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ABSTRACT

We study the existence of pure strategy Markov perfect equilibria in two-person perfect information games. There is a state space *X* and each period player's possible actions are a subset of *X*. This set of feasible actions depends on the current state, which is determined by the choice of the other player in the previous period. We assume that *X* is a compact Hausdorff space and that the action correspondence has an acyclic and asymmetric graph. For some states there may be no feasible actions and then the game ends. Payoffs are either discounted sums of utilities of the states visited, or the utility of the state where the game ends. We give sufficient conditions for the existence of equilibrium *e.g.* in case when either feasible action sets are finite or when players' payoffs are continuously dependent on each other. The latter class of games includes zero-sum games and pure coordination games.

JEL Classification: C72, C73

Keywords: dynamic games, Markov perfect equilibrium

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# 1 Introduction

We study the existence of pure strategy Markov perfect equilibria in two-person perfect information games. There is a state space $X$ and each period player's possible actions are a subset of $X$. This set of feasible actions depends on the current state, which is determined by the choice of the other player in the previous period. We assume that $X$ is a compact Hausdorff space and that action correspondence has an acyclic and asymmetric graph. For some states there may be no feasible actions and then the game ends. Payoffs are either discounted sums of utilities of the states visited, or the utility of the state where the game ends. We give sufficient conditions for the existence of equilibrium when either feasible action sets are finite or when players' payoffs are continuously dependent on each other. The latter class of games includes zero-sum games and pure coordination games.

Given an initial state $x_0 \in X$, player $i_0$ starts the game by choosing some action $x_1$ from the set $A(x_0)$ of feasible actions. After that his opponent chooses an action from $A(x_1)$, and so on. Hence given an initial state $x_0$ and a first mover $i_0$, we have a perfect information extensive form game. A (pure) *Markov strategy* of player $i$ selects one feasible action to each state (whenever there are feasible actions). In a *Markov perfect equilibrium* ($MPE$), player's Markov strategy is a best reply against the Markov strategy of the opponent.

We find Markov equilibrium attractive as a solution concept. It is simple and usually easy to interpret. Here we discuss the existence of such equilibria. Of course, one would like to get a deeper understanding if, and when, restriction to Markov strategies makes sense. We will not deal with such foundational issues here, but see *e.g.* Bhaskar *et.al* (2010) and Doraszelski and Escobar (2009).

Well-known papers dealing with the existence of a pure strategy *subgame perfect equilibrium* ($SPE$) in perfect information games include Harris (1985a,b), Hellwig and Leininger (1987), Hellwig *et.al* (1990). Harris (1985a) is a representative paper. In his paper terminal histories are infinitely long. The main assumptions for the existence of a pure $SPE$ are:

1. the set of terminal histories is compact,

2. payoffs over terminal histories are continuous.

So discounting is a special case but other payoff structures such as limit of means (of time averages) or quitting games are not dealt with. Markov equilibria are also not studied.

Fink (1964) shows the existence of a *mixed strategy* $MPE$ in a finite action, finite states case with discounting. Solan and Vieille (2003) show the existence of an $\varepsilon$ -$SPE$ in mixed strategies for "quitting games". In such games the active player $i_n$ at stage $n \in \mathbb{N}$ has two options: to stop the game in which case payoffs are realized, or to let the game to continue to stage $n+1$. Kuipers *et.al.* (2009) study a version of this game in which the active player can either quit or give the move to any other player (in effect there are $n$ states). They show that there exists a pure strategy $SPE$ although a Markov perfect equilibrium need not exist.

By analyzing the examples where a pure $MPE$ does not exist, we can often find a technical reason that explains such an anomaly. Then we can make assumptions to get around these problematic cases and find conditions that are sufficient for the existence of a pure $MPE$.

Besides acyclicity and irreflexivity of the action correspondence, another important assumption is that to each uncountable subset $Y \subset X$ there exists a state in $Y$ such that the next state cannot be in $Y$ (Assumption 2). That is, either there exists a state (a *terminal state*) in $Y$ where there are no actions available, or there is a state in $Y$ such that the next state is necessarily outside of $Y$. Actually this property can be seen as a generalization of acyclicity and irreflexivity to uncountable subsets. Namely, if we we formulate Assumption 2 for finite subsets, then this property boils down to irreflexivity and acyclicity.

