Quaderni di Dipartimento

A Skorohod Representation Theorem for Uniform Distance

Patrizia Berti
(Università di Modena e Reggio Emilia)

Luca Pratelli
(Accademia Navale di Livorno)

Pietro Rigo
(Università di Pavia)

109 (01-10)

Dipartimento di economia politica
e metodi quantitativi
Università degli studi di Pavia
Via San Felice, 5
I-27100 Pavia

Gennaio 2010

A SKOROHOD REPRESENTATION THEOREM FOR UNIFORM DISTANCE

PATRIZIA BERTI, LUCA PRATELLI, AND PIETRO RIGO

Abstract. Let μ_n be a probability measure on the Borel σ-field on $D[0,1]$ with respect to Skorohod distance, $n \geq 0$. Necessary and sufficient conditions for the following statement are provided. On some probability space, there are $D[0,1]$-valued random variables X_n such that $X_n \sim \mu_n$ for all $n \geq 0$ and $\|X_n - X_0\| \to 0$ in probability, where $\|\cdot\|$ is the sup-norm. Such conditions do not require μ_0 separable under $\|\cdot\|$. Applications to exchangeable empirical processes and to pure jump processes are given as well.

1. Introduction

Let D be the set of real cadlag functions on $[0,1]$ and

$$\|x\| = \sup_t |x(t)|, \quad u(x,y) = \|x - y\|, \quad x, y \in D.$$

Also, let d be Skorohod distance and $\mathcal{B}_d$, $\mathcal{B}_u$ the Borel σ-fields on D with respect to (w.r.t.) d and u, respectively.

In real problems, one usually starts with a sequence $(\mu_n : n \geq 0)$ of probabilities on $\mathcal{B}_d$. If $\mu_n \to \mu_0$ weakly (under d), Skorohod representation theorem yields $d(X_n, X_0) \xrightarrow{a.s.} 0$ for some D-valued random variables X_n such that $X_n \sim \mu_n$ for all $n \geq 0$. However, X_n can fail to approximate X_0 uniformly. A trivial example is $\mu_n = \delta_{x_n}$, where $(x_n) \subset D$ is any sequence such that $x_n \to x_0$ according to d but not according to u.

Lack of uniform convergence is sometimes a trouble. Thus, given a sequence $(\mu_n : n \geq 0)$ of laws on $\mathcal{B}_d$, it is useful to have conditions for:

$$\begin{gathered}\text{On some probability space } (\Omega, \mathcal{A}, P), \text{ there are random variables} \\ X_n : \Omega \to D \text{ such that } X_n \sim \mu_n \text{ for all } n \geq 0 \text{ and } \|X_n - X_0\| \xrightarrow{P} 0.\end{gathered} \tag{1}$$

Convergence in probability cannot be strengthened into a.s. convergence in condition (1). In fact, it may be that (1) holds, and yet there are not D-valued random variables Y_n such that $Y_n \sim \mu_n$ for all n and $\|Y_n - Y_0\| \xrightarrow{a.s.} 0$; see Example 7.

This paper is concerned with (1). The main result is Theorem 4, which states that (1) holds if and only if

$$\lim_n \sup_{f \in L} |\mu_n(f) - \mu_0(f)| = 0, \tag{2}$$

2000 *Mathematics Subject Classification.* 60B10, 60A05, 60A10.

Key words and phrases. Cadlag function – Exchangeable empirical process – Separable probability measure – Skorohod representation theorem– Uniform distance – Weak convergence of probability measures.

where L is the set of functions $f : D \to \mathbb{R}$ satisfying

$$\sigma(f) \subset \mathcal{B}_d, \quad -1 \leq f \leq 1, \quad |f(x) - f(y)| \leq \|x - y\| \text{ for all } x, y \in D.$$

Theorem 4 can be commented as follows. Say that a probability μ, defined on $\mathcal{B}_d$ or $\mathcal{B}_u$, is *u-separable* in case $\mu(A) = 1$ for some u-separable $A \in \mathcal{B}_d$. Suppose μ_0 is u-separable and define $\mu_0^*(H) = \mu_0(A \cap H)$ for $H \in \mathcal{B}_u$, where $A \in \mathcal{B}_d$ is u-separable and $\mu_0(A) = 1$. Since μ_n is defined only on $\mathcal{B}_d$ for $n \geq 1$, we adopt Hoffmann-Jørgensen's definition of convergence in distribution for non measurable random elements; see e.g. [7] and [9]. Let I_0 be the identity map on $(D, \mathcal{B}_u, \mu_0^*)$ and I_n the identity map on $(D, \mathcal{B}_d, \mu_n)$, $n \geq 1$. Further, let D be regarded as a metric space under u. Then, since μ_0^* is u-separable, one obtains:

(i) Condition (1) holds (with $\|X_n - X_0\| \xrightarrow{a.s.} 0$) provided $I_n \to I_0$ in distribution;

(ii) $I_n \to I_0$ in distribution if and only if $\lim_n \sup_{f \in L} |\mu_n(f) - \mu_0(f)| = 0$.

Both (i) and (ii) are known facts; see Theorems 1.7.2, 1.10.3 and 1.12.1 of [9].

The spirit of Theorem 4, thus, is that one can dispense with u-separability of μ_0 to get (1). This can look surprising, as separability of the limit law is crucial in Skorohod representation theorem; see [5]. However, $X_n \sim \mu_n$ is asked only on $\mathcal{B}_d$ and not on $\mathcal{B}_u$. Indeed, X_n can even fail to be measurable w.r.t. $\mathcal{B}_u$.

Non u-separable laws on $\mathcal{B}_d$ are quite usual. A cadlag process Z, with jumps at random time points, has typically a non u-separable distribution on $\mathcal{B}_d$. One example is $Z(t) = B_{M(t)}$, where B is a standard Brownian bridge, M an independent random distribution function and the jump-points of M have a non discrete distribution. Such a Z is the limit in distribution, under d, of certain exchangeable empirical processes; see [1] and [3].

In applications, unless μ_0 is u-separable, checking condition (2) is usually difficult. In this sense, Theorem 4 can be viewed as a "negative" result, as it states that condition (1) is quite hard to reach. This is partly true. However, there are also meaningful situations where (2) can be proved with a reasonable effort. Two examples are exchangeable empirical processes, which motivated Theorem 4, and a certain class of jump processes. Both are discussed in Section 4.

Our proof of Theorem 4 is admittedly long and it is confined in a final appendix. Some preliminary results, of possible independent interest, are needed. We mention Proposition 2 and Lemma 13 in particular.

A last remark is that Theorem 4 is still valid if D is replaced by $D\big([0,1], \mathcal{X}\big)$, the space of cadlag functions from $[0,1]$ into a separable Banach space $\mathcal{X}$.

