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Abstract

Copula-GARCH models have been recently proposed in the financial literature as a statistical tool to build flexible multivariate distributions. Our extensive simulation studies investigate the small sample properties of these models and examine how misspecification in the marginals may affect the estimation of the dependence function represented by the copula. We show that the use of normal marginals when the true Data Generating Process is leptokurtic or asymmetric, produces negatively biased estimates of the normal copula correlations. A striking result is that these biases reach their highest value when correlations are strongly negative, and viceversa. This result remains unchanged with both positively skewed and negatively skewed data, while no biases are found if the variables are uncorrelated. Besides, the effect of marginals asymmetry on correlations is smaller than that of leptokurtosis. We finally analyse the performance of these models in terms of numerical convergence and positive definiteness of the estimated copula correlation matrix.[1]

Keywords: Copulas, Copula-GARCH models, Maximum Likelihood, Simulation, Small Sample Properties.

JEL classification: C15, C32, C51, C63.

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[1] The Technical Appendix, which is represented by pages 26-75 and contains most of the figures describing the simulation results, is intended only and exclusively as a separate supplement which will be posted online on the corresponding author's webpage.

1 Introduction

The increasing globalisation of world economies raises the issue of multivariate joint modelling. At the same time, the growing importance of financial markets amplifies the time-varying volatility and leptokurtosis of economic and financial variables. Given these stylised facts, the assumption of joint normality can be no longer realistic. Copula theory can be a solution to this problem. Indeed, the essential idea of the copula approach is that a joint distribution can be factored into the marginals and a dependence function called copula. The dependence relationship is entirely determined by the copula, while position, scaling and shape (mean, standard deviation, skewness and kurtosis) are entirely determined by the marginals. Copulas have been successfully used in finance, and we refer the interested reader to the outstanding book by Cherubini et al. (2004) for a detailed treatment of many financial applications. Recently, copulas have also been proposed to model operational risks (cf. Chernobai et al. (2007) and Fantazzini et al. (2008)).

The first contribution of this paper is a Monte Carlo study of the finite sample properties of the marginals and copula estimators under different hypotheses about the Data Generating Process (DGP). We choose to analyze a multivariate Copula-GARCH model because it is a convenient choice for economic and financial applications. More precisely, we use Normal or Generalized-t distributions (see Hansen, 1994) for the marginal noise terms. The latter distribution is suitable to model skewed and leptokurtic data.

As for marginal parameters, when the true DGP is skewed and leptokurtic, our simulation studies highlight the strong biases affecting the parameter estimates, and particularly the GARCH parameters. Besides, we show that the empirical distribution for these parameters is strongly asymmetric and the mean and median estimates can be rather different. In this case, the biases are largest when normal marginals are considered, but even the use of the correctly specified Generalized-t marginals results in very poor estimates when the sample dimension T is small. When T increases, the biases and the t-tests decrease for all parameters (except for the symmetric Student's t degrees of freedom, when the DGP is skewed). However, the t-tests for the null hypothesis that the empirical mean across simulations is equal to the true value still reject the null hypothesis for most of the parameters even when $T = 2000$. Interestingly, no qualitative differences are found across different copula dimensions as well as across different correlation levels.

In general, these results point out the difficulties of estimating GARCH models with small samples, thus extending previous simulation evidence in Hwang and Valls Pereira (2006) who, however, considered only univariate models with normally distributed errors and did not examine the effect of different joint distributions.

As for the dependence parameters, when there is skewness in the data and symmetric marginals are used, the estimated correlations are negatively biased, and the bias increases when moving from the Student's t to the normal (marginal) distribution, reaching values as high as 25% of the true values. A striking result is that this bias reaches its highest value when the correlation is strongly negative ($\rho_0 < -0.5$), and viceversa. This result still holds with both positively skewed and negatively skewed data, while no biases are found if the variables are uncorrelated. When the marginal leptokurtosis decreases, the biases in correlations decrease as expected, but the previous conclusions remain unchanged: the biases for negative correlations are almost double compared with positive correlations, and they are highest when the marginals used are normally distributed. No major differences in the biases are found when moving from small samples to large samples (even though the

t-statistics are slightly smaller), as well as when moving from bivariate to 10-variate normal copulas. Besides, these results remain mostly the same even when using an ill-specified correlation matrix for the normal copula, whose lowest eigenvalue is close to zero.
The second contribution of this paper is an analysis of copula-GARCH models in terms of numerical convergence and positive definiteness of the estimated copula correlation matrix. The numerical maximization of the log-likelihood fails to converge mostly when the marginals DGP is leptokurtic and asymmetric and we deal with small samples. In this case, the use of misspecified normal marginals determines the highest number of convergence failures (close to 50%), while the correctly specified Generalized-t marginals perform slightly better (over 30%). Similarly to the biases for the copula correlations, leptokurtosis has a stronger effect on numerical convergence than asymmetry, and convergence improves when leptokurtosis decreases.
Interestingly, we observe that the type of correlation matrix used for the normal copula does not affect the number of convergence failures, which remains constant across different specifications. Furthermore, our simulation studies show that even when the true correlation matrix is ill-conditioned, the estimated correlation matrix is always positive definite without any constraints. These results confirm recent evidence in Fantazzini (2008) who compares different estimation methods for the T-copula and finds that the estimated correlation matrix can be non positive definite only when dealing with very small samples ($T < 100$) and when the true underlying process has an ill-conditioned correlation matrix.
The rest of the paper is organized as follows. Section 2 presents the copula-GARCH models, while Section 3 gathers simulation studies purporting to assess the small sample properties of these models under different DGPs. Section 4 concludes.

2 Copula-GARCH Modelling

Consider a general copula-GARCH model, where the n endogenous variables $x_{i,t}$ are explained by an intercept μ_i and an error term $\sqrt{h_{i,t}}\eta_{i,t}$ [2]

$$\begin{aligned} x_{1,t} &= \mu_1 + \sqrt{h_{1,t}}\,\eta_{1,t} \\ \vdots \quad & \qquad \vdots \\ x_{n,t} &= \mu_n + \sqrt{h_{n,t}}\,\eta_{n,t}. \end{aligned} \tag{1}$$

Let the standardized innovations $\eta_{i,t}$ have mean zero and variance one, while $\sqrt{h_{i,t}}$ can be constant or time-varying like in GARCH(1,1) models[3]:

$$\begin{aligned} h_{1,t} &= \omega_1 + \alpha_1(\eta_{1,t-1}\sqrt{h_{1,t-1}})^2 + \beta_1 h_{1,t-1} \\ \vdots \quad & \qquad \vdots \qquad\qquad\qquad \vdots \\ h_{n,t} &= \omega_n + \alpha_n(\eta_{n,t-1}\sqrt{h_{n,t-1}})^2 + \beta_n h_{n,t-1} \end{aligned} \tag{2}$$

Furthermore, the innovations $\eta_{i,t}$ have a conditional joint distribution $H_t(\eta_{1,t}, \ldots, \eta_{n,t}; \theta)$ with the parameters vector θ, which can be expressed as follows, thanks to the so-called

[2]We also considered models with autoregressive lags, but they delivered similar qualitative results to the case where only the constant is present. Therefore, we do not consider these models here. We thank an anonymous referee for highlighting this point.

[3]See Bollerslev, Engle, and Nelson (1994) for a detailed survey of GARCH models.

Sklar's theorem (1959):

$$(\eta_{1,t},\ldots,\eta_{n,t}) \sim H_t\left(\eta_{1,t},\ldots,\eta_{n,t};\theta\right) = C_t\left(F_1(\eta_{1,t};\alpha_1),\ldots,F_n(\eta_{n,t};\alpha_n);\gamma\right) \tag{3}$$

that is the joint distribution H_t of a vector of innovations $\eta_{i,t}$ is the copula $C_t\left(\,\cdot\,;\gamma\right)$ of the cumulative distribution functions of the innovations marginals $F_1(\eta_{1,t};\alpha_1),\ldots,$ $F_n(\eta_{n,t};\alpha_n)$, where $\gamma,\alpha_1,\ldots,\alpha_n$ are the copula and marginals parameters, respectively.
Since a copula is a function that links together two or more marginals distributions to form a multivariate joint distribution, copulas allow us to model the dependence structure between different variables in a flexible way and, at the same time, to use marginals distributions not necessarily identical. For example, $F_1(\eta_{1,t};\nu_1)$ may follow a Student's t distribution with ν_1 degrees of freedom, $F_2(\eta_{2,t})$ a standard normal distribution, while $F_3(\eta_{3,t};\lambda_3,\nu_3)$ may be a Generalized-t distribution (see Hansen (1994)) with ν_3 degrees of freedom, where λ_3 is the skewness parameter. For more details about copulas and Sklar's Theorem, the interested reader is referred to the methodological overviews by Nelsen (1999) and Cherubini et al. (2004).
By applying Sklar's theorem and using the relationship between the distribution and the density function for the case of the multivariate joint normal, we can derive the Normal copula, whose probability density function is:

$$\begin{aligned} c^{Normal}(\Phi(x_1),\ldots,\Phi(x_n);\mathbf{\Sigma}) &= \frac{f^{Normal}(x_1,\ldots,x_n)}{\prod\limits_{i=1}^{n} f_i^{Normal}(x_i)} = \frac{\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}x'\mathbf{\Sigma}^{-1}x\right)}{\prod\limits_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_i^2\right)} = \\ &= \frac{1}{|\mathbf{\Sigma}|^{1/2}}\exp\left(-\frac{1}{2}\zeta'(\mathbf{\Sigma}^{-1}-I)\zeta\right), \end{aligned} \tag{4}$$

where $\zeta = (\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n))'$ is the vector of univariate Gaussian inverse distribution functions, $u_i = \Phi\ (x_i)$, while $\mathbf{\Sigma}$ is the correlation matrix.
A similar procedure can be followed to derive the *t-copula*, which is the copula of the multivariate Student's t-distribution. Moreover, a model can allow for a time-varying dependence structure. However, recent literature (see Chen et al. (2004)) has shown that for daily financial data a simple normal copula with no dynamics is sufficient to describe the joint dependence structure in most cases. Only when the number of considered variables is higher than 20, statistically significant differences start to emerge and more complicated copulas than the Normal one may be required. Fantazzini et al. (2008) found similar evidence with monthly operational risk data. Besides, macroeconomic analysis usually works with a small number of endogenous variables sampled at a monthly or lower frequency, which are known to have a much simpler dependence structure than daily financial data; for this reason we stick to a constant normal copula $C_t^{Normal} = C^{Normal}$.
This copula belongs to the class of Elliptical copulas. An alternative to Elliptical copulas is given by Archimedean copulas: however, they present the serious limitation of modelling only positive dependence (or only partial negative dependence), while their multivariate extensions involve strict restrictions on bivariate dependence parameters. That is why we do not consider them here.

