geq f(x)$ for all $x, y \in X$ with $y \geq x$.

For each h, let

$$\Delta_h = \{\lambda_h = (\lambda_{h1}, \cdots, \lambda_{hM(h)}) \in \mathbb{R}_+^{M(h)} : \textstyle\sum_m \lambda_{hm} = 1\}.$$

Let $W_h : \mathcal{X}_h \times \Delta_h \to \mathbb{R}$ be the function defined by

$$W_h(\mathbf{x}_h, \lambda_h) = \sum_m \lambda_{hm} U_{hm}(\mathbf{x}_h).$$

Moreover, for each $\lambda_h \in \Delta_h$, we define a function $W_h^{\lambda_h} : \mathcal{X}_h \to \mathbb{R}$ by

$$W_h^{\lambda_h}(\mathbf{x}_h) = W_h(\mathbf{x}_h, \lambda_h).$$

The function $W_h^{\lambda_h}$ is a weighted sum of the household members' utility functions, where λ_{hm} is the weight of the m-th member's utility. We call this $W_h^{\lambda_h}$ *household h's welfare function with weight λ_h*.

In a household, a maximization of the welfare function leads to an efficient consumption choice. Indeed, it is easy to see that for $\lambda_h \in \Delta_h$ with $\lambda_h \gg 0$, if $\mathbf{x}_h \in \mathcal{X}_h$ maximizes $W_h^{\lambda_h}(\cdot)$ over $B_h(p)$, then, $\mathbf{x}_h \in EB_h(p)$. Note that welfare functions with different weights may yield different efficient consumption bundles.

Let $\Delta = \prod_h \Delta_h$ with a generic element $\lambda = (\lambda_h)_h$. Each $\lambda \in \Delta$ defines the list of welfare functions $(W_h^{\lambda_h})_h$ of the households.

The following lemma is a preliminary result to be used in the proofs of our main existence results.

Lemma 1. *Let $\lambda \in \Delta$. Suppose that*

- $\omega_h \gg 0$ *for all h,*
- $W_h^{\lambda_h}$ *is continuous on $\mathcal{X}_h$ for all h, and*
- $W_h^{\lambda_h}$ *is concave on $\mathcal{X}_h$ for all h.*

Then, there exist an element $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ and a non-negative real number $\alpha^* \in \mathbb{R}_+$ such that

(i) $p^* * \mathbf{x}_h^* \leq p^* \cdot \omega_h + \alpha^*$ *for all h,*

(ii) *For each h, if $W_h^{\lambda_h}(\mathbf{y}_h) > W_h^{\lambda_h}(\mathbf{x}_h^*)$, then $p^* * \mathbf{y}_h > p^* \cdot \omega_h + \alpha^*$,*

(iii) $\sum_h \sum_m x_{hm}^* = \sum_h \omega_h$.

4 Existence under budget exhaustion

In this section, we address the existence of a competitive equilibrium under the following assumption.

(BE). Budget Exhaustion

For all h and $p \in \mathbb{R}^\ell$, if $\mathbf{x}_h \in EB_h(p)$, then $p * \mathbf{x}_h = p \cdot \omega_h$.

This assumption is introduced in Gersbach and Haller (1999) as one of the sufficient conditions for the existence of a competitive equilibrium. [7] Under (BE) and some standard assumptions, they first prove the existence of a competitive equilibrium among households with free disposal (therefore with nonnegative equilibrium price) and then prove the existence of a strictly positive equilibrium price with exact market clearing by adding several conditions (Gersbach and Haller 1999, Proposition 1).

However, it should be noted that in Gersbach and Haller's proof, they implicitly assume that (BE) holds in the truncated economy rather than in the original economy (they employ the truncation technique in their proof, as we do in the proof of Lemma 1). Indeed, (BE) may not hold in the truncated economy even if (BE) holds in the original economy (and vice versa). [8]

In contrast, in this paper, we derive the existence of a competitive equilibrium among households (with exact market clearing) by assuming (BE) in the original economy:

Theorem 1. *Suppose that*

- $\omega_h \gg 0$ *for all* h*,*
- U_{hm} *is continuous and concave on* $\mathcal{X}_h$ *for all* h *and* m*, and,*
- *(BE) holds.*

Then, there exists a competitive equilibrium among households $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ *with* $p^* \neq 0$*.*

The first two assumptions in Theorem 1 are also used in the first part of Proposition 1 in Gersbach and Haller (1999).

