# CESifo Working Papers

www.cesifo.org/wp

# Expectations, Liquidity, and Short-term Trading

## Giovanni Cespa
## Xavier Vives

# Expectations, Liquidity, and Short-term Trading

## Abstract

In a market with short term agents and heterogeneous information, when liquidity trading displays persistence, prices reflect average expectations about fundamentals and liquidity trading. Informed investors exploit a private learning channel to infer the demand of liquidity traders from the order flow to anticipate the evolution of the future aggregate demand for the stock. This yields multiple equilibria which can be ranked in terms of liquidity and informational efficiency. Our results have implications for the impact of High Frequency Trading (HFT) on market quality and for the role of average expectations in asset pricing. We show that with persistence HFT may enhance informational efficiency and liquidity but only by creating an unstable equilibrium. In the equilibrium with high (low) informational efficiency, prices are closer to (farther away from) fundamentals compared to consensus estimates.

JEL-Code: G100, G120, G140.

Keywords: expected returns, multiple equilibria, average expectations, reliance on public information, momentum and reversal, price crash, Beauty Contest, High Frequency Trading.

*Giovanni Cespa*
*Cass Business School*
*106, Bunhill Row*
*UK – London EC1Y 8TZ*
*giovanni.cespa@gmail.com*

*Xavier Vives*
*IESE Business School*
*Avinguda Pearson, 21*
*Spain – 08034 Barcelona*
*xvives@iese.edu*

# Introduction

We study the drivers of asset prices in a two-period market where short-term, informed, competitive, risk-averse agents trade on account of private information and to accommodate liquidity supply, facing a persistent demand from liquidity traders.

Short term speculation ranks high on the regulatory agenda, testifying policy makers' concern with the possibly destabilizing impact it has on the market. For instance, the report on the causes of the "Flash-Crash" issued by the staffs of the CFTC-SEC highlights the role of High Frequency Trading (HFT) – a class of market players who engage in extremely short-term strategies – in exacerbating the sharp price drop that characterized the crash. Relatedly, policy makers' concern over the role played by financial markets in the current crisis has reanimated the debate over the means to curb short termism via the introduction of a transaction tax (the so-called "Tobin-tax"). At the same time it is claimed that HFT improves liquidity (see, e.g., Hendershott, Jones, and Menkveld (2010)), thus playing a beneficial role in the market. The jury is still out on the effects of HFT and, more in general, on the impact of short term speculation. The issue has a long tradition in economic analysis. Indeed, short term trading is at the base of Keynes' beauty contest view of financial markets, according to which what matters are the average expectations of the average expectations of investors in an infinite regress of higher order beliefs. In this context it has been claimed that traders tend to put a disproportionately high weight on public information in their forecast of asset prices (see Allen, Morris, and Shin (2006)).

In this paper we present a two-period model of short term trading with asymmetric information in the tradition of dynamic noisy rational expectations models (see, e.g., Singleton (1987), Brown and Jennings (1989)). We advance the understanding of the effects of short horizons on market quality, presenting a conciliation of the conflicting evidence on HFT. We also establish the limits of the beauty contest analogy for financial markets, and deliver sharp predictions on asset pricing which are consistent with the received empirical evidence (including noted anomalies).

Assuming that informed investors have short horizons and that, due to persistence, the demand of liquidity traders is predictable, allows us to capture important features of actual markets. Indeed, a short term investor's concern over the price at which he unwinds, will make him more sensitive to the possibility to extrapolate patterns on the evolution of the *future* aggregate demand from the observation of the *current* aggregate demand for the stock. With correlated liquidity trading, this implies that not only "fundamentals" information, but also any information on the orders placed by uninformed investors becomes relevant to predict the future price.[1] The fact that both information on fundamentals and on liquidity trading matters is a crucial determinant of equilibrium and market quality properties.

We find that with heterogeneous information and short horizons, liquidity trading persis-

---

[1] This appears to be especially true for HFTs that use "order anticipation" algorithms to uncover patterns within market data which provide information into large institutional order flows (see the January 2010 *SEC Concept Release on Equity Market Structure*, and Scott Patterson, June 30, 2010, "Fast Traders Face Off with Big Investors Over 'Gaming,'" *Wall Street Journal*).

tence yields multiple equilibria. This is so since informed investors use their private signals on the fundamentals *also* to infer the demand of liquidity traders from the first period price, to anticipate the impact that liquidity traders have on the price at which they unwind their positions. In this way investors exploit a private learning channel from the price (as in Amador and Weill (2010) and Manzano and Vives (2011)).

The dual role of fundamentals information is at the base of equilibrium multiplicity. The intuition for this result is as follows. First period informed investors use their private signal to anticipate the impact of fundamentals and, when liquidity trading is persistent, that of the first period liquidity trading on the second period price, at which they unwind. The more the second period price reflects first period liquidity trading, the less their private signal is useful to predict the liquidation price, and the lower is their response to private information. This, in turn, makes the first period price reflect more liquidity trading, and less private information. As a result the effect of private information impounded in the price by second period investors prevails over that coming from first period investors. This generates adverse selection, magnifying the second period price impact of trades and augmenting first period investors' uncertainty over the liquidation price, further lowering their response to private information. In this equilibrium, the response to private information is low, and the second period market is thin. Conversely, the more the second period price reflects the fundamentals, the more investors' private signal is useful to predict the liquidation price, and the larger is the response to private information. This, makes the first period price reflect less liquidity trading, and more private information. As a result, the effect of private information impounded in the price by first period investors prevails over that coming from second period investors. In this case there is favorable selection, which enhances second period liquidity and lowers investors' uncertainty over the liquidation price, and further boosts their reaction to private information. In this equilibrium the response to private information is high and the price impact of a unit of liquidity demand is negative due to favorable selection. When liquidity trading is transient, the dual role of private information vanishes, and a unique equilibrium arises.

Thus, with persistent liquidity trading, two self-fulfilling equilibria arise: in one equilibrium the market is thin, and prices are poorly informationally efficient. In the other equilibrium, the opposite occurs, with a thick market and highly informationally efficient prices. We show that in the latter (former) equilibrium, first period investors engage in a "conditional" momentum (reversal) strategy. Indeed, in equilibrium investors anticipate the price at which they trade, so that factoring out the impact of first period public information, the covariation of future returns could be either due to liquidity trading or fundamentals information. In the equilibrium with high (low) liquidity, as prices are closer to (farther away from) fundamentals, the second (first) effect prevails, and returns positively (negatively) covary around their means. As a consequence, when estimating a positive order imbalance investors chase the market (take the other side of the market), anticipating a trend (reversal) in the price at which they unwind their positions.

Finally, studying the stability of the equilibrium solutions, we show that the high liquidity

equilibrium is unstable according to the best reply dynamics. The results are robust to introducing residual uncertainty about the liquidation value. In this case there are potentially three equilibria and it is possible to show that an unexpected increase in residual uncertainty may induce a price crash.[2]

Other authors find that in the presence of short horizons multiple equilibria can arise (see, e.g., Spiegel (1998) and Watanabe (2008)). However, in these cases multiplicity arises out of the bootstrap nature of expectations in the steady state equilibrium of an *infinite horizon* model with overlapping generations of two-period lived investors. Spiegel (1998) studies the model with *no* asymmetric information.[3] Watanabe (2008) extends the model of Spiegel (1998) to account for the possibility that investors have heterogeneous short-lived private information. However, in his case too the analysis concentrates on the steady state equilibrium, which does not make his results directly comparable to ours.[4]

Our results are related to and have implications for four strands of the literature.

First, our paper is related to the literature that investigates the relationship between the impact of short-term investment horizons on prices and investors' reaction to their private signals (see, e.g. Singleton (1987), Brown and Jennings (1989), Froot, Scharfstein, and Stein (1992), Dow and Gorton (1994), Vives (1995), Cespa (2002), Albagli (2011) and Vives (2008) for a survey). If prices are semi-strong efficient (as in Vives (1995)) then there is no private learning channel from prices since the price is a sufficient statistic for public information. Brown and Jennings (1989), instead analyze a model in which prices are not semi-strong efficient, with short term investors and where liquidity trading can be correlated. Their work provides a rationale for "technical analysis," showing how in the absence of semi-strong efficiency the *sequence* of transaction prices provides more information than the *current* stock price to forecast the final payoff. We argue that lacking semi-strong efficiency, in the presence of correlated liquidity trading, first period investors have a private learning channel from the price which provides them with additional information on the future stock price. We also provide a closed form characterization of the equilibrium, emphasizing the role of this private learning channel in generating equilibrium multiplicity.

Second, our paper is also related to the growing literature on HFT. Indeed, our short term investors can be seen as high frequency traders who use private information to read the public order flow which, in view of liquidity traders' demand correlation, provides insights into the evolution of the future aggregate demand.[5] Hendershott and Riordan (2010) find that high frequency traders orders have a permanent price impact which is larger than that of

---

[2]Multiple, self-fulfilling equilibria can also arise because of participation externalities, as e.g. in Admati and Pfleiderer (1988), and Pagano (1989). In our context multiplicity is due to a purely informational effect.

[3]Our model with no private information is akin to a finite horizon version of Spiegel (1998) and as we show in Corollary 5, in this case we obtain a unique equilibrium.

[4]Relatedly, Dennert (1991) studies an OLG extension of Grossman and Stiglitz (1980), concentrating on the steady state solution. In his setup too private information is short-lived, which impedes second period investors' inference about the information held by first period investors.

[5]In this respect, our results are also related to the literature that studies the ability of the non-informational component of total imbalances to predict stock returns (see, e.g. Coval and Stafford (2007), and Hendershott and Seasholes (2009)).

"slow" human traders. This allows the price to adjust more rapidly toward the (information-ally) efficient price. Based on this evidence, Biais, Foucault, and Moinas (2011) assume that high frequency traders are able to process information before "slow" traders. This generates a negative externality which is responsible for excessive investment in HFT compared to a utilitarian welfare-maximization benchmark. Our model provides an explanation for high frequency traders' superior ability to impound fundamentals information. In the high liquidity equilibrium due to the the private learning channel from prices high frequency traders escalate their response to private information, making prices more informationally efficient. However, as argued above, this equilibrium is unstable, which implies that in our setup HFT may induce fragility.

Third, the paper is related to the work that studies the influence of Higher Order Expecta-tions (HOEs) on asset prices (see Allen, Morris, and Shin (2006), Bacchetta and van Wincoop (2008), Kondor (2009), and Nimark (2007)). Allen, Morris, and Shin (2006) finds that when prices are driven by HOEs about fundamentals, they underweight private information (with respect to the optimal statistical weight) and are farther away from fundamentals compared to consensus. We show that in the unique equilibrium that obtains when liquidity trading is transient, investors dampen their response to private information and this result holds. A sim-ilar result also holds in the equilibrium with low liquidity when liquidity trading is persistent. However, along the equilibrium with high liquidity the price is more strongly tied to funda-mentals compared to consensus, and overweights average private information (compared to the optimal statistical weight).[6] Bacchetta and van Wincoop (2006) study the role of HOEs in the FX market. They show that HOEs worsen the signal extraction problem that investors face when observing changes in the exchange rate that originate from trades based on fundamen-tals information and hedging motives. In our setup this happens in the equilibrium with low liquidity, whereas in the equilibrium with high liquidity, investors' strong reaction to private information eases off the signal extraction problem.

Finally, the paper is also related to the literature on limits to arbitrage. In this respect, our multiplicity result is reminiscent of De Long et al. (1990), but in a model with fully rational traders, and a finite horizon. Thus, our paper naturally relates to the strand of this literature that views limits to arbitrage as the analysis of how "non-fundamental demand shocks" impact asset prices in models with rational agents (Gromb and Vayanos (2010), Vayanos and Woolley (2008)). Our contribution in this respect is twofold: first we prove that when such shocks display persistence, they impact in a non-trivial way the information extraction process of rational investors, generating implications for price efficiency and market liquidity. Second, we relate these findings to the literature on return predictability. In fact, along the high liquidity equilibrium, we show that momentum arises at short horizons, while at long horizons reversal occurs in any equilibrium. Intuitively, momentum is the result of two forces. On the one hand, with persistence, the impact sign of any anticipated first period order imbalance on second and third period expected returns is the *same*; on the other hand, as we argued above, along the

---

[6]In a related paper, we show that a similar conclusion holds in a model with long term investors (see Cespa and Vives (2011)).

equilibrium with high liquidity the conditional covariance of returns is positive. Our results are thus in line with the empirical findings on return anomalies that document the existence of positive return autocorrelation at short horizons, and negative autocorrelation at long horizons (Jegadeesh and Titman (1993), and De Bondt and Thaler (1985)). Our model also predicts that momentum is related to a high volume of informational trading, in line with the evidence in Llorente, Michaely, Saar, and Wang (2002).

The rest of the paper is organized as follows. In the next section we analyze the static benchmark. In the following section, we study the two-period extension and present the multiplicity result, relating it to liquidity traders' persistence. In the following sections we relate our results to the literature on HFT, Higher Order Expectations and asset pricing. The final section summarizes our results and discusses their empirical implications. Most of the proofs are relegated to the appendix.

# 1   The static benchmark

Consider a one-period stock market where a single risky asset with liquidation value $v$, and a risk-less asset with unitary return are traded by a continuum of risk-averse, informed investors in the interval $[0, 1]$ together with liquidity traders. We assume that $v \sim N(\bar{v}, \tau_v^{-1})$. Investors have CARA preferences (denote with $\gamma$ the risk-tolerance coefficient) and maximize the expected utility of their wealth: $W_i = (v - p)x_i$.[7] Prior to the opening of the market every informed investor $i$ obtains private information on $v$, receiving a signal $s_i = v + \epsilon_i$, $\epsilon_i \sim N(0, \tau_\epsilon^{-1})$, and submits a demand schedule (generalized limit order) to the market $X(s_i, p)$ indicating the desired position in the risky asset for each realization of the equilibrium price.[8] Assume that $v$ and $\epsilon_i$ are independent for all $i$, and that error terms are also independent across investors. Liquidity traders submit a random demand $u$ (independent of all other random variables in the model), where $u \sim N(0, \tau_u^{-1})$. We denote by $E_i[Y]$, $\mathrm{Var}_i[Y]$ the expectation and the variance of the random variable $Y$ formed by an investor $i$, conditioning on the private and public information he has: $E_i[Y] = E[Y|s_i, p]$, $\mathrm{Var}_i[Y] = \mathrm{Var}[Y|s_i, p]$. Finally, we make the convention that, given $v$, the average signal $\int_0^1 s_i di$ equals $v$ almost surely (i.e. errors cancel out in the aggregate: $\int_0^1 \epsilon_i di = 0$),[9] and denote by $\bar{E}[v] = \int_0^1 E_i[v] di$ investors' average opinion (the "consensus" opinion) about $v$.

In the above CARA-normal framework, a symmetric rational expectations equilibrium (REE) is a set of trades contingent on the information that investors have, $\{X(s_i, p)$ for $i \in [0, 1]\}$ and a price functional $P(v, u)$ (measurable in $(v, u)$), such that investors in $[0, 1]$ optimize given their information and the market clears:

$$\int_0^1 x_i di + u = 0.$$

---

[7]We assume, without loss of generality with CARA preferences, that the non-random endowment of informed investors is zero.

[8]The unique equilibrium in linear strategies of this model is symmetric.

[9]See Section 3.1 in the Technical Appendix of Vives (2008) for a justification of the convention.