2.1 Copula and Marginals Estimation

Let us suppose to have a set of T empirical data of n economic and financial series, and $\theta = (\alpha_1,\ldots,\alpha_n;\gamma)$ is the parameters vector to estimate, where $\alpha_i, i = 1,\ldots,n$ are the parameters of the marginal distribution F_i and γ is the vector of the copula parameters. It follows from (3) that the log-likelihood function for the joint conditional distribution $H_t(\ \cdot\ ;\ \theta\ )$ is given by:

$$l(\theta) = \sum_{t=1}^{T} \log(c(F_1(\eta_{1,t};\alpha_1), ..., F_n(\eta_{n,t};\alpha_n);\gamma)) + \sum_{t=1}^{T}\sum_{i=1}^{n} \log f_i(\eta_{i,t};\alpha_i) \tag{5}$$

Hence, the log likelihood of the joint distribution is just the sum of the log likelihoods of the margins and the log likelihood of the copula. Standard ML estimates may be obtained by maximizing the above expression with respect to the parameters $(\alpha_1, \ldots, \alpha_n\ ;\ \gamma)$. In practice this can involve a large numerical optimization problem with many parameters which can be difficult to solve. However, given the partitioning of the parameter vector into separate parameters for each margin and parameters for the copula, one may use (5) to break up the optimization problem into several small optimizations, each with fewer parameters. This multi-step procedure is known as the method of Inference Functions for Margins (IFM), see Joe and Xu (1996) and Joe (1997) for more details.
According to the IFM method, the parameters of the marginal distributions are estimated separately from the parameters of the copula. In other words, the estimation process is divided into the following two steps:

1. Estimate the parameters α_i, $i = 1, \ldots, n$ of the marginal distributions F_i using the ML method:
$$\hat{\alpha}_i = \arg\max l^i(\alpha_i) = \arg\max \sum_{t=1}^{T} \log f_i(\eta_{i,t};\alpha_i), \tag{6}$$
where l^i is the log-likelihood function of the marginal distribution F_i;

2. Estimate the copula parameters γ, given the estimations performed in step 1):
$$\hat{\gamma} = \arg\max l^c(\gamma) = \arg\max \sum_{t=1}^{T} \log(c(F_1(\eta_{1,t};\hat{\alpha}_1), \ldots, F_n(\eta_{n,t};\hat{\alpha}_n);\gamma)), \tag{7}$$
where l^c is the log-likelihood function of the copula.

Joe (1997) compares the efficiency of the IFM method relatively to full maximum likelihood for a number of multivariate models and finds the IFM method to be highly efficient. Therefore, we think it is safe to use the IFM method and benefit from the huge reduction in complexity implied for numerical optimization.

3 Simulation Studies

In this section we present the results of the simulation studies concerning a multivariate copula-GARCH(1,1) model. This model is tractable and flexible enough to fit many economic and financial applications. The model specification is given by:

$$Y_t = \mu + \sqrt{h_t}\eta_t, \tag{8}$$

where Y_t and μ are vectors, while the matrix h is diagonal and contains the variances:

$$h_{i,t} = \omega_i + \alpha_i(\eta_{i,t-1}\sqrt{h_{i,t-1}})^2 + \beta_i h_{i,t-1}, \quad i = 1, \ldots, n.$$

The possible Data Generating Processes (DGPs) we consider are specified below:

1. We examine six different types of marginals for the innovations η_t:

   - A Generalized-t (see Hansen (1994)) with skewness parameter $\lambda = -0.5$ and degrees of freedom $\nu = 3$;
   - A Generalized-t with skewness parameter $\lambda = 0$ and degrees of freedom $\nu = 3$, that is a symmetric Student's t-distribution.
   - A Generalized-t with skewness parameter $\lambda = 0.5$ and degrees of freedom $\nu = 3$;
   - A Generalized-t with skewness parameter $\lambda = -0.5$ and degrees of freedom $\nu = 10$;
   - A Generalized-t with skewness parameter $\lambda = 0$ and degrees of freedom $\nu = 10$, that is a symmetric Student's–t distribution quite close to a standard normal distribution.
   - A Generalized-t with skewness parameter $\lambda = 0.5$ and degrees of freedom $\nu = 10$;

2. We examine four types of normal copulas to model the joint dependence (3) of the innovations η_t. Particularly, we consider:

   (a) the case where two variables have a bivariate Normal copula, with the copula linear correlation ρ_0 ranging between -0.9 and 0.9 (step 0.1).

   (b) the case where ten variables are uncorrelated, i.e. they have a multivariate Normal copula whose linear correlation matrix $\mathbf{\Sigma}$ is diagonal:

1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1

Table 1: Normal copula Correlation matrix: Diagonal case

(c) the case where ten variables have a multivariate Normal copula, with the copula correlation matrix $\mathbf{\Sigma}$ equal to:

1	0.21	0.33	0.22	0.36	0.30	0.37	0.34	0.31	0.47
0.21	1	0.20	0.15	0.27	0.18	0.18	0.31	0.20	0.21
0.33	0.20	1	0.16	0.32	0.28	0.40	0.33	0.17	0.42
0.22	0.15	0.16	1	0.20	0.16	0.18	0.20	0.27	0.20
0.36	0.27	0.32	0.20	1	0.32	0.33	0.55	0.33	0.35
0.30	0.18	0.28	0.16	0.32	1	0.28	0.32	0.26	0.31
0.37	0.18	0.40	0.18	0.33	0.28	1	0.35	0.23	0.40
0.34	0.31	0.33	0.20	0.55	0.32	0.35	1	0.31	0.35
0.31	0.20	0.17	0.27	0.33	0.26	0.23	0.31	1	0.30
0.47	0.21	0.42	0.20	0.35	0.31	0.40	0.35	0.30	1

Table 2: Normal copula Correlation matrix: Dow Jones Industrial Index returns

This is the correlation matrix for the returns of the first 10 stocks belonging to the Dow Jones Industrial Index, observed between 11/18/1988 and 11/20/2003.

(d) the case where ten variables have a multivariate Normal copula, with the copula correlation matrix $\mathbf{\Sigma}$ equal to:

1	-0.15	-0.15	-0.15	-0.15	-0.14	-0.09	-0.03	0.05	0.13
-0.15	1	-0.15	-0.15	-0.15	-0.13	-0.08	-0.02	0.06	0.14
-0.15	-0.15	1	-0.15	-0.15	-0.12	-0.07	-0.01	0.07	0.15
-0.15	-0.15	-0.15	1	-0.15	-0.11	-0.06	0.01	0.08	0.15
-0.15	-0.15	-0.15	-0.15	1	-0.10	-0.05	0.02	0.09	0.15
-0.14	-0.13	-0.12	-0.11	-0.10	1	-0.04	0.03	0.10	0.15
-0.09	-0.08	-0.07	-0.06	-0.05	-0.04	1	0.04	0.11	0.15
-0.03	-0.02	-0.01	0.01	0.02	0.03	0.04	1	0.12	0.15
0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	1	0.15
0.13	0.14	0.15	0.15	0.15	0.15	0.15	0.15	0.15	1

Table 3: Normal copula Correlation matrix with lowest eigenvalue equal to 0.0768

We choose this correlation matrix because it is ill-specified and it allows us to study the effect that ill-specified correlation matrices have on numerical convergence and positive definitiveness of the estimated correlation matrices.

3. We consider two possible data alternatives: $T = 500$, $T = 2000$.

4. We consider the same conditional mean and variance specification for all marginals:

$$\mu = \begin{bmatrix} 0.05 \\ \dots \\ 0.05 \end{bmatrix}, \omega = \begin{bmatrix} 0.01 \\ \dots \\ 0.01 \end{bmatrix}, \alpha = \begin{bmatrix} 0.05 \\ \dots \\ 0.05 \end{bmatrix}, \beta = \begin{bmatrix} 0.9 \\ \dots \\ 0.9 \end{bmatrix}.$$

This choice is justified for the sake of simplicity and again to keep the number of simulated DGPs still tractable. However, we choose values to mimic the most common stylized facts of financial markets, such as the strong persistence in conditional variances (see, for example, Tsay (2002) and references therein).

We generated 1000 Monte Carlo samples for each marginal and copula specification previously described and then we estimated the following models: 1) Generalized-t / Normal copula, 2) Student's t / Normal copula, 3) Normal / Normal copula .

3.1 Effects of Marginals Misspecifications on Marginals Parameters Estimation

We report in Figures 25-54 in the Appendix the mean bias in percentage, the median bias in percentage, and the t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Figures 25-32 report the results when the empirical marginals are Normal, Figures 33-42 report the results when the empirical marginals are symmetric Student's t, and Figures 43-54 report the results when the empirical marginals are Generalized-t.
The simulation studies highlight some interesting results:

- ***Normal marginals parameters estimation***:
  - *Conditional Mean*: If the true DGP has skewed marginals but we use the Normal, the biases in the conditional mean parameters are close to 8% of the true values for the parameter μ, when $T = 500$ and $\nu = 3$. Particularly, if there is negative skewness, the estimated $\hat{\mu}$ is positively biased and viceversa. This bias remains constant across different correlation levels as well as copula dimensions (bivariate or 10-variate). If the sample dimension or the degrees of freedom of the true DGP increases, the biases decrease and the t-tests become not significant or weakly significant at the 5% level.

    If the true DGP is symmetric, the empirical parameters are not statistically different from the true values in all cases.
  - *Conditional Variance*: the lack of a parameter modelling fat tails induces a large positive bias in the parameters ω and α and a negative one in β. The mean biases of these parameters can be larger than 150%, 50% and 10%, respectively, when $T = 500$, $\nu = 3$ and data are skewed, while they are slightly smaller when data are symmetric. Instead, if we consider median estimates, biases are much smaller and α is negatively biased, thus highlighting a strong asymmetric distribution for the variance parameters. These results point out the difficulties of estimating GARCH models with small samples, thus confirming previous simulation evidence in Hwang and Valls Pereira (2006) who, however, considered only univariate models with normally distributed errors and did not examine the effect of different joint distributions. Moreover, strong biases were expected, since it has been shown by Newey and Steigerwald (1997) that a QML estimator can be biased when data are not symmetric. When the dimension T increases, the biases and the t-tests decrease for all parameters, but they still remain significant (apart for $\hat{\alpha}$ when $\nu = 10$ and considering a bivariate copula).
  - *Sample dimension*: when the T dimension increases, biases and t-statistics decrease for all parameters. However, the t-tests still reject the null hypothesis for all parameters, except for $\hat{\mu}$.
  - *Copula dimension*: no qualitative differences are found across different copula dimensions.

- ***Student's t marginals parameters estimation***:
  - *Conditional Mean*: If the true DGP has skewed marginals but we use the Student's t, the biases in the conditional mean parameters can be as high as 140% of the true values for the parameter μ, when $T = 500$ and $\nu = 3$. Again, if there is negative skewness, the estimated $\hat{\mu}$ is positively biased and viceversa. As expected, no significant bias is found when the true DGP is symmetric. Biases decrease when ν increases but they remain significant. Interestingly, the biases do not change much when the sample dimension increases. Biases remain constant across different correlation levels as well as copula dimensions (bivariate or 10-variate).
  - *Conditional Variance*: The use of this symmetric distribution causes severe (positive) biases in the ω and α estimates, when $T = 500$ and $\nu = 3$ and the data are skewed: up to 130% and 30% of the true values for the former two parameters, respectively. Instead, the parameter β is estimated more precisely, with (negative) biases smaller than 7%. Similarly to the normal distribution, the median estimates are more precise: up to 30% and 10% for the ω and α, whereas the negative bias for β decreases to 2%. Interestingly, all these biases remain almost unchanged across different values for λ and ν when $T = 500$. Instead, when the T dimension increases, biases and t-statistics decrease as well. Besides, even though the biases for $\hat{\beta}$ and $\hat{\alpha}$ are smaller than 1%-2% (in absolute values) of the true values when $T = 2000$ and $\lambda = 0$, nevertheless the t-statistics still reject the null hypothesis.
  - *Degrees of freedom*: When data are skewed, there is a negative bias that increases both with the sample dimension and the magnitude of ν. It can be higher than 30% of the true value when $T = 2000$.
  - *Sample dimension*: Except for the degrees of freedom previously discussed, when the T dimension increases, biases and t-statistics decrease as well. Besides, the t-tests still reject the null hypothesis for all parameters, except for $\hat{\mu}$ when the true DGP is symmetric.
  - *Copula dimension*: All the previous biases remain constant across different correlation levels as well as copula dimensions (bivariate or 10-variate).