With some additional assumptions, we can obtain the existence of a strictly positive equilibrium price, which is the same conclusion as (the corrected version of) the second part of Proposition 1 in Gersbach and Haller (1999).

Corollary 1. *Suppose that all the assumptions of Theorem 1 hold. Suppose further that there exists at least one* h *such that* U_{hm} *is strictly monotonic and exhibits non-negative externalities for all* m*. Then, there exists a competitive equilibrium among households* $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ *with* $p^* \gg 0$*.*

Corollary 1 differs technically from the second part of Proposition 1 in Gersbach and Haller (1999) since, as noted above, Corollary 1 is based on (BE) for the original economy, whereas Gersbach and Haller's result is based on (BE) for the truncated economy.

5 Existence under (local) nonsatiation

In this section, we provide another existence result by replacing (BE) with the following assumption.

(PNSM). Presence of nonsatiated members

For all h*, there exists at least one member* m *whose utility function* U_{hm} *is nonsatiated on* $\mathcal{F}_h$*.*

[7] This assumption is also used in Haller (2000) to establish the first welfare theorem.

[8] The author is grateful to an anonymous referee for providing these observations.

The next lemma asserts that if a household h has at least one member whose utility function is nonsatiated on $\mathcal{F}_h$, the nonsatiation property is inherited by h's welfare function with a certain weight.

Lemma 2. *For a household h, suppose that*

- *U_{hm} is continuous on $\mathcal{X}_h$ for all m, and*
- *U_{hm} is nonsatiated on $\mathcal{F}_h$ for at least one m.*

Then, there exists $\lambda_h \in \Delta_h$ such that $\lambda_h \gg 0$ and $W_h^{\lambda_h}$ is nonsatiated on $\mathcal{F}_h$.

The proof is provided in the Appendix.

We now state the existence of a competitive equilibrium among households.

Theorem 2. *Suppose that*

- *$\omega_h \gg 0$ for all h,*
- *U_{hm} is continuous and concave on $\mathcal{X}_h$ for all hm, and,*
- *(PNSM) holds.*

Then, there exists a competitive equilibrium among households $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ with $p^* \neq 0$.

6 Concluding Remarks

The assumption (BE) asserts that every efficient consumption bundle of a household lies on the budget line. In a one-person households model, i.e., the standard Arrow–Debreu model, (BE) is implied, for example, by the local nonsatiation assumption. In contrast, in the multi-member households model, as shown in Haller (2000), [9] (BE) may not hold even if every consumer has a monotonic preference with respect to his or her individual consumption. However, Theorem 2 demonstrates that even if (BE) fails, the existence of a competitive equilibrium is still ensured, provided that every household has at least one member whose preference is nonsatiated on the feasible consumption set.

Unfortunately, unlike (BE), the assumption (PNSM) does not ensure strong Pareto optimality (in the usual sense) of a competitive equilibrium allocation (see Example 1 below). However, it is worth noting that every competitive equilibrium allocation among households is weakly Pareto optimal without any further assumptions. Moreover, one can easily verify that every equilibrium allocation $\mathbf{x}^*$ satisfies the following welfare property, which is weaker than weak Pareto optimality, but stronger than strong Pareto optimality:

There is no $\mathbf{y} \in \mathcal{X}$ such that

$$U_{hm}(\mathbf{y}_h) \geq U_{hm}(\mathbf{x}_h^*) \quad \textit{for all } hm \in I, \textit{ and}$$
$$U_{hm}(\mathbf{y}_h) > U_{hm}(\mathbf{x}_h^*) \quad \textit{for at least one } m \textit{ for each } h.$$

In other words, for an equilibrium allocation $\mathbf{x}^*$, there is no allocation $\mathbf{y}$ in which every individual achieves at least the same utility as in $\mathbf{x}^*$, and at least one member in each household achieves a strictly higher utility.

[9]Haller 2000, Example 3.3, p.844.

Example 1. Let $\ell = 1$, $H = 2$. There exist three consumers, simply labeled $i = 1, 2, 3$, with generic consumption bundles $x_i \in \mathbb{R}^2$. Consumers 1 and 2 form a household denoted by h. Consumer 3 forms a second household denoted by k. Let $\omega_h = 2$ and $\omega_k = 1$. The utility functions of consumers are

$$U_1(\mathbf{x}_h) = x_1 - x_2, \quad U_2(\mathbf{x}_h) = x_2 - x_1,$$
$$U_3(\mathbf{x}_k) = x_3.$$

Note that every U_i satisfies the local nonsatiation property.