2. A preliminary result

Let $(\Omega, \mathcal{A}, P)$ be a probability space. The outer and inner measures are

$$P^*(H) = \inf\{P(A) : H \subset A \in \mathcal{A}\}, \quad P_*(H) = 1 - P^*(H^c), \quad H \subset \Omega.$$

Given a metric space (S, ρ) and maps $X_n : \Omega \to S$, $n \geq 0$, say that X_n *converges to X_0 in (outer) probability*, written $X_n \xrightarrow{P} X_0$, in case

$$\lim_n P^*\big(\rho(X_n, X_0) > \epsilon\big) = 0 \quad \text{for all } \epsilon > 0.$$

In the sequel, d_{TV} denotes *total variation distance* between two probabilities defined on the same σ-field.

Proposition 1. *Let $(F,\mathcal{F})$ be a measurable space and μ_n a probability on $(F,\mathcal{F})$, $n \geq 0$. Then, on some probability space $(\Omega,\mathcal{A},P)$, there are measurable maps $X_n : (\Omega,\mathcal{A}) \to (F,\mathcal{F})$ such that*

$$P_*(X_n \neq X_0) = P^*(X_n \neq X_0) = d_{TV}(\mu_n,\mu_0) \text{ and } X_n \sim \mu_n \text{ for all } n \geq 0.$$

Proposition 1 is well known, even if in a slightly different form; see Theorem 2.1 of [8]. A proof of the present version is in Section 3 of [5].

Next proposition is fundamental for proving our main result. Among other things, it can be viewed as an improvement of Proposition 1.

Proposition 2. *Let λ_n be a probability on $(F \times G,\, \mathcal{F} \otimes \mathcal{G})$, $n \geq 0$, where $(F,\mathcal{F})$ is a measurable space and $(G,\mathcal{G})$ a Polish space equipped with its Borel σ-field. The following conditions are equivalent:*

(a) *There are a probability space $(\Omega,\mathcal{A},P)$ and measurable maps $(Y_n,Z_n) : (\Omega,\mathcal{A}) \longrightarrow (F \times G,\, \mathcal{F} \otimes \mathcal{G})$ such that*

$$(Y_n,Z_n) \sim \lambda_n \text{ for all } n \geq 0, \quad P^*(Y_n \neq Y_0) \longrightarrow 0, \quad Z_n \xrightarrow{P} Z_0;$$

(b) *For each bounded Lipschitz function $f : G \to \mathbb{R}$,*

$$\lim_n \sup_{A\in\mathcal{F}} \Big| \int I_A(y)\, f(z)\, \lambda_n(dy,dz) - \int I_A(y)\, f(z)\, \lambda_0(dy,dz) \Big| = 0.$$

To prove Proposition 2, we first recall a result of Blackwell and Dubins [6].

Theorem 3. *Let G be a Polish space, $\mathcal{M}$ the collection of Borel probabilities on G, and m the Lebesgue measure on $(0,1)$. There is a Borel measurable map*

$$\Phi : \mathcal{M} \times (0,1) \longrightarrow G$$

such that, for every $\nu \in \mathcal{M}$,
(i) $\Phi(\nu,\cdot) \sim \nu$ under m;
(ii) There is a Borel set $A_\nu \subset (0,1)$ such that $m(A_\nu) = 1$ and

$$\Phi(\nu_n,t) \longrightarrow \Phi(\nu,t) \quad \text{whenever } t \in A_\nu,\ \nu_n \in \mathcal{M} \text{ and } \nu_n \to \nu \text{ weakly.}$$

We also need to recall disintegrations. Let λ be a probability on $(F \times G, \mathcal{F} \otimes \mathcal{G})$, where $(F,\mathcal{F})$ and $(G,\mathcal{G})$ are arbitrary measurable spaces. In this paper, λ is said to be *disintegrable* if there is a collection $\alpha = \{\alpha(y) : y \in F\}$ such that:
− $\alpha(y)$ is a probability on $\mathcal{G}$ for $y \in F$;
− $y \mapsto \alpha(y)(C)$ is $\mathcal{F}$-measurable for $C \in \mathcal{G}$;
− $\lambda(A \times C) = \int_A \alpha(y)(C)\, \mu(dy)$ for $A \in \mathcal{F}$ and $C \in \mathcal{G}$, where $\mu(\cdot) = \lambda(\cdot \times G)$.
Such an α is called a *disintegration* for λ. For λ to admit a disintegration, it suffices that G is a Borel subset of a Polish space and $\mathcal{G}$ the Borel σ-field on G.

Proof of Proposition 2. "(a) $\Rightarrow$ (b)". Under (a), for each $A \in \mathcal{F}$ and bounded Lipschitz $f : G \to \mathbb{R}$, one obtains

$$\begin{aligned}
\Big| \int I_A(y)\, f(z)\, \lambda_n(dy,dz) - \int I_A(y)\, f(z)\, \lambda_0(dy,dz) \Big| &= \Big| E_P\{ I_A(Y_n)\, f(Z_n) - I_A(Y_0)\, f(Z_0) \} \Big| \\
&\leq E_P \Big| f(Z_n)\, (I_A(Y_n) - I_A(Y_0)) \Big| + E_P \Big| I_A(Y_0)\, (f(Z_n) - f(Z_0)) \Big| \\
&\leq \sup|f|\, P^*(Y_n \neq Y_0) + E_P \Big| f(Z_n) - f(Z_0) \Big| \longrightarrow 0.
\end{aligned}$$

"(b) ⇒ (a)". Let $\mu_n(A) = \lambda_n(A \times G)$, $A \in \mathcal{F}$. By (b), $d_{TV}(\mu_n, \mu_0) \to 0$. Hence, by Proposition 1, on a probability space $(\Theta, \mathcal{E}, Q)$ there are measurable maps $h_n : (\Theta, \mathcal{E}) \to (F, \mathcal{F})$ satisfying $h_n \sim \mu_n$ for all n and $Q^*(h_n \neq h_0) \to 0$. Let

$$\Omega = \Theta \times (0,1), \quad \mathcal{A} = \mathcal{E} \otimes \mathcal{B}_{(0,1)}, \quad P = Q \times m,$$

where $\mathcal{B}_{(0,1)}$ is the Borel σ-field on $(0,1)$ and m the Lebesgue measure.