- ***Generalized-t marginals parameters estimation***:
  - *Conditional Mean*: the estimated parameters of the conditional mean show a small mean bias around 3% (5% if we consider the median), when $T = 500$ and $\nu = 3$, which is statistically significant. The bias decreases and becomes no more significant when ν or T increase.
  - *Conditional Variance*: When there is skewness in the data, the parameters are estimated more precisely than when the observations are symmetric: in the latter case, for example, the constant ω shows a strong positive bias which can be close to 90% (mean) when $T = 500$, while in the former case the bias is around 60% (mean). These biases are smaller if median estimates are considered, instead. This difference can be explained by the fact that the Generalized-t is not the most efficient model when data are symmetric. However, these positive biases are present for all considered distributions, and clearly highlight the remarkable difficulty of estimating a GARCH model in small samples.

- *Degrees of freedom*: As expected, the degrees of freedom ν are estimated much more precisely when they are low than viceversa (positive mean bias around 40% when $T = 500$ and $\nu = 10$), given that this parameter is much more difficult to identify when it is high in magnitude. If the sample dimension increases, biases and t-statistics decrease as expected.
- *Skewness parameter*: The estimates show a negative mean bias around 10% in small samples with fat tails, i.e. $\nu = 3$, which disappears as the sample dimension increases. Small samples and thinner tails, i.e. $\nu = 10$, slightly mitigate the bias to 8%. Similar but smaller biases hold when the median is considered.
- *Sample dimension*: when the T dimension increases, biases and t-statistics decrease for all parameters. However, the t-tests still reject the null hypothesis for all parameters, except for $\hat{\mu}$, $\hat{\nu}$ when $\nu = 3$ and $\hat{\lambda}$ when $\lambda = 0$.
- *Copula dimension*: No difference is found across different copula dimensions (and correlation levels): this result was expected since the Generalized-t is the true marginal specification and, in this case, its parameters are variation free with respect to the copula parameters (see Patton (2006)).

3.2 Effects of Marginals Misspecifications on Copula Parameters Estimation

Figures 1-4 report the mean bias in percentage, and the t-test for the null hypothesis that the empirical mean across simulations is equal to the true value, in this case $H_0 : \hat{\rho} = \rho_0$. Figure 1 reports the results for the bivariate copula correlations; Figure 2 for the 10-variate Normal copula with the correlation matrix reported in Table 1 (that is diagonal); Figure 3 for the 10-variate Normal copula with the correlation matrix reported in Table 2 (that is the one referring to DJI stocks returns); Figure 4 for the 10-variate Normal copula with the correlation matrix reported in Table 3 (the one with the lowest eigenvalue close to zero). For the sake of interest and space, we report in the manuscript only the case when $T = 500$ and the empirical marginals are normally distributed, while we put in the Appendix in Figures 5-12 the cases with the other marginal distributions and in Figures 13-24 all the cases with $T = 2000$. Differently from the coefficients of the marginal distributions, we do not report the median biases in the figures, because they are very close to the mean biases and do not provide any additional information.

- ***Bivariate Normal Copula***:
  - *Correlation Matrix*: when there is skewness in the data and symmetric marginals are used, the estimated correlations are negatively biased, and the bias increases when moving from the Student's t to the normal (marginal) distribution, reaching values as high as 25% of the true values. A striking result is that this bias reaches its highest value when the correlation is strongly negative ($\rho_0 < -0.5$), and viceversa. This result remains unchanged with both positively skewed and negatively skewed data, while no biases are found if the variables are uncorrelated: the latter case is the only one when the t-statistics are not significant and the null hypothesis is not rejected[4].

[4]Figures 1,5,6 and 13-15 do not report the case where $\rho_0 = 0$, because the mean and median biases in percentage involve a division by zero in this case.

If the true Generalized-t is used, the estimation biases are equal or smaller than 1% even with $T = 500$. However, the t-tests reject the null hypothesis that $\hat{\rho} = \rho_0$ across simulations when the correlation is strongly negative or positive ($|\rho| > 0.5$). Interestingly, the t-statistics (in absolute value) take again higher values when ρ is negative than when ρ is positive: for example, when $T = 500$, the true DGP is a Generalized-t with $\nu = 3$ and $\lambda = -0.5$ with a bivariate copula linear correlation $\rho_0 = -0.8$, the t-statistic for the null that $\hat{\rho} = \rho_0$ is equal to 9.85. If the true correlation is $\rho_0 = 0.8$, everything else kept the same, the t-statistic (in absolute value) is equal to 5.33.

When the degrees of freedom of the true marginals increase from 3 to 10, the biases in the correlations decrease as expected (ranging between -10% and 0%), but the previous conclusions remain unchanged: the biases for negative correlations are almost double compared to positive correlations, and they are highest when the marginals used are normally distributed.

- *Sample dimension*: no major differences in the biases are found when moving from $T = 500$ to $T = 2000$, even though the t-statistics are slightly smaller.

- ***10-variate Normal Copula (Diagonal case)***:
  - *Correlation Matrix*: Similarly to the result obtained in the bivariate case, if the variables are uncorrelated we can use misspecified marginals with no harm. The t-tests do not reject the hypothesis that $\hat{\rho} = \rho_0$ in almost all cases.
  - *Sample dimension*: no major differences in the biases are found when moving from $T = 500$ to $T = 2000$.

- ***10-variate Normal Copula (DJI returns)***:
  - *Correlation Matrix*: If normal marginals are used and the true DGP is skewed, the estimated correlations are again negatively biased and this bias decreases if the true positive correlations increase: the bias is close to -20% if $\rho_0 = 0.15$ and decreases to -12% if $\rho_0 = 0.55$. Biases decrease when ν increases and/or the process is symmetric but they remain significant. If the Student's–t is used biases decrease, but the qualitative results do not change. Similarly to the bivariate case, t-statistics increase with the magnitude of the (positive) correlations when the true DGP is skewed. This result holds also when using the Generalized-t distribution, when the mean bias is close to zero.
  - *Sample dimension*: no major differences in the biases are found when moving from $T = 500$ to $T = 2000$, even though the t-statistics are smaller.

- ***10-variate Normal Copula (Ill-conditioned correlation matrix)***:
  - *Correlation Matrix*: The previous results are confirmed also when the correlation matrix is ill specified: the correlations show a negative bias and t-statistics are higher for negative correlations than for positive correlations. However, we observe two interesting facts: biases tend to be almost constant across positive and negative correlations (also if the true DGP is skewed). Besides, when the true correlation $\rho_0 \to 0$, the bias tends to explode (but the t-tests do not reject the null hypothesis).

- *Sample dimension*: no major differences in the biases are found when moving from $T = 500$ to $T = 2000$, even though the biases around $\rho_0 = 0$ tend to decrease.

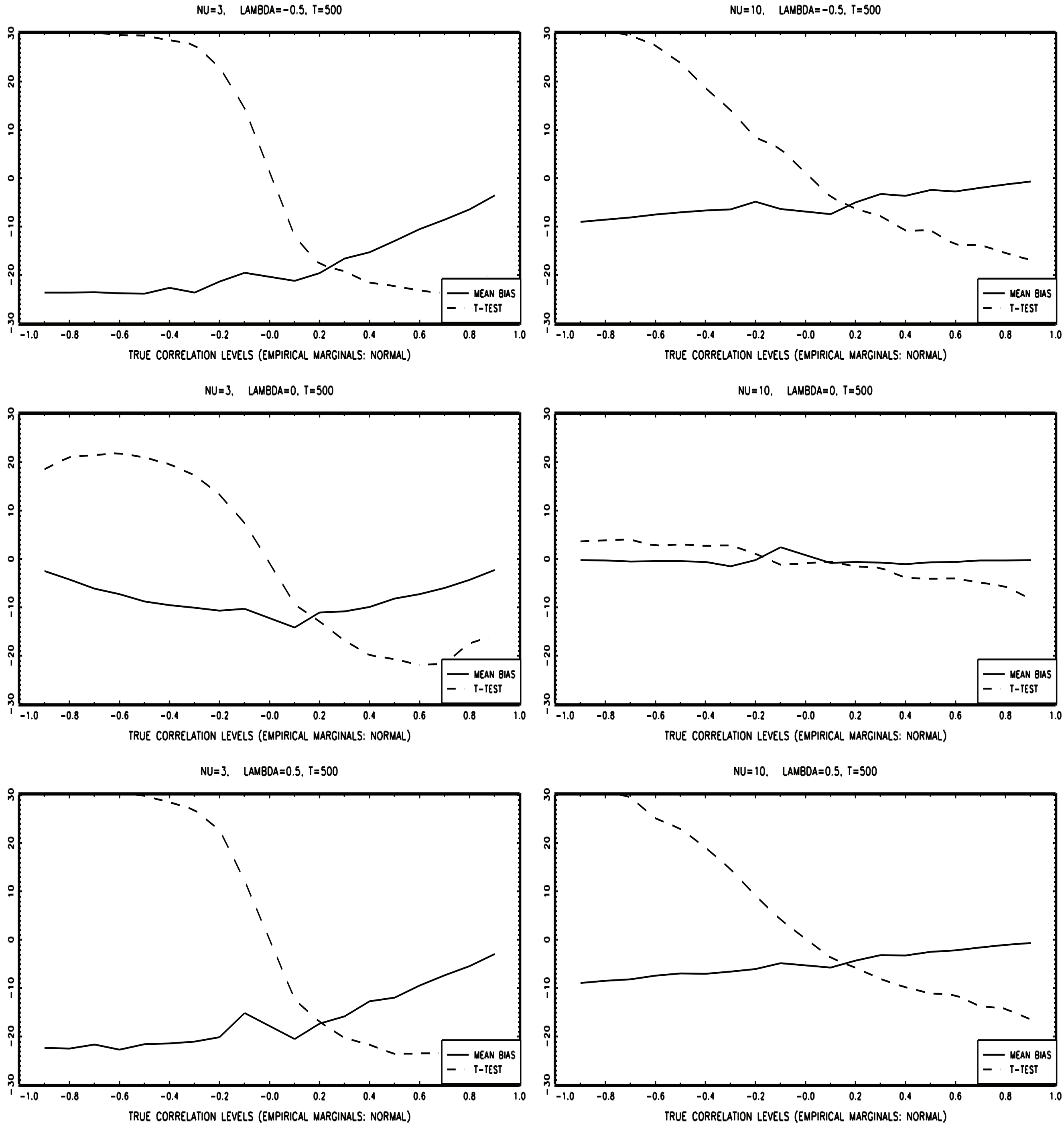

Figure 1: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the bivariate copula correlations when T=500 and the empirical marginals are Normal. The DGP values of ν and λ are reported on top of the plots.