Then, it is easy to check that $\mathbf{x}^* = (\mathbf{x}_h^*, \mathbf{x}_k^*) = [(1,1), 1]$ and $p^* = 1$ constitute a competitive equilibrium among households, and $\mathbf{x}^*$ is strongly Pareto dominated by the allocation $\mathbf{y} = [(1/2, 1/2), 2]$.

Finally, the following observation also clarifies the difference between the existence result under (BE) and that under (PNSM): If (BE) holds, then *any* choice of utilitarian welfare weights $\lambda = (\lambda_h)_h \in \Delta$ with $\lambda \gg 0$ leads to the existence of a competitive equilibrium among households $(\mathbf{x}^*, p^*)$ (in which each household h maximizes $W_h^{\lambda_h}$ on $B_h(p^*)$). In contrast, if (PNSM) holds, then *some* utilitarian welfare weights $\lambda = (\lambda_h)_h \in \Delta$ with $\lambda \gg 0$ lead to the existence of a competitive equilibrium among households, but others may not. More specifically, from the proof of Lemma 2, each household must assign a weight that is sufficiently close to 1 to a nonsatiated member.

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Appendix

In the proof of Lemma 1, we use the following theorem by Geistdoerfer-Florenzano (1982).

Theorem 3. *Let P be a closed convex cone (with vertex 0) of $\mathbb{R}^\ell$, and $\overline{B} = \{p \in \mathbb{R}^\ell : \|p\| \leq 1\}$, $S = \{p \in \mathbb{R}^\ell : \|p\| = 1\}$ be respectively the (closed) unit ball and the unit-sphere of $\mathbb{R}^\ell$. Let $\zeta : \overline{B} \cap P \to \mathbb{R}^\ell$ be an upper semicontinuous correspondence with nonempty compact convex values. Assume that*

$$\forall p \in S \cap P, \quad \exists z \in \zeta(p), \quad p \cdot z \leq 0.$$

Then, there exists $\overline{p} \in \overline{B} \cap P$ such that $\zeta(\overline{p}) \cap P^0 \neq \varnothing$, where

$$P^0 = \{z \in \mathbb{R}^\ell : z \cdot q \leq 0, \quad \forall q \in P\}$$

is the polar cone of P.

Proof of Lemma 1. Let $\overline{B} = \{p \in \mathbb{R}^\ell : \|p\| \leq 1\}$ and $S = \{p \in \mathbb{R}^\ell : \|p\| = 1\}$.

Suppose that $\lambda \in \Delta$ is the element such that $W_h^{\lambda_h}$ is continuous and concave on $\mathcal{X}_h$ for all h.

For each h, we define h's relaxed budget correspondence $\beta_h : \overline{B} \to \mathcal{X}_h$ by

$$\beta_h(p) = \{\mathbf{x}_h \in \mathcal{X}_h^r : p * \mathbf{x}_h \leq p \cdot \omega_h + (1 - \|p\|)\}.$$

Let $r = 2\| \sum_h \omega_h \|$ and $\mathcal{X}_h^r = \mathcal{X}_h \cap \operatorname{cl} B(0, r)$, where $B(0, r) = \{x \in \mathbb{R}^{\ell M(h)} : \|x\| < r\}$. From the definition, $\mathcal{F}_h \subset \mathcal{X}_h \cap B(0, r)$ for all h.

Let $\beta_h^r : \overline{B} \to \mathcal{X}_h^r$ be the correspondence defined by $\beta_h^r(p) = \beta_h(p) \cap \mathcal{X}_h^r$. It is easy to see that β_h^r is nonempty convex valued. Moreover, since $\omega_h \gg 0$ and $\mathcal{X}_h^r$ is compact, the correspondence β_h^r is continuous on $\overline{B}$. [10]

For each h, we define h's demand correspondence $\varphi_h : \overline{B} \to \mathcal{X}_h^r$ by

$$\varphi_h(p) = \{\mathbf{x}_h \in \mathcal{X}_h^r : \mathbf{x}_h \in \beta_h^r(p) \text{ and } W_h^{\lambda_h}(\mathbf{x}_h) \geq W_h^{\lambda_h}(\mathbf{y}_h) \text{ for all } \mathbf{y}_h \in \beta_h^r(p)\}.$$

By the concavity of $W_h^{\lambda_h}$, the correspondence φ_h is convex valued for all h. Moreover, by Berge's Maximum Theorem, φ_h is upper semicontinuous with nonempty compact values.