Since G is Polish, each λ_n admits a disintegration $\alpha_n = \{\alpha_n(y) : y \in F\}$. By Theorem 3, there is a map $\Phi : \mathcal{M} \times (0,1) \longrightarrow G$ satisfying conditions (i)-(ii). Let

$$Y_n(\theta, t) = h_n(\theta) \quad \text{and} \quad Z_n(\theta, t) = \Phi\big\{\alpha_n(h_n(\theta)), t\big\}, \quad (\theta, t) \in \Theta \times (0,1).$$

For fixed θ, condition (i) yields $Z_n(\theta, \cdot) = \Phi\big\{\alpha_n(h_n(\theta)), \cdot\big\} \sim \alpha_n(h_n(\theta))$ under m. Since α_n is a disintegration for λ_n, for all $A \in \mathcal{F}$ and $C \in \mathcal{G}$ one has

$$P(Y_n \in A, Z_n \in C) = \int_\Theta I_A(h_n(\theta))\, m\big\{t : Z_n(\theta, t) \in C\big\}\, Q(d\theta)$$

$$= \int_{\{h_n \in A\}} \alpha_n(h_n(\theta))(C)\, Q(d\theta) = \int_A \alpha_n(y)(C)\, \mu_n(dy) = \lambda_n(A \times C).$$

Also, $P^*(Y_n \neq Y_0) = Q^*(h_n \neq h_0) \longrightarrow 0$ by Lemma 1.2.5 of [9].

Finally, we prove $Z_n \xrightarrow{P} Z_0$. Write $\alpha_n(y)(f) = \int f(z)\, \alpha_n(y)(dz)$ for all $y \in F$ and $f \in L_G$, where L_G is the set of Lipschitz functions $f : G \to [-1, 1]$. Since $Q^*(h_n \neq h_0) \to 0$, there are $A_n \in \mathcal{F}$ such that $Q(A_n^c) \to 0$ and $h_n = h_0$ on A_n. Given $f \in L_G$,

$$E_Q\Big|\alpha_n(h_n)(f) - \alpha_0(h_0)(f)\Big| - 2\, Q(A_n^c) \leq E_Q\big\{I_{A_n}\Big|\alpha_n(h_0)(f) - \alpha_0(h_0)(f)\Big|\big\}$$

$$\leq E_Q\Big|\alpha_n(h_0)(f) - \alpha_0(h_0)(f)\Big| = \int \Big|\alpha_n(y)(f) - \alpha_0(y)(f)\Big|\, \mu_0(dy).$$

Using condition (b), it is not hard to see that $\int |\alpha_n(y)(f) - \alpha_0(y)(f)|\, \mu_0(dy) \longrightarrow 0$. Therefore, $\alpha_n(h_n)(f) \xrightarrow{Q} \alpha_0(h_0)(f)$ for each $f \in L_G$, and this is equivalent to

$$\text{each subsequence } (n') \text{ contains a further subsequence } (n'')$$
$$\text{such that } \alpha_{n''}(h_{n''}(\theta)) \longrightarrow \alpha_0(h_0(\theta)) \quad \text{weakly for } Q\text{-almost all } \theta;$$

see Remark 2.3 and Corollary 2.4 of [2]. Thus, by property (ii) of Φ, each subsequence (n') contains a further subsequence (n'') such that $Z_{n''} \xrightarrow{a.s.} Z_0$. That is, $Z_n \xrightarrow{P} Z_0$ and this concludes the proof.

□

3. Existence of cadlag processes, with given distributions on the Skorohod Borel σ-field, converging uniformly in probability

As in Section 1, $\mathcal{B}_d$ and $\mathcal{B}_u$ are the Borel σ-fields on D w.r.t. d and u. Also, L is the class of functions $f : D \to [-1, 1]$ which are measurable w.r.t. $\mathcal{B}_d$ and Lipschitz w.r.t. u with Lipschitz constant 1. We recall that, for $x, y \in D$, the Skorohod distance $d(x, y)$ is the infimum of those $\epsilon > 0$ such that

$$\|x - y \circ \gamma\| \leq \epsilon \quad \text{and} \quad \sup_{s \neq t} \Big|\log \frac{\gamma(s) - \gamma(t)}{s - t}\Big| \leq \epsilon$$

for some strictly increasing homeomorphism $\gamma : [0,1] \to [0,1]$. The metric space (D, d) is separable and complete.

We write $\mu(f) = \int f \, d\mu$ whenever μ is a probability on a σ-field and f a real bounded function, measurable w.r.t. such a σ-field.

Motivations for the next result have been given in Section 1.

Theorem 4. *Let μ_n be a probability measure on $\mathcal{B}_d$, $n \geq 0$. Then, conditions* (1) *and* (2) *are equivalent, that is,*

$$\lim_n \sup_{f \in L} |\mu_n(f) - \mu_0(f)| = 0$$

if and only if there are a probability space $(\Omega, \mathcal{A}, P)$ and measurable maps $X_n : (\Omega, \mathcal{A}) \to (D, \mathcal{B}_d)$ such that $X_n \sim \mu_n$ for each $n \geq 0$ and $\|X_n - X_0\| \xrightarrow{P} 0$.

The proof of Theorem 4 is given in the Appendix. Here, we state a corollary and an open problem and we make two examples.

In applications, the μ_n are often probability distributions of random variables, all defined on some probability space $(\Omega_0, \mathcal{A}_0, P_0)$. In the spirit of [4], a (minor) question is whether condition (1) holds with the X_n defined on $(\Omega_0, \mathcal{A}_0, P_0)$ as well.

Corollary 5. *Let $(\Omega_0, \mathcal{A}_0, P_0)$ be a probability space and $Z_n : (\Omega_0, \mathcal{A}_0) \to (D, \mathcal{B}_d)$ a measurable map, $n \geq 1$. Suppose $\lim_n \sup_{f \in L} |E_{P_0}\{f(Z_n)\} - \mu_0(f)| = 0$ for some probability measure μ_0 on $\mathcal{B}_d$. If P_0 is nonatomic, there are measurable maps $X_n : (\Omega_0, \mathcal{A}_0) \to (D, \mathcal{B}_d)$, $n \geq 0$, such that*

$$X_0 \sim \mu_0, \quad X_n \sim Z_n \text{ for each } n \geq 1, \quad \|X_n - X_0\| \xrightarrow{P_0} 0.$$

Also, P_0 is nonatomic if $\mu_0\{x\} = 0$ for all $x \in D$, or if $P_0(Z_n = x) = 0$ for some $n \geq 1$ and all $x \in D$.