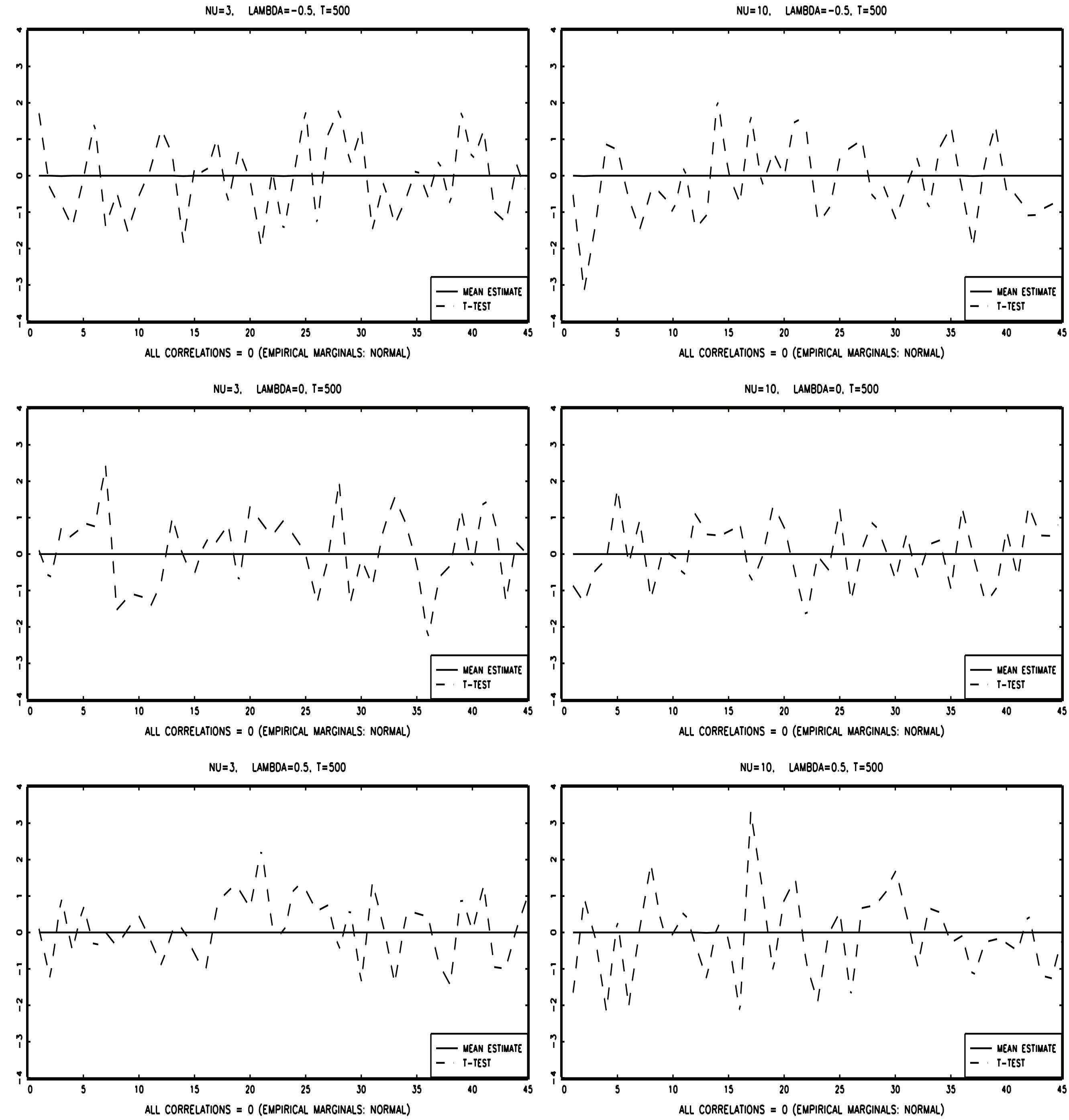

Figure 2: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 1, when T=500 and the empirical marginals are Normal. The DGP values of ν and λ are reported on top of the plots.

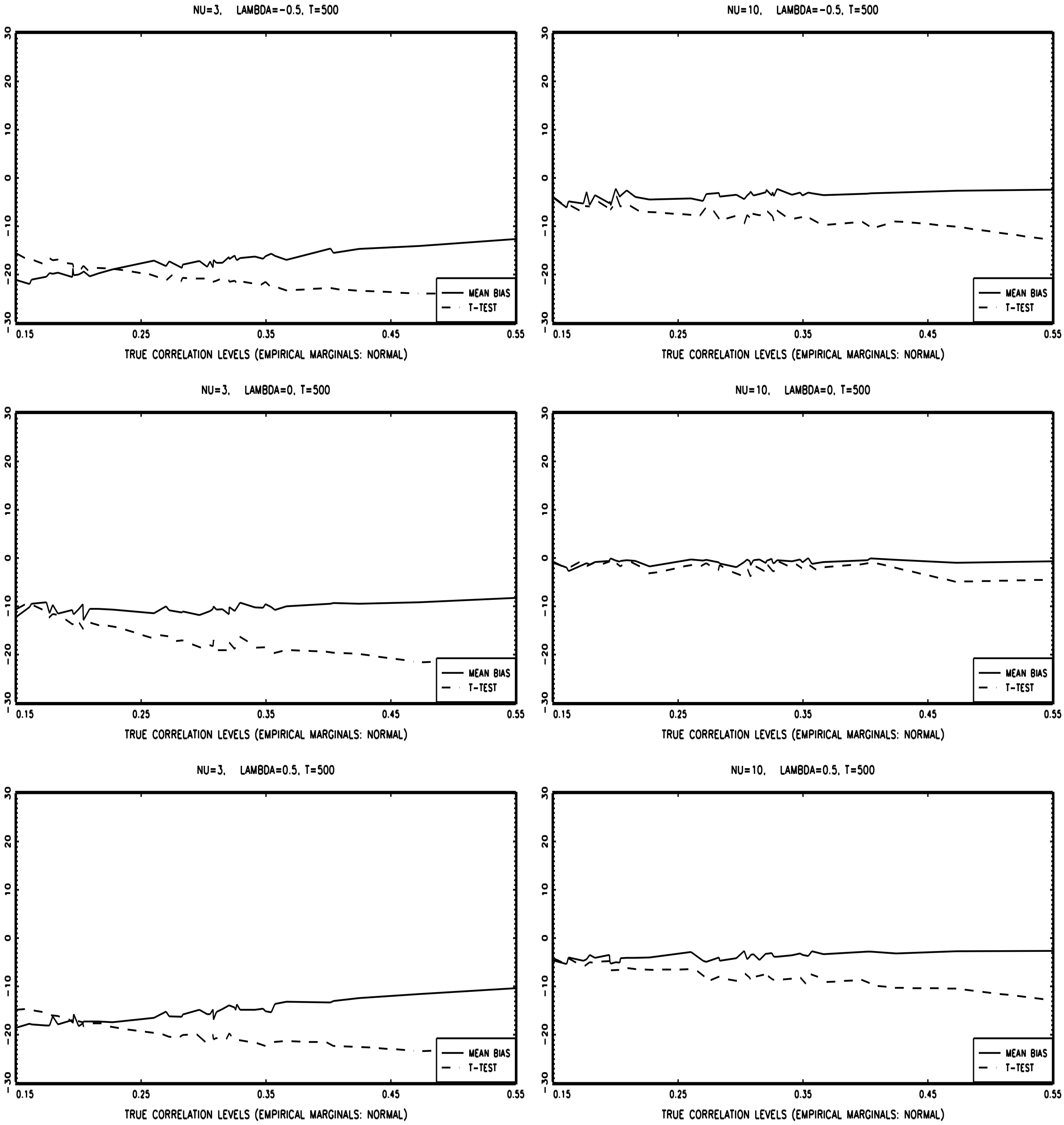

Figure 3: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 2, when T=500 and the empirical marginals are Normal. The DGP values of ν and λ are reported on top of the plots.

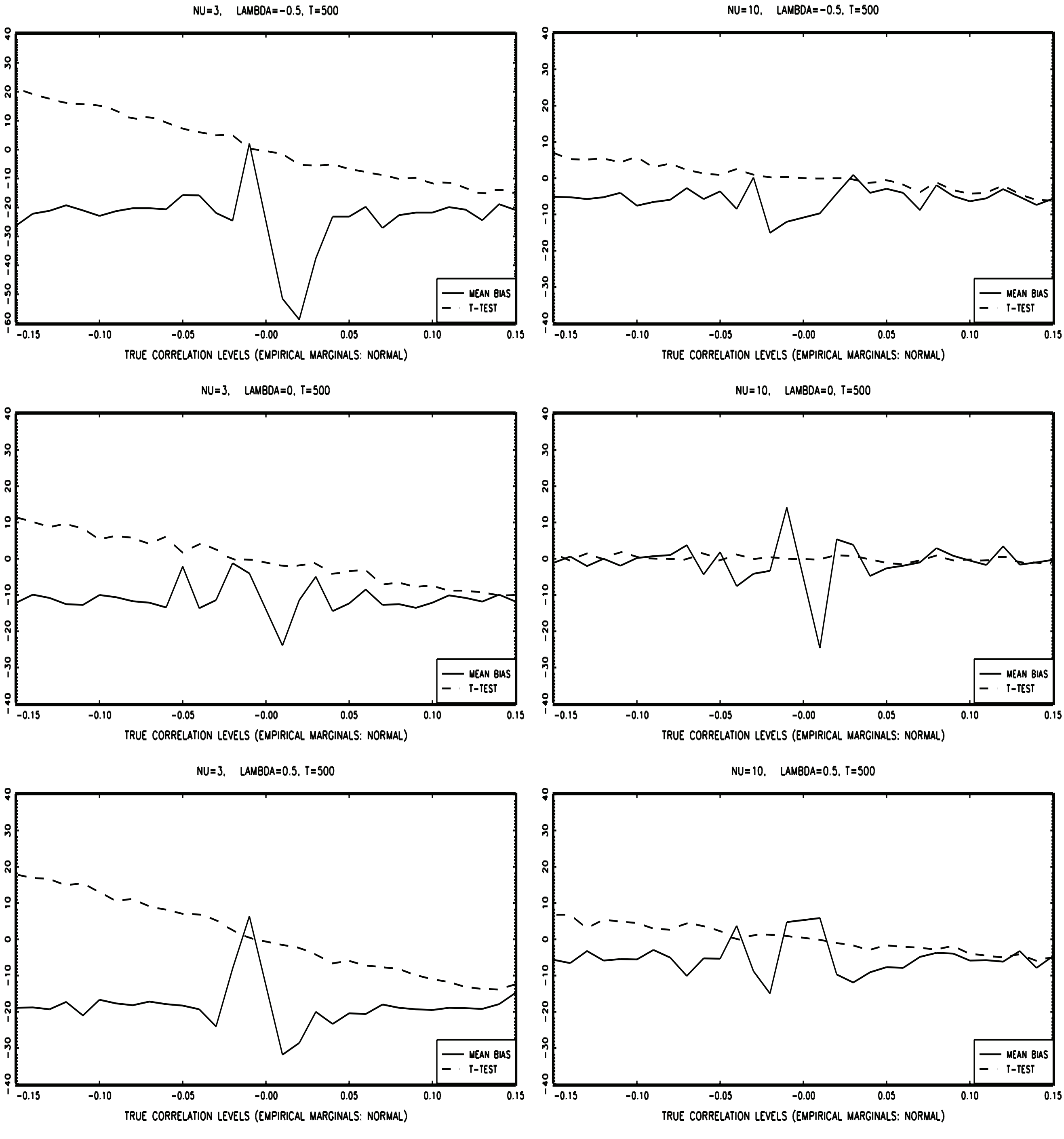

Figure 4: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 3, when T=500 and the empirical marginals are Normal. The DGP values of ν and λ are reported on top of the plots.