Given the above definition, it is easy to verify that a unique, symmetric equilibrium in linear strategies exists in the class of equilibria with a price functional of the form $P(v, u)$ (see, e.g. Admati (1985), Vives (2008)). The equilibrium strategy of an investor $i$ is given by

$$X(s_i, p) = \frac{a}{\alpha_E}(E_i[v] - p),$$

where

$$a = \gamma \tau_\epsilon, \tag{1}$$

denotes the responsiveness to private information, $\tau_i \equiv (\mathrm{Var}_i[v])^{-1}$, and $\alpha_E = \tau_\epsilon / \tau_i$ is the optimal statistical (Bayesian) weight to private information. Imposing market clearing the equilibrium price is given by

$$p = \bar{E}[v] + \frac{\alpha_E}{a}u \tag{2}$$

$$= E[v|p] + \Lambda E[u|p], \tag{3}$$

where $E[u|p] = a(v - E[v|p]) + u$, and

$$\Lambda = \frac{\mathrm{Var}_i[v]}{\gamma}. \tag{4}$$

Equations (2), and (3) show that the price can be given two alternative representations. According to the first one, the price reflects the consensus opinion investors hold about the liquidation value plus the impact of the demand from liquidity traders (multiplied by the risk-tolerance weighted uncertainty over the liquidation value). Indeed, in a static market owing to CARA and normality, an investor's demand is proportional to the expected gains from trade $E_i[v] - p$. As the price aggregates all investors' demands, it reflects the consensus opinion $\bar{E}[v]$ shocked by the orders of liquidity traders.

According to (3), the anticipated impact of liquidity traders' demand moves the price away from the semi-strong efficient price. Therefore, $\Lambda$ captures the "inventory" related component of market liquidity.[10] Liquidity traders' demand has an additional impact on the price, through the effect it produces on $E[v|p]$. This is an adverse selection effect which adds to the inventory effect, implying that the (reciprocal of the) liquidity of the market is measured by:

$$\lambda \equiv \frac{\partial p}{\partial u} = \Lambda + (1 - \alpha_E)\frac{a\tau_u}{\tau},$$

where $\tau = 1/\mathrm{Var}[v|p] = \tau_v + a^2 \tau_u$.[11]

---

[10]When risk averse investors accommodate an expectedly positive demand of liquidity traders, they require a compensation against the possibility that the liquidation value is higher than the public expectation (if instead $E[u|p] < 0$, investors require to pay a price lower than $E[v|p]$ to cover the risk that $v < E[v|p]$). Such a compensation is larger, the higher is the uncertainty investors face (captured by $\Lambda$) and the wider is their expected exposure to the liquidity traders' shock (their expected inventory, $E[u|p]$).

[11]The adverse selection effect comes from the signal extraction problem dealers face in this market: since $a > 0$, if investors on average have good news they buy the asset, and $E[v|p]$ increases, reflecting this information. However, this effect cannot be told apart from the buying pressure of liquidity traders, which also makes $E[v|p]$

Finally, note that the private signal in this case only serves to forecast the liquidation value $v$. In the next section we will argue that due to persistence, liquidity traders' demand impacts the order flow across different trading dates. In this case investors also use their private signals to extrapolate the demand of liquidity traders from the order flow to anticipate the impact this has on future prices. This additional use of private information will be responsible for equilibrium multiplicity.

## 2 A two-period market with short term investors

Consider now a two-period extension of the market analyzed in the previous section. At date 1 (2), a continuum of short-term investors in the interval $[0, 1]$ enters the market, loads a position in the risky asset which it unwinds in period 2 (3). Investor $i$ has CARA preferences (denote with $\gamma$ the common risk-tolerance coefficient) and maximizes the expected utility of his short term profit $\pi_{in} = (p_{n+1} - p_n)x_{in}$, $n = 1, 2$ (we set $p_0 = \bar{v}$ and $p_3 = v$).[12] The short term horizons of investors can be justified on grounds of incentive reasons related to performance evaluation, or because of difficulties associated with financing long-term investment in the presence of capital market imperfections (see Holmström and Ricart i Costa (1986), and Shleifer and Vishny (1990)). An investor $i$ who enters the market in period 1 receives a signal $s_i = v + \epsilon_i$ which he recalls in the second period, where $\epsilon_i \sim N(0, \tau_\epsilon^{-1})$, $v$ and $\epsilon_i$ are independent for all $i$.[13] We make the convention that, given $v$, the average signal $\int_0^1 s_i di$ equals $v$ almost surely (i.e., errors cancel out in the aggregate $\int_0^1 \epsilon_i di = 0$). We also assume that informed investors observe equilibrium prices and submit a linear demand schedule (generalized limit order): $X_1(s_i, p_1) = a_2 s_i - \varphi_1(p_1)$, and $X_2(s_i, p_1, p_2) = a_2 s_i - \varphi_2(p_1, p_2)$, indicating the desired position in the risky asset for each realization of the equilibrium price. The constant $a_n$ denotes the private signal responsiveness, while $\varphi_n(\cdot)$ is a linear function of the equilibrium prices.[14]

The position of liquidity traders is assumed to follow an AR(1) process:

$$
\begin{aligned}
\theta_1 &= u_1 \\
\theta_2 &= \beta\theta_1 + u_2,
\end{aligned}
\tag{5}
$$

where $\beta \in [0, 1]$ and $\{u_1, u_2\}$ is an i.i.d. normally distributed random process (independent of all other random variables in the model) with $u_n \sim N(0, \tau_u^{-1})$. If $\beta = 1$, $\{\theta_n\}_{n=1}^2$ follows a random walk and we are in the usual case of independent liquidity trade increments: $u_2 = \theta_2 - \theta_1$ is independent from $u_1$ (e.g., Kyle (1985), Vives (1995)). If $\beta = 0$, then liquidity trading is i.i.d. across periods (this is the case considered by Allen et al. (2006)).[15]

---

increase.

[12]We assume, without loss of generality, that the non-random endowment of investors is zero.

[13]The model can be extended to the case in which investors receive a new private signal in the second period. However, this complicates the analysis without substantially changing its qualitative results.

[14]The equilibria in linear strategies of this model are symmetric.

[15]See Chordia and Subrahmanyam (2004), Easley et al. (2008), and Hendershott and Seasholes (2009) for empirical evidence on liquidity traders' demand predictability.

We denote by $p^n = \{p_t\}_{t=1}^n$, and by $E_{in}[Y] = E[Y|s_i, p^n]$, $E_n[Y] = E[Y|p^n]$, $\text{Var}_{in}[Y] = \text{Var}[Y|s_i, p^n]$, and $\text{Var}_n[Y] = \text{Var}[Y|p^n]$, respectively the expectation and variance of the random variable $Y$ formed by a trader conditioning on the private and public information he has at time $n$, and that obtained conditioning on public information only. The variables $\tau_n$ and $\tau_{in}$ denote instead the precisions of the investors' forecasts of $v$ based only on public and on public and private information: $\tau_n = (1/\text{Var}_n[v])$, and $\tau_{in} = (1/\text{Var}_{in}[v])$.

The consensus opinion about the fundamentals at time $n$ is denoted by $\bar{E}_n[v] \equiv \int_0^1 E_{in}[v]di$. Letting $\alpha_{E_n} = \tau_\epsilon/\tau_{in}$, we have $E_{in}[v] = \alpha_{E_n}s_{in} + (1 - \alpha_{E_n})E_n[v]$, and due to our convention $\bar{E}_n[v] = \alpha_{E_n}v + (1 - \alpha_{E_n})E_n[v]$.

## 2.1 Equilibrium analysis

We start by giving a general description of the equilibrium. The following proposition characterises equilibrium prices:

**Proposition 1.** *At a linear equilibrium of the market the price is given by*

$$p_n = \alpha_{P_n}\left(v + \frac{\theta_n}{a_n}\right) + (1 - \alpha_{P_n})E_n[v], \tag{6}$$

*where $\theta_n = u_n + \beta\theta_{n-1}$, and $a_n$, $\alpha_{P_n}$ denote the responsiveness to private information displayed by investors and by the price at time $n$. We have that $\alpha_{P_2} = \alpha_{E_2} < 1$.*

According to (6), at period $n$ the equilibrium price is a weighted average of the market expectation about the fundamentals $v$, and the (noisy) average private information held by investors. Rearranging this expression yields

$$p_n - E_n[v] = \frac{\alpha_{P_n}}{a_n}\left(a_n\left(v - E_n[v]\right) + \theta_n\right) \tag{7}$$

$$= \Lambda_n E_n[\theta_n],$$

where $\Lambda_n \equiv \alpha_{P_n}/a_n$, implying that there is a discrepancy between $p_n$ and $E_n[v]$ which, as in (3), captures a premium which is proportional to the expected stock of liquidity trading that investors accommodate at $n$:

**Corollary 1.** *At a linear equilibrium, the price incorporates a premium above the semi-strong efficient price:*

$$p_n = E_n[v] + \Lambda_n E_n[\theta_n], \tag{8}$$

*where $\Lambda_2 = \text{Var}_{i2}[v]/\gamma$, and*

$$\Lambda_1 = \frac{\text{Var}_{i1}[p_2]}{\gamma} + \beta\Lambda_2. \tag{9}$$

Comparing (9) with (4), shows that short term trading affects the inventory component of liquidity. In a static market when investors absorb the demand of liquidity traders, they

9

are exposed to the risk coming from the randomness of $v$. In a dynamic market, short term investors at date 1 face instead the risk due to the randomness of the following period price (at which they unwind). As liquidity trading displays persistence, second period informed investors absorb part of first period liquidity traders' position and this contributes to first period investors' uncertainty over $p_2$, yielding (9).

As in the static benchmark, besides the impact of the inventory component $\Lambda_n$, with differential information the price impact also reflects an asymmetric information component as the following corollary shows:

**Corollary 2.** *Let $\Delta a_2 = a_2 - \beta a_1$ and $\Delta a_1 = a_1$. At a linear equilibrium, the price impact of trades is measured by*

$$\lambda_n \equiv \frac{\partial p_n}{\partial u_n} = \Lambda_n + (1 - \alpha_{P_n})\frac{\Delta a_n \tau_u}{\tau_n}, n = 1, 2. \tag{10}$$

According to (10) the asymmetric information component of liquidity at $n$ is captured by

$$(1 - \alpha_{P_n})\frac{\Delta a_n \tau_u}{\tau_n}. \tag{11}$$

Differently from the static benchmark, in a dynamic market this effect depends on the $\beta$-weighted *net* position of informed investors yielding trading intensity $\Delta a_2 = a_2 - \beta a_1$. Indeed, at equilibrium:

$$x_1 + \theta_1 = 0, \text{ and } x_2 + \beta\theta_1 + u_2 = 0 \Rightarrow x_2 - \beta x_1 + u_2 = 0.$$

As a result, the impact of private information depends on the *change* in informed investors' position as measured by $\Delta a_2 = a_2 - \beta a_1$. When $a_2 > \beta a_1$, the effect of private information impounded in the price by second period investors prevails over that coming from first period investors. In this case, asymmetric information generates adverse selection in the second period, and augments the price impact of trades. This is always the case when $\beta = 0$ in which case the position of first period informed investors does not matter. However, when $a_2 < \beta a_1$, the opposite occurs and the effect of the information impounded in the price by first period investors prevails, yielding *favorable* selection. In this case, asymmetric information *reduces* the price impact of trades. We will see that in equilibrium both possibilities may arise.

In the first period investors use their private signal on the fundamentals to anticipate the second period price, insofar as the latter is related to $v$. However, as argued above, due to liquidity trading persistence when $\beta > 0$, $p_2$ also reflects the demand of first period liquidity traders. This leads investors to use their private information also to infer $\theta_1$ from the order

10

flow. Using (6) and (7):

$$E_{i1}[p_2] = E_{i1}\left[\alpha_{P_2}\left(v + \frac{\theta_2}{a_2}\right) + (1 - \alpha_{P_2})E_2[v]\right]$$

$$= \left(\alpha_{P_2} + (1 - \alpha_{P_2})\frac{(\Delta a_2)^2 \tau_u}{\tau_2}\right)E_{i1}[v] + (1 - \alpha_{P_2})\frac{\tau_1}{\tau_2}E_1[v] + \beta\Lambda_2 E_{i1}[\theta_1]. \quad (12)$$

According to the above expression, when $\beta > 0$ the private signal serves two purposes: it allows to predict the impact of fundamentals on $p_2$ and it creates a private learning channel from the first period price (as in Amador and Weill (2010) and Manzano and Vives (2011)) that allows investors to recover information on $\theta_1$ from the observation of $p_1$ to predict the impact of $\theta_2$ on $p_2$.

It is worth noting that the individual assessment of $\theta_1$, $E_{i1}[\theta_1] = a_1(v - E_{i1}[v]) + \theta_1$, is decreasing in $E_{i1}[v]$. For a given $p_1$ a higher assessment of the fundamentals $E_{i1}[v]$ goes together with a lower assessment of liquidity trading $E_{i1}[\theta_1]$. The consequence of this fact is that an increase in $\beta$ will push towards a lower response to private information ($E_{i1}[v]$) to forecast $p_2$ since from (12) the weight to $E_{i1}[v]$ in $E_{i1}[p_2]$ is

$$\left(\alpha_{P_2} + (1 - \alpha_{P_2})\frac{(\Delta a_2)^2 \tau_u}{\tau_2}\right) - \beta\Lambda_2 a_1. \quad (13)$$

The first part of the weight corresponds to the usual response to private information because of market making ($\alpha_{P_2}$) and because of speculation on fundamentals ($(1 - \alpha_{P_2})(\Delta a_2)^2 \tau_u/\tau_2$). The second part ($\beta\Lambda_2$) corresponds to the private learning channel from prices which detracts from the weight to private information, the more so when $\beta$ grows.

The dual role of private information yields multiple equilibria.

**Proposition 2.** *Linear equilibria always exist. In equilibrium, $a_2 = \gamma\tau_\epsilon$, and $a_1$ is implicitly defined by the equation $\phi(a_1) \equiv a_1(1 + \gamma\tau_u\Delta a_2) - \gamma a_2\Delta a_2\tau_u = 0$. If $\beta \in (0, 1]$:*

1. *There are two equilibria $a_1^*$, $a_1^{**}$, where $a_1^* < a_1^{**}$ (see (A.17), and (A.18), in the appendix for explicit expressions).*

2. *We have that $a_2 - \beta a_1^* > 0$, and $\lambda_2^* > 0$, while $a_2 - \beta a_1^{**} < 0$, and $\lambda_2^{**} < 0$. Furthermore, $|\lambda_2^{**}| < \lambda_2^*$, and prices are more informative along the equilibrium with high second period liquidity: $\tau_n^{**} > \tau_n^*$.*

*If $\beta = 0$, the equilibrium is unique:*

$$a_1 = \lim_{\beta \to 0} a_1^* = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}. \quad (14)$$

In the first period informed investors use their private signal to anticipate the impact of fundamentals and, when $\beta > 0$, that of first period liquidity trading on the second period price, at which they unwind. The more $p_2$ reflects $\theta_1$, the less their private signal is useful to predict

11

the liquidation price, and the lower is $a_1$. This, in turn, makes the first period price reflect more liquidity trading, and less private information. As a result the effect of private information impounded in the price by second period investors prevails over that coming from first period investors, $a_2 > \beta a_1^*$, and $\lambda_2^* > 0$. Hence, *adverse selection* magnifies the second period price impact of trades and augments first period investors' uncertainty over the liquidation price, further lowering their response to private information. In this equilibrium, $a_1^*$ is low, and the second period market is thin with $\lambda_2^* > 0$. Conversely, the more $p_2$ reflects $v$, the more investors' private signal is useful to predict the liquidation price, and the larger is $a_1$. This makes the first period price reflect less liquidity trading, and more private information. As a result, the effect of private information impounded in the price by first period investors prevails over that coming from second period investors, $a_2 < \beta a_1^{**}$, and $\lambda_2^{**} < 0$. In this case, *favorable* selection enhances second period liquidity, which lowers investors' uncertainty over the liquidation price, and further boosts their reaction to private information.