Proof. Since (D, d) is separable, P_0 is nonatomic if $P_0(Z_n = x) = 0$ for some $n \geq 1$ and all $x \in D$. By Corollary 5.4 of [4], $(\Omega_0, \mathcal{A}_0, P_0)$ supports a D-valued random variable Z_0 with $Z_0 \sim \mu_0$. Hence, P_0 is nonatomic even if $\mu_0\{x\} = 0$ for all $x \in D$. Next, by Theorem 4, on a probability space $(\Omega, \mathcal{A}, P)$ there are D-valued random variables Y_n such that $Y_0 \sim \mu_0$, $Y_n \sim Z_n$ for $n \geq 1$ and $\|Y_n - Y_0\| \xrightarrow{P} 0$. Let $(D^\infty, \mathcal{B}_d^\infty)$ be the countable product of $(D, \mathcal{B}_d)$ and

$$\nu(A) = P\big((Y_0, Y_1, \ldots) \in A\big), \quad A \in \mathcal{B}_d^\infty.$$

Then, ν is a Borel probability on a Polish space. Thus, if P_0 is nonatomic, $(\Omega_0, \mathcal{A}_0, P_0)$ supports a D^∞-valued random variable $X = (X_0, X_1, \ldots)$ with $X \sim \nu$; see e.g. Theorem 3.1 of [4]. Since $(X_0, X_1, \ldots) \sim (Y_0, Y_1, \ldots)$, this concludes the proof. □

Let (S, ρ) be a metric space such that $(x, y) \mapsto \rho(x, y)$ is measurable w.r.t. $\mathcal{E} \otimes \mathcal{E}$, where $\mathcal{E}$ is the ball σ-field on S. This is actually true in case $(S, \rho) = (D, u)$ and it is very useful to prove Theorem 4. Thus, a question is whether (D, u) can be replaced by (S, ρ) in Theorem 4. Precisely, let $(\mu_n : n \geq 0)$ be a sequence of laws on $\mathcal{E}$ and L_S the class of functions $f : S \to [-1, 1]$ such that $\sigma(f) \subset \mathcal{E}$ and $|f(x) - f(y)| \leq \rho(x, y)$ for all $x, y \in S$. Then,

Conjecture: $\lim_n \sup_{f \in L_S} |\mu_n(f) - \mu_0(f)| = 0$ if and only if $\rho(X_n, X_0) \longrightarrow 0$ in probability for some S-valued random variables X_n such that $X_n \sim \mu_n$ for all n.

We finally give two examples. The first shows that condition (2) cannot be weakened into $\mu_n(f) \to \mu_0(f)$ for each *fixed* $f \in L$.

Example 6. For each $n \geq 0$, let $h_n : (0,1) \to [0,\infty)$ be a Borel function such that $\int_0^1 h_n(t)\,dt = 1$. Suppose that $h_n \to h_0$ in $\sigma(L_1, L_\infty)$ but not in L_1 under Lebesgue measure m on $(0,1)$, that is,

$$\begin{gathered} \limsup_n \int_0^1 |h_n(t) - h_0(t)|\,dt > 0, \\ \lim_n \int_0^1 h_n(t)\,g(t)\,dt = \int_0^1 h_0(t)\,g(t)\,dt \quad \text{for all bounded Borel functions } g. \end{gathered} \tag{3}$$

Take a sequence $(T_n : n \geq 0)$ of $(0,1)$-valued random variables, on a probability space $(\Theta, \mathcal{E}, Q)$, such that each T_n has density h_n w.r.t. m. Define

$$Z_n = I_{[T_n,1]} \quad \text{and} \quad \mu_n(A) = Q(Z_n \in A) \text{ for } A \in \mathcal{B}_d.$$

Then $Z_n = \phi(T_n)$, with $\phi : (0,1) \to D$ given by $\phi(t) = I_{[t,1]}$, $t \in (0,1)$. Hence, for fixed $f \in L$, one obtains

$$\mu_n(f) = E_Q\big\{f \circ \phi(T_n)\big\} = \int_0^1 h_n(t)\,f \circ \phi(t)\,dt \longrightarrow \int_0^1 h_0(t)\,f \circ \phi(t)\,dt = \mu_0(f).$$

Suppose now that $X_n \sim \mu_n$ for all $n \geq 0$, where the X_n are D-valued random variables on some probability space $(\Omega, \mathcal{A}, P)$. Since

$$P\big\{\omega : X_n(\omega)(t) \in \{0,1\} \text{ for all } t\big\} = Q\big\{\theta : Z_n(\theta)(t) \in \{0,1\} \text{ for all } t\big\} = 1,$$

it follows that

$$P\big(\|X_n - X_0\| > \frac{1}{2}\big) = P(X_n \neq X_0) \geq d_{TV}(\mu_n, \mu_0) = \frac{1}{2}\int_0^1 |h_n(t) - h_0(t)|\,dt.$$

Therefore, X_n fails to converge to X_0 in probability.

A slight change in Example 6 shows that convergence in probability cannot be strengthened into a.s. convergence in condition (1). Precisely, it may be that (1) holds, and yet there are not D-valued random variables Y_n satisfying $Y_n \sim \mu_n$ for all n and $\|Y_n - Y_0\| \xrightarrow{a.s.} 0$.

Example 7. In the notation of Example 6, instead of (3) assume

$$\lim_n \int_0^1 |h_n(t) - h_0(t)|\,dt = 0 \quad \text{and} \quad m\big(\liminf_n h_n < h_0\big) > 0$$

where m is Lebesgue measure on $(0,1)$. Since

$$d_{TV}(\mu_n, \mu_0) = \frac{1}{2}\int_0^1 |h_n(t) - h_0(t)|\,dt \longrightarrow 0,$$

condition (1) trivially holds by Proposition 1. Suppose now that $Y_n \sim \mu_n$ for all $n \geq 0$, where the Y_n are D-valued random variables on a probability space $(\Omega, \mathcal{A}, P)$. As $m\big(\liminf_n h_n < h_0\big) > 0$, Theorem 3.1 of [8] yields $P(Y_n = Y_0 \text{ ultimately}) < 1$. On the other hand, since $P\big(Y_n(t) \in \{0,1\} \text{ for all } t\big) = 1$, one obtains

$$P\big(\|Y_n - Y_0\| \longrightarrow 0\big) = P(Y_n = Y_0 \text{ ultimately}) < 1.$$

4. Applications

Condition (2) is not always hard to be checked, even if μ_0 is not u-separable. We illustrate this fact by two examples. To this end, we first note that conditions (1)-(2) are preserved under certain mixtures.