3.3 Computational Aspects

We now investigate the computational aspects of copula-GARCH models in terms of numerical convergence[5]. Table 4 reports the numerical convergence failures across different DGPs and sample dimensions.
Table 4 highlights some interesting results:

- *If the true marginal DGP is skewed, strongly leptokurtic and* $T = 500$, the number of numerical failures is extremely high, being close to 33% for the Generalized-t, to 35% for the symmetric t, and up to 48% when normal marginals are used.

- *If the true process is leptokurtic but symmetric*, numerical failures decrease but still remain very high, ranging between 22% (Generalized-t) and 35% (Normal).

- *If the sample dimension increases to* $T = 2000$, convergence errors decrease substantially for the Generalized-t and the Symmetric t, being smaller than 1% if the process is symmetric, while close to 7% for the Generalized-t and to 3% for the symmetric t if the true process is skewed. Interestingly, convergence failures are still very high if normal marginals are used and the process is leptokurtic: over 40% if the DGP is skewed, over 20% if it is symmetric.

- Instead, *if the true DGP shows low leptokurtosis*, then the use of normal marginals determine the lowest convergence errors, both in large ($\leq 1\%$) and in small samples ($\leq 16\%$), thus confirming the results with univariate normal GARCH reported in Hwang and Valls Pereira (2006)[6].

- Finally, we observe that *the type of correlation matrix for the normal copula does not affect the number of convergence failures*, which remains constant across the three different specifications. Furthermore, *all estimated correlation matrices are positive definite without any constraints*: these results confirm recent evidence in Fantazzini (2008) who compares different estimation methods for the T-copula and finds that the correlation matrix can be non positive definite only when dealing with very small samples ($n < 100$) and when the true underlying process has the lowest eigenvalue close to zero.

4 Conclusions

This paper investigated the small sample properties of copula-GARCH models. We analyzed a model specification suitable for many economic and financial applications. This model can account for GARCH variance and skewed and asymmetric noise terms. We performed a Monte Carlo study to assess the potential impact of misspecified margins on the estimation of the marginals and copula parameters under different hypotheses for the Data Generating Process.

[5] We used the `maxlik` library of the GAUSS software and a convergence tolerance for the gradient of the estimated parameters equal to 1e-5;

[6] We investigated the properties of the ML estimates imposing the standard Bollerslev's (1986) non-negativity conditions for the conditional volatility. In order to keep the number of simulated DGPs still tractable, we did not consider here the weaker non-negativity conditions proposed by Hwang and Valls Pereira (2006). However, we expect the estimates of these variations to show similar properties. We leave this topic as an avenue for further research.

With regards to marginals parameters, our simulation studies highlight the strong biases affecting the parameter estimates when the true DGP is skewed and leptokurtic, particularly in the case of GARCH parameters. Besides, they show that the empirical distribution of these parameters is strongly asymmetric and that the mean and median estimates can be rather different. Biases are largest when normal marginals are considered, but even the use of the correctly specified Generalized-t marginals results in very poor estimates when the sample dimension is small. When the T dimension increases, biases and the t-statistics decrease for all parameters (except for the symmetric Student's t degrees of freedom, when the DGP is skewed). Interestingly, no qualitative differences are found across different copula dimensions as well as across different correlation levels. In general, these results point out the difficulties in estimating GARCH models with small samples, thus extending previous simulation evidence in Hwang and Valls Pereira (2006) who, however, considered only univariate models with normally distributed errors and did not examine the effect of different joint distributions.

Copula corr. matrix:	*DIAGONAL*		*DOW JONES returns*		*ILL-SPECIFIED*	
Sample dimension	*True Marginal DGP*	*Failure rate*	*True Marginal DGP*	*Failure rate*	*True Marginal DGP*	*Failure rate*
	Empirical: Generalized-t / Normal copula					
T=500	ν=3,λ=-0.5	33.96%	ν=3,λ=-0.5	33.80%	ν=3,λ=-0.5	34.49%
T=500	ν=3,λ=0	23.19%	ν=3,λ=0	22.63%	ν=3,λ=0	21.96%
T=500	ν=3,λ=0.5	31.63%	ν=3,λ=0.5	32.89%	ν=3,λ=0.5	32.50%
T=500	ν=10,λ=-0.5	27.69%	ν=10,λ=-0.5	28.82%	ν=10,λ=-0.5	28.57%
T=500	ν=10,λ=0	25.52%	ν=10,λ=0	25.96%	ν=10,λ=0	24.74%
T=500	ν=10,λ=0.5	27.54%	ν=10,λ=0.5	29.10%	ν=10,λ=0.5	28.89%
T=2000	ν=3,λ=-0.5	8.51%	ν=3,λ=-0.5	8.51%	ν=3,λ=-0.5	8.13%
T=2000	ν=3,λ=0	0.90%	ν=3,λ=0	1.01%	ν=3,λ=0	0.89%
T=2000	ν=3,λ=0.5	6.78%	ν=3,λ=0.5	6.89%	ν=3,λ=0.5	7.29%
T=2000	ν=10,λ=-0.5	2.99%	ν=10,λ=-0.5	3.01%	ν=10,λ=-0.5	3.19%
T=2000	ν=10,λ=0	0.88%	ν=10,λ=0	0.93%	ν=10,λ=0	0.73%
T=2000	ν=10,λ=0.5	3.07%	ν=10,λ=0.5	3.02%	ν=10,λ=0.5	3.18%
	Empirical: Student's t / Normal copula					
T=500	ν=3,λ=-0.5	34.78%	ν=3,λ=-0.5	35.58%	T=500,ν=3,λ=-0.5	35.40%
T=500	ν=3,λ=0	22.05%	ν=3,λ=0	22.32%	T=500,ν=3,λ=0	21.81%
T=500	ν=3,λ=0.5	35.09%	ν=3,λ=0.5	35.36%	T=500,ν=3,λ=0.5	35.64%
T=500	ν=10,λ=-0.5	26.12%	ν=10,λ=-0.5	26.67%	T=500,ν=10,λ=-0.5	26.55%
T=500	ν=10,λ=0	23.70%	ν=10,λ=0	23.59%	T=500,ν=10,λ=0	24.24%
T=500	ν=10,λ=0.5	25.69%	ν=10,λ=0.5	26.90%	T=500,ν=10,λ=0.5	26.39%
T=2000	ν=3,λ=-0.5	3.76%	ν=3,λ=-0.5	3.43%	T=2000,ν=3,λ=-0.5	3.63%
T=2000	ν=3,λ=0	0.95%	ν=3,λ=0	0.95%	T=2000,ν=3,λ=0	0.96%
T=2000	ν=3,λ=0.5	3.16%	ν=3,λ=0.5	3.40%	T=2000,ν=3,λ=0.5	3.50%
T=2000	ν=10,λ=-0.5	2.14%	ν=10,λ=-0.5	1.81%	T=2000,ν=10,λ=-0.5	2.18%
T=2000	ν=10,λ=0	0.83%	ν=10,λ=0	0.76%	T=2000,ν=10,λ=0	0.68%
T=2000	ν=10,λ=0.5	1.96%	ν=10,λ=0.5	1.84%	T=2000,ν=10,λ=0.5	2.10%
	Empirical: Normal / Normal copula					
T=500	ν=3,λ=-0.5	48.39%	ν=3,λ=-0.5	48.02%	ν=3,λ=-0.5	48.25%
T=500	ν=3,λ=0	34.72%	ν=3,λ=0	34.31%	ν=3,λ=0	34.84%
T=500	ν=3,λ=0.5	48.68%	ν=3,λ=0.5	48.67%	ν=3,λ=0.5	48.26%
T=500	ν=10,λ=-0.5	16.57%	ν=10,λ=-0.5	16.92%	ν=10,λ=-0.5	16.56%
T=500	ν=10,λ=0	14.59%	ν=10,λ=0	14.99%	ν=10,λ=0	14.33%
T=500	ν=10,λ=0.5	16.52%	ν=10,λ=0.5	15.99%	ν=10,λ=0.5	16.65%
T=2000	ν=3,λ=-0.5	42.36%	ν=3,λ=-0.5	43.00%	ν=3,λ=-0.5	42.17%
T=2000	ν=3,λ=0	26.89%	ν=3,λ=0	27.76%	ν=3,λ=0	27.24%
T=2000	ν=3,λ=0.5	42.90%	ν=3,λ=0.5	43.39%	ν=3,λ=0.5	41.95%
T=2000	ν=10,λ=-0.5	0.15%	ν=10,λ=-0.5	0.13%	ν=10,λ=-0.5	0.15%
T=2000	ν=10,λ=0	0.08%	ν=10,λ=0	0.13%	ν=10,λ=0	0.13%
T=2000	ν=10,λ=0.5	0.13%	ν=10,λ=0.5	0.15%	ν=10,λ=0.5	0.08%

Table 4: Numerical convergence failures across different DGPs and sample dimensions

With regards to the dependence parameters, when there is skewness in the data and symmetric marginals are used, the estimated correlations are negatively biased, and the bias increases when moving from the Student's t to the normal (marginals) distribution, reaching values as high as 25% of the true values. A striking result is that this bias reaches its highest value when the correlation is strongly negative, and viceversa. This result remains unchanged with both positively skewed and negatively skewed data, while no biases are found if the variables are uncorrelated. When the marginals leptokurtosis decreases, the biases in the correlations decrease as expected, but the previous conclusions remain unaltered. No major differences in the biases are found when moving from small samples to large samples (even though the t-statistics are slightly smaller), as well as when moving from bivariate to 10-variate normal copulas. Interestingly, these results remain mostly unchanged even when using an ill-specified correlation matrix for the normal copula, whose lowest eigenvalue is close to zero.

The second contribution of this paper is an analysis of copula-GARCH models in terms of numerical convergence and positive definiteness of the estimated copula correlation matrix. The numerical maximization of the log-likelihood function fails to converge mostly when the marginals DGP is leptokurtic and asymmetric and we deal with small samples. In this case, the use of misspecified normal marginals determines the highest number of convergence failures (close to 50%), but the correctly specified Generalized-t marginals do not perform much better (over 30%). Similarly to the biases in the copula correlations, leptokurtosis has a stronger effect on numerical convergence than asymmetry, and convergence improves when leptokurtosis decreases.

Interestingly, we observe that the type of correlation matrix for the normal copula does not affect the number of convergence failures, which remains constant across different specifications. Furthermore, our simulation studies show that even when the true correlation matrix is ill-conditioned, the estimated correlation matrix is always positive definite without any constraints. These results confirm recent evidence in Fantazzini (2008) who compares different estimation methods for the T-copula and finds that the correlation matrix can be non positive definite only when dealing with very small samples ($n < 100$) and when the true underlying process has the lowest eigenvalue close to zero.