When $\beta = 0$, liquidity trading is transient, and first period investors cannot use their information to forecast its impact on $p_2$. This eliminates the private learning channel from the first period price and yields a unique equilibrium.[16]

The next result characterizes investors' strategies:

**Corollary 3.** *At a linear equilibrium, the strategies of an informed investor are given by*

$$X_1(s_i, p_1) = \frac{a_1}{\alpha_{E_1}}(E_{i1}[v] - p_1) + \frac{\alpha_{P_1} - \alpha_{E_1}}{\alpha_{E_1}} E_1[\theta_1] \tag{15}$$

$$X_2(s_i, p_1, p_2) = \frac{a_2}{\alpha_{E_2}}(E_{i2}[v] - p_2). \tag{16}$$

*When $\beta \in (0, 1]$ and (i) $a_1 = a_1^*$, $\alpha_{P_1} < \alpha_{E_1}$, and when (ii) $a_1 = a_1^{**}$, $\alpha_{P_1} > \alpha_{E_1}$. For $\beta = 0$, $\alpha_{P_1} < \alpha_{E_1}$.*

According to (16), in the second period an investor acts like in a static market. In the first period, instead, he loads his position anticipating the second period price, and scaling it down according to the uncertainty he faces on $p_2$:

$$X_1(s_i, p_1) = \gamma \frac{E_{i1}[p_2] - p_1}{\text{Var}_{i1}[p_2]}. \tag{17}$$

In this case, his strategy can be expressed as the sum of two components (see (15)). The first component captures the investor's activity based on his *private* estimation of the difference between the fundamentals and the equilibrium price. This may be seen as akin to "long-term" speculative trading, aimed at taking advantage of the investor's superior information on the liquidation value of the asset, since $p_2$ is correlated with $v$. The second component captures the investor's activity based on the extraction of order flow, i.e. *public*, information. This trading is instead aimed at timing the market by exploiting short-run movements in the asset price determined by the evolution of the future aggregate demand. Indeed, using the expressions in

---

[16]As we show in the proof of the proposition, $\lim_{\beta \to 0} a_1^{**} = +\infty$.

Corollaries 1, 2:

$$\text{Cov}_1[v - p_2, p_2 - p_1] = -\frac{\lambda_2}{\gamma \tau_{i1} \tau_u} > 0, \tag{18}$$

and due to Proposition 2, the sign of this expression depends on the equilibrium that arises. In particular, along the equilibrium with high (low) liquidity, $\lambda_2 < (>)0$, implying that based on public information the investor expects that returns display momentum (reversal). As a consequence, when observing

$$E_1[\theta_1] = a_1(v - E_1[v]) + \theta_1 > 0,$$

the investor infers that this realization is more driven by fundamentals information (liquidity trading) and goes long (short) in the asset, "chasing the trend" ("making" the market), anticipating that second period investors will bid the price up (down) when he unwinds his position.

As the persistence in liquidity trading ($\beta$) increases, in both equilibria first period informed investors speculate less aggressively on their private information. This is so since the importance of the private learning channel from prices, which detracts from the weight to private information, grows with $\beta$.

**Corollary 4.** *At equilibrium for all $\beta \in (0, 1)$, $\partial a_1 / \partial \beta < 0$.*

**Proof.** The equation that yields the first period responsiveness to private information is given by:

$$\phi(a_1) \equiv \lambda_2 \tau_{i2} a_1 - \gamma \Delta a_2 \tau_u \tau_\epsilon = 0.$$

The result then follows immediately, since from implicit differentiation of the above with respect to $\beta$:

$$\frac{\partial a_1}{\partial \beta} = -\frac{\gamma \tau_u a_1 (a_2 - a_1)}{(1 + \gamma \tau_u \Delta a_2) + \gamma \beta \tau_u (a_2 - a_1)} < 0,$$

independently of the equilibrium that arises; in the low liquidity equilibrium we have $\beta a_1 < a_1 < a_2 \equiv \gamma \tau_\epsilon$, and in the high liquidity equilibrium $a_1 > a_2 / \beta > a_2$ and $1 + \gamma \tau_u \Delta a_2 < 0$. $\quad \square$

We conclude this section, showing that absent private information, equilibrium multiplicity disappears when $\beta > 0$. In this case prices are invertible in the demand of liquidity traders, and the model is akin to Grossman and Miller (1988):

**Corollary 5.** *When $\tau_\epsilon = 0$, for all $\beta \in [0, 1]$ there exists a unique equilibrium where*

$$p_n = \bar{v} + \Lambda_n \theta_n \tag{19}$$

$$X_n(p_n) = -\Lambda_n^{-1}(p_n - \bar{v}), \tag{20}$$

$\Lambda_2 = \text{Var}[v]/\gamma$, *and*

$$\Lambda_1 = \frac{\text{Var}_1[p_2]}{\gamma} + \beta \Lambda_2.$$

According to (20), informed investors always take the other side of the order flow, buying the asset at a discount when $\theta_n < 0$, and selling it at a premium otherwise. For a given realization of the innovation in liquidity traders' demand $u_n$, the larger is $\Lambda_n$, the larger is the adjustment in the price investors require in order to absorb it. Therefore, $\Lambda_n$ proxies for the liquidity of the market. Uniqueness follows from the absence of private information. Indeed, in this case prices are invertible in the demand of liquidity traders and the feedback loop responsible for equilibrium multiplicity disappears.

## 2.2 Stability

In this section we analyze the stability of the equilibrium solutions. We start by defining the best response function which determines the equilibrium responsiveness to private information of a first period investor, given the average responsiveness of his peers.

Owing to CARA and normality, an investor in the first period trades according to (17). In the proof of Proposition 1, we use (17) to show that an investor's best response obtains as the ratio between the risk-adjusted weight to private information in the investor's private forecast $\gamma E_{i1}[p_2]$, and the investor's uncertainty over the liquidation price $\mathrm{Var}_{i1}[p_2]$:[17]

$$\psi(a_1) = \gamma \frac{a_2 \Delta a_2 \tau_u}{1 + \gamma \Delta a_2 \tau_u}. \tag{21}$$

Inspection of (21) yields that for $a_1 = \hat{a}_1 \equiv \beta^{-1}(a_2 + (\gamma \tau_u)^{-1}) \in (a_1^*, a_1^{**})$, the best response is discontinuous: $\lim_{a_1 \to \hat{a}_1^-} \psi(a_1) = -\infty$, and $\lim_{a_1 \to \hat{a}_1^+} \psi(a_1) = +\infty$. Indeed, in this case an individual investor in period 1 would like to take an unbounded position since the favorable selection effect exactly offsets the inventory risk effect, implying that he does not face any price risk. This discontinuity implies that the best response has two branches and two equilibria appear (see Figure 1).

Differentiating (21) yields

$$\psi'(a_1) = -\gamma \frac{\beta a_2 \tau_u}{(1 + \gamma \Delta a_2 \tau_u)^2} < 0, \tag{22}$$

for all $\beta > 0$ and both branches. The decisions on the weight assigned to private information in the first period are strategic substitutes. This is the outcome of the interaction of the usual Grossman-Stiglitz type forces for strategic susbtitutability in the use of information together with the influence of the private learning channel from prices.

To analyze stability, consider the following argument. Assume that the market is at an equilibrium point $\bar{a}_1$, so that $\bar{a}_1 = \psi(\bar{a}_1)$. Suppose now that a small perturbation to $\bar{a}_1$ occurs. As a consequence, first period investors modify their weights to private information so that the aggregate weight becomes $\bar{a}_1' = \psi(\bar{a}_1')$. If the market goes back to the original $\bar{a}_1$ according to the best reply dynamics with the best response function $\psi(\cdot)$, then the equilibrium is stable.
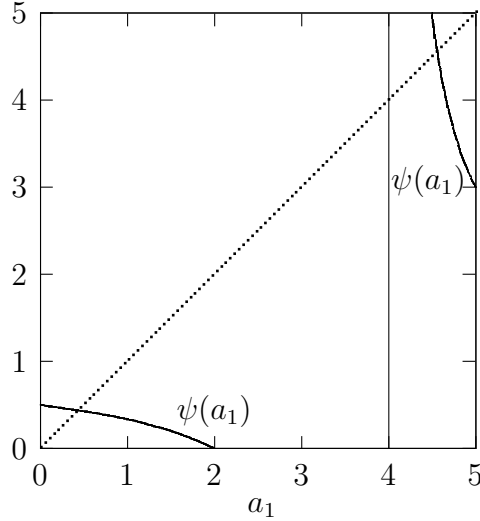
---

[17]See (A.11).

Figure 1: Equilibrium determination and stability. The figure displays the best reply function $\psi(a_1)$ (solid line) and the 45-degree line $a_1$ (dotted line). Equilibria obtain at the points where the two intersect. The vertical line (drawn at the point $\hat{a}_1 = 4$) shows the value of $a_1$ for which the best reply mapping is discontinuous. Parameters' values are as follows: $\tau_v = \tau_u = \tau_\epsilon = \gamma = 1$, and $\beta = .5$. For these values the equilibria are $a_1^* = 0.438$ and $a_1^{**} = 4.561$. Inspection of the equilibria shows that $|\psi'(a_1^*)| < 1$, while $|\psi'(a_1^{**})| > 1$.

Otherwise it is unstable. Formally, we have the following definition:

**Definition 1** (Stability). *An equilibrium is stable (unstable) if and only if its corresponding value for $a_1$ is a stable (unstable) fixed point for the best response function $\psi(\cdot)$ (i.e., if and only if its corresponding value for $a_1$ satisfies $|\psi'(a_1) < 1|$).*

With this definition we obtain the following.

**Corollary 6.** *The low (high) liquidity equilibrium is stable (unstable) with respect to the best response dynamics:*

$$|\psi'(a_1^{**})| > 1 > |\psi'(a_1^*)|. \tag{23}$$

The above result implies that the equilibrium with high responsiveness to private information and high liquidity is unstable according to the best reply dynamics (see Figure 1) while the low responsiveness one is stable. Strategic substitutability is much stronger in the second branch of the best response, leading to instability (see appendix B for an explanation of the degree of strategic substitutability in terms of the decomposition of the traditional and private learning channel effects on the best response along the lines of (13)).

## 3  The effect of residual uncertainty

In this section we perform a robustness exercise and assume that investors face residual uncertainty over the final liquidation value. Therefore, we model the final payoff as $\hat{v} = v + \delta$, where $\delta \sim N(0, \tau_\delta^{-1})$ is a random term orthogonal to all the random variables in the market,

15

and about which no investor is informed. The addition of the random term $\delta$ allows to study the effect of an increase in the residual uncertainty that surrounds investors' environment in periods of heightened turbulence, and shows that a price crash can occur within our framework.

With residual uncertainty, the expressions for prices and investors' strategies do not change (that is, expressions (8), (15), and (16) hold). However, the equilibrium obtains as the solution of a system of two cubic equations and is therefore no longer closed form solvable.

**Proposition 3.** *When investors face residual uncertainty over the final payoff, there always exists a linear equilibrium, where $a_1$ and $a_2$ obtain as a solution to the following system of cubic equations:*

$$\phi_2(a_1, a_2) \equiv a_2(1 + \kappa) - \gamma\tau_\epsilon = 0 \tag{24}$$

$$\phi_1(a_1, a_2) \equiv a_1(1 + \kappa + \gamma\Delta a_2\tau_u) - \gamma a_2\Delta a_2\tau_u(1 + \kappa) = 0, \tag{25}$$

*where $\kappa \equiv \tau_{i2}/\tau_\delta$.*

Studying (24) shows that for any $a_1$ there exists a unique real solution to $\phi_2(\cdot) = 0$, which simplifies the numerical analysis.[18] Let $a_2^*(a_1)$ be the unique real solution to (24), then the first period responsiveness to private information obtains as a fixed point of the following map:

$$\psi_1(a_1, a_2^*(a_1)) \equiv \gamma\frac{a_2\Delta a_2\tau_u(1 + \kappa)}{1 + \kappa + \gamma\Delta a_2\tau_u}. \tag{26}$$

Numerical analysis of the solution to the above fixed point shows that for low values of $1/\tau_\delta$ there are typically three equilibria, which can be ranked in terms of responsiveness to private information: $a_1^* < a_1^{**} < a_1^{***}$, and second period liquidity (see Figure 2).

Depending on parameters' values equilibria can be stable or unstable. In Figure 2, panel (a) we show an example in which both the low and the intermediate liquidity equilibria are stable according to the best reply dynamics. As residual uncertainty increases ($\tau_\delta^{-1}$ grows larger), only $a_1^*$ survives. Intuitively, a higher residual uncertainty, increases $\kappa$, and lowers second period investors' response to private signals. Thus, first period investors anticipate that the second period price is less related to the fundamentals, which lowers their reliance on private information. As a consequence, the first period price reflects less $v$ and more $\theta_1$, further lowering first period investors' reliance on private information. Therefore, even though a high liquidity equilibrium can be made stable, it is bound to disappear in periods of heavy market turbulence (when $\tau_\delta^{-1}$ increases).

Based on the above results, we now show that an increase in residual uncertainty may yield a price crash.[19] To see this, let's assume that $\bar{u} < 0$, so that informed investors in period 1

---

[18]This is immediate, since owing to (24), at equilibrium $a_2 > 0$, which implies that we can restrict attention to the positive orthant. Then, $\phi_2(a_1, 0) = -\gamma\tau_\delta\tau_\epsilon < 0$, and $(\partial\phi_2/\partial a_2) = 3a_2^2\tau_u - 4\beta a_1 a_2\tau_u + \tau_{i1} + (\beta a_1)^2\tau_u + \tau_\delta$, which is a quadratic with discriminant $\Delta = 4\tau_u((\beta a_1)^2\tau_u - 3(\tau_{i1} + \tau_\delta)) < 0$, and thus is always positive. Therefore, for any $a_1$, the second period optimal responsiveness to private information is uniquely determined.

[19]Other authors have investigated price crashes in markets with asymmetric information. See Gennotte and Leland (1990), Romer (1993), and Barlevy and Veronesi (2003).
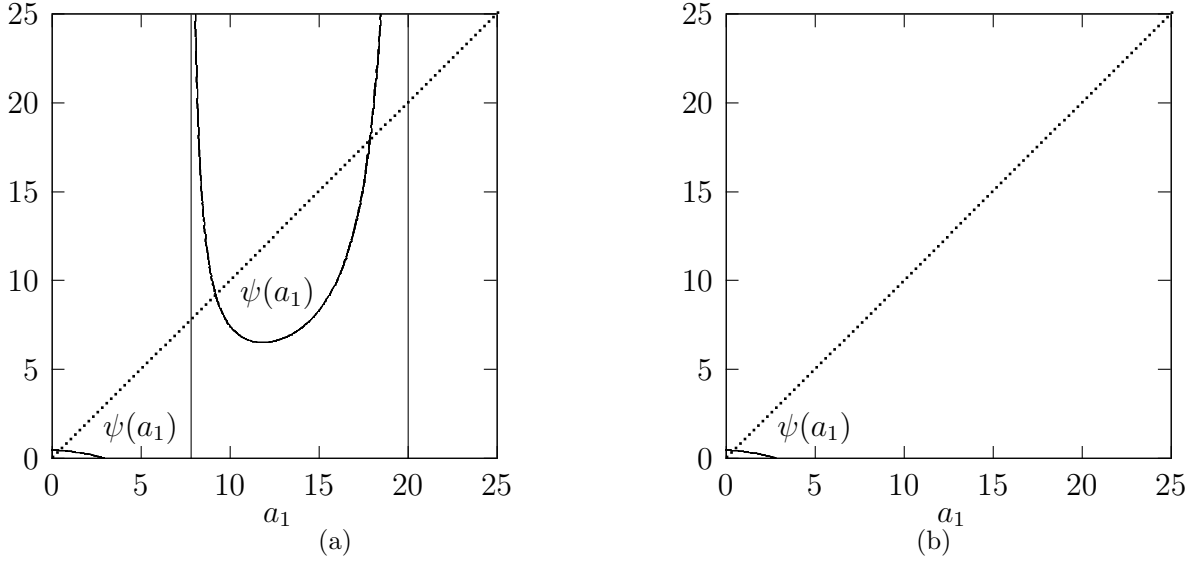
Figure 2: Equilibrium determination and stability with residual uncertainty. The figure displays the best reply function $\psi(a_1)$ (solid line) and the 45-degree line $a_1$ (dotted line). Equilibria obtain at the points where the two intersect. The vertical lines in panel (a) show the values of $a_1$ for which the best reply mapping is discontinuous. Parameters' values are as follows: $\tau_v = \tau_u = \tau_\epsilon = \gamma = 1$, $\beta = .3$, $1/\tau_\delta = 1/90$ in panel (a), and $1/\tau_\delta = 1/60$ in panel (b). For these values the equilibria in panel (a) are $(a_1^*, a_2(a_1^*)) = (0.447, 0.968)$, $(a_1^{**}, a_2(a_1^{**})) = (9.928, 0.494)$, and $(a_1^{***}, a_2(a_1^{***})) = (17.88, 0.205)$. Numerical evaluation of the slope of the reaction function yields: $\psi'(a_1^*, a_2(a_1^*)) = -0.088$, $\psi'(a_1^{**}, a_2(a_1^{**})) = -0.781$, and $\psi'(a_1^{***}, a_2(a_1^{***})) = 10.789$. In panel (b) we have a unique equilibrium with $(a_1^*, a_2(a_1^*)) = (0.439, 0.954)$, and $\psi'(a_1^*, a_2(a_1^*)) = -0.088$.