Corollary 8. *Let G be the set of distribution functions on $[0,1]$ and $\mathcal{G}$ the σ-field on G generated by the maps $g \mapsto g(t)$, $0 \le t \le 1$. Let π be a probability on $\mathcal{G}$ and μ_n and λ_n probabilities on $\mathcal{B}_d$. Then, condition (1) holds provided*

$$\sup_{f \in L} |\lambda_n(f) - \lambda_0(f)| \longrightarrow 0 \quad \textit{and}$$

$$\mu_n(A) = \int \lambda_n\big\{x : x \circ g \in A\big\}\,\pi(dg) \quad \textit{for all } n \ge 0 \textit{ and } A \in \mathcal{B}_d.$$

Proof. By Theorem 4, there are a probability space $(\Theta, \mathcal{E}, Q)$ and measurable maps $Z_n : (\Theta, \mathcal{E}) \longrightarrow (D, \mathcal{B}_d)$ such that $Z_n \sim \lambda_n$ for all n and $\|Z_n - Z_0\| \xrightarrow{Q} 0$. Define $\Omega = \Theta \times G$, $\mathcal{A} = \mathcal{E} \otimes \mathcal{G}$, $P = Q \times \pi$, and $X_n(\theta, g) = Z_n(\theta) \circ g$ for all $(\theta, g) \in \Theta \times G$. It is routine to check that $X_n \sim \mu_n$ for all n and $\|X_n - X_0\| \xrightarrow{P} 0$. □

Example 9. (Exchangeable empirical processes). Let $(\xi_n : n \ge 1)$ be a sequence of $[0,1]$-valued random variables on the probability space $(\Omega_0, \mathcal{A}_0, P_0)$. Suppose (ξ_n) *exchangeable* and define

$$F(t) = E_{P_0}\big(I_{\{\xi_1 \le t\}} \mid \tau\big)$$

where τ is the tail σ-field of (ξ_n). Take F to be regular, i.e., each F-path is a distribution function. Then, the n-th empirical process can be defined as

$$Z_n(t) = \frac{\sum_{i=1}^n \big\{I_{\{\xi_i \le t\}} - F(t)\big\}}{\sqrt{n}}, \quad 0 \le t \le 1,\, n \ge 1.$$

Since $Z_n : (\Omega_0, \mathcal{A}_0) \to (D, \mathcal{B}_d)$ is measurable, one can define $\mu_n(\cdot) = P_0(Z_n \in \cdot)$. Also, let μ_0 be the probability distribution of

$$Z_0(t) = B_{M(t)}$$

where B is a standard Brownian bridge on $[0,1]$ and M an independent copy of F (with B and M defined on some probability space). Then, $\mu_n \to \mu_0$ weakly (under d) but μ_0 can fail to admit any extension to $\mathcal{B}_u$; see [3] and Example 11 of [1]. Thus, Z_n can fail to converge in distribution, under u, according to Hoffmann-Jørgensen's definition. However, Corollaries 5 and 8 grant that:

On $(\Omega_0, \mathcal{A}_0, P_0)$, there are measurable maps $X_n : (\Omega_0, \mathcal{A}_0) \to (D, \mathcal{B}_d)$ such that $X_n \sim Z_n$ for each $n \ge 0$ and $\|X_n - X_0\| \xrightarrow{P_0} 0$.

Define in fact $B_n(t) = n^{-1/2} \sum_{i=1}^n \big\{I_{\{u_i \le t\}} - t\big\}$, where $u_1, u_2, \ldots$ are i.i.d. random variables (on some probability space) with uniform distribution on $[0,1]$. Then, $B_n \to B$ in distribution, under u, according to Hoffmann-Jørgensen's definition. Let λ_n and λ_0 be the probability distributions of B_n and B, respectively. Since λ_0 is u-separable, $\sup_{f \in L} |\lambda_n(f) - \lambda_0(f)| \longrightarrow 0$ (see Section 1). Thus, the first condition of Corollary 8 holds. The second condition follows from de Finetti's representation theorem, by letting $\pi(A) = P_0(F \in A)$ for $A \in \mathcal{G}$. Hence, condition (1) holds.

It remains to see that the X_n can be defined on $(\Omega_0, \mathcal{A}_0, P_0)$. To this end, it can be assumed $\mathcal{A}_0 = \sigma(\xi_1, \xi_2, \ldots)$. If P_0 is nonatomic, it suffices to apply Corollary 5. Suppose P_0 has an atom A. Since $\mathcal{A}_0 = \sigma(\xi_1, \xi_2, \ldots)$, up to P_0-null sets, A is of the form $A = \{\xi_n = t_n \text{ for all } n \geq 1\}$ for some constants t_n. Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be a permutation of $1, 2, \dots$ and $A_\sigma = \{\xi_n = t_{\sigma_n} \text{ for all } n \geq 1\}$. By exchangeability,

$$P_0(A_\sigma) = P_0(A) > 0 \quad \text{for all permutations } \sigma,$$

and this implies $t_n = t_1$ for all $n \geq 1$. Let H be the union of all P_0-atoms. Up to P_0-null sets, one obtains

$$H \subset \{\xi_n = \xi_1 \text{ for all } n \geq 1\} \subset \{Z_n = 0 \text{ for all } n \geq 1\}.$$

If $P_0(H) = 1$, thus, it suffices to let $X_n = 0$ for all $n \geq 0$. If $0 < P_0(H) < 1$, since $P_0(\cdot \mid H^c)$ is nonatomic and (ξ_n) is still exchangeable under $P_0(\cdot \mid H^c)$, it is not hard to define the X_n on $(\Omega_0, \mathcal{A}_0, P_0)$ in such a way that $X_n \sim Z_n$ for all $n \geq 0$ and $\|X_n - X_0\| \xrightarrow{P_0} 0$.

Example 10. (Pure jump processes). For each $n \geq 0$, let

$$C_n = (C_{n,j} : j \geq 1) \quad \text{and} \quad Y_n = (Y_{n,j} : j \geq 1)$$

be sequences of real random variables, defined on the probability space $(\Omega_0, \mathcal{A}_0, P_0)$, such that

$$0 \leq Y_{n,j} \leq 1 \quad \text{and} \quad \sum_{j=1}^{\infty} |C_{n,j}| < \infty.$$

Define

$$Z_n(t) = \sum_{j=1}^{\infty} C_{n,j} \, I_{\{Y_{n,j} \leq t\}}, \quad 0 \leq t \leq 1, \, n \geq 0.$$

Since $Z_n : (\Omega_0, \mathcal{A}_0) \longrightarrow (D, \mathcal{B}_d)$ is measurable, one can define $\mu_n(\cdot) = P_0(Z_n \in \cdot)$. Then, condition (1) holds provided

$$C_n \text{ is independent of } Y_n \text{ for every } n \geq 0,$$

$$\sum_{j=1}^{\infty} |C_{n,j} - C_{0,j}| \xrightarrow{P_0} 0 \quad \text{and} \quad d_{TV}\big(\nu_{n,k}, \, \nu_{0,k}\big) \longrightarrow 0 \text{ for all } k \geq 1,$$

where $\nu_{n,k}$ denotes the probability distribution of $(Y_{n,1}, \dots, Y_{n,k})$.