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A Technical appendix

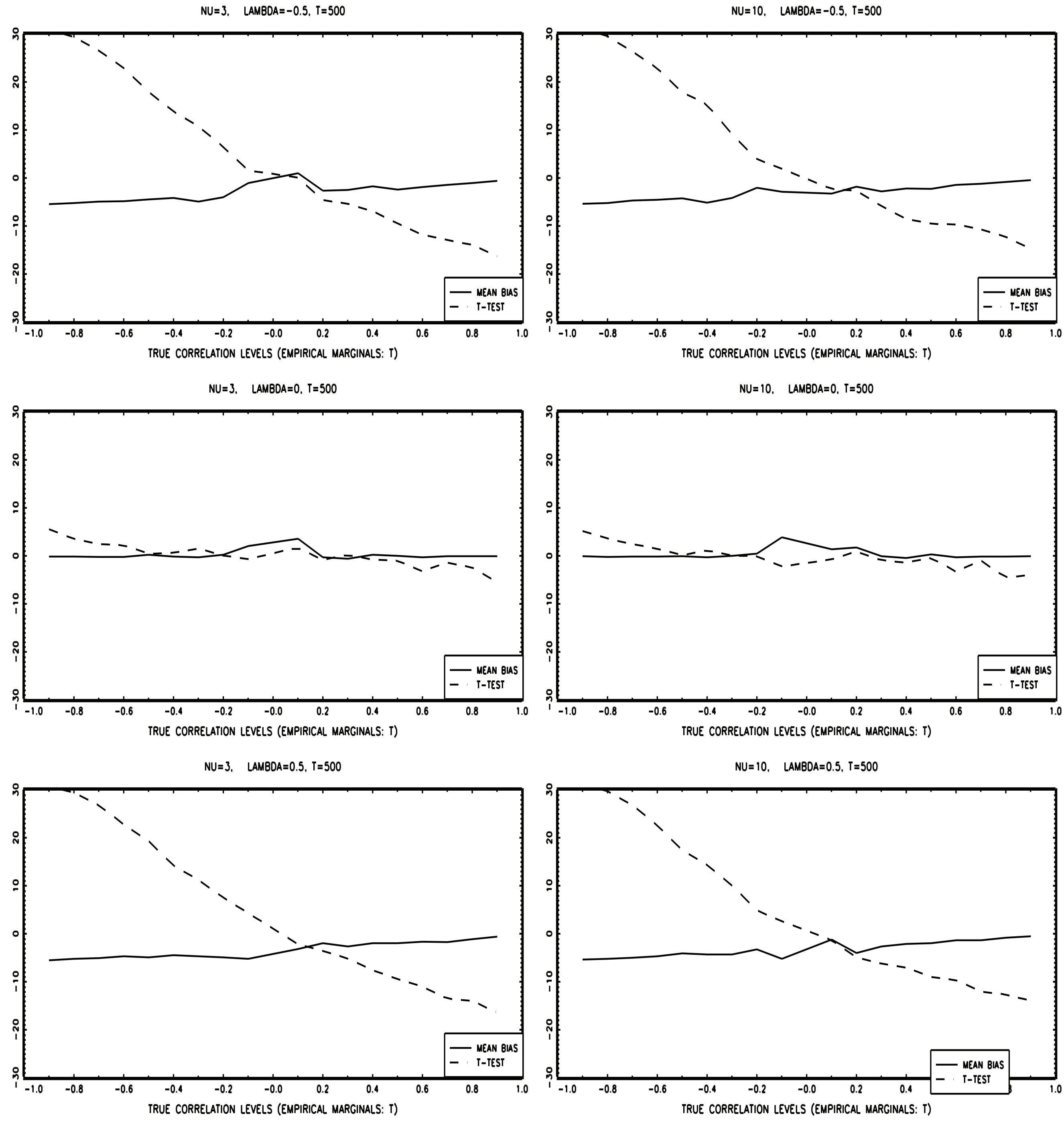

Figure 5: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the bivariate copula correlations when T=500 and the empirical marginals are symmetric Student's t. The DGP values of ν and λ are reported in the top of the plots.

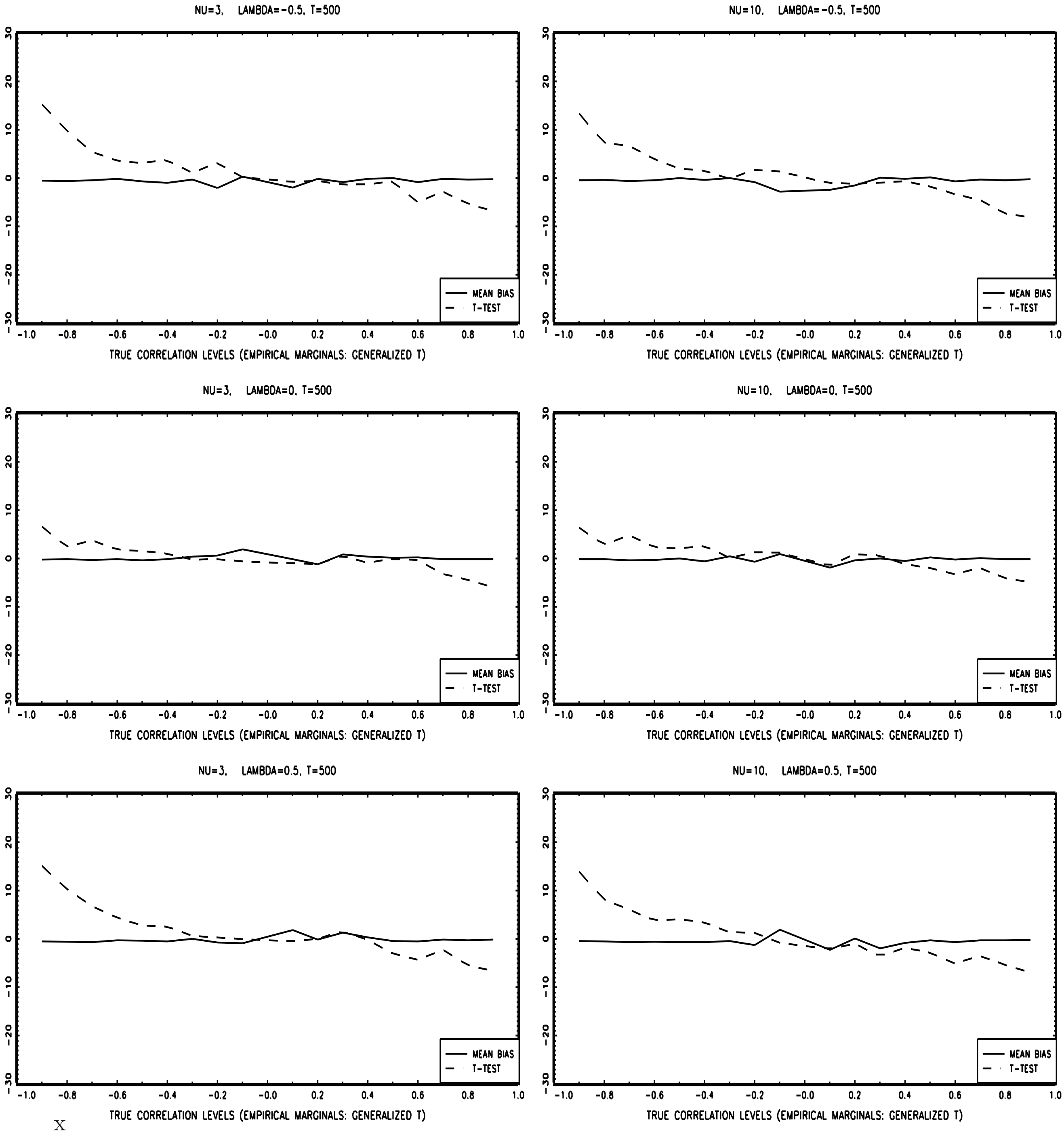

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Figure 6: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the bivariate copula correlations when T=500 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

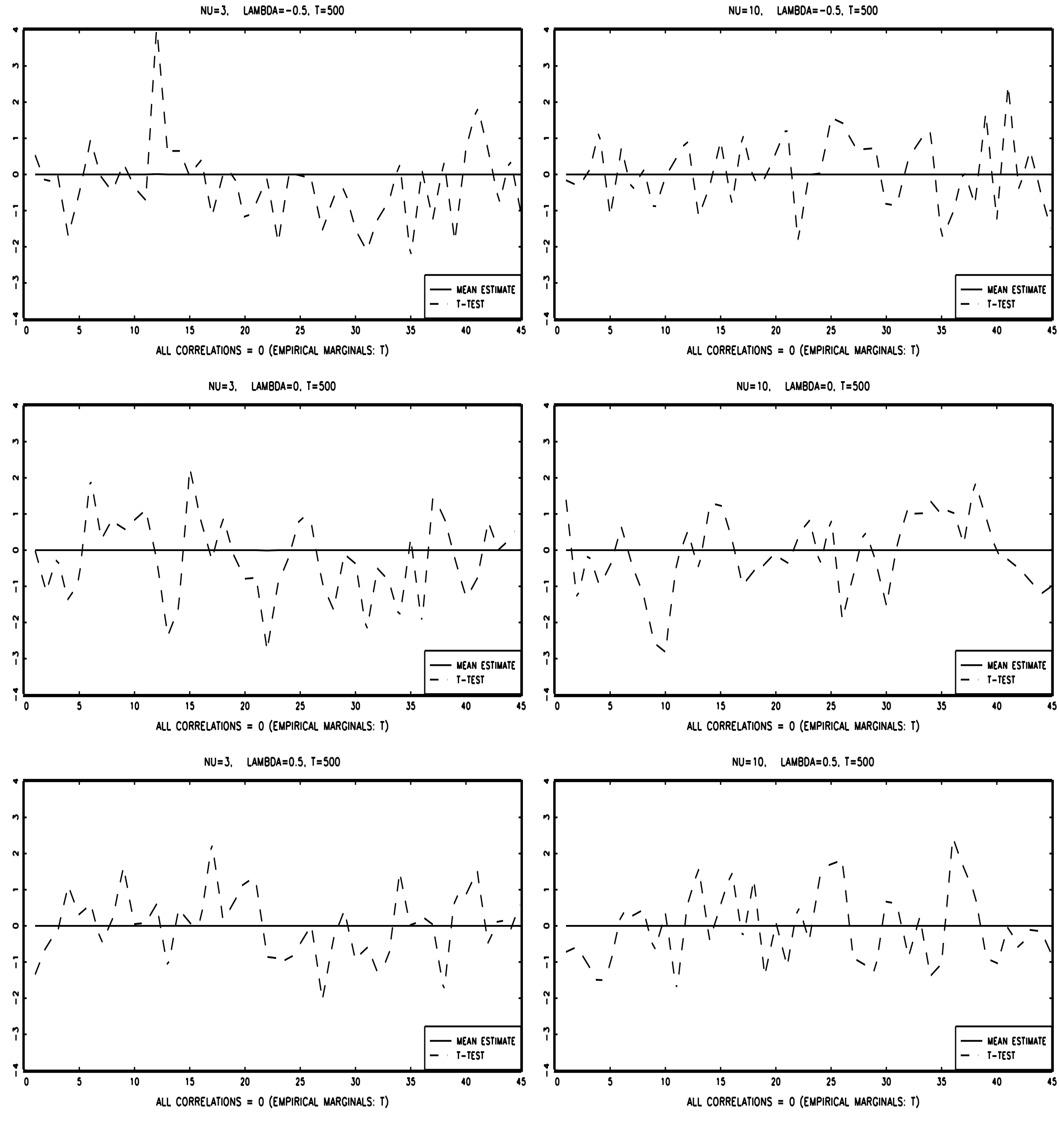

Figure 7: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 1, when T=500 and the empirical marginals are symmetric Student's t. The DGP values of ν and λ are reported in the top of the plots.