expect to hold a positive amount of the asset. In appendix C we show that this assumption implies

$$E[p_1] \equiv \bar{p}_1 = \bar{v} + \Lambda_1 \bar{u}. \tag{27}$$

Indeed, when $\bar{u} < 0$ first period investors anticipate absorbing a positive supply of the asset at equilibrium and thus require a compensation on the price they pay which lowers the expected price below the unconditional expectation of the payoff the more the higher is $\Lambda_1$: $E[p_1] < \bar{v}$. Now, assume the same set of parameters of Figure 2, and $\bar{v} = -\bar{u} = 1$. Start with $1/\tau_\delta = 1/90$, so that there are three equilibria, which imply three expected price levels at respectively

$$\bar{p}_1^* = 0.3479, \ \bar{p}_1^{**} = 0.9966, \ \bar{p}_1^{***} = 0.9991, \tag{28}$$

reflecting the fact that in the high liquidity equilibrium (the one with three stars) the uncertainty on $p_2$ is very small, and thus the expected price is very high since investors demand a small compensation to absorb the expected selling pressure from liquidity traders. Conversely, in the low liquidity equilibrium (one star) the opposite occurs. Now if $1/\tau_\delta = 1/60$ we know

that the number of equilibria narrows down to 1 and

$$\bar{p}_1^* = 0.337991.$$

If we let $1/\tau_\delta$ increase gradually, we obtain Figure 3. According to the figure, for low values of the residual uncertainty parameter, three equilibria arise with expected prices that rank from farther away to very close to $\bar{v} = 1$, respectively for the low, intermediate and high liquidity equilibrium (expected prices in the latter two equilibria are very close to each other, as testified by the values in (28)). Suppose that investors have coordinated on the equilibrium with intermediate liquidity and that the market is suddenly shocked by an increase in residual uncertainty. As a result the equilibrium set becomes a singleton, and the expected price crashes to a much lower level, even though the fundamentals of the economy have not changed.
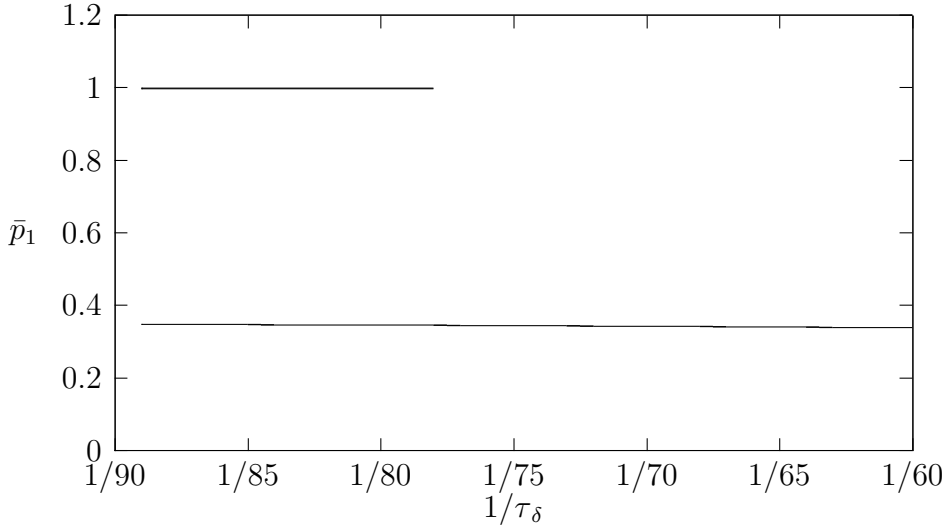


Figure 3: Example of a price crash. For $1/\tau_\delta < 1/78$, there are three equilibria, which narrow down to one when $1/\tau_\delta \geq 1/78$. Parameters' values are as follows: $\gamma = \tau_\epsilon = \tau_v = \tau_u = 1$, $\beta = .3$, $\bar{v} = -\bar{u} = 1$, and $1/\tau_\delta \in \{1/90, 1/89, \ldots, 1/60\}$.

## 4   HFT

HFT accounts for a rapidly increasing proportion of the trades carried out in today's exchanges (according to Tabb Group, in 2010 HFT represented roughly two thirds of all equity trades in the US and slightly more than one third in Europe). High Frequency Traders (HFTs) scan market data using computer algorithms to detect trading opportunities and rapidly react to such opportunities to lock-in profitable trades (the "latency" between the time at which the information is obtained by the computer and the time at which the trade is executed at the exchange is in the order of milliseconds and even microseconds).

The previous sections have established that when liquidity traders' demand is persistent, short term investors exploit a private learning channel from the first period price to learn the

18

demand of liquidity traders. This generates multiple equilibria with strikingly different features. In this section we interpret our short term, informed investors as HFTs and apply our results to the analysis of HFT's impact on the market.

Two related reasons make our analysis well suited to capture HFT. The first one stems from our joint assumptions that informed investors have a short term horizon *and* that liquidity traders' demand displays persistence. Indeed, these two assumptions allow us to model the possibility that informed investors' holding period is *shorter* than liquidity traders' one. To see this, consider the following argument. In the first period, market clearing implies:

$$x_1 + \theta_1 = 0.$$

Suppose now that $0 < \beta < 1$, then in the second period market clearing involves (i) the reverting position of first period informed investors $-x_1$, (ii) the position of second period informed investors $x_2$, (iii) a fraction $1 - \beta$ of the first period liquidity traders' position $\theta_1$ (who revert), and (iv) the new generation of liquidity traders with demand $u_2$. Letting $\Delta x_2 \equiv x_2 - x_1$, $\Delta \theta_2 \equiv \theta_2 - \theta_1 = u_2 - (1 - \beta)\theta_1$, market clearing implies

$$x_2 - x_1 + u_2 - (1 - \beta)\theta_1 = 0 \Leftrightarrow \Delta x_2 + \Delta \theta_2 = 0 \Leftrightarrow x_2 + \beta \theta_1 + u_2 = 0.$$

Note that while due to short horizons, the *entire* position of first period informed investors reverts at time 2, assuming persistent liquidity trading implies that *only* a fraction $(1 - \beta)\theta_1$ of first period liquidity trades reverts (while the complementary fraction is held until the liquidation date). The lower is $\beta$, the higher is this fraction. Thus, when informed investors have a short horizon, persistent liquidity trades allow to model in a parsimonious way the possibility that agents have *different holding periods*: when $\beta = 0$ informed investors and liquidity traders have the same holding period; as $\beta$ grows, liquidity traders' holding period becomes increasingly longer than that of informed investors (see Table 1). It is precisely this holding period discrepancy that allows informed investors to exploit the private learning channel, to anticipate the impact that liquidity traders' future demand has on the price at which they unwind.[20]

The existence of a private learning channel from prices leads informed investors to use private signals to infer the evolution of the second period aggregate demand from the first period (public) order flow (see Corollary 3). This offers the second reason to interpret our informed investors as HFTs since our informed investors have a privileged channel to interpret the order flow and therefore can anticipate it. Along the high liquidity equilibrium, investors speculate on price continuation, whereas along the low liquidity equilibrium they speculate on reversal. Thus, the behavior of informed investors in our model runs parallel to the claimed order anticipation with HFT.[21]

---

[20]In this way we can interpret the case $\beta = 0$ as capturing the extreme situation in which the technological features of HFT are available to all liquidity traders too, whereas when $\beta = 1$ the technological gap between HFT and liquidity trading is maximal.

[21]Indeed, the strategies described in Corollary 3 are akin to HFT "order anticipation". According to the January 2010 SEC Concept release, an order anticipation strategy "...involves any means to ascertain the

| Trading Date | | 1 | 2 |
|---|---|---|---|
| Liquidity traders | Holding | – | $\beta\theta_1$ |
| | New shock | $u_1$ | $u_2$ |
| | Position | $\theta_1 = u_1$ | $\theta_2 = \beta\theta_1 + u_2$ |
| | Reverting | – | $(1 - \beta)\theta_1$ |
| Informed investors | Position | $x_1$ | $x_2$ |
| | Reverting | – | $x_1$ |

Table 1: The evolution of liquidity trades and informed investors' positions in the two periods. The position of liquidity traders in every period is given by the sum of "Holding" and "New shock." Market clearing at $n$ requires that the sum of liquidity traders' and informed investors' positions offset each other.

Empirically, a number of authors find that HFT substantially boosts market quality, in particular enhancing price (informational) efficiency, and improving market depth (see, e.g., Hendershott, Jones, and Menkveld (2010), Brogaard (2010), and Hendershott and Riordan (2010)). At the same time, however, concern has been voiced over the potentially destabilizing role of HFT (Kirilenko et al. (2010)). Our results in Proposition 2, Corollary 6, and in Section 3 appear to be in line with the above evidence. Indeed, we find that it is precisely the activity of using *private* information to infer the demand of liquidity trading from *public* information that, via the private learning channel, can generate an equilibrium with high informational efficiency, and a thick market. However, such equilibrium turns out to be unstable (in the absence of residual uncertainty, with respect to the best reply dynamics) and even if stable is likely to disappear when market conditions deteriorate and private information is a poor guide to investment decisions. Thus, the presence of HFTs leads to equilibrium multiplicity with the outcome of either instability at a high liquidity equilibrium, or a stable equilibrium with low liquidity.

Our results also cast a different light on order anticipation. Order anticipators are commonly interpreted as "parasitic traders," who profit from the exploitation of other traders' orders without contributing to the informational efficiency of prices, nor improving market liquidity.[22] This seems to be at odds with the above mentioned empirical evidence. Our model clarifies that order anticipation can enhance market quality, via the effect of the private learning channel from prices on investors' use of private information.

---

existence of a large buyer (seller) that does not involve violation of a duty, misappropriation of information, or other misconduct. Examples include the employment of sophisticated pattern recognition software to ascertain from *publicly* available information the existence of a large buyer (seller). . ." (emphasis added). This description matches the intuition for first period investors' strategies (see (15)).

[22]See Harris (2002). Cartea and Penalva (2011) introduce HFT in a symmetric information model of liquidity provision à la Grossman-Miller, assuming that these investors have a superior ability to anticipate the evolution of the future aggregate demand for the asset compared to professional traders. In this context, they show that HFT has a negative impact on market quality, increasing microstructure noise and magnifying the price impact of trades.

Finally, we conclude our analysis of HFT by considering its impact on the expected losses of liquidity traders. As argued above, with persistence, a fraction $\beta$ of first period liquidity traders hold their position until the event date, while the remaining unwind at date 2. This implies that first period liquidity traders' expected profits are given by

$$\Pi_{\theta_1} \equiv E[\beta\theta_1(v - p_1) + (1 - \beta)\theta_1(p_2 - p_1)]$$

$$= -\left(\beta\lambda_1 + (1 - \beta)\lambda_2\Delta a_2\left(\frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}}\right)\right)\tau_u^{-1} < 0. \tag{29}$$

It is easy to see that along the high liquidity equilibrium $\lim_{\beta \to 0} \Pi_{\theta_1} = 0$, since in this case the first period signal responsiveness diverges at $\beta = 0$ (see Proposition 2). On the contrary, along the equilibrium with low liquidity

$$\lim_{\beta \to 0} \Pi_{\theta_1} = \Pi_{\theta_1}\big|_{a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}, \beta = 0} < 0.$$

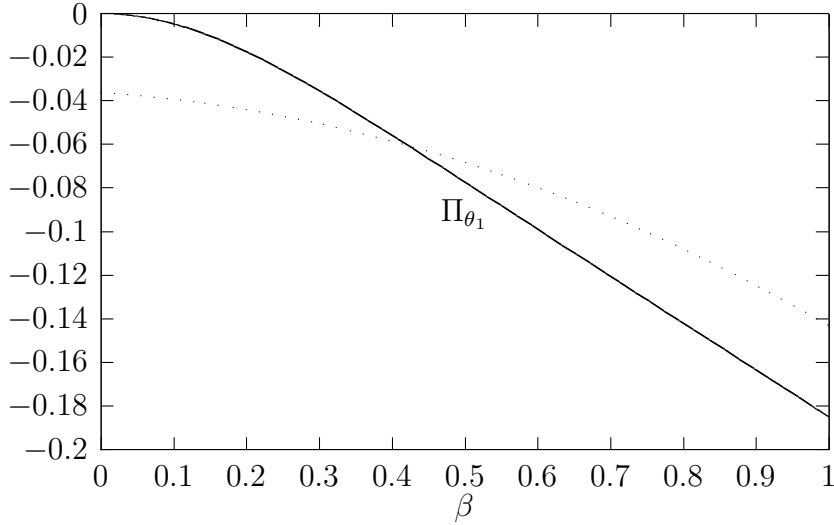In general, plotting (29) along the two equilibria that arise with $\beta > 0$ one obtains Figure 4.



Figure 4: Expected profit along the low liquidity equilibrium (dotted line) and along the high liquidity equilibrium (continuous line) as a function of $\beta$. Other parameter values are $\tau_v = 10$, $\tau_u = 1$, $\tau_\epsilon = 1$, $\gamma = 1$.

Depending on parameter values the two plots intersect or not, but the bottomline is that for $\beta$ small, liquidity traders' expected losses are always smaller along the high liquidity equilibrium. This is again in line with the view that the introduction of algorithmic trading and HFT have generated a large improvement in market liquidity accompanied by a reduction in adverse selection risk (see, e.g. Hendershott, Jones, and Menkveld (2010)). Indeed, in the high liquidity equilibrium, first period investors response to private information is strong, and in fact it generates favorable selection in the second period (see Proposition 2). However, as already argued above, in our framework this result is a weak one, as the equilibrium with high liquidity is unstable. Furthermore, looking again at the figure, we see that in both equilibria a higher

technological advantage of HFT (higher $\beta$) increases the expected losses of first period liquidity traders.

# 5  Average expectations and reliance on public information

In this section we use our model to investigate the claim that when investors have a short horizon, prices reflect the latter HOEs about fundamentals and are farther away from the final payoff compared to average expectations (Allen, Morris, and Shin (2006)). We show here that as with liquidity trading persistence investors use their private information *also* to infer the demand of liquidity traders from the first period order flow, the first period price is driven by investors' HOEs about fundamentals *and* by their average expectations about liquidity trading. This, in turn, has implications for price reliance on public information.