For each h, we define the correspondence $\overline{\varphi}_h : \overline{B} \to \mathbb{R}^\ell$ by

$$\overline{\varphi}_h(p) = \{\overline{\mathbf{x}}_h \in \mathbb{R}^\ell : \overline{\mathbf{x}}_h = \textstyle\sum_m x_{hm}, \ \ \mathbf{x}_h \in \varphi_h(p)\}.$$

Then, it is readily verified that $\overline{\varphi}_h$ is upper semicontinuous with nonempty compact convex values.

Finally, we define the market excess demand correspondence $\zeta : \overline{B} \to \mathbb{R}^\ell$ by

$$\zeta(p) = \sum_h \overline{\varphi}_h(p) - \sum_h \omega_h.$$

[10] It is readily verified that β_h^r is upper semicontinuous on $\overline{B}$. To see that β_h^r is lower semicontinuous on $\overline{B}$, note first that the correspondence $\gamma_h : \overline{B} \to \mathcal{X}_h^r$ defined by

$$\gamma_h(p) = \{\mathbf{x}_h \in \mathcal{X}_h^r : p * \mathbf{x}_h < p \cdot \omega_h + (1 - \|p\|)\}$$

has an open graph in $\overline{B} \times \mathcal{X}_h^r$. Moreover, since $\omega_h \gg 0$, we have $\beta_h^r(p) = \operatorname{cl}(\gamma_h(p))$ for all $p \in \overline{B}$, where $\operatorname{cl}(\gamma_h(p))$ denotes the closure of $\gamma_h(p)$ in X_h^r. Since the closure of a correspondence with an open graph is lower semicontinuous, β_h^r is lower semicontinuous.

The correspondence ζ is clearly upper semicontinuous with nonempty compact convex values, and satisfies

$$p \cdot \mathbf{z} \leq 0 \quad \text{for all} \quad p \in S \quad \text{and} \quad \mathbf{z} \in \zeta(p).$$

Applying Theorem 3 as $P = \mathbb{R}^\ell$, we obtain $p^* \in \overline{B}$ with $0 \in \zeta(p^*)$. From the definition of ζ, there exists $\mathbf{x}^* = (\mathbf{x}_h^*)_h \in \mathcal{X}^r$ such that

$$\mathbf{x}_h^* \in \varphi_h(p^*) \quad \text{for all} \quad h, \tag{1}$$

and

$$0 = \sum_h \sum_m x_{hm}^* - \sum_h \omega_h. \tag{2}$$

Note that equation (2) implies that $\mathbf{x}_h^* \in \mathcal{F}_h$ for all h.

Let $\alpha^* = (1 - \|p^*\|) \geq 0$. Then, it is clear that $(\mathbf{x}^*, p^*) \in \mathcal{X}^r \times \overline{B}$ and $\alpha^* \in \mathbb{R}_+$ satisfy conditions (i) and (iii) in the statement of the lemma. Thus, it remains to show that $\mathbf{x}_h^*$ maximizes $W_h^{\lambda_h}$ on $\beta_h(p^*)$ (not just on $\beta_h^r(p^*)$) for all h.

Suppose that for some h, there exists $\mathbf{y}_h \in \beta_h(p^*)$ with $W_h^{\lambda_h}(\mathbf{y}_h) > W_h^{\lambda_h}(\mathbf{x}_h^*)$. Since $\mathbf{x}_h^* \in \beta_h^r(p^*) \cap \mathcal{F}_h \subset \mathcal{X}_h \cap B(0, r)$, there exists $t \in (0,1)$ with $t\,\mathbf{x}_h^* + (1-t)\,\mathbf{y}_h \in \beta_h^r(p^*)$. Moreover, by the concavity of $W_h^{\lambda_h}$, we have

$$W_h^{\lambda_h}(t\,\mathbf{x}_h^* + (1-t)\,\mathbf{y}_h) > W_h^{\lambda_h}(\mathbf{x}_h^*),$$

which contradicts (1). □

Proof of Theorem 1. Take an arbitrary $\lambda = (\lambda_h)_h \in \Delta$ with $\lambda \gg 0$.

For each h, since U_{hm} is continuous and concave on $\mathcal{X}_h$ for all m, the function $W_h^{\lambda_h}$ is continuous and concave on $\mathcal{X}_h$.