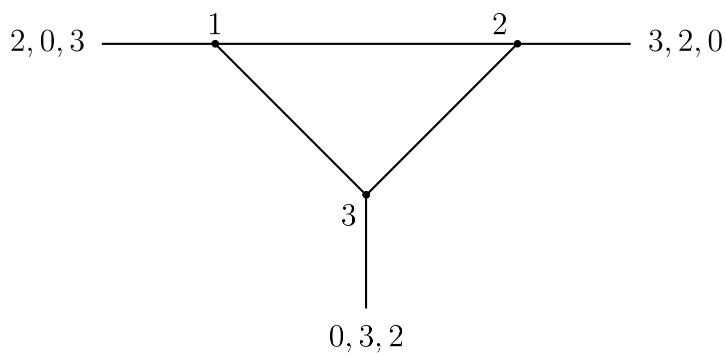
We show that if the set of feasible actions is finite, and the *closure* of the action correspondence satisfies Assumptions 1 and 2, then there exists a Markov perfect equilibrium (Theorem 1). Utility functions over states can be arbitrary.

When the feasible action sets may be infinite, we assume that the action correspondence and utilities over states are continuous. If players utilities are *continuously dependent* and Assumption 2 holds, then there exists a Markov perfect equilibrium, given that a relatively weak technical assumption (Assumption 3, p. 9) is satisfied (Theorem 2; Theorem 3). Players' utilities are continuously dependent for example in zero-sum games and in pure coordination games.

In Section 2 examples of games with no pure $MPE$ are studied. The model and notation is introduced in Section 3. The results are given in Section 4. In Section 5 the assumptions of the model are discussed.

# 2 Examples with no pure MPE

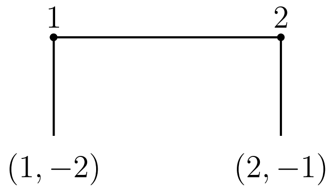
*EXAMPLE 1.* [Adapted from Flesch *et.al.* (1997); Solan-Vieille (2003); Kuipers *et.al.* (2009).]




The state space is $X = \{1, 2, 3\}$, player $i \in \{1, 2, 3\}$ has the move at state $i$ and can either quit or give the move to player $i+1$ (where $3+1 = 1$). Utilities from states $i$ are zero, no discounting. There is no pure $MPE$. Staying in the cycle cannot be an $MPE$. If $i$ should quit, then $i-1$ would not quit, in which case $i-2$ would certainly quit. But then $i$ would not quit. A pure $SPE$ exists: if $i$ starts the game, $i$ should quit. If after some history $j$ should quit but deviates, then as a punishment $j+1$ must not quit and $j+2$ must quit.

*EXAMPLE 2.* [Solan-Vieille (2003).] The state space is $X = \{1, 2\}$, player $i \in \{1, 2\}$ has the move at state $i$ and can either quit or give the move to player $i+1$. Utilities from states $i = 1, 2$ are zero, discounting $1/2 < \delta \leq 1$.

No pure $MPE$. If 1 should quit, then 2 would quit. But then 1 would not quit, and 2 would not quit. But then 1 would quit. No mixed $MPE$ if




$\delta = 1$. A pure $SPE$ when $1/2 < \delta < 1$. (Solan-Vieille have $\mathbb{N}$ as a state space (how many periods the game has lasted), so their Markov strategy is not Markov in our model.)

When action sets are infinite, there need not exist an optimal policy even if utility function and action correspondence are continuous and time horizon is finite.

*EXAMPLE 3.* A one-person game, $X = [-1, 1]$, the action correspondence is $A(x) = [x + 1, 1]$ for $x \leq 0$; $A(x) = \emptyset$ for $x > 0$. Utility from state $x$ is $u(x) = -x^2$. Either discounted sum of utilities from states, or utility only from the terminal states ($x > 0$). No optimal actions. Assume discounting, $0 < \delta \leq 1$, and initial state $-1$. Then by choosing $x = 0$ player gets $-1 + 0 - \delta^2 = -1 - \delta^2$. By choosing $x > 0$ player gets $-1 - \delta x^2$, which increases to $-1$ as $x$ goes to zero. Hence no optimal strategy. The same holds for the payoff structure such that non-zero payoffs are available at the terminal states only.

# 3 The Model

We study the following kind two-person games on a nonempty set $X$. An *initial state* $x_0 \in X$ of the game is given, and player $i_0 \in \{1, 2\}$ is called to make a choice $x_1$ from a set of actions $A(x_0) \subset X$ (this assumption is not restrictive as demonstrated in Section 5). If $A(x_0)$ is empty, then the game is over. If $A(x_0) \neq \emptyset$, the choice $x_1$ is the state of the game in period 2, and then player $i_1 \neq i_0$ makes a choice from a set $A(x_1) \subset X$, if $A(x_1) \neq \emptyset$. If the state of the game is $x_t$ after $t$ stages, player $i_t \in \{1, 2\}$ makes a choice from a set $A(x_t) \subset X$, if $A(x_t) \neq \emptyset$, and otherwise the game is over. If $t$ is odd, then $i_t = i_1$, and if $t$ is even then $i_t = i_0$. A state $x \in X$ is a *terminal state*, if $A(x) = \emptyset$.