Starting from the second period, and imposing market clearing yields

$$\int_0^1 X_2(s_i, p_1, p_2)di + \theta_2 = 0. \tag{30}$$

Due to CARA and normality, we have

$$X_2(s_i, p_1, p_2) = \gamma \frac{E_{i2}[v] - p_2}{\text{Var}_{i2}[v]}.$$

Replacing the above in (30) and solving for the equilibrium price we obtain

$$p_2 = \bar{E}_2[v] + \Lambda_2 \theta_2.$$

Similarly, in the first period, imposing market clearing yields:

$$\int_0^1 X_1(s_i, p_1)di + \theta_1 = 0,$$

and solving for the equilibrium price we obtain

$$p_1 = \bar{E}_1[p_2] + \frac{\text{Var}_{i1}[p_2]}{\gamma}\theta_1. \tag{31}$$

Substituting the above obtained expression for $p_2$ in (31) yields

$$\begin{aligned} p_1 &= \bar{E}_1\left[\bar{E}_2[v] + \Lambda_2\theta_2\right] + \frac{\text{Var}_{i1}[p_2]}{\gamma}\theta_1 \\ &= \bar{E}_1\left[\bar{E}_2[v]\right] + \beta\Lambda_2\bar{E}_1\left[\theta_1\right] + \frac{\text{Var}_{i1}[p_2]}{\gamma}\theta_1. \end{aligned} \tag{32}$$

According to (32), there are three terms that form the second period price: investors' second order average expectations over the liquidation value ($\bar{E}_1[\bar{E}_2[v]]$), the risk-adjusted impact of the

first period stock of liquidity trades $(\theta_1)$, and investors' average expectations over first period liquidity trades $(\bar{E}_1[\theta_1])$. The latter arises since $p_2$ depends on $\theta_2$, which in turn is correlated with $\theta_1$ when $\beta > 0$. Thus, investors in period 1 are interested in estimating $\theta_1$.

Expression (32) implies that due to persistence in liquidity trading, the weight placed by the price on investors' average information is the sum of two terms: the first term captures the impact of HOEs on $v$, the second term reflects the impact of investors' average expectations over $\theta_1$. Computing

$$\bar{E}_1\left[\bar{E}_2[v]\right] = \bar{\alpha}_{E_1}v + (1 - \bar{\alpha}_{E_1})E_1[v]$$
$$\bar{E}_1[\theta_1] = a_1(1 - \alpha_{E_1})(v - E_1[v]) + \theta_1,$$

where

$$\bar{\alpha}_{E_1} = \alpha_{E_1}\left(1 - \frac{\tau_1}{\tau_2}(1 - \alpha_{E_2})\right).$$

Given (32), this implies that the *total* weight the price places on average private information is given by

$$\alpha_{P_1} = \bar{\alpha}_{E_1} + \beta\Lambda_2 a_1(1 - \alpha_{E_1}).$$

Note that $\forall\beta$, $\bar{\alpha}_{E_1} < \alpha_{E_1}$. Thus, when liquidity trading is transient $(\beta = 0)$ the first period price places a larger weight on public information than the optimal statistical weight. This finding is in line with Morris and Shin (2002), and Allen, Morris, and Shin (2006). The latter prove that with heterogeneous information, prices reflect investors' HOEs about the final payoff. In this case, the law of iterated expectations does not hold, and investors' forecasts overweight public information because these anticipate the average market opinion knowing that this also depends on the public information observed by other investors. The price is then systematically farther away from fundamentals compared to investors' consensus.

However, when liquidity trading is persistent, the price also reflects investors' average expectations about the impact that the demand of first period liquidity traders has on the second period price. Thus, an additional term adds to $\bar{\alpha}_{E_1}$ which for

$$a_1 > \frac{\alpha_{E_1}}{\beta\Lambda_2(1 - \alpha_{E_1})},$$

can increase the weight placed on average private information *above* the optimal statistical weight. Due to Corollary 3 we then have

**Corollary 7.** *At equilibrium,*

*1. When $\beta \in (0, 1]$, if*

$$a_1 = \begin{cases} a_1^*, & then \quad \alpha_{P_1} < \alpha_{E_1}, \quad and \quad \text{Cov}[p_1, v] < \text{Cov}[\bar{E}_1[v], v] \\ a_1^{**}, & then \quad \alpha_{P_1} > \alpha_{E_1}, \quad and \quad \text{Cov}[p_1, v] > \text{Cov}[\bar{E}_1[v], v]. \end{cases}$$

*2. When $\beta = 0$, $\alpha_{P_1} < \alpha_{E_1}$ and $\text{Cov}[p_1, v] < \text{Cov}[\bar{E}_1[v], v]$.*

With persistence liquidity traders' positions in the first and second period are positively correlated. Thus, investors use their private signals *also* to infer the demand of liquidity traders from the order flow (the private learning channel), anticipating the impact this has on the next period price. As private signals are informative about the fundamentals, this increases the weight the first period price assigns to the average private information. Along the equilibrium with high liquidity, investors escalate their response to private information. In this case the extra weight that adds to $\bar{\alpha}_{E_1}$ is high enough to draw the price closer to fundamentals compared to consensus. In view of the results obtained in Section 2.2 this equilibrium is, however, unstable. Along the equilibrium with low liquidity the price is farther away from fundamentals compared to consensus. This equilibrium, which shares the same properties of the one found by Allen, Morris, and Shin (2006), is instead stable.

# 6    Asset pricing implications

In this section we investigate the implications of our analysis for the asset pricing literature. In particular, we show that liquidity trading persistence can generate positive autocovariance of returns, without the need to impose heterogenous beliefs (as in Banerjee, Kaniel, and Kremer (2009)) or to assume that investors' preferences display a behavioral bias (as in, e.g., Daniel, Hirshleifer, and Subrahmanyam (1998)). We then look at the expected volume of informational trading and, consistently with the evidence presented in Llorente, Michaely, Saar, and Wang (2002), we find that in our setup a high volume of informational trading predicts momentum.

As we argue below, these results are consistent with a large body of evidence that points at the existence of patterns in the autocorrelation of returns. Typically these studies encompass slower trading frequencies than those adopted by HFT. However, as argued by Khandani and Lo (2011), strategies that exploit such patterns can be implemented at different frequencies, including the very high ones that characterize HFT.

## 6.1    Momentum and reversal

We start by computing the return autocovariance at different horizons:

**Corollary 8** (Autocovariance of returns). *At equilibrium:*

1. *For all $\beta \in [0,1]$, $\mathrm{Cov}[p_2 - p_1, p_1 - \bar{v}] < 0$.*

2. *For $\beta \in (0,1]$, $\mathrm{Cov}[v - p_2, p_1 - \bar{v}] < 0$. For $\beta = 0$, $\mathrm{Cov}[v - p_2, p_1 - \bar{v}] = 0$.*

3. *For $\beta \in (0,1]$, along the equilibrium with high liquidity $\mathrm{Cov}[v - p_2, p_2 - p_1] > 0$. Along the equilibrium with low liquidity, for $\tau_v < \hat{\tau}_v$, there exists a value $\hat{\beta}$ such that for all $\beta > \hat{\beta}$, $\mathrm{Cov}[v - p_2, p_2 - p_1] > 0$ (the expression for $\hat{\tau}_v$ is given in the appendix, see equation (A.36)). If $\beta = 0$, $\mathrm{Cov}[v - p_2, p_2 - p_1] < 0$.*

According to the above result, along the equilibrium with high liquidity, momentum occurs at short horizons (close to the end of the trading horizon), whereas at a longer horizon, returns display reversal.[23] This is in line with the empirical findings on return anomalies that document the existence of positive return autocorrelation at short horizons (ranging from six to twelve months, see Jegadeesh and Titman (1993)), and negative autocorrelation at long horizons (from three to five years, see De Bondt and Thaler (1985)).

The first two results derive from the fact that a given estimated first period imbalance, $E_1[\theta_1]$, has an opposite effect on $p_1 - \bar{v}$, and $p_2 - p_1$, $v - p_2$.[24] For the third result, a covariance decomposition (and the normality of returns) yields:

$$\text{Cov}[v - p_2, p_2 - p_1] = \text{Cov}\left[E_1[v - p_2], E_1[p_2 - p_1]\right] + \text{Cov}_1[v - p_2, p_2 - p_1]$$
$$= \beta \frac{\text{Var}_{i1}[p_2]}{\gamma} \text{Var}[E_1[\theta_1]] + \left(-\frac{\lambda_2}{\gamma \tau_{i1} \tau_u}\right), \tag{33}$$

implying that the short-term interim returns' autocovariance can be decomposed in two terms. The first term captures the returns' covariation due to the fact that both $E_1[v - p_2]$ and $E_1[p_2 - p_1]$ vary with $p_1$. The second term captures the returns' covariation due to the fact that for each $p_1$ both second and third period returns jointly vary around their corresponding conditional expectations. All else equal, with persistence the anticipated impact of the first period imbalance has the same sign on both the second and third period expected returns, so that the first term is always positive when $\beta > 0$.[25]

For the second term, factoring out the impact of first period information, the joint covariation of returns around their expectations could be driven either by liquidity trading or by fundamentals information. In the high liquidity equilibrium, as prices are close to fundamentals, the second effect predominates and returns positively covary around their means. Conversely, in the low liquidity equilibrium, prices are more driven by liquidity trades, so that returns tend to covary around their means in opposite directions.

Equation (33) shows that optimal investment behavior in our model departs in a substantial way from the one of an outside observer that relies on the sign of the *unconditional* return covariance to trade. Indeed, an investor in our model engages in momentum trading only for a stock that displays a very strong positive autocovariance. Equivalently, he may adopt a contrarian strategy even when an outside observer would see $\text{Cov}[v - p_2, p_2 - p_1] > 0$. This is because, as argued in Corollary 3, informed investors base their decision to chase the trend or act as contrarians on the sign of $\text{Cov}_1[v - p_2, p_2 - p_1]$.

It is interesting to relate our result on momentum with Daniel, Hirshleifer, and Subrahmanyam (1998) who assume that *overconfident* investors underestimate the dispersion of the

---

[23]Numerical simulations show that in a model with three periods, in the equilibrium with high liquidity, both $\text{Cov}[v - p_3, p_3 - p_2]$ and $\text{Cov}[p_3 - p_2, p_2 - p_1]$ are positive.

[24]As one can verify $\text{Cov}[v - p_2, p_1 - \bar{v}] = \text{Cov}[E_1[v - p_2], E_1[p_1 - \bar{v}]] = -\beta \Lambda_2 \text{Cov}[E_1[\theta_1], p_1] < 0$, and $\text{Cov}[p_2 - p_1, p_1 - \bar{v}] = \text{Cov}[E_1[p_2 - p_1], E_1[p_1 - \bar{v}] = (\beta \Lambda_2 - \Lambda_1) \text{Cov}[E_1[\theta_1], p_1] < 0$.

[25]If $p_1$ decreases compared to $p_2$, both $p_2 - p_1$ and $v - p_2$ are expected to increase, given that the selling pressure could come from liquidity traders which have a persistent supply. Thus, liquidity trades persistence offsets the mean reversion effect due to first period short-term investors' unwinding at date 2.

error term affecting their signals and "overreact" to private information. This, in turn, generates long term reversal and, in the presence of confirming public information which due to *biased self attribution* boosts investors' confidence, also lead to short term positive return autocorrelation. This pattern of overreaction, continuation, and correction is likely to affect stocks which are more difficult to value (e.g., growth stocks). In such a context, momentum is thus a symptom of mispricing and hence related to prices wandering away from fundamentals (conversely, reversal is identified with price corrections). In our model, along the equilibrium with high liquidity, investors rationally react more strongly to their private signals compared to the static benchmark, in contrast to the overreaction effect of the behavioral literature.[26] However, this heavy reaction to private information leads to stronger information impounding and to prices that track better the fundamentals (see Proposition 2). Momentum at short horizons in this case is therefore associated with a rapid convergence of the price to the full information value. To illustrate this fact, in Figure 5 we plot the mean price paths along the equilibrium with low liquidity (thick line), with high liquidity (thin line), and along the "static" equilibrium, that is the one that would obtain if investors reacted to information as if they were in a static market (dotted line). From the plot it is apparent that in the equilibrium with high liquidity the price displays a faster adjustment to the full information value than in the equilibrium with low liquidity (and the static equilibrium). This shows that the occurrence of momentum is not at odds with price (informational) efficiency.
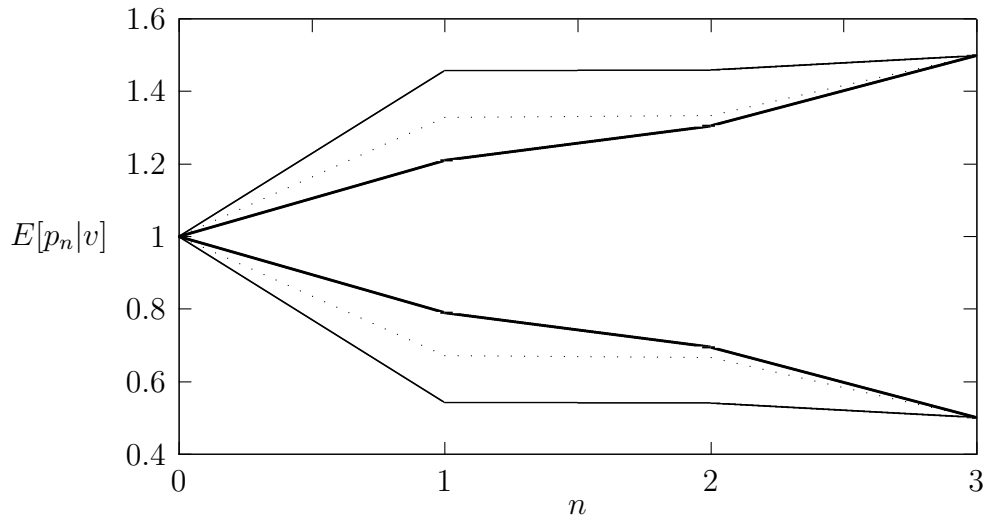


Figure 5: Mean price paths along the equilibrium with low liquidity (thick line), high liquidity (thin line), and assuming that first period investors react to private information as if they were in a static market (i.e., setting $a_1 = \gamma \tau_\epsilon$). Parameters' values are as follows: $\tau_v = \tau_\epsilon = \tau_u = \gamma = 1$, $\bar{v} = 1$, $\beta = .9$ and $v \in \{1.5, .5\}$.

As stated in the corollary, momentum can also occur along the equilibrium with low liquid-

---

[26]Indeed, the static solution calls for $a_1 = \gamma \tau_\epsilon$ (see, e.g, Admati (1985), or Vives (2008)), and it is easy to verify that $0 < a_1^* < \gamma \tau_\epsilon < a_1^{**}$. In Daniel, Hirshleifer, and Subrahmanyam (1998) overconfident investors overweight private information in relation to the prior.

ity, provided that investors are sufficiently uncertain about the liquidation value prior to trading (that is, $\tau_v$ is low) and that liquidity trading is sufficiently persistent ($\beta$ high). In that equilibrium, investors respond less to private information, information impounding is staggered, and prices adjust more slowly to the full information value (see Figure 5). However, if sufficiently persistent, liquidity trading exerts a continuous price pressure which can eventually outweigh the former effect. Therefore, along this equilibrium momentum arises with slow convergence to the full information value, implying that the occurrence of a positive autocorrelation at short horizons *per se* does not allow to unconditionally identify the informational properties of prices.

Finally, at long horizons, the effect of private information on the correlation of returns washes out and the only driver of the autocovariance is the persistence in liquidity trading, which generates reversal.

## 6.2 Expected volume and return predictability

We now turn our attention to the implications of our results for the expected volume of informational trading and the predictability of returns along the two equilibria. We show that the expected volume of informational trading is high (low) along the high (low) liquidity equilibrium. This implies that a high volume of informational trading predicts momentum, in line with the evidence presented by Llorente, Michaely, Saar, and Wang (2002). However, as we have argued in the previous section, also along the equilibrium with low liquidity momentum can occur, provided liquidity trading displays sufficiently strong persistence (and the ex-ante uncertainty about the liquidation value is sufficiently high). This implies that a low volume of informational trading can also predict continuation. In this case, though, momentum is a signal of slow price convergence to the liquidation value. In sum, momentum is compatible with both a high and a low volume of informational trading, but the implications that return continuation has for price informativeness are markedly different in the two situations.