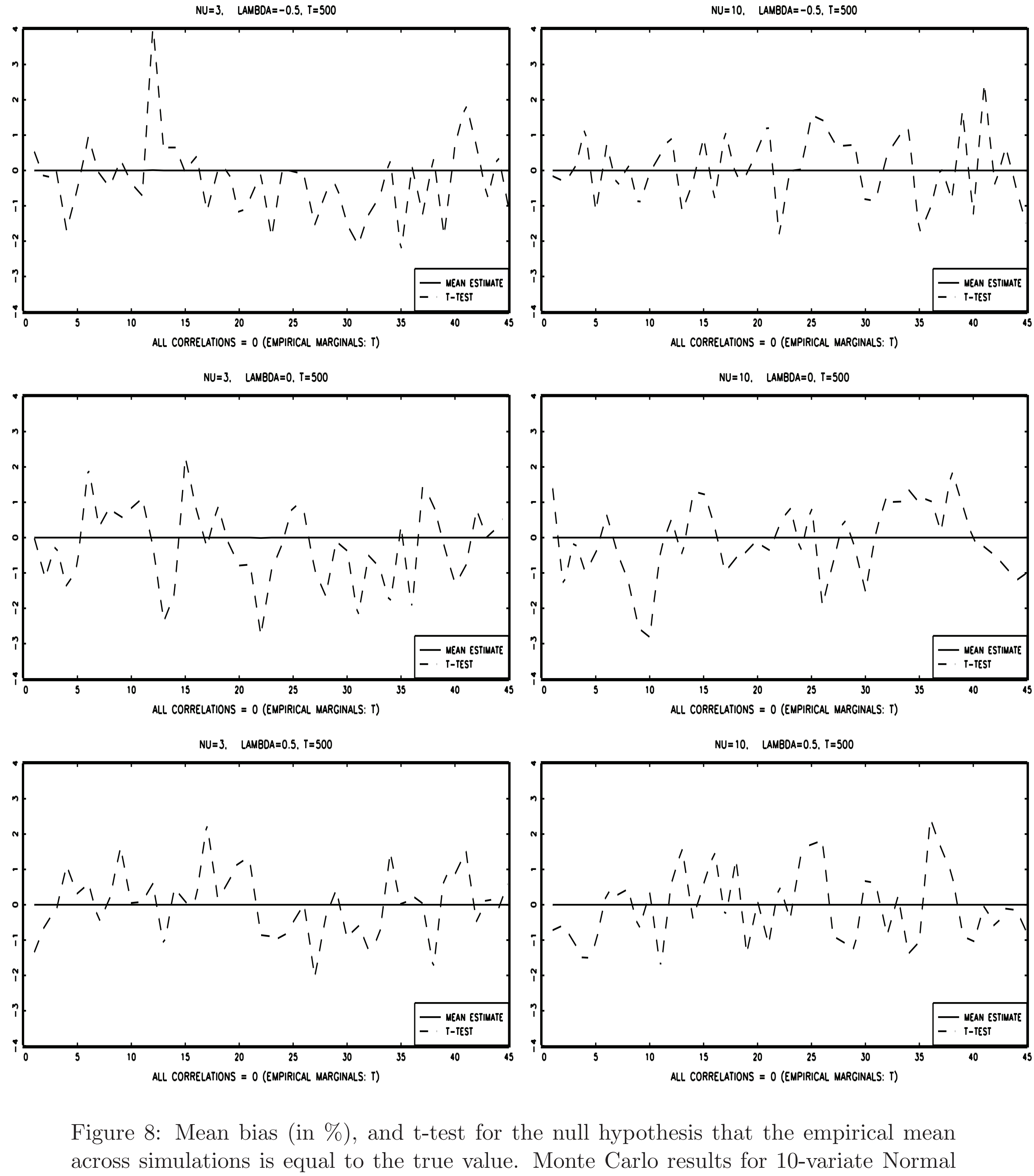

Figure 8: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 1, when T=500 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

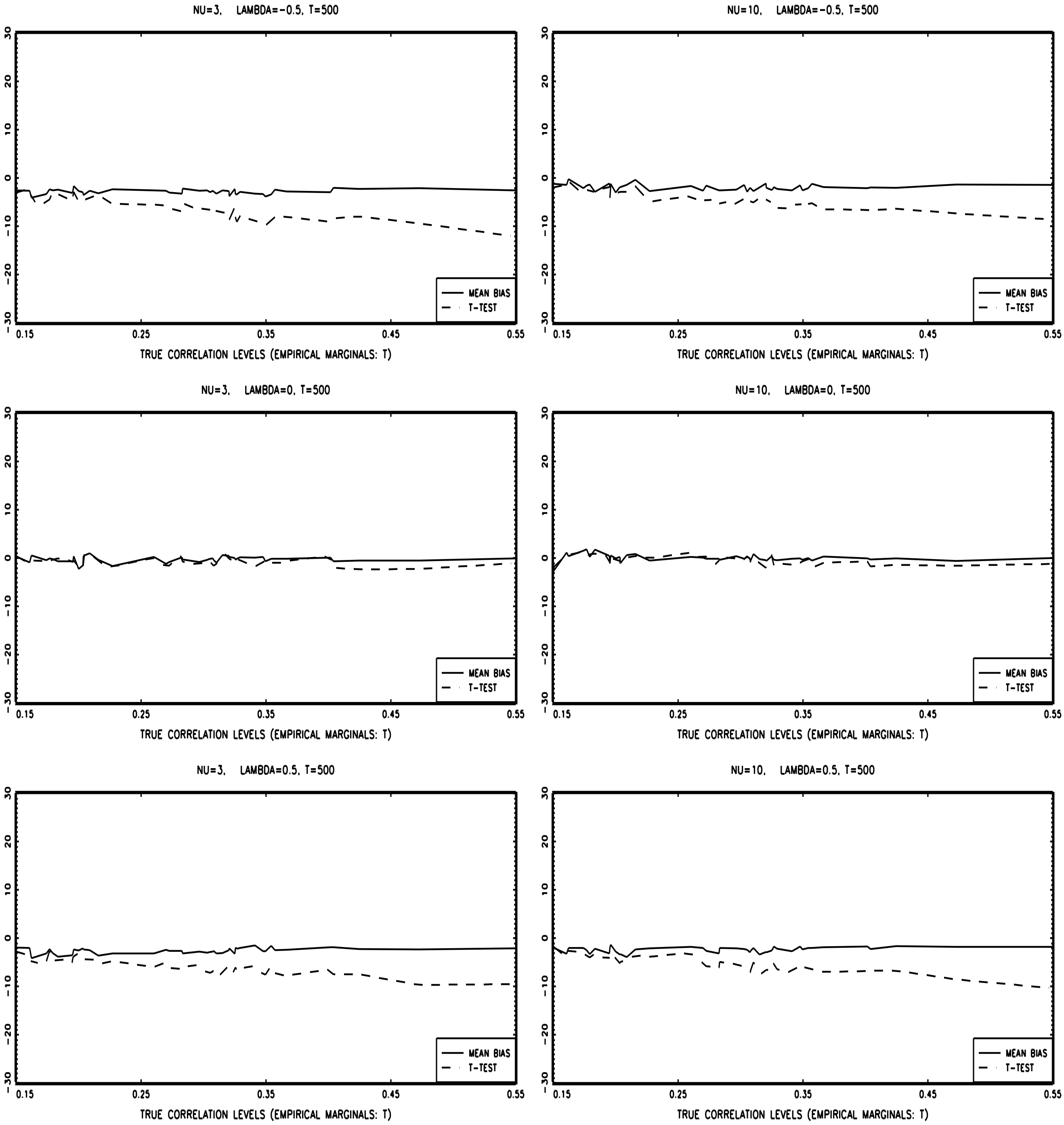

Figure 9: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 2, when T=500 and the empirical marginals are symmetric Student's t. The DGP values of ν and λ are reported in the top of the plots.

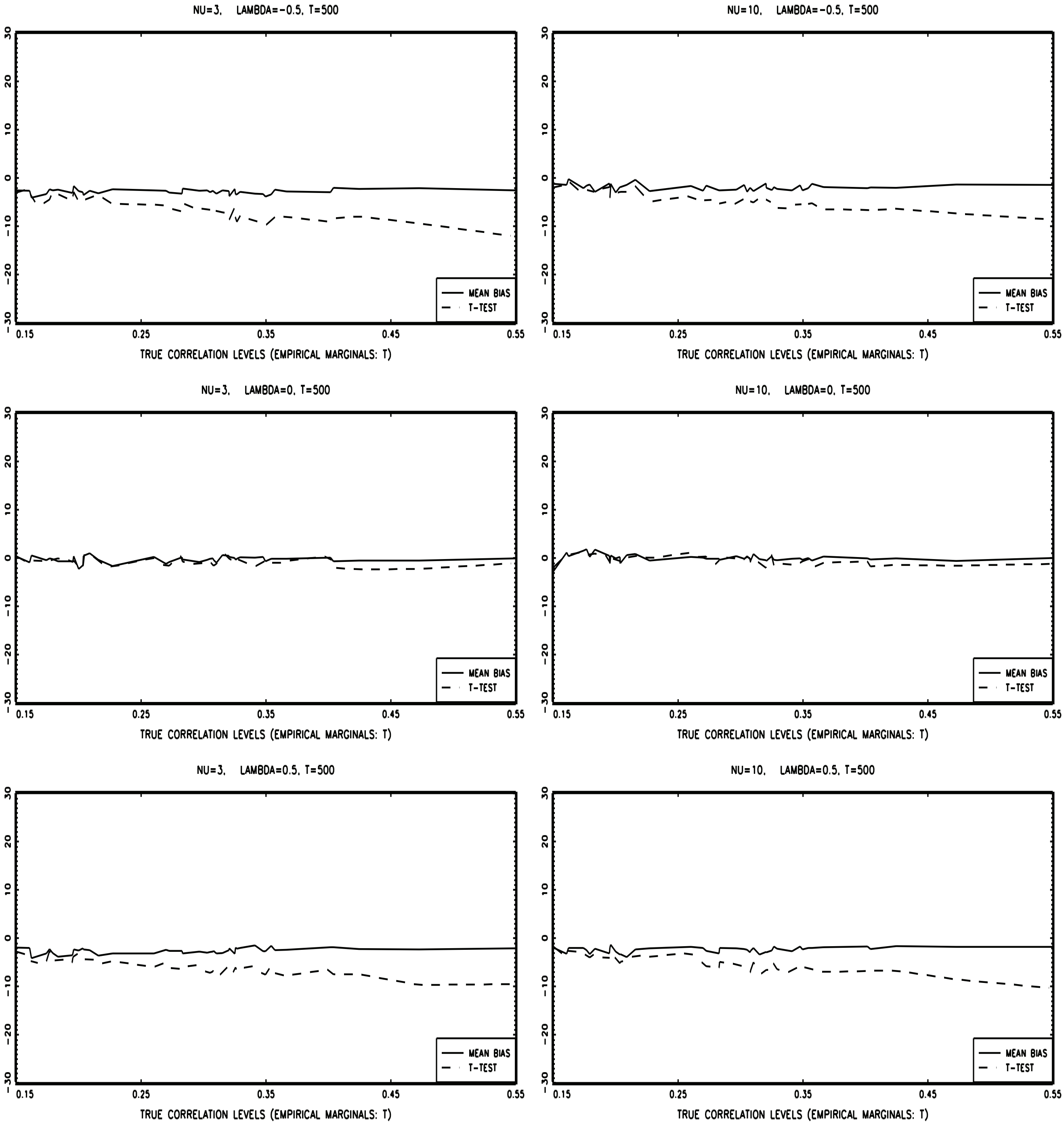

Figure 10: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 2, when T=500 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