For instance, $\nu_{n,k} = \nu_{0,k}$ for all n and k in case $Y_{n,j} = V_{n+j}$ with $V_1, V_2, \dots$ a stationary sequence. Also, independence between C_n and Y_n can be replaced by

$$\sigma(C_{n,j}) \subset \sigma(Y_{n,1}, \dots, Y_{n,j}) \quad \text{for all } n \geq 0 \text{ and } j \geq 1.$$

To prove (1), define $Z_{n,k}(t) = \sum_{j=1}^{k} C_{n,j} \, I_{\{Y_{n,j} \leq t\}}$. For each $f \in L$,

$$\begin{aligned}
|\mu_n(f) - \mu_0(f)| &\leq |Ef(Z_n) - Ef(Z_{n,k})| + |Ef(Z_{n,k}) - Ef(Z_{0,k})| + |Ef(Z_{0,k}) - Ef(Z_0)| \\
&\leq E\big\{2 \wedge \|Z_n - Z_{n,k}\|\big\} + |Ef(Z_{n,k}) - Ef(Z_{0,k})| + E\big\{2 \wedge \|Z_0 - Z_{0,k}\|\big\} \\
&\leq E\Big\{2 \wedge \sum_{j>k} |C_{n,j}|\Big\} + |Ef(Z_{n,k}) - Ef(Z_{0,k})| + E\Big\{2 \wedge \sum_{j>k} |C_{0,j}|\Big\}
\end{aligned}$$

where $E(\cdot) = E_{P_0}(\cdot)$. Given $\epsilon > 0$, take $k \geq 1$ such that $E\big\{2 \wedge \sum_{j>k} |C_{0,j}|\big\} < \epsilon$. Then,

$$\limsup_n \sup_{f \in L} |\mu_n(f) - \mu_0(f)| < 2\,\epsilon + \limsup_n \sup_{f \in L} |Ef(Z_{n,k}) - Ef(Z_{0,k})|.$$

It remains to show that $\sup_{f\in L}|Ef(Z_{n,k}) - Ef(Z_{0,k})| \longrightarrow 0$. Since C_n is independent of Y_n, up to changing $(\Omega_0, \mathcal{A}_0, P_0)$ with some other probability space, it can be assumed

$$P_0\big(Y_{n,j} \neq Y_{0,j} \text{ for some } j \leq k\big) = d_{TV}\big(\nu_{n,k},\, \nu_{0,k}\big);$$

see Proposition 1. The same is true if $\sigma(C_{n,j}) \subset \sigma(Y_{n,1}, \dots, Y_{n,j})$ for all n and j. Then, letting $A_{n,k} = \{Y_{n,j} = Y_{0,j} \text{ for all } j \leq k\}$, one obtains

$$\begin{aligned}\sup_{f\in L}|Ef(Z_{n,k}) - Ef(Z_{0,k})| &\leq E\big\{I_{A_{n,k}}\, 2 \wedge \|Z_{n,k} - Z_{0,k}\|\big\} + 2\, P_0(A^c_{n,k}) \\ &\leq E\big\{2 \wedge \sum_{j=1}^{\infty}|C_{n,j} - C_{0,j}|\big\} + 2\, d_{TV}\big(\nu_{n,k},\, \nu_{0,k}\big) \longrightarrow 0.\end{aligned}$$

Thus, condition (2) holds, and an application of Theorem 4 concludes the proof.

APPENDIX

Three preliminary lemmas are needed to prove Theorem 4. The first is part of the folklore about Skorohod distance, and we state it without a proof. Let $\Delta x(t) = x(t) - x(t-)$ denote the jump of $x \in D$ at $t \in (0, 1]$.

Lemma 11. *Fix $\epsilon > 0$ and $x_n \in D$, $n \geq 0$. Then, $\limsup_n \|x_n - x_0\| \leq \epsilon$ whenever $d(x_n, x_0) \longrightarrow 0$ and*

$$|\Delta x_n(t)| > \epsilon \quad \textit{for all large } n \textit{ and } t \in (0,1) \textit{ such that } |\Delta x_0(t)| > \epsilon.$$

The second lemma is a consequence of Remark 6 of [5], but we give a sketch of its proof as it is basic for Theorem 4. Let μ, ν be laws on $\mathcal{B}_d$ and $\mathcal{F}(\mu, \nu)$ the class of probabilities λ on $\mathcal{B}_d \otimes \mathcal{B}_d$ such that $\lambda(\cdot \times D) = \mu(\cdot)$ and $\lambda(D \times \cdot) = \nu(\cdot)$. Since the map $(x, y) \mapsto \|x - y\|$ is measurable w.r.t. $\mathcal{B}_d \otimes \mathcal{B}_d$, one can define

$$W_u(\mu, \nu) = \inf_{\lambda \in \mathcal{F}(\mu,\nu)} \int 1 \wedge \|x - y\|\, \lambda(dx, dy).$$

Lemma 12. *For a sequence $(\mu_n : n \geq 0)$ of probabilities on $\mathcal{B}_d$, condition (1) holds if and only if $W_u(\mu_0, \mu_n) \longrightarrow 0$.*

Proof. The "only if" part is trivial. Suppose $W_u(\mu_0, \mu_n) \to 0$. Let $\Omega = D^\infty$, $\mathcal{A} = \mathcal{B}_d^\infty$ and $X_n : D^\infty \to D$ the n-th canonical projection, $n \geq 0$. Take $\lambda_n \in \mathcal{F}(\mu_0, \mu_n)$ such that $\int 1 \wedge \|x - y\|\, \lambda_n(dx, dy) < \frac{1}{n} + W_u(\mu_0, \mu_n)$. Since (D, d) is Polish, λ_n admits a disintegration $\alpha_n = \{\alpha_n(x) : x \in D\}$ (see Section 2). By Ionescu-Tulcea theorem, there is a unique probability P on $\mathcal{B}_d^\infty$ such that $X_0 \sim \mu_0$ and

$$\beta_n(x_0, x_1, \dots, x_{n-1})(A) = \alpha_n(x_0)(A), \quad (x_0, x_1, \dots, x_{n-1}) \in D^n,\, A \in \mathcal{B}_d,$$

is a regular version of the conditional distribution of X_n given $(X_0, X_1, \dots, X_{n-1})$ for all $n \geq 1$. Under such P, one obtains $(X_0, X_n) \sim \lambda_n$ (so that $X_n \sim \mu_n$) and

$$\epsilon\, P\big(\|X_0 - X_n\| > \epsilon\big) \leq E_P\big\{1 \wedge \|X_0 - X_n\|\big\} < \frac{1}{n} + W_u(\mu_0, \mu_n) \longrightarrow 0 \quad \text{for all } \epsilon \in (0, 1).$$

□

The third lemma needs some more effort. Let $\phi_0(x, \epsilon) = 0$ and

$$\phi_{n+1}(x, \epsilon) = \inf\{t :\, \phi_n(x, \epsilon) < t \leq 1,\, |\Delta x(t)| > \epsilon\}$$

where $n \geq 0$, $\epsilon > 0$, $x \in D$ and $\inf \emptyset := 1$. The map $x \mapsto \phi_n(x, \epsilon)$ is universally measurable w.r.t. $\mathcal{B}_d$ for all n and ϵ.