We start by defining the volume of informational trading as the expected traded volume in the market with heterogeneous information net of the expected volume that obtains in the market with no private information analyzed in Corollary 5. This yields:[27]

$$
\begin{aligned}
V_1 &\equiv \int_0^1 E\left[|X_1(s_i, p_1)|\right] di - \int_0^1 E\left[|X_1(p_1)|\right] di \\
&= \int_0^1 \sqrt{\frac{2}{\pi} \mathrm{Var}\left[X_1(s_i, p_1)\right]} di - \int_0^1 \sqrt{\frac{2}{\pi} \mathrm{Var}\left[X_1(p_1)\right]} di \\
&= \sqrt{\frac{2}{\pi}} \left( \sqrt{a_1^2 \tau_\epsilon^{-1} + \tau_u^{-1}} - \sqrt{\tau_u^{-1}} \right),
\end{aligned}
\tag{34}
$$

---

[27]This is consistent with He and Wang (1995).

and

$$V_2 \equiv \int_0^1 E\left[|X_2(s_i, p_1, p_2) - X_1(s_1, p_1)|\right] di - \int_0^1 E\left[|X_2(p_2) - X_1(p_1)|\right] di$$

$$= \int_0^1 \sqrt{\frac{2}{\pi} \mathrm{Var}\left[X_2(s_i, p_1, p_2) - X_1(s_i, p_1)\right]} di - \int_0^1 \sqrt{\frac{2}{\pi} \mathrm{Var}\left[X_2(p_2) - X_1(p_1)\right]} di$$

$$= \sqrt{\frac{2}{\pi}} \left( \sqrt{(a_1^2 + a_2^2)\tau_\epsilon^{-1} + (1 + (\beta - 1)^2)\tau_u^{-1}} - \sqrt{(1 + (\beta - 1)^2)\tau_u^{-1}} \right). \tag{35}$$

We measure the total volume of informational trading with $V_1 + V_2$, and obtain

**Corollary 9** (Expected volume of informational trading). *At equilibrium, for all $\beta \in (0, 1]$ the expected volume of informational trading is higher along the high liquidity equilibrium. When $\beta = 0$ only the equilibrium with a low volume of informational trading survives.*

**Proof.** Rearranging the expressions for investors' strategies obtained in Corollary 3 yields $x_{in} = a_n \epsilon_{in} - \theta_n$, for $n = 1, 2$. Owing to the fact that for a normally distributed random variable $Y$ we have

$$E[|Y|] = \sqrt{\frac{2}{\pi} \mathrm{Var}[Y]},$$

which implies (see (34), and (35)), that $V_1 + V_2$ is an increasing function of $a_1$. Recall that while $a_2 = \gamma \tau_\epsilon$, in the first period the response to private information is higher along the equilibrium with high liquidity: $a_1^{**} > a_1^*$, and the result follows. Finally, from Proposition 2 when $\beta = 0$,

$$a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u} < a_1^{**}.$$

$\square$

The intuition for the above result is straightforward: as along the equilibrium with high liquidity investors step up the response to their signals, the position change due to private information is higher along such equilibrium.

Taking together Corollaries 33 and 9 imply that a high volume of informational trading in the second period predicts return continuation, no matter what the persistence in liquidity trading is. A low volume of informational trading, on the other hand, can also be associated with momentum, provided liquidity trading is sufficiently persistent.

# 7 Conclusions

When liquidity traders' positions are positively correlated across trading dates, investors exploit a private learning channel to infer the demand of liquidity traders from the price at which they load their positions, and anticipate the price at which they unwind them. We show that this effect generates multiple equilibria which can be ranked in terms of investors' responsiveness to private information, liquidity, and informational efficiency.

In our framework short term investors can be seen as HFTs that, due to the predictability of liquidity traders' demand implied by persistence, engage in order anticipation. This provides an economic rationale to the positive impact of HFT on market quality that is widely documented in the empirical literature. Indeed, we show that HFT can induce an equilibrium with high informational efficiency and liquidity. However, this equilibrium is unstable (in the absence of residual uncertainty, with respect to the best reply dynamics), and even if stable is likely to disappear when market conditions deteriorate and private information is a poor guide to investment decisions. Thus, even though HFT can have a positive impact on market quality, the benefits it creates are likely to be fleeting.

Our analysis also allows to clarify the role of HOEs in asset pricing. As with liquidity trading persistence investors use their private information *also* to infer the demand of liquidity traders from the order flow, we show that prices are driven by average expectations about *fundamentals* and *liquidity trading*. Along the equilibrium with low liquidity, prices are driven by average expectations about fundamentals, and therefore over-rely on public information (compared to the optimal statistical weight). Conversely, along the equilibrium with high liquidity, the price is driven by average expectations about liquidity trading, investors step up their response to private signals, and prices under-rely on public information (compared to the optimal statistical weight).

Our paper provides an alternative interpretation for empirically documented regularities on the patterns of return autocorrelation. As we have argued, at long horizons returns display reversal. However, return correlation at short horizons depends on the equilibrium that prevails in the market. In the equilibrium with high liquidity, investors escalate their response to private information and momentum arises. Conversely, in the low liquidity equilibrium investors scale down their response to private signals and, when liquidity trading is not very persistent, returns tend to revert. While this offers an explanation for returns' predictability which departs from behavioral assumptions, our analysis also makes the empirical prediction that both a high or a low volume of informational trading can predict momentum. In the former case, this is a signal that prices rely poorly on public information and accurately reflect fundamentals starting from the earlier stages of the trading process. In this case momentum at short horizons proxies for a rapid price convergence to the full information value. In the latter case, instead, prices heavily rely on public information and offer a poor signal of fundamentals. In this case, therefore, momentum proxies for a continuing, liquidity-driven, price pressure.

# References

Admati, A. R. (1985). A noisy rational expectations equilibrium for multi-asset securities markets. *Econometrica 53*(3), 629–657.

Admati, A. R. and P. Pfleiderer (1988). A theory of intraday patterns: volume and price variability. *Review of Financial Studies 1*(1), 3–40.

Albagli, E. (2011). Amplification of uncertainty in illiquid markets. *Working Paper*.

Allen, F., S. Morris, and H. S. Shin (2006). Beauty contests and iterated expectations in asset markets. *Review of Financial Studies 19*(3), 719–752.

Amador, M. and P. O. Weill (2010). Learning from prices: Public communication and welfare. *Journal of Political Economy 118*(5), 866–907.

Bacchetta, P. and E. van Wincoop (2006). Can information heterogeneity explain the exchange rate determination puzzle? *American Economic Review 96*(3), 552–576.

Bacchetta, P. and E. van Wincoop (2008). Higher order expectations in asset pricing. *Journal of Money, Credit and Banking 40*(5), 837–866.

Banerjee, S., R. Kaniel, and I. Kremer (2009). Price drift as an outcome of differences in higher order beliefs. *Review of Financial Studies 22*(9), 3707–3734.

Barlevy, G. and P. Veronesi (2003). Rational panics and stock market crashes. *Journal of Economic Theory 110*(2), 234–263.

Biais, B., T. Foucault, and S. Moinas (2011). Equilibrium High Frequency Trading. *Working Paper*.

Brogaard (2010). High Frequency Trading and its impact on market quality. *Working Paper*.

Brown, D. P. and R. H. Jennings (1989). On technical analysis. *Review of Financial Studies 2*(4), 527–551.

Cartea, A. and J. Penalva (2011). Where is the value in high frequency trading? *Working Paper*.

Cespa, G. (2002). Short-term investment and equilibrium multiplicity. *European Economic Review 46*(9), 1645–1670.

Cespa, G. and X. Vives (2011). Dynamic trading and asset prices: Keynes vs. Hayek. *Review of Economic Studies (forthcoming)*.

Chordia, T. and A. Subrahmanyam (2004). Order imbalance and individual stock returns: Theory and evidence. *Journal of Financial Economics 72*(3), 485–518.

Coval, J. and E. Stafford (2007). Asset fire sales (and purchases) in equity markets. *Journal of Financial Economics 86*(2), 479–512.

Daniel, K., D. Hirshleifer, and A. Subrahmanyam (1998). Investor psychology and security market under- and overreactions. *Journal of Finance 53*, 1839–1884.

De Bondt, W. F. M. and R. Thaler (1985). Does the stock market overreact? *Journal of Finance 40*(3), 793–805.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann (1990). Noise trader risk in financial markets. *Journal of Political Economy 98*(4), 703–738.

Dennert, J. (1991). Insider trading and the cost of capital in a multi-period economy. *London School of Economics Financial Market Group Discussion Paper 128*.

Dow, J. and G. Gorton (1994). Arbitrage chains. *Journal of Finance 49*(3), 819–49.

Easley, D., R. F. Engle, M. O'Hara, and L. Wu (2008). Time-varying arrival rates of informed and uninformed trades. *Journal of Financial Econometrics 6*(2), 171–207.

Froot, K. A., D. S. Scharfstein, and J. C. Stein (1992). Herd on the street: Informational inefficiencies in a market with short-term speculation. *Journal of Finance 47*(4), 1461–1484.

Gennotte, G. and H. Leland (1990). Market liquidity, hedging, and crashes. *American Economic Review 80*(5), 999–1021.

Gromb, D. and D. Vayanos (2010). Limits of arbitrage: The state of the theory. NBER Working Papers 15821, National Bureau of Economic Research, Inc.

Grossman, S. J. and M. H. Miller (1988). Liquidity and market structure. *Journal of Finance 43*(3), 617–37.

Grossman, S. J. and J. E. Stiglitz (1980). On the impossibility of informationally efficient markets. *American Economic Review 70*(3), 393–408.

Harris, L. (2002). *Trading and Exchanges: Market Microstructure for Practitioners*. Oxford University Press.

He, H. and J. Wang (1995). Differential information and dynamic behavior of stock trading volume. *Review of Financial Studies 8*(4), 919–972.

Hendershott, T., C. M. Jones, and A. Menkveld (2010). Does algorithmic trading improve liquidity? *Journal of Finance 66*(1), 1–33.

Hendershott, T. and R. Riordan (2010). Algorithmic trading and information. *Working Paper*.

Hendershott, T. and M. S. Seasholes (2009). Market predictability and non-informational trading. *Working Paper*.

Holmström, B. and J. Ricart i Costa (1986). Managerial incentives and capital management. *Quarterly Journal of Economics 101*, 835–860.

Jegadeesh, N. and S. Titman (1993). Returns to buying winners and selling losers: Implications for stock market efficiency. *Journal of Finance 48*, 65–91.

Khandani, A. E. and A. W. Lo (2011). What happened to the quants in august 2007? evidence from factors and transactions data. *Journal of Financial Markets 14*(1), 1–46.

Kirilenko, A., A. S. Kyle, S. Mehrdad, and T. Tuzun (2010). The flash crash: The impact of high frequency trading on an electronic market. *Working Paper, SSRN*.

Kondor, P. (2009). The more we know, the less we agree: higher-order expectations, public announcements and rational inattention. *Working Paper*.

Kyle, A. (1985). Continuous auctions and insider trading. *Econometrica 53*(6), 1315–1336.

Llorente, G., R. Michaely, G. Saar, and J. Wang (2002). Dynamic volume-return relation of individual stocks. *Review of Financial Studies 15*(4), 1005–1047.

Manzano, C. and X. Vives (2011). Public and private learning from prices, strategic substitutability and complementarity, and equilibrium multiplicity. *Journal of Mathematical Economics 47*, 346–369.

Morris, S. and H. S. Shin (2002). The social value of public information. *American Economic Review 92*, 1521–1534.

Nimark, K. P. (2007). Dynamic higher order expectations. *Working Paper*.

Pagano, M. (1989). Trading volume and asset liquidity. *Quarterly Journal of Economics 104*, 255–274.

Romer, D. (1993, December). Rational asset-price movements without news. *American Economic Review 83*(5), 1112–30.

Shleifer, A. and R. Vishny (1990). Equilibrium short horizons of investors and firms. *American Economic Review 80*, 148–153.

Singleton, K. J. (1987). Asset prices in a time-series model with disparately informed, competitive traders. In W. Barnett and K. Singleton (Eds.), *New approaches to monetary economics*. Cambridge University Press, Cambridge.

Spiegel, M. (1998). Stock price volatility in a multiple security overlapping generations model. *Review of Financial Studies 11*(2), 419–447.

Vayanos, D. and P. Woolley (2008). An institutional theory of momentum and reversal. *Working Paper, LSE*.

Vives, X. (1995). Short-term investment and the informational efficiency of the market. *Review of Financial Studies 8*(1), 125–160.

Vives, X. (2008). *Information and Learning in Markets: The Impact of Market Microstructure*. Princeton University Press.

Watanabe, M. (2008). Price volatility and investor behavior in an overlapping generations model with information asymmetry. *Journal of Finance 63*(1), 229–272.

# A    Appendix

Consider a candidate linear (symmetric) equilibrium and let $z_1 \equiv a_1 v + \theta_1$ be the "informational content" of the first period order flow and simlarly $z_2 \equiv \Delta a_2 v + u_2$ where $\Delta a_2 \equiv a_2 - \beta a_1$ for the second period. Then, it is easy to see that $p_1$ is observationally equivalent (o.e.) to $z_1$ and that the sequence $\{z_1, z_2\}$ is o.e. to $\{p_1, p_2\}$. Consider a candidate linear (symmetric) equilibrium $x_{i1} = a_1 s_i - \varphi_1(p_1)$, $x_{i2} = a_2 s_i - \varphi_2(p_1, p_2)$, where $\varphi_n(\cdot)$ is a linear function. Letting $x_n \equiv \int_0^1 x_{in} di$, and imposing market clearing in the first period implies (due to our convention):

$$x_1 + \theta_1 = 0 \Leftrightarrow a_1 v + \theta_1 = \varphi_1(p_1). \tag{A.1}$$

In the second period the market clearing condition is

$$
\begin{aligned}
x_2 + \beta\theta_1 + u_2 = 0 &\Leftrightarrow x_2 - \beta x_1 + u_2 = 0 \\
&\Leftrightarrow a_2 v - \varphi_2(p_1, p_2) - \beta(a_1 v - \varphi_1(p_1)) + u_2 = 0 \\
&\Leftrightarrow \Delta a_2 v + u_2 = \varphi_2(p_1, p_2) - \beta\varphi_1(p_1), \tag{A.2}
\end{aligned}
$$

where in the second line we use (A.1). From (A.1) and (A.2) it is easy to see that $z_1$ is o.e. to $p_1$ and that $\{z_1, z_2\}$ is o.e. to $\{p_1, p_2\}$. It thus follows that $E_n[v] = \tau_n^{-1}(\tau_v + \tau_u \sum_{t=1}^n \Delta a_t z_t)$, $\mathrm{Var}_n[v] = (\tau_v + \tau_u \sum_{t=1}^n (\Delta a_t)^2)^{-1}$, $E_{in}[v] = \tau_{in}^{-1}(\tau_n E_n[v] + \tau_\epsilon s_i)$, and $\mathrm{Var}_{in}[v] = (\tau_n + \tau_\epsilon)^{-1} \equiv \tau_{in}^{-1}$.

To prove our argument, we proceed by backwards induction. In the last trading period traders act as in a static model and owing to CARA and normality we have

$$X_2(s_{i1}, z^2) = \gamma \frac{E_{i2}[v] - p_2}{\mathrm{Var}_{i2}[v]}, \tag{A.3}$$

and

$$
\begin{aligned}
p_2 &= \bar{E}_2[v] + \frac{\mathrm{Var}_{i2}[v]}{\gamma} \\
&= \alpha_{P_2}\left(v + \frac{\theta_2}{a_2}\right) + (1 - \alpha_{P_2}) E_2[v], \tag{A.4}
\end{aligned}
$$

where $a_2 = \gamma\tau_\epsilon$, and $\alpha_{P_2} = \alpha_{E_2}$. Rearranging (A.4) we obtain

$$
\begin{aligned}
p_2 &= \frac{\alpha_{P_2}}{a_2}(a_2 v - \beta a_1 v + \beta a_1 v + \theta_2) + (1 - \alpha_{P_2})E_2[v] \\
&= \left(\underbrace{\frac{\alpha_{P_2}}{a_2} + (1 - \alpha_{P_2})\frac{\Delta a_2 \tau_u}{\tau_2}}_{\lambda_2}\right) z_2 + (1 - \lambda_2 \Delta a_2)\hat{p}_1, \tag{A.5}
\end{aligned}
$$

where $\hat{p}_1 \equiv (\gamma\tau_1 + \beta a_1)^{-1}(\gamma\tau_1 E_1[v] + \beta z_1)$, which provides an alternative expression for $p_2$

which separates the impact on second period "news" from the information contained in the first period order flow.