Applying Lemma 1 for $\lambda = (\lambda_h)_h \in \Delta$, we obtain an element $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ and a non-negative real number $\alpha^* \in \mathbb{R}_+$ such that

(i) $p^* * \mathbf{x}_h^* \leq p^* \cdot \omega_h + \alpha^*$ for all h,

(ii) For each h, if $W_h^{\lambda_h}(\mathbf{y}_h) > W_h^{\lambda_h}(\mathbf{x}_h^*)$, then $p^* * \mathbf{y}_h > p^* \cdot \omega_h + \alpha^*$,

(iii) $\sum_h \sum_m x_{hm}^* = \sum_h \omega_h$.

We divide the proof into two cases according to the value of p^*.

Case 1. $p^* = 0$.

Note first that (ii) implies that $W_h^{\lambda_h}(\mathbf{x}_h^*) \geq W_h^{\lambda_h}(\mathbf{y}_h)$ for all $\mathbf{y}_h \in \mathcal{X}_h$ and h. Then, we must have $\sum_m x_{hm}^* = \omega_h$ for each h. Indeed, if $\sum_m x_{hm}^* \neq \omega_h$ for some h, then there exists $p \neq 0$ such that $p \cdot \sum_m x_{hm}^* < p \cdot \omega_h$, which contradicts (BE).

It is now clear that for any $p \neq 0$, the element $(\mathbf{x}^*, p) \in \mathcal{X} \times \mathbb{R}^\ell \setminus \{0\}$ constitutes a competitive equilibrium among households.

Case 2. $p^* \neq 0$.

We prove that $(\mathbf{x}^*, p^*)$ is a competitive equilibrium among households. In view of (iii), it suffices to show that $\mathbf{x}_h^* \in EB_h(p^*)$ for all h.

We first prove

$$p^* * \mathbf{x}_h^* \geq p^* \cdot \omega_h \quad \text{for all} \quad h. \tag{3}$$

Suppose that $p^* * \mathbf{x}_h^* < p^* \cdot \omega_h$ for some h. Since $\lambda_h \gg 0$ and (ii) holds, we have $\mathbf{x}_h^* \in EB_h(p^*)$. Then, (BE) implies that $p^* * \mathbf{x}_h^* = p^* \cdot \omega_h$, which is a contradiction.

We next prove that $p^* * \mathbf{x}_h^* = p^* \cdot \omega_h$ for all h. Suppose that $p^* * \mathbf{x}_h^* > p^* \cdot \omega_h$ for some h. Then, summing up the equations in (3) for h, we obtain

$$p^* \cdot \sum_h \sum_m x_{hm}^* > p^* \cdot \sum_h \omega_h,$$

which contradicts (iii).

Then, since $\lambda_h \gg 0$ for all h, (ii) implies that $\mathbf{x}_h^* \in EB_h(p^*)$ for all h.

Therefore, we conclude that $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ is a competitive equilibrium among households. □

Proof of Corollary 1. From Theorem 1, there exists a competitive equilibrium among households $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$.

Suppose that $p_j^* \leq 0$ for some $j = 1, \cdots, \ell$.

Let h be the household such that for all m, the utility function U_{hm} is strictly monotonic in x_{hm} and exhibits non-negative externalities.

For each m, define hm's consumption bundle $y_{hm} = (y_{hm1}, \cdots, y_{hm\ell}) \in X_{hm}$ by

$$y_{hms} = \begin{cases} x_{hms}^* + 1 & \text{if} \quad s = j, \\ x_{hms}^* & \text{if} \quad s \neq j. \end{cases}$$

Let $\mathbf{y}_h = (y_{hm})_m \in \mathcal{X}_h$. Since each U_{hm} is strictly monotonic and exhibits non-negative externalities, we have

$$U_{hm}(\mathbf{y}_h) > U_{hm}(\mathbf{x}_h^*) \quad \text{for all} \quad m.$$

Moreover, since $p_j^* \leq 0$, we have $p^* * \mathbf{y}_h < p^* * \mathbf{x}_h^* \leq p^* \cdot \omega_h$. However, this contradicts the fact that $\mathbf{x}_h^* \in EB_h(p^*)$. □

Proof of Lemma 2. Let $h1$ be the consumer whose utility function U_{h1} is nonsatiated on $\mathcal{F}_h$. Note that $W_h : \mathcal{X}_h \times \Delta_h \to \mathbb{R}$ is continuous on $\mathcal{X}_h \times \Delta_h$ since U_{hm} is continuous for all m.