We assume that the set $X$ is a compact Hausdorff space. We may view the action sets $A(x)$ as images of a relation $A \subset X \times X$: $A(x) = \{y \mid (x, y) \in A\}$. The relation $A$ is *asymmetric*, if for all $x \in X$, $x \notin A(x)$. The relation $A$ is *acyclic* if for all paths $(x_0, \ldots, x_t)$ such that $x_{n+1} \in A(x_n)$, $n < t$, it holds that $x_0 \notin A(x_t)$.

Recall that a relation $A$ is closed if $A \subset X \times X$ is closed, when $X \times X$ has the product topology. The relation $A$ may also be viewed as a correspondence $x \twoheadrightarrow A(x)$. The correspondence (or relation) $A$ has closed values, if $A(x) \subset X$ is closed for every $x \in X$. Closed correspondences have closed values. Since $X$ is compact Hausdorff, $A$ is closed iff $A$ is an upper semicontinuous correspondence with closed values. The correspondence $A$ is *continuous*, if it is both upper semicontinuous and lower semicontinuous.

The game has perfect information: each stage $t$ the player $i_t$ observes the *history* $h^t = (x_0, \ldots, x_{t-1})$. Denote by $H^t$ the set of all histories of length $t$, and let $H = \cup_t H^t$ be the set of all histories. We consider feasible histories only: $h^t = (x_0, \ldots, x_{t-1})$ is such that $x_k \in A(x_{k-1})$, for all $k = 1, \ldots, t-1$. We may denote the feasible set of actions after history $h^t = (x_0, \ldots, x_{t-1})$ by $A(h^t)$ or by $A(x_{t-1})$.

A *strategy* of player $i \in \{1, 2\}$ is a function $s_i : H \longrightarrow X$ such that $s_i(h^t) \in A(h^t_{t-1})$. A *Markov strategy* $s_i$ is such that $s_i(h^t)$ depends only on the state $h^t_{t-1}$ of the game in period $t$. That is, a Markov strategy is a function $s_i$ on $X$ such that $s_i(x) \in A(x)$ if $A(x)$ is nonempty. (One may wonder if the perfect information assumption is in contradiction with the Markov property since action for both players is defined on states where actions are available. It is demonstrated in Section 5 that this is not the case.)

Given a *strategy profile* $s = (s_1, s_2)$, let $h(s)$ be the *path* or *play* generated by it, *i.e.*, either $h(s) = h^t = (x_0, \ldots, x_{t-1})$ for some $t$, or else $h(s) = \{x_t\}_{t=0}^{\infty}$ is an infinite sequence of elements $x_t \in X$. In the former case, let $T(s) = t-1$, so $T(s)$ is the time index of the terminal state. In the latter case $A(x_t) \neq \emptyset$ for all $t$, and then we define $T(s) = \infty$. If $T(s) < \infty$, the last action taken is $h(s)_{T(s)}$ and this is also the terminal state $x_{t-1}$ of the game.

Let $u_i : X \longrightarrow \mathbb{R}$ be a utility function of player $i \in \{1, 2\}$. We study the game with two different specifications of payoffs over strategies. In the first specification, the payoff of $i \in \{1, 2\}$ is the discounted sum of his future payoffs:

$$U_i(s) = \sum_{t=0}^{T(s)} \delta^t u_i(h_t(s)), \tag{1}$$

where $\delta$ is the discount factor, $0 < \delta < 1$.

In the second specification, the payoff of $i \in \{1, 2\}$ is

$$U_i(s) = \begin{cases} u_i(h_{T(s)}(s)) & \text{if } T(s) < \infty \\ 0 & \text{if } T(s) = \infty \end{cases} \tag{2}$$

So in this case players get zero if $s$ generates an infinite history, and otherwise they get the payoff of the terminal state $h_{T(s)}(s) = x_{T(s)}$. In (1), if the game ends in finite time, the path that leads to a terminal state also affects payoffs.