In the first period owing to CARA and normality, an agent $i$ trades according to

$$X_1(s_{i1}, z_1) = \gamma \frac{E_{i1}[p_2] - p_1}{\text{Var}_{i1}[p_2]},\tag{A.6}$$

where, using (A.5),

$$E_{i1}[p_2] = \lambda_2 \Delta a_2 E_{i1}[v] + (1 - \lambda_2 \Delta a_2)\hat{p}_1\tag{A.7}$$

$$\text{Var}_{i1}[p_2] = \lambda_2^2 \left( \frac{\tau_{i2}}{\tau_{i1}\tau_u} \right).\tag{A.8}$$

Replacing (A.7) and (A.8) in (A.6) yields

$$X_1(s_{i1}, z_1) = \frac{\gamma \Delta a_2 \tau_{i1} \tau_u}{\lambda_2 \tau_{i2}}(E_{i1}[v] - \hat{p}_1) + \frac{\gamma \tau_{i1} \tau_u}{\lambda_2^2 \tau_{i2}}(\hat{p}_1 - p_1).$$

Imposing market clearing and identifying equilibrium coefficients yields

$$p_1 = \alpha_{P_1} \left( v + \frac{\theta_1}{a_1} \right) + (1 - \alpha_{P_1})E_1[v],\tag{A.9}$$

where

$$\alpha_{P_1} \equiv \alpha_{E_1} \left( 1 + \frac{(\beta\rho - 1)\tau_1}{\tau_{i2}} \right),\tag{A.10}$$

$a_1$ obtains as a fixed point of the mapping

$$\begin{aligned}\psi(a_1) &= \gamma \frac{\lambda_2 \Delta a_2 \alpha_{E_1}}{\text{Var}_{i1}[p_2]} \\ &= \gamma \frac{a_2 \Delta a_2 \tau_u}{1 + \gamma \Delta a_2 \tau_u},\end{aligned}\tag{A.11}$$

and $\rho \equiv a_1/(\gamma\tau_\epsilon)$. Finally, note that using (A.10) and (A.11) and rearranging the expression for the first period strategy yields

$$X_1(s_{i1}, z_1) = \frac{a_1}{\alpha_{E_1}}(E_{i1}[v] - p_1) + \frac{\alpha_{P_1} - \alpha_{E_1}}{\alpha_{E_1}} \frac{a_1}{\alpha_{P_1}}(p_1 - E_1[v]).$$

$\square$

## Proof of Corollary 1

In the second period, rearranging (A.4), $p_2 = E_2[v] + \Lambda_2 E_2[\theta_2]$, where $\Lambda_2 = \text{Var}_{i2}[v]/\gamma$. In the

first period due to short term horizons, we have

$$p_1 = \bar{E}_1[p_2] + \frac{\text{Var}_{i1}[p_2]}{\gamma}\theta_1$$

$$= \bar{E}_1\left[\bar{E}_2[v]\right] + \frac{\text{Var}_{i2}[v]}{\gamma}\beta\bar{E}_1[\theta_1] + \frac{\text{Var}_{i1}[p_2]}{\gamma}\theta_1$$

$$= \alpha_{P_1}v + (1 - \alpha_{P_1})E_1[v] + \left(\beta\frac{\text{Var}_{i2}[v+\delta]}{\gamma} + \frac{\text{Var}_{i1}[p_2]}{\gamma}\right)\theta_1, \tag{A.12}$$

where

$$\alpha_{P_1} = \bar{\alpha}_{E_1} + \frac{\text{Var}_{i2}[v]}{\gamma}\beta a_1(1 - \alpha_{E_1}),$$

and

$$\bar{\alpha}_{E_1} = \alpha_{E_1}\left(1 - \frac{\tau_1}{\tau_2}(1 - \alpha_{E_2})\right).$$

Now, we know that at a linear equilibrium

$$p_1 = \alpha_{P_1}\left(v + \frac{\theta_1}{a_1}\right) + (1 - \alpha_{P_1})E_1[v]. \tag{A.13}$$

Comparing (A.12) and (A.13), we then see that an alternative expression for $a_1$ is the following:

$$a_1 = \gamma\frac{\alpha_{P_1}}{\text{Var}_{i1}[p_2] + \beta\text{Var}_{i2}[v]}.$$

Given the definition of the inventory component market depth, from the last equation we conclude that

$$\Lambda_2 = \frac{\text{Var}_{i1}[p_2] + \beta\text{Var}_{i2}[v]}{\gamma}. \tag{A.14}$$

$\square$

## Proof of Corollary 2

For the second period price, see (A.5). For the first period price, we rearrange (A.9) to obtain

$$p_1 = \left(\underbrace{\frac{\alpha_{P_1}}{a_1} + (1 - \alpha_{P_1})\frac{a_1\tau_u}{\tau_1}}_{\lambda_1}\right)z_1 + (1 - \alpha_{P_1})\frac{\tau_v}{\tau_1}\bar{v}. \tag{A.15}$$

$\square$

## Proof of Proposition 2

For any $\beta \in [0, 1]$, in the second period an equilibrium must satisfy $a_2 = \gamma\tau_\epsilon$. In the first

period, using (A.11), an equilibrium must satisfy

$$\phi_1(a_1, a_2) \equiv a_1 \lambda_2(\tau_2 + \tau_\epsilon) - \gamma \tau_\epsilon \Delta a_2 \tau_u$$

$$= a_1(1 + \gamma \tau_u \Delta a_2) - \gamma^2 \tau_\epsilon \Delta a_2 \tau_u = 0. \tag{A.16}$$

The above equation is a quadratic in $a_1$ which for any $a_2 > 0$ and $\beta > 0$ possesses two positive, real solutions:

$$a_1^* = \frac{1 + \gamma \tau_u a_2 (1 + \beta) - \sqrt{(1 + \gamma \tau_u a_2 (1 + \beta))^2 - 4\beta(\gamma \tau_u a_2)^2}}{2\beta \gamma \tau_u} \tag{A.17}$$

$$a_1^{**} = \frac{1 + \gamma \tau_u a_2 (1 + \beta) + \sqrt{(1 + \gamma \tau_u a_2 (1 + \beta))^2 - 4\beta(\gamma \tau_u a_2)^2}}{2\beta \gamma \tau_u}, \tag{A.18}$$

with $a_1^{**} > a_1^*$. This proves that for $\beta > 0$ there are two linear equilibria.

Inspection of the above expressions for $a_1$ shows that $\beta a_1^* < a_2$, while $\beta a_1^{**} > a_2$. This implies that $\beta \rho > 1$ for $a_1 = a_1^{**}$ and $\beta \rho < 1$ otherwise. Thus, using (A.10), we obtain $\alpha_{P_1}(a_1^*, a_2) < \alpha_{E_1}(a_1^*, a_2)$, and $\alpha_{P_1}(a_1^{**}, a_2) > \alpha_{E_1}(a_1^{**}, a_2)$. The result for second period illiquidity follows from substituting (A.17) and (A.18) in $\lambda_2$. To see that prices are more informative along the low illiquidity equilibrium note that in the first period $\mathrm{Var}[v|z_1]^{-1} = \tau_1 = \tau_v + a_1^2 \tau_u$. In the second period, the price along the low illiquidity equilibrium is more informative than along the high illiquidity equilibrium if and only if

$$\frac{(1 + \beta^2 + \gamma a_2 \tau_u((1 - \beta^2) + \beta(1 + \beta^2)))\sqrt{(1 + \gamma a_2 \tau_u(1 + \beta))^2 - 4\beta(\gamma a_2 \tau_u)^2}}{\gamma^2 \beta^2 \tau_u} > 0,$$

which is always true.

When $\beta \to 0$, along the high liquidity equilibrium we have

$$\lim_{\beta \to 0} \frac{1 + \gamma \tau_u(a_2 + \beta \gamma \tau_\epsilon) + \sqrt{1 + \gamma \tau_u(2(a_2 + \beta \gamma \tau_\epsilon) + \gamma \tau_u(a_2 - \beta \gamma \tau_\epsilon)^2)}}{2\beta \gamma \tau_u} = \infty,$$

while along the low liquidity equilibrium, using l'Hospital's rule,

$$\lim_{\beta \to 0} \frac{1 + \gamma \tau_u(a_2 + \beta \gamma \tau_\epsilon) - \sqrt{1 + \gamma \tau_u(2(a_2 + \beta \gamma \tau_\epsilon) + \gamma \tau_u(a_2 - \beta \gamma \tau_\epsilon)^2)}}{2\beta \gamma \tau_u} = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}.$$

From (A.10) it then follows that in this case $\alpha_{P_1} < \alpha_{E_1}$. Finally, defining

$$a_{10}^* = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u},$$

and taking the limit of $\lambda_2$ as $\beta \to 0$ when $a_1 = a_1^*$ yields

$$\lim_{\beta \to 0} \lambda_2(a_1^*, a_2) = \frac{1 + \gamma \tau_u a_2}{\gamma(\tau_v + (a_{10}^*)^2 \tau_u + a_2^2 \tau_u + \tau_\epsilon)} > 0,$$

whereas $\lim_{\beta \to 0} \lambda_2(a_1^{**}, a_2) = 0$. □

## Proof of Corollary 3

See the proof of Proposition 1.

□

## Proof of Corollary 5

Assume that in a linear equilibrium $x_n = -\phi_n(p_n)$, with $\phi_n(\cdot)$ a linear function of $p_n$. This implies that the market clearing equation at time $n$ reads as follows:

$$x_n + \theta_n = 0 \Leftrightarrow -\phi_n(p_n) + \theta_n = 0.$$

Hence, at equilibrium $p_n$ is observationally equivalent to $\theta_n$, i.e. at time $n$ investors *know* the realisation of the noise stock $\theta_n$. To solve for the equilibrium we proceed by backward induction and start from the second trading round, where due to CARA and normality, we have

$$X_2(p_2) = \gamma \frac{E_2[v] - p_2}{\text{Var}_2[v]} \tag{A.19}$$
$$= -\Lambda_2^{-1}(p_2 - \bar{v}),$$

with

$$\Lambda_2 \equiv \frac{1}{\gamma \tau_v}, \tag{A.20}$$

and

$$p_2 = \bar{v} + \Lambda_2 \theta_2. \tag{A.21}$$

In the first period, we have

$$X_1(p_1) = \gamma \frac{E_1[p_2] - p_1}{\text{Var}_1[p_2]} \tag{A.22}$$
$$= -\Lambda_1^{-1}(p_1 - \bar{v}),$$

with

$$\Lambda_1 \equiv \frac{1 + \gamma \beta \Lambda_2^{-1} \tau_u}{\gamma \Lambda_2^{-2} \tau_u}, \tag{A.23}$$

and

$$p_1 = \bar{v} + \Lambda_1 \theta_1. \tag{A.24}$$

□

## Proof of Corollary 6

Starting from the low liquidity equilibrium, we need to verify that $|\psi'(a_1^*)| < 1$, or that when $a_1 = a_1^*$,

$$\gamma\beta a_2 \tau_u < (1 + \gamma\tau_u\Delta a_2)^2.$$

Substituting (A.17) on R.H.S. of the above inequality and rearranging yields

$$|\psi'(a_1^*)| < 1 \Leftrightarrow -2(1+a_2\gamma\tau_u(1-\beta))(1+a_2\gamma\tau_u(1-\beta)+\sqrt{(1 + \gamma\tau_u a_2(1 + \beta))^2 - 4\beta(\gamma\tau_u a_2)^2}) < 0,$$

which is always satisfied. For the high liquidity equilibrium, we need instead to verify that $|\psi'(a_1^{**})| > 1$, or that when $a_1 = a_1^{**}$,

$$\gamma\beta a_2 \tau_u > (1 + \gamma\tau_u\Delta a_2)^2.$$

Substituting (A.18) on R.H.S. of the above inequality and rearranging yields

$$|\psi'(a_1^{**})| > 1 \Leftrightarrow 2(1+a_2\gamma\tau_u(1-\beta))(-(1+a_2\gamma\tau_u(1-\beta))+\sqrt{(1 + \gamma\tau_u a_2(1 + \beta))^2 - 4\beta(\gamma\tau_u a_2)^2}) > 0,$$

which is always satisfied, since the first factor in the product on the R.H.S. of the above expression is positive, while manipulating the second factor shows that

$$\sqrt{(1 + \gamma\tau_u a_2(1 + \beta))^2 - 4\beta(\gamma\tau_u a_2)^2} > (1 + a_2\gamma\tau_u(1 - \beta)) \Leftrightarrow 4a_2\beta\gamma\tau_u > 0.$$

$\square$

## Proof of Proposition 3

With residual uncertainty, in the second period investors trade according to

$$X_2(s_i, z^2) = \gamma\frac{E_{i2}[v] - p_2}{\text{Var}_{i2}[v + \delta]}, \tag{A.25}$$

which implies that at equilibrium

$$a_2 = \frac{\gamma\tau_\epsilon}{1 + \kappa}, \tag{A.26}$$

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2\Delta a_2)\hat{p}_1,$$

where

$$\lambda_2 = \frac{1 + \kappa + \gamma\tau_u\Delta a_2}{\gamma\tau_{i2}} \tag{A.27}$$

$$\hat{p}_1 = \frac{\gamma\tau_1 E_1[v] + \beta(1 + \kappa)z_1}{\gamma\tau_1 + \beta a_1(1 + \kappa)}, \tag{A.28}$$

38

with $\kappa \equiv \tau_{i2}/\tau_\delta$. In the first period, we then have

$$X_1(s_i, z_1) = \gamma \frac{E_{i1}[p_2] - p_1}{\text{Var}_{i1}[p_2]},$$

where $E_{i1}[p_2] = \lambda_2 \Delta a_2 E_{i1}[v] + (1 - \lambda_2 \Delta a_2)\hat{p}_1$, and $\text{Var}_{i1}[p_2] = \lambda_2^2 \text{Var}_{i1}[z_2]$. Identifying the first period signal responsiveness yields

$$a_1 = \gamma \frac{\Delta a_2^2 \tau_u (1 + \kappa)}{1 + \kappa + \gamma \Delta a_2 \tau_u (1 - \beta(1 + \kappa))}. \tag{A.29}$$

The equilibrium obtains as a solution to the system (A.26)-(A.29) which corresponds to (24)-(25). $\qquad\square$

## PROOF OF COROLLARY 7

Using (6) the covariance between $p_1$ and $v$ is given by

$$\text{Cov}[v, p_1] = \alpha_{P_1} \frac{1}{\tau_v} + (1 - \alpha_{P_1}) \left( \frac{1}{\tau_v} - \frac{1}{\tau_1} \right), \tag{A.30}$$

and carrying out a similar computation for the first period consensus opinion

$$\text{Cov}\left[\bar{E}_1[v], v\right] = \alpha_{E_1} \frac{1}{\tau_v} + (1 - \alpha_{E_1}) \left( \frac{1}{\tau_v} - \frac{1}{\tau_1} \right). \tag{A.31}$$

We can now subtract (A.31) from (A.30) and obtain

$$\text{Cov}\left[p_1 - \bar{E}_1[v], v\right] = \frac{\alpha_{P_1} - \alpha_{E_1}}{\tau_1}, \tag{A.32}$$

implying that the price at time 1 over relies on public information (compared to the optimal statistical weight) if and only if the covariance between the price and the fundamentals falls short of that between the consensus opinion and the fundamentals. $\qquad\square$