Lemma 13. *Let $\mathcal{F}_k$ be the Borel σ-field on $\mathbb{R}^k$ and $I \subset (0,1)$ a dense subset. For a sequence $(\mu_n : n \geq 0)$ of probabilities on $\mathcal{B}_d$, condition (1) holds provided*

$$\sup_{A \in \mathcal{F}_k} \Big| \int f(x)\, I_A\big(\phi_1(x,\epsilon), \dots, \phi_k(x,\epsilon)\big)\, \mu_n(dx) - \int f(x)\, I_A\big(\phi_1(x,\epsilon), \dots, \phi_k(x,\epsilon)\big)\, \mu_0(dx) \Big| \longrightarrow 0$$

for each $k \geq 1$, $\epsilon \in I$ and function $f : D \to [-1,1]$ such that $|f(x) - f(y)| \leq d(x,y)$ for all $x, y \in D$.

Proof. Fix $\epsilon \in I$ and write $\phi_n(x)$ instead of $\phi_n(x, \epsilon)$. As each ϕ_n is universally measurable w.r.t. $\mathcal{B}_d$, there is a set $T \in \mathcal{B}_d$ such that

$$\mu_n(T) = 1 \quad \text{and} \quad I_T\, \phi_n \text{ is } \mathcal{B}_d\text{-measurable for all } n \geq 0.$$

Thus, ϕ_n can be assumed $\mathcal{B}_d$-measurable for all n. Let k be such that

$$\mu_0\{x : \phi_r(x) \neq 1 \text{ for some } r > k\} < \epsilon.$$

For such a k, define $\phi(x) = (\phi_1(x), \dots, \phi_k(x))$, $x \in D$, and

$$\lambda_n(A) = \mu_n\big\{x : (\phi(x), x) \in A\big\}, \quad A \in \mathcal{F}_k \otimes \mathcal{B}_d.$$

Since (D,d) is Polish, Proposition 2 applies to such λ_n with $(F, \mathcal{F}) = (\mathbb{R}^k, \mathcal{F}_k)$ and $(G, \mathcal{G}) = (D, \mathcal{B}_d)$. Condition (b) holds by the assumption of the Lemma. Thus, by Proposition 2, on a probability space $(\Omega, \mathcal{A}, P)$ there are measurable maps $(Y_n, Z_n) : (\Omega, \mathcal{A}) \to (\mathbb{R}^k \times D,\, \mathcal{F}_k \otimes \mathcal{B}_d)$ satisfying

$$(Y_n, Z_n) \sim \lambda_n \text{ for all } n \geq 0, \quad P(Y_n \neq Y_0) \longrightarrow 0, \quad d(Z_n, Z_0) \xrightarrow{P} 0.$$

Since $P\big(Y_n = \phi(Z_n)\big) = \lambda_n\{(\phi(x), x) : x \in D\} = 1$, one also obtains

$$\lim_n P\big(\phi(Z_n) = \phi(Z_0)\big) = 1. \tag{4}$$

Next, by (4) and $d(Z_n, Z_0) \xrightarrow{P} 0$, there is a subsequence (n_j) such that

$$\limsup_n P\big(\|Z_n - Z_0\| > \epsilon\big) = \lim_j P\big(\|Z_{n_j} - Z_0\| > \epsilon\big),$$

$$d(Z_{n_j}, Z_0) \xrightarrow{a.s.} 0, \quad P\big(\phi(Z_{n_j}) = \phi(Z_0) \text{ for all } j\big) > 1 - \epsilon.$$

Define $U = \limsup_j \|Z_{n_j} - Z_0\|$ and

$$H = \{\phi_r(Z_0) = 1 \text{ for all } r > k\} \cap \{\phi(Z_{n_j}) = \phi(Z_0) \text{ for all } j\} \cap \{d(Z_{n_j}, Z_0) \longrightarrow 0\}.$$

For each $\omega \in H$, Lemma 11 applies to $Z_0(\omega)$ and $Z_{n_j}(\omega)$, so that $U(\omega) \leq \epsilon$. Further,

$$\begin{aligned} P(H^c) &\leq P\big(\phi_r(Z_0) \neq 1 \text{ for some } r > k\big) + P\big(\phi(Z_{n_j}) \neq \phi(Z_0) \text{ for some } j\big) \\ &< \mu_0\{x : \phi_r(x) \neq 1 \text{ for some } r > k\} + \epsilon < 2\,\epsilon. \end{aligned}$$

Since $U \leq \epsilon$ on H,

$$\begin{aligned} \limsup_n P\big(\|Z_n - Z_0\| > \epsilon\big) &= \lim_j P\big(\|Z_{n_j} - Z_0\| > \epsilon\big) \leq P(U \geq \epsilon) \\ &\leq P(U = \epsilon) + P(H^c) < P(U = \epsilon) + 2\,\epsilon. \end{aligned}$$

On noting that $E_P\big\{1 \wedge \|Z_0 - Z_n\|\big\} \leq \epsilon + P\big(\|Z_n - Z_0\| > \epsilon\big)$, one obtains

$$\limsup_n W_u(\mu_0, \mu_n) \leq \limsup_n E_P\big\{1 \wedge \|Z_0 - Z_n\|\big\} < P(U = \epsilon) + 3\,\epsilon.$$

Since I is dense in $(0,1)$, then $P(U=\epsilon)+3\,\epsilon$ can be made arbitrarily small for a suitable $\epsilon\in I$. Thus, $\limsup_n W_u(\mu_0,\mu_n)=0$. An application of Lemma 12 concludes the proof. $\square$

We are now ready for the last attack to Theorem 4.

Proof of Theorem 4. "$(1)\Rightarrow(2)$". Just note that

$$\begin{aligned}|\,\mu_n(f)-\mu_0(f)|&=|\,E_P\big\{f(X_n)\big\}-E_P\big\{f(X_0)\big\}|\leq E_P|\,f(X_n)-f(X_0)|\\&\leq E_P\big\{2\wedge\|X_n-X_0\|\big\}\longrightarrow 0,\quad\text{for each } f\in L,\ \text{under (1)}.\end{aligned}$$

"$(2)\Rightarrow(1)$". Let $B_\epsilon=\{x:|\Delta x(t)|=\epsilon \text{ for some } t\in(0,1]\}$. Then, B_ϵ is universally measurable w.r.t. $\mathcal{B}_d$ and $\mu_0(B_\epsilon)>0$ for at most countably many $\epsilon>0$. Hence, $I=\{\epsilon\in(0,1):\mu_0(B_\epsilon)=0\}$ is dense in $(0,1)$.