We denote by $\Gamma(x_0, i_0) = (X, A, x_0, i_0, u_1, u_2)$, $x_0 \in X, i_0 \in \{1, 2\}$, any game such that a) the initial state is $x_0$; b) player $i_0$ makes the first move; and c) payoffs over strategies are given either by equation (1) or by equation (2). The assumption that $X$ is nonempty compact Hausdorff is maintained throughout the paper. We denote by $\Gamma$ the set of all such games when $x_0 \in X$ and $i_0 \in \{1, 2\}$: $\Gamma = \{\Gamma(x_0, i_0) \mid x_0 \in X, i_0 \in \{1, 2\}\}$. The sets $X, A$ and functions $u_i$ are the same for all games in $\Gamma$.

A strategy profile $\bar{s} = (\bar{s}_1, \bar{s}_2)$ is a *subgame perfect equilibrium* for the set $\Gamma$ of games, if for any initial state $x_0 \in X$, $\bar{s}_1$ maximizes $u_1(s_1, \bar{s}_2)$ and $\bar{s}_2$ maximizes $u_2(\bar{s}_1, s_2)$, no matter which player starts the game. A subgame perfect equilibrium $\bar{s}$ is called *Markov perfect* if the strategies $\bar{s}_i$ are Markovian, $i = 1, 2$.

# 4 Results

We make the following assumptions.

**Assumption 1** *The graph of $A$ is acyclic and irreflexive.*

We saw in Examples 1 and 2 that cycles may cause the nonexistence of an $MPE$. Livshits (2002) has an example with three players and finitely many states such that the action correspondence is acyclic but *not* irreflexive and there are no pure $MPE$.

Our first result deals with a special case when a) payoffs are calculated as in equation (1), and (ii) the state space is a compact *metric* space.

**Proposition 1** *Suppose $X$ is compact metric, the set $\Gamma$ of games $\Gamma(x_0, i_0)$ satisfies Assumption 1, the functions $u_i$ are continuous, and that $A(x)$ is finite for each $x \in X$. Then there exists a Markov perfect equilibrium $s$, if payoffs are calculated as in equation* (1)*.*

**Proof.** See Appendix. ■

*REMARK 1.* Note that closedness of the action correspondence $A$ was not needed.

**Assumption 2** *Every uncountable closed $Y \subset X$ contains an element $y$ such that $Y \cap A(y) = \emptyset$.*

Note that $Y \cap A(y) = \emptyset$ is satisfied in particular when $A(y) = \emptyset$. So if $X$ is uncountable, Assumption 2 implies that some action sets $A(x)$ are empty.

**Lemma 1** *Suppose $A$ is a closed relation on a compact Hausdorff space $X$ satisfying Assumptions 1 and 2. Then there is $K > 0$ such that all histories $h^t$ have lenght $t \leq K$.*

**Proof.** If there is a nonempty closed $Y \subset X$ such that $A(y) \cap Y \neq \emptyset$ for all $y \in Y$, then there is a nonempty perfect $Z \subset Y$ such that $A(z) \cap Z \neq \emptyset$. This follows since $A$ is a closed asymmetric and acyclic relation (Salonen and Vartiainen 2010, Lemma 2). Since perfect subsets are uncountable, it follows that every (uncountable or countable) closed $Y \subset X$ contains $y \in Y$ such that $A(y) \cap Y = \emptyset$.

Define $A^{-1}[Z] = \{x \in X \mid z \in A(x) \text{ for some } z \in Z\}$, for all nonempty $Z \subset X$. Let $X_0 = X$, and $X_{n+1} = A^{-1}[X_n]$ for $n = \{0, 1, \ldots\}$. Since $A$ is a closed relation, each $A^{-1}[X_n]$ is closed. We show that for some $n > 0$, $X_n$ is empty.

If $X_n \neq \emptyset$ for all $n$, then $Y = \cap_n A^{-1}[X_n]$ is a nonempty closed subset, since $X$ is compact Hausdorff and $X_{n+1} \subset X_n$. Hence there exists $y \in Y$ such that $A(y) \cap Y = \emptyset$. Since $A$ is a closed relation, $A(y)$ is closed. Since $y \in A^{-1}[X_n]$ for every $n$, it follows that $A(y) \cap X_n \neq \emptyset$. But then $A(y) \cap Y =$

$\cap_n(A(y) \cap A^{-1}[X_n]) \neq \emptyset$, a contradiction. Hence there exists a least integer $K$ such that $X_K \neq \emptyset$ and $X_n = \emptyset$ for all $n > K$.