## PROOF OF COROLLARY 33

To compute $\text{Cov}[p_2 - p_1, p_1 - \bar{v}]$ we rearrange expression (6) in the paper to obtain

$$\begin{aligned} p_2 - p_1 &= \lambda_2 u_2 + \lambda_2 \Delta a_2 \left( (1 - \alpha_{E_1})(v - E_1[v]) - \frac{\alpha_{E_1}}{a_1} \theta_1 \right) \\ &= \lambda_2 u_2 + \lambda_2 \Delta a_2 \left( \frac{\tau_v}{\tau_{i1}}(v - \bar{v}) - \frac{\tau_{i1} - \tau_v}{a_1 \tau_{i1}} u_1 \right). \end{aligned} \tag{A.33}$$

Next, using expression (8) we obtain

$$\text{Cov}[p_2 - p_1, p_1 - \bar{v}] = -\frac{\lambda_1 \lambda_2 \Delta a_2 \tau_\epsilon}{a_1 \tau_u \tau_{i1}} < 0, \forall \beta \in (0, 1].$$

Computing the limit for $\beta \to 0$ of the above expression yields different results depending on whether we concentrate on the high or low liquidity equilibrium. Indeed,

$$\lim_{\beta \to 0} \text{Cov}[p_2 - p_1, p_1 - \bar{v}]|_{a_1 = a_1^*} = -\lambda_1 \frac{(1 + \gamma \tau_u a_2)^2 \tau_\epsilon}{\gamma a_2 \tau_u^2 \tau_{i1}} < 0$$

$$\lim_{\beta \to 0} \text{Cov}[p_2 - p_1, p_1 - \bar{v}]|_{a_1 = a_1^{**}} = 0.$$

To compute the expression for $\text{Cov}[v - p_2, p_1 - \bar{v}]$ we use (A.34) and (8), and obtain

$$\text{Cov}[v - p_2, p_1 - \bar{v}] = -\frac{\beta \lambda_1}{\gamma \tau_{i2} \tau_u} < 0 \text{ for all } \beta > 0.$$

In this case the taking the limit of the above covariance for $\beta \to 0$ yields the same result across the two equilibria that arise:

$$\lim_{\beta \to 0} \text{Cov}[v - p_2, p_1 - \bar{v}]|_{a_1 = a_1^*} = \lim_{\beta \to 0} \text{Cov}[v - p_2, p_1 - \bar{v}]|_{a_1 = a_1^{**}} = 0.$$

To compute $\text{Cov}[v - p_2, p_2 - p_1]$ we use (A.33) and using again expression (8) we get

$$
\begin{aligned}
v - p_2 &= (1 - \alpha_{E_2})(v - E_2[v]) - \frac{\alpha_{P_2}}{a_2} \theta_2 \\
&= \frac{\tau_v(v - \bar{v})}{\tau_{i2}} - \frac{\beta + \gamma \tau_u a_1}{\gamma \tau_{i2}} u_1 - \frac{1 + \gamma \tau_u \Delta a_2}{\gamma \tau_{i2}} u_2.
\end{aligned}
\tag{A.34}
$$

Using (A.33) and (A.34) we can now compute the autocovariance of returns and get

$$
\begin{aligned}
\text{Cov}[v - p_2, p_2 - p_1] &= \lambda_2 \left( \frac{\Delta a_2 \tau_v}{\tau_{i1} \tau_{i2}} - \frac{1 + \gamma \tau_u \Delta a_2}{\gamma \tau_{i2} \tau_u} + \Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1 \tau_{i1}} \frac{\beta + \gamma \tau_u a_1}{\gamma \tau_{i2} \tau_u} \right) \\
&= -\frac{\lambda_2}{\gamma \tau_{i2} \tau_u} \left( 1 - \beta \Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1 \tau_{i1}} \right).
\end{aligned}
\tag{A.35}
$$

Looking at (A.35) we can immediately say that along the equilibrium with high liquidity there is momentum. This is true because in that equilibrium $\lambda_2 < 0$ and $\Delta a_2 < 0$. Along the equilibrium with low liquidity momentum can occur, *depending on the persistence of liquidity trades*. To see this, note that since in this equilibrium $\lambda_2 > 0$ and $\Delta a_2 > 0$, from (A.35) momentum needs

$$1 - \beta \Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1 \tau_{i1}} < 0,$$

which can be rearranged as an (implicit) condition on the magnitude of $\beta$:

$$\frac{a_1 \tau_{i1}}{\Delta a_2 (\tau_{i1} - \tau_v)} < \beta < 1.$$

If $\beta = 0$, the above condition is never satisfied. Indeed, in this case there exists a unique equilibrium in which $\Delta a_2 = a_2 > 0$. Therefore, when $\beta = 0$ returns always display reversal. If

$\beta = 1$, the condition is satisfied if

$$a_1 \tau_v + a_1(\tau_\epsilon + a_1^2 \tau_u) < \Delta a_2 \tau_u (\tau_\epsilon + a_1^2 \tau_u).$$

Isolating $\tau_v$ in the above expression yields:

$$\tau_v < \hat{\tau}_v \equiv \frac{(\Delta a_2 - a_1)(\tau_\epsilon + a_1^2 \tau_u)}{a_1}, \tag{A.36}$$

which, since $a_1$ does not depend on $\tau_v$ (see (A.17)), gives an explicit upper bound on $\tau_v$. Hence, if $\tau_v < \hat{\tau}_v$, there exists a $\hat{\beta}$ such that for all $\beta \geq \hat{\beta}$ momentum occurs between the second and third period returns along the equilibrium with high illiquidity. $\qquad \square$

# B    Appendix

In this appendix we decompose the best response function to analyse the effects that impinge on a first period investor's responsiveness to private information. We start by using (8) to compute an investor $i$'s expectation of $p_2$:

$$
\begin{aligned}
E_{i1}[p_2] &= E_{i1}\left[E_2[v] + \Lambda_2 E_2[\theta_2]\right] && \text{(B.1)}\\
&= E_{i1}[E_2[v]] + \Lambda_2 E_{i1}[E_2[\theta_2]]\\
&= E_{i1}\left[\frac{\Delta a_2 \tau_u}{\tau_2}z_2 + \frac{\tau_1}{\tau_2}E_1[v]\right] + \Lambda_2 E_{i1}[a_2(v - E_2[v]) + \theta_2]\\
&= \frac{(\Delta a_2)^2 \tau_u}{\tau_2}E_{i1}[v] + \frac{\tau_1}{\tau_2}E_1[v] + \Lambda_2 a_2(E_{i1}[v] - E_{i1}[E_2[v]]) + \beta\Lambda_2 E_{i1}[\theta_1]\\
&= \frac{(\Delta a_2)^2 \tau_u}{\tau_2}E_{i1}[v] + \frac{\tau_1}{\tau_2}E_1[v] + \Lambda_2 a_2\left(E_{i1}[v] - \frac{(\Delta a_2)^2 \tau_u}{\tau_2}E_{i1}[v] - \frac{\tau_1}{\tau_2}E_1[v]\right)\\
&\quad + \beta\Lambda_2(a_1(v - E_{i1}[v]) + \theta_1).
\end{aligned}
$$

If we collect the terms that multiply $E_{i1}[v]$ in the last row we have

$$
\left(\frac{(\Delta a_2)^2 \tau_u}{\tau_2} + \Lambda_2 a_2\frac{\tau_1}{\tau_2} - \beta\Lambda_2 a_1\right),
$$

which implies that the weight an investor places on private information in the first period is given by

$$
\underbrace{\frac{\gamma\alpha_{E_1}}{\mathrm{Var}_{i1}[p_2]} \times \frac{(\Delta a_2)^2 \tau_u}{\tau_2}}_{1} + \underbrace{\frac{\gamma\alpha_{E_1}}{\mathrm{Var}_{i1}[p_2]} \times \Lambda_2 a_2\frac{\tau_1}{\tau_2}}_{2} + \underbrace{\left(-\frac{\gamma\alpha_{E_1}}{\mathrm{Var}_{i1}[p_2]} \times \beta\Lambda_2 a_1\right)}_{3},
$$

where $\alpha_{E_1} = \tau_\epsilon/\tau_{i1}$ denotes the optimal statistical weight to private information in the first period. The above expression shows that an investor's response to private information can be decomposed in three terms:

- Term 1 captures the response to private signals that reflects the anticipated impact of the fundamentals information arriving at time 2 (i.e., how $z_2$ affects $E_2[v]$).

- Term 2 captures the response to private signals that reflects the anticipated impact that the innovation in liquidity trading has on the second period price (i.e., how $u_2$ affects $E_2[\theta_2]$; note that while at date 1 an investor cannot predict $u_2$, he can predict how the market in period 2 will react to $u_2$, since this is recorded by $E_2[\theta_2] = a_2(v - E_2[v]) + \theta_2$).

- Term 3 captures the response to private signals that reflects the anticipated impact of first period liquidity trading on the second period price (i.e., how $\theta_1$ affects $E_2[\theta_2]$). This reflects investors' private learning channel from the first period price.

Now, the three terms above behave differently as $a_1$ increases depending on the value of $\beta$. For $\beta = 0$, term 3 disappears while term 1 decreases in $a_1$, reflecting the fact that the more

aggressively investors respond to private information in the first period, the more the second and first period prices reflect the fundamentals. This, in turn, leads investors to lower their reliance on private information, consistently with the standard, strategic substitutability effect between private and public information of the Grossman-Stiglitz setup. Term 2, instead, increases in $a_1$, reflecting the fact that the more aggressively investors respond to private information, the closer is the second period price to the fundamentals, and the lower is the uncertainty faced by second period investors and the compensation they require for accommodating the demand of liquidity traders in the second period. This latter effect runs counter to the one coming from term 1 but is not strong enough to offset it. As a result, $\varphi'(a_1^*) < 0$, $a_1^* < \gamma\tau_\epsilon$ in this case (see Figure 6, parameters' values are the same of Figure 1, except for $\beta$ which in this case is null).
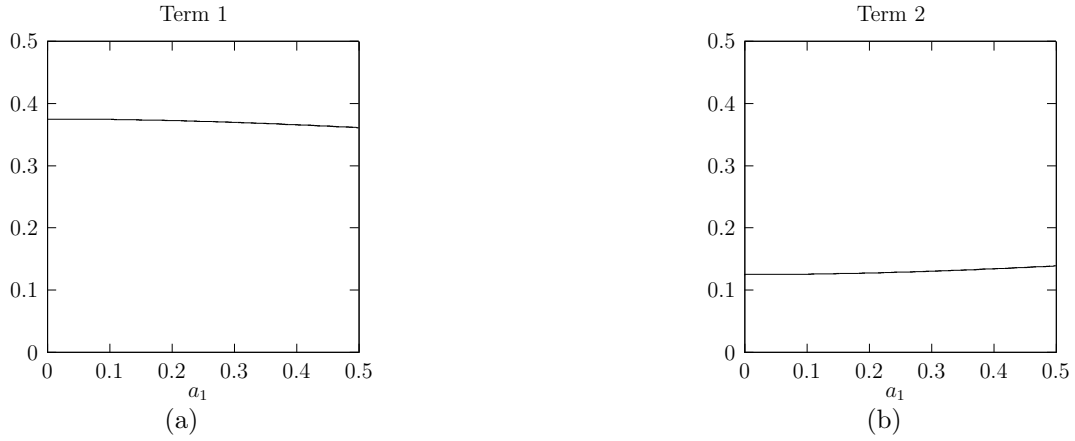


Figure 6: Case $\beta = 0$.

When $\beta > 0$ the private learning channel from $p_1$ affects the response to private information in period 1. This, in turn, implies that positive selection can occur in the second period *and* that $\text{Var}_{i1}[p_2]$ can be null.

The implication of this latter effect is to create two regions in the space of solutions to the equation $a_1 = \varphi(a_1)$: $[0, \hat{a}_1)$ and $(\hat{a}_1, +\infty)$, where $\hat{a}_1$ is such that $\text{Var}_{i1}[p_2]$ is null for $a_1 = \hat{a}_1$. Points to the left of $\hat{a}_1$ yield the solution $a_1^*$ and points to the right of $\hat{a}_1$ yield the solution $a_1^{**}$. We display the behavior of the three terms in Figure 7 (parameters' values are the same of Figure 1). For $a_1 < \hat{a}_1$ we have that term 1 is non monotone (first decreasing, then increasing) in $a_1$, term 2 is increasing in $a_1$, and term 3 is negative and decreasing in $a_1$. For $a_1 = \hat{a}_1$ all three terms diverge (the first and second to $+\infty$, the third to $-\infty$). For $a_1 > \hat{a}_1$ terms 1 and 2 are decreasing in $a_1$ while term 3 is increasing in $a_1$ (that is it grows towards 0).
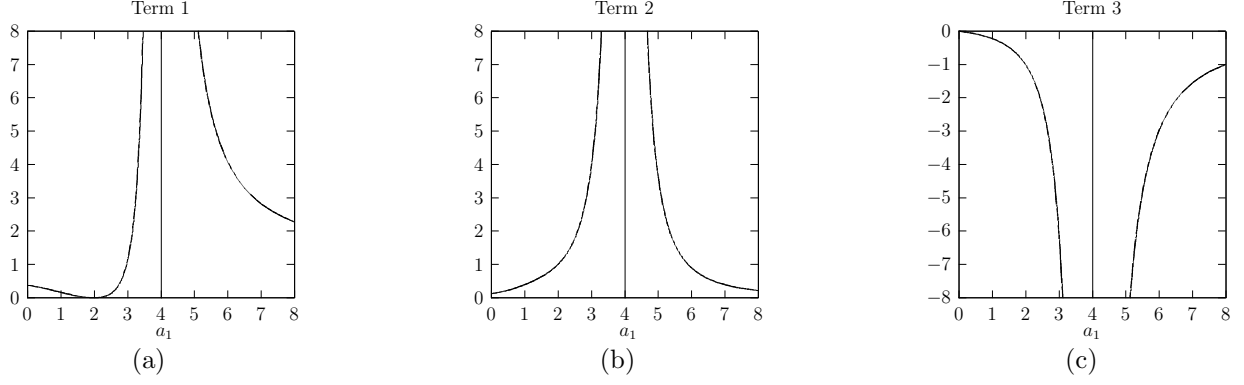
Figure 7: Case $\beta = 0.5$.

# C    Appendix

In this appendix we show that if $u_n \sim N(\bar{u}, \tau_u^{-1})$, with $\bar{u} \neq 0$, then the expected price in the first period can be written as in (27). To see this, suppose $\bar{u} \neq 0$, then, it is easy to that that $\{z_1, z_2\}$ is observationally equivalent to $\{p_1, p_2\}$, while

$$a_1^{-1}(z_1 - \bar{u}) \equiv v + a_1^{-1}(u_1 - \bar{u})|v \sim N(v, a_1^{-2}\tau_u^{-1})$$
$$(\Delta a_2)^{-1}(z_2 - \bar{u}) \equiv v + (\Delta a_2)^{-1}(u_2 - \bar{u})|v \sim N(v, (\Delta a_2)^{-2}\tau_u^{-1}).$$

Therefore, nothing changes for the precisions in the projection expressions while

$$E_{in}[v] = \frac{\tau_v \bar{v} + \overbrace{\tau_u \sum_{t=1}^{n} \Delta a_t (z_t - \bar{u})}^{\tau_2 E_2[v]} + \tau_\epsilon s_{in}}{\tau_{in}}$$
$$= \frac{\tau_2 E_2[v] + \tau_\epsilon s_{in}}{\tau_{in}}.$$

As a result, everything works as in the model with $\bar{u} = 0$, except that now when $\bar{u} < 0$ first period investors anticipate absorbing a positive supply of the asset at equilibrium and thus require a compensation on the price they pay which lowers the expected price below the unconditional expectation of the payoff the more the higher is $\Lambda_1$:

$$E[p_1] = \bar{v} + \Lambda_1 \bar{u} < \bar{v}.$$

$\square$