Fix $\epsilon\in I$, $k\geq1$, and a function $f:D\to[-1,1]$ such that $|f(x)-f(y)|\leq d(x,y)$ for all $x,\,y\in D$. By Lemma 13, for condition (1) to be true, it is enough that

$$\lim_n\sup_{A\in\mathcal{F}_k}|\,\mu_n\big\{f\,I_A(\phi)\big\}-\mu_0\big\{f\,I_A(\phi)\big\}\,|=0 \tag{5}$$

where $\phi(x)=\big(\phi_1(x),\ldots,\phi_k(x)\big)$, $x\in D$, and $\phi_j(x)=\phi_j(x,\epsilon)$ for all j.

In order to prove (5), given $b\in(0,\frac{\epsilon}{2})$, define

$$F_b=\{x:|\Delta x(t)|\notin(\epsilon-2b,\,\epsilon+2b)\ \text{ for all } t\in(0,1]\},\quad G_b=\{x:d(x,F_b)\geq\frac{b}{2}\}.$$

Then,

(i) $G_b^c\subset F_{b/2}$; (ii) $\phi(x)=\phi(y)$ whenever $x,\,y\in F_b$ and $\|x-y\|<b$.

Statement (ii) is straightforward. To check (i), fix $x\notin G_b$ and take $y\in F_b$ with $d(x,y)<b/2$. Let $\gamma:[0,1]\to[0,1]$ be a strictly increasing homeomorphism such that $\|x-y\circ\gamma\|<b/2$. For all $t\in(0,1]$,

$$|\Delta x(t)|\leq|\Delta\,y\circ\gamma(t)|+2\,\|x-y\circ\gamma\|<|\Delta y(\gamma(t))|+b.$$

Similarly, $|\Delta x(t)|>|\Delta y(\gamma(t))|-b$. Since $y\in F_b$, it follows that $x\in F_{b/2}$.

Next, define

$$\psi_b(x)=\frac{d(x,\,G_b)}{d(x,\,F_b)+d(x,\,G_b)},\quad x\in D.$$

Then, $\psi_b=0$ on G_b and ψ_b is Lipschitz w.r.t. d with Lipschitz constant $2/b$. Hence, ψ_b is Lipschitz w.r.t. u with Lipschitz constant $2/b$ (since $d\leq u$). Basing on (i)-(ii) and such properties of ψ_b, it is not hard to check that $\psi_b\,I_A(\phi)$ is Lipschitz w.r.t. u, with Lipschitz constant $2/b$, for every $A\in\mathcal{F}_k$. In turn, since $d\leq u$ and f is Lipschitz w.r.t. d with Lipschitz constant 1,

$$f_A=f\,\psi_b\,I_A(\phi),\quad A\in\mathcal{F}_k,$$

is Lipschitz w.r.t. u with Lipschitz constant $(1+2/b)$. Moreover,

$$|\,\mu_n\big\{f\,I_A(\phi)\big\}-\mu_n(f_A)|\leq\mu_n|f\,I_A(\phi)\,(1-\psi_b)|\leq\mu_n(1-\psi_b).$$

On noting that $(1+2/b)^{-1}f_A\in L$ for every $A\in\mathcal{F}_k$, condition (2) yields

$$\begin{aligned}&\limsup_n\sup_{A\in\mathcal{F}_k}|\,\mu_n\big\{f\,I_A(\phi)\big\}-\mu_0\big\{f\,I_A(\phi)\big\}\,|\\&\leq\limsup_n\big\{\mu_n(1-\psi_b)+\sup_{A\in\mathcal{F}_k}|\mu_n(f_A)-\mu_0(f_A)|+\mu_0(1-\psi_b)\big\}\\&=2\,\mu_0(1-\psi_b)\leq2\,\mu_0(F_b^c).\end{aligned}$$

Since $\epsilon \in I$ and $\bigcap_{b>0} F_b^c = \{x : |\Delta x(t)| = \epsilon \text{ for some } t\} = B_\epsilon$, one obtains

$$\limsup_n \sup_{A \in \mathcal{F}_k} |\,\mu_n\big\{f\,I_A(\phi)\big\} - \mu_0\big\{f\,I_A(\phi)\big\}| \leq 2 \lim_{b \to 0} \mu_0(F_b^c) = 2\,\mu_0(B_\epsilon) = 0.$$

Therefore, condition (5) holds and this concludes the proof.

□

References

[1] Berti P., Rigo P. (2004) Convergence in distribution of non measurable random elements, *Ann. Probab.*, 32, 365-379.

[2] Berti P., Pratelli L., Rigo P. (2006) Almost sure weak convergence of random probability measures, *Stochastics*, 78, 91-97.

[3] Berti P., Pratelli L., Rigo P. (2006) Asymptotic behaviour of the empirical process for exchangeable data, *Stoch. Proc. Appl.*, 116, 337-344.

[4] Berti P., Pratelli L., Rigo P. (2007) Skorohod representation on a given probability space, *Prob. Theo. Rel. Fields*, 137, 277-288.

[5] Berti P., Pratelli L., Rigo P. (2009) Skorohod representation theorem via disintegrations, *Submitted*, currently available at: `http://economia.unipv.it/pagp/pagine_personali/prigo/skorsankhya.pdf`

[6] Blackwell D., Dubins L.E. (1983) An extension of Skorohod's almost sure representation theorem, *Proc. Amer. Math. Soc.*, 89, 691-692.

[7] Dudley R.M. (1999) *Uniform central limit theorems*, Cambridge University Press.

[8] Sethuraman J. (2002) Some extensions of the Skorohod representation theorem, *Sankhya*, 64, 884-893.

[9] van der Vaart A., Wellner J.A. (1996) *Weak convergence and empirical processes*, Springer.

Patrizia Berti, Dipartimento di Matematica Pura ed Applicata "G. Vitali", Universita' di Modena e Reggio-Emilia, via Campi 213/B, 41100 Modena, Italy
E-mail address: `berti.patrizia@unimore.it`

Luca Pratelli, Accademia Navale, viale Italia 72, 57100 Livorno, Italy
E-mail address: `pratel@mail.dm.unipi.it`

Pietro Rigo (corresponding author), Dipartimento di Economia Politica e Metodi Quantitativi, Universita' di Pavia, via S. Felice 5, 27100 Pavia, Italy
E-mail address: `prigo@eco.unipv.it`