Let $A_n = X_n \setminus X_{n+1}$ for $n < K$, and $A_K = X_K$. Then each $A_n$ is nonempty, and $A(x) \cap X_n = \emptyset$ for each $x \in A_n$. So for example, $A_0$ contains all states $x$ such that $A(x) = \emptyset$, that is, $A_0$ is the set of all end states of $\Gamma$. The set $A_1$ contains all states $x$ such that $A(x) \neq \emptyset$ and $A(x) \subset A_0$. So $A_1$ contains all states such that there is exactly one move left before the game ends. By the same reasoning, $A_k$ contains all states $x$ such that $A(x) \neq \emptyset$ and $k$ is the maximum number of moves that are needed to end the game, $k \leq K$. Note that *less* than $k$ moves may suffice to end the game when $k > 1$, but the longest path to the end state has $k$ moves. ■

By using Lemma 1 we can prove the existence of a Markov perfect equilibrium in games with finite action sets $A(x)$.

**Theorem 1** *Suppose that the games* $\Gamma(x_0, i_1) = (X, A, x_0, i_0, u_1, u_2)$ *in the set* $\Gamma$ *have finite action sets* $A(x), x \in X$. *If the closure* $clA$ *of the relation* $A$ *satisfies Assumptions 1 and 2, then there exists a Markov perfect equilibrium* $s$, *if payoffs are computed either by equation* (1) *or by equation* (2).

**Proof.** If the action sets were $clA(x)$ instead of $A(x)$ in the games in $\Gamma$, then the lengths of all histories would have a common upper bound by Lemma 1. Since $A(x) \subset clA(x)$, all histories $h^t$ in games in $\Gamma$ must satisfy $t \leq K$, for some $K > 0$. We may assume that some history has lenght $K$.

Like in the proof of Lemma 1, $X$ is partitioned into nonempty sets $A_0, \ldots, A_K$ such that (1) $A(x) = \emptyset$ iff $x \in A_0$, and (2) $A(x) \cap A_t = \emptyset$ for all $t \geq k$ if $x \in A_k$. Given any $x \in A_k$, it takes at most $k$ steps to reach a terminal state $x \in A_0$, and there is some $x \in A_k$ and some choices such that it takes $k$ steps to reach a terminal state $x \in A_0$.

We can solve a Markov perfect equilibrium by applying backwards induction.

*Step 1.* Given $x \in A_1$, solve to each player $i \in \{1, 2\}$ a utility maximizing choice $s_i(x) \in A(x)$. Since $A(x)$ is nonempty and finite, these maximizers exist. After that choice has been made, the game is over.

*Step n.* Suppose that a Markov perfect equilibrium in continuous strategies $s = (s_1, s_2)$ has been solved for initial states in $x \in A_1 \cup \cdots \cup A_{n-1}$, $n > 1$, no matter who makes the first move.

Given $x \in A_n$, solve to each player $i \in \{1, 2\}$ a utility maximizing initial choice $s_i(x) \in A(x)$, given that equilibrium strategies are followed in the future. Since $A(x)$ is nonempty and finite, these maximizers exist.

Continue backwards until $s_i(x)$ is solved for each $x \in A_k, 0 < k \leq K$. By construction, the profile $s = (s_1, s_2)$ is a Markov perfect equilibrium. ■

*REMARK 2.* Note that Theorem 1 would hold if payoffs over strategies were given by functions $U_i(s) = V_i(y_0, \ldots, y_n)$, where $y_k = u_i(h(s)_k)$, given some functions $V_i$ over vectors $(y_0, \ldots, y_n) \in \mathbb{R}^{n+1}, n \geq 0$.

*REMARK 3.* Continuity of $u_i$ on $X$ was no need in Theorem 1.

If action sets $A(x)$ are not necessarily finite, Theorem 1 fails even when the action correspondence $A$ and utility functions $u_i$ are continuous. This was demonstrated in Example 3 in Section 2.

The problem in Example 3 is that the set of terminal histories that last two periods is not closed. There was a sequence of two-period long terminal histories whose limit was not a terminal history. This non-closedness caused that there was a jump in the payoff function at this limit. The next assumption takes care of such anomalies.

**Assumption 3** *For any $t > 0$, the set of feasible terminal histories $(x_0, \ldots, x_t)$ is a closed subset of $X^{t+1}$.*

The subset of those states $y$ that can be reached from $x_0$ by $t$ steps but *not* by $t + 1$ steps is closed (possibly empty).

Our second main result gives sufficient conditions for the existence of a Markov perfect equilibrium for games where players utilities are dependent in the following way.

**Assumption 4** *For all $x, y \in X$, $u_1(x) = u_1(y)$ iff $u_2(x) = u_2(y)$, .*

Let $Y_i = u_i[X]$, $i = 1, 2$. We leave the proof of the following Lemma to the reader.