Let $f : \Delta_h \to \mathbb{R}$ be the function defined by

$$f(\lambda_h) = \max_{\mathbf{x}_h \in \mathcal{F}_h} W_h(\mathbf{x}_h, \lambda_h).$$

By Berge's Maximum Theorem, f is continuous on Δ_h.

Let $\mathbf{x}_h^* = \text{argmax}_{\mathbf{x}_h \in \mathcal{F}_h} U_{h1}(\mathbf{x}_h)$. Since U_{h1} is nonsatiated on $\mathcal{F}_h$, there exists $\mathbf{y}_h \in \mathcal{X}_h \setminus \mathcal{F}_h$ such that $U_{h1}(\mathbf{y}_h) > U_{h1}(\mathbf{x}_h^*)$. Thus, for $(1, 0, \cdots, 0) \in \Delta_h$,

$$W_h(\mathbf{y}_h, (1, 0, \cdots, 0)) = U_{h1}(\mathbf{y}_h) > U_{h1}(\mathbf{x}_h^*) = f(1, 0, \cdots, 0).$$

Since W_h and f are continuous in λ_h, there exists $\lambda_h^* \in \Delta_h$ such that $\lambda_h^* \gg 0$ and $W_h(\mathbf{y}_h, \lambda_h^*) > f(\lambda_h^*)$.

Then, $W_h^{\lambda_h^*}$ is nonsatiated on $\mathcal{F}_h$. Indeed, for all $\mathbf{x}_h \in \mathcal{F}_h$,

$$W_h^{\lambda_h^*}(\mathbf{y}_h) = W_h(\mathbf{y}_h, \lambda_h^*) > f(\lambda_h^*) \geq W_h^{\lambda_h^*}(\mathbf{x}_h),$$

which completes the proof. □

Proof of Theorem 2. For each h, by Lemma 2, there exists $\lambda_h \in \Delta_h$ such that $\lambda_h \gg 0$ and $W_h^{\lambda_h}$ is continuous and concave on $\mathcal{X}_h$, and nonsatiated on $\mathcal{F}_h$. Note that together with the concavity, the nonsatiation property of $W_h^{\lambda_h}$ implies the local nonsatiation property of $W_h^{\lambda_h}$.

Applying Lemma 1 for $\lambda = (\lambda_h)_h \in \Delta$, we obtain an element $(\mathbf{x}^*, p^*) \in \mathcal{X} \times \mathbb{R}^\ell$ and a non-negative real number $\alpha^* \in \mathbb{R}_+$ such that

(i) $p^* * \mathbf{x}_h^* \leq p^* \cdot \omega_h + \alpha^*$ for all h,

(ii) For each h, if $W_h^{\lambda_h}(\mathbf{y}_h) > W_h^{\lambda_h}(\mathbf{x}_h^*)$, then $p^* * \mathbf{y}_h > p^* \cdot \omega_h + \alpha^*$,

(iii) $\sum_h \sum_m x_{hm}^* = \sum_h \omega_h$.

We prove that $(\mathbf{x}^*, p^*)$ is a competitive equilibrium among households. In view of (iii), it suffices to show that $\mathbf{x}_h^* \in EB_h(p^*)$ for all h.

Note first that since $W_h^{\lambda_h}$ is (locally) nonsatiated on $\mathcal{F}_h$ for all h, we must have $p^* \neq 0$.

We next prove that $\mathbf{x}_h^* \in B_h(p^*)$ for all h. For all h, since $\mathbf{x}_h^* \in \mathcal{F}_h$ and $W_h^{\lambda_h}$ is locally nonsatiated on $\mathcal{F}_h$, we have

$$p^* \cdot \sum_m x_{hm}^* = p^* \cdot \omega_h + \alpha^*. \tag{4}$$

Summing up the equations in (4) for h, we obtain

$$p^* \cdot \sum_h \sum_m x_{hm}^* = p^* \cdot \sum_h \omega_h + H\alpha^*.$$

However, since (iii) implies

$$p^* \cdot \sum_h \sum_m x_{hm}^* = p^* \cdot \sum_h \omega_h,$$

we must have $\alpha^* = 0$. Thus, $\mathbf{x}_h^* \in B_h(p^*)$ for all h.

Since $\lambda_h \gg 0$ and (ii) holds, it is clear that $\mathbf{x}_h^* \in EB_h(p^*)$ for all h. □