**Lemma 2** *If utility functions* $u_1$ *and* $u_2$ *are continuous, then Assumption 3 holds iff there exists a continuous bijection* $f : Y_1 \longrightarrow Y_2$.

We can now prove our second main result.

**Theorem 2** *Suppose that the set* $\Gamma$ *of games* $\Gamma(x_0, i_0)$ *satisfies Assumptions 1, 2, 3 and 4, and that the correspondence* $A$ *and functions* $u_i$ *are continuous. Then there exists a Markov perfect equilibrium* $s$*, if payoffs are calculated as in equation* (1).

**Proof.** Lemma 1 implies that that all histories $h^t$ have length $t \leq K$, for some $K > 0$, and we assume that $K$ is the least such integer. Like in the proof of Lemma 1, $X$ is partitioned into nonempty sets $A_0, \ldots, A_K$ such that (1) $A(x) = \emptyset$ iff $x \in A_0$, and (2) $A(x) \cap A_t = \emptyset$ for all $t \geq k$ if $x \in A_k$. Given any $x \in A_k$, it takes at most $k$ steps to reach a terminal state $x \in A_0$, and there is some $x \in A_k$ and some choices such that it takes $k$ steps to reach a terminal state $x \in A_0$.

We apply the backward induction principle to solve for a Markov perfect equilibrium.

*Step 1.* Given $x \in A_1$, solve to each player $i \in \{1, 2\}$ a utility maximizing last choice $s_i(x) \in A(x)$. Since $u_i$ is continuous and $A(x)$ is nonempty and closed, these maximizers exist. Since $A$ is continuous, the maximized utility $u_i(s_i(x))$ is a continuous function of $x$ by the Berge's maximum theorem. [To see that Berge's theorem applies here, note that since $A$ is closed, the subset $A_0$ is open. Hence $X \setminus A_0$ is closed and compact, and a choice $y_i(x)$ maximizing $u_i$ would exists for every $x \in X \setminus A_0$. By Berge's theorem, $u_i(y_i(x))$ is continuous. Since $y_i = s_i$ on the subset $A_1$, $u_i(s_i(x))$ is a continuous function of $x$.] Then also $u_j(s_i(x)) = g(u_i(s_i(x)))$ is a continuous function of $x$, $j \neq i$, where $g$ is either the continuous bijection $f$ of Lemma 2 or its inverse $f^{-1}$.

*Step 2.* Given $x \in A_2$ and player $i \in \{1, 2\}$, let $A^i_{20}(x) = A(x) \cap A_0$ and $A^i_{21}(x) = A(x) \cap A_1$. So $A^i_{20}(x)$ contains those choices for $i$ that will end the game, and $A^i_{21}(x)$ contains those choices that will give the player $j \neq i$ one more opportunity to choose. By Assumption 2, these subsets are closed.

If $A^i_{20}(x)$ is nonempty, it contains a nonempty closed subset of elements $y$ that maximize $u_i(y)$. If $A^i_{21}(x)$ is nonempty, it contains a nonempty closed subset of elements $z$ that maximize $u_i(z) + \delta u_i(s_j(z))$ since $u_i(s_j(z))$ is continuous in $z$ by *Step 1.* If both $A^i_{20}(x)$ and $A^i_{21}(x)$ are nonempty, we find a nonempty closed set of maximizers of the continuous function $\max\{u_i(y), u_i(z) + \delta u_i(s_j(z))\}$.

Note that the correspondence $A$ restricted to domain $A_2$ is continuous, and hence correspondences $A^i_{20}$ and $A^i_{21}$ are continuous on $A_2$ as well. By the Berge's maximum theorem, player $i$'s maximized utility depends continuously on $x \in A_2$. Since $u_i = f \circ u_j$ (or $u_i = f^{-1} \circ u_j$) for the continuous bijection $f$ of Lemma 2, player $j$'s utility depends continuously on $x \in A_2$ as well, via the equilibrium strategy $s_i(x)$ of $i$.

Hence a Markov perfect equilibrium strategies $s = (s_1, s_2)$ have been solved for initial states in $A_1 \cup A_2$, no matter who makes the first move. In order to keep the notation as simple as possible, we do not index the equilibria by the name of the player who starts the game. Notice however that actually we have solved so far *two* equilibria: one if player 1 starts the game and one if player 2 starts the game.

*Step n.* Suppose that a Markov perfect equilibrium strategies $s = (s_1, s_2)$ has been solved for initial states in $x \in A_1 \cup \cdots \cup A_{n-1}$, $n > 1$, no matter who makes the first move.

Given $x \in A_n$ and player $i \in \{1, 2\}$, let $A^i_{nm}(x) = A(x) \cap A_m$ for $m = 0, \ldots, n-1$. So a choice $y \in A^i_{nm}(x)$ means that after $y$, at most $m$ choices can be made before the game ends. The proof is exactly the same as in *Step 2* except that there are more subsets $A^i_{nm}(x)$.

We find that if $A^i_{nm}(x)$ is nonempty, there exists a nonempty closed subset of elements $y \in A^i_{nm}(x)$ that maximize the function $u_i(y) + \delta u_i(y_1) + \cdots + \delta^m u_i(y_m)$, where $y_1 = s_j(y)$, $y_2 = s_i(y_1), \ldots$, and $y_m$ is the state where the game ends when the equilibrium strategies $s_i, s_j$ solved in steps $n-1, \ldots, 1$ are applied. Since there are only finitely many nonempty closed subsets $A^i_{nm}(x)$, a nonempty closed subset of maximizers of the discounted sum of utilities can be found from $A(x)$.

As in *Step 2.*, the conditions of Berge's Maximum Theorem are satisfied, so we can find a maximizer $s_i(x) \in A(x)$, $i \in \{1, 2\}$, and players' maximized payoffs depend continuously on $x$. So a Markov perfect equilibrium exists when payoffs are calculated as in equation (1). ■

A similar existence result holds also when payoffs are calculated according to equation (2).

**Theorem 3** *Suppose that the set* $\Gamma$ *of games* $\Gamma(x_0, i_0)$ *satisfies Assumptions 1, 2, 3 and 4, and that the correspondence* $A$ *and functions* $u_i$ *are continuous. Then there exists a Markov perfect equilibrium* $s$*, if payoffs are calculated as in equation* (2)*.*

**Proof.** The proof is the same as the proof of Theorem 2 up to *Step 2*.

*Step 2.* Given $x \in A_2$ and player $i \in \{1, 2\}$, let $A^i_{20}(x) = A(x) \cap A_0$ and $A^i_{21}(x) = A(x) \cap A_1$. So $A^i_{20}(x)$ contains those choices for $i$ that will end the game, and $A^i_{21}(x)$ contains those choices that will give the player $j \neq i$ one more opportunity to choose. By Assumption 2, these subsets are closed.

If $A^i_{20}(x)$ is nonempty, it contains a nonempty closed subset of elements $y$ that maximize $u_i(y)$. If $A^i_{21}(x)$ is nonempty, it contains a nonempty closed subset of elements $z$ that maximize $u_i(s_j(z))$ since $u_i(s_j(z))$ depends continuously on $z$. If both $A^i_{20}(x)$ and $A^i_{21}(x)$ are nonempty, we find a nonempty closed set of maximizers of the continuous function $\max\{u_i(y), u_i(s_j(z))\}$. Note that the correspondence $A$ restricted to domain $A_2$ is continuous, and hence correspondences $A^i_{20}$ and $A^i_{21}$ are continuous on $A_2$ as well. By the Berge's Maximum Theorem, player $i \in \{1, 2\}$ has a maximizer $s_i(x) \in A(x)$ and his maximized payoff depends continuously on $x \in A_2$. Since $u_i = f \circ u_j$ (or $u_i = f^{-1} \circ u_j$) for the continuous bijection $f$ of Lemma 2, player $j$'s payoff depends continuously on $x$ as well.

Hence a Markov perfect equilibrium $s = (s_1, s_2)$ has been solved for initial states in $A_1 \cup A_2$, no matter who makes the first move.

The rest of the proof is the same as *Step n* in the proof of Theorem 2, except that now payoffs depend only on the states $x \in A_0$ where the game ends (in the same way as outlined in *Step 2* above). ∎

The following result follows immediately from Theorems 2 and 3.

**Corollary 1** *Suppose that the set* $\Gamma$ *of games* $\Gamma(x_0, i_0)$ *satisfies Assumptions 1 and 2 and that the correspondence* $A$ *and functions* $u_i$ *are continuous. Then there exists a Markov perfect equilibrium* $s$*, if* $u_1 = -u_2$ *and payoffs are calculated as in equation* (1) *or as in equation* (2)*.*

# 5 Discussion

We assume that 1) actions are states: $A(x) \subset X$, and that 2) utility at the current state depends on the current state $u(x)$. None of our results would change if we assume that

1. utility depends on the action taken at the current state: $u(y)$, for $y \in A(x)$;

2. utility depends on the current state and the action taken at this state: $u(x, y)$, for $y \in A(x)$;

3. actions are not states: $A(x) \subset A$ for each $x \in X$, where $A$ is a compact Hausdorff space, and given current state and action $(x, a)$ the new state is $g(x, a) \in X$ where $g$ is a continuos function. Take $X' = X \times A$. At each state $x' = (x, a)$ define action subset by $A'(x') = \{g(x')\} \times A(x)$ if $a \in A(x)$ and $A'(x') = \emptyset$ if $a \notin A(x)$. So at each state $x' = (x, a)$ new states $(g(x'), b) \in \{g(x')\} \times A(x)$ may be chosen. It is easy to show that if $A$ is a closed correspondence, then $A'$ is a closed correspondence on the compact Hausdorff space $X'$.

One may wonder if the perfect information assumption is not in contradiction with the Markov property of strategies. We may construct state spaces in such a way that this is not the case. For example, given the original state space $X$, form two identical copies of it by defining $X_1 = X \times \{1\}$ and $X_2 = X \times \{2\}$. Then $X_1$ and $X_2$ are disjoint compact Hausdorff spaces. Let the new state space be $X' = X_1 \cup X_2$. Define a new action correspondence so that $A'(x_1) \subset X_2$ for each $x_1 \in X_1$ and $A'(x_2) \in X_1$ for each $x_2 \in X_2$. The new correspondence $A'$ differs from the original $A$ only because it is defined on tuples $x' = (x, i)$, and its values are of the form $A(x, i) = A(x) \times \{j\}, i \neq j$.

APPENDIX

**Proof of Proposition 1.** Begin by indexing by ordinals $\alpha$ those states $x$ for which $A(x) \neq \emptyset$, and denote them by $x_\alpha$, $\alpha < \kappa$ where $\kappa$ is the cardinality of $X$. Apply transifinite induction as follows.

*The initial step.* Take the state $x_0$ and nominate one of the players as the first mover. Build a pseudo game to each $T > 0$ such that all feasible histories from $x_0$ are at most $T$ periods long, and from that on the action is always $x$ and both players get payoff 0. This is done except in cases when the terminal history already has length at most $T$, and these cases are left as they are. This pseudo game has a pure $MPE$, $s^T$, and it is the same as in the extensive form game with at most $T$ period histories that starts from $x_0$. This holds since nobody actually makes any moves after $T$ periods.

Let $T$ go to infinity (and keep $x_0$ the same as above). Let $Y^T$ denote the product of all nonempty action sets at nodes of this tree that have a $T \geq 0$ period history. This product set is finite, and we equip it with the usual topology. Let $Y = \prod_{T=0}^{\infty} Y^T$ with the product topology. Then $Y$ is a compact metric space. Let $x^T \in Y$ be such that the choices are the same as in the profile $s^T$ when the length of the history is $t \leq T$ periods. From period $T$ onwards the same constant $x$ is always chosen independently of the state.

Then the sequence $\{x^T\}_{T=0}^{\infty}$ has a convergent subsequence, and w.l.o.g. we assume that the sequence itself converges to $s \in Y$. By continuity of payoffs, $s$ is an $MPE$. Solve similarly an $MPE$ when $i \neq i_0$ is the first mover. Denote by $N(x_0)$ the (decision and terminal) nodes that can be reached from $x_0$, including $x_0$. Then an $MPE$ has been solved for the case when $N(x_0)$ is the state space and $A$ is the original action correspondence restricted to $N(x_0)$.

*The induction step.* Let $\alpha$ be the least ordinal such that $x_\alpha$ hasn't yet been given an action in any $MPE$. Denote by $N(x_\alpha)$ the nodes (with $A(x) \neq \emptyset$) that can be reached from $x_\alpha$, including $x_\alpha$, and denote by $N^\alpha$ the nodes that have already been given an action in an $MPE$ at an earlier stage $\beta < \alpha$ of induction.

Then, as above, solve an $MPE$ (for both players being first movers) in the extensive game starting from $x_\alpha$ when the decision nodes in $N^\alpha \cap N(x_\alpha)$ are given the actions that have already been assigned to these nodes. Then an $MPE$ has been solved in the case $N^\alpha \cup N(x_\alpha)$ is the state space and the action correspondence is $A$ restricted to this set.

Therefore an action $s_i(x)$ is assigned to both players $i = 1, 2$ at every decision node $x \in \cup_{\beta \leq \alpha} N(x_\beta)$ such that these actions form an $MPE$ $s$ when the state space is $\cup_{\beta \leq \alpha} N(x_\beta)$. ■

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