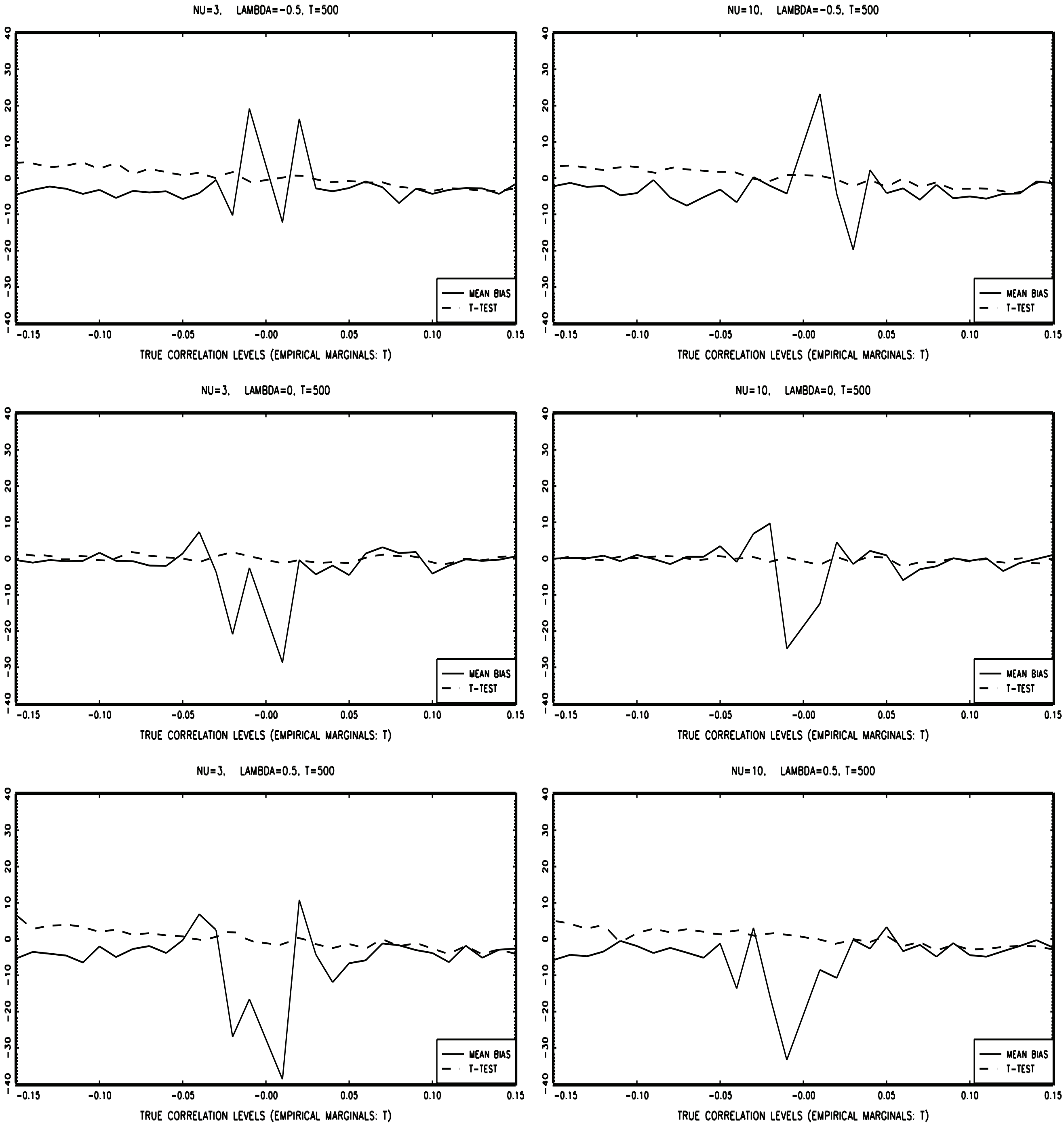

Figure 11: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 3, when T=500 and the empirical marginals are symmetric Student's t.

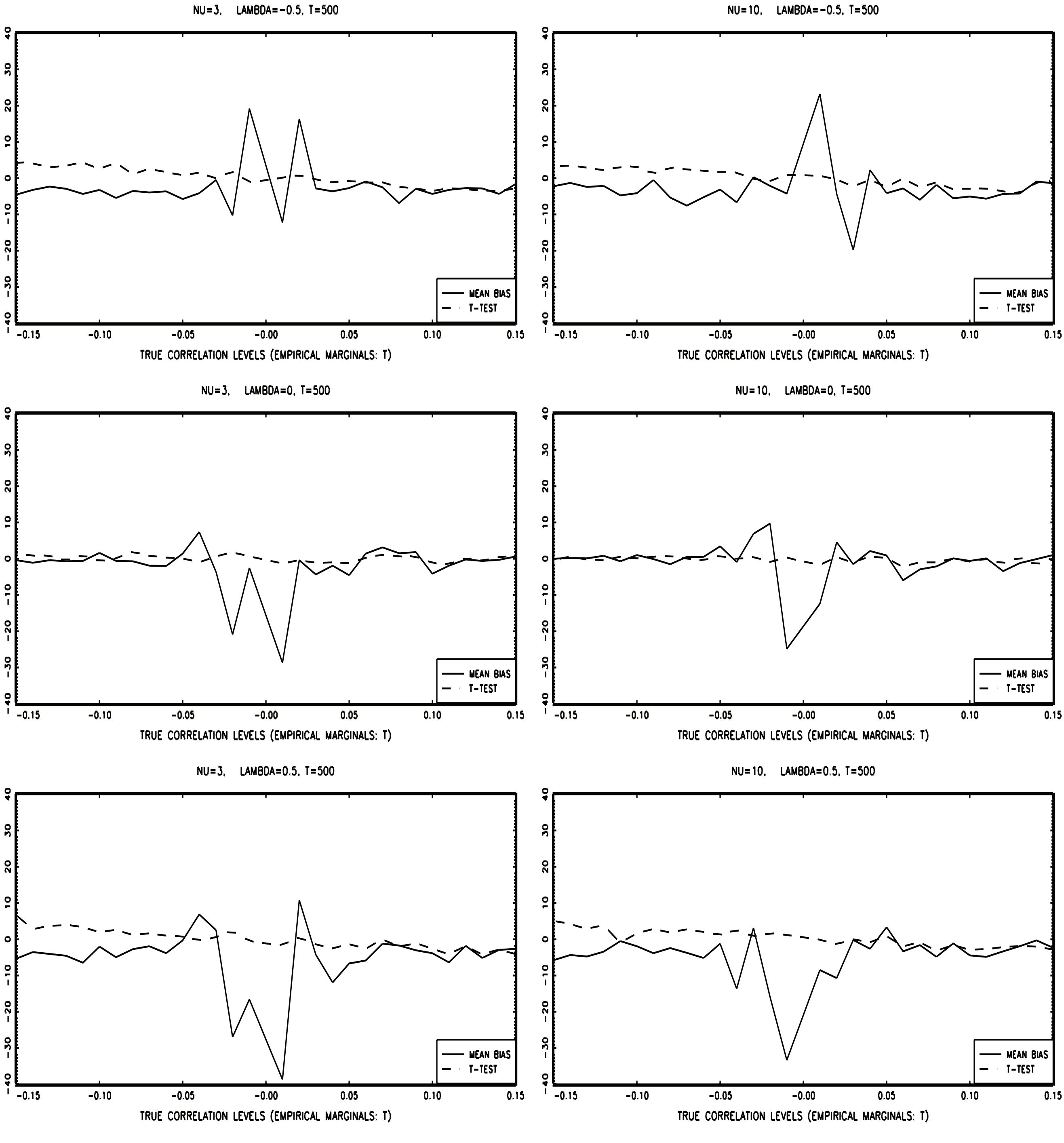

Figure 12: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with the correlation matrix reported in Table 3, when T=500 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

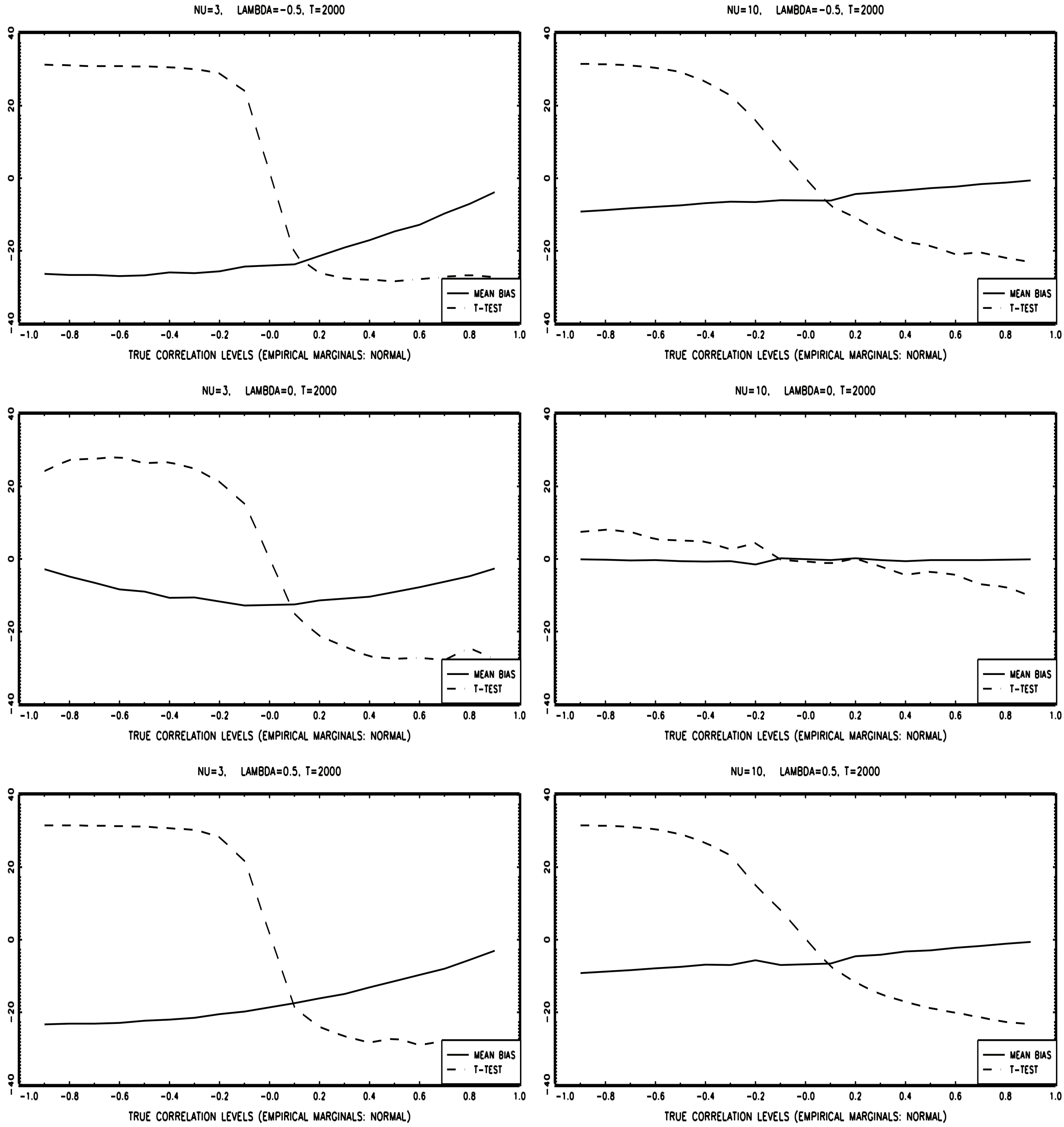

Figure 13: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the bivariate copula correlation when T=2000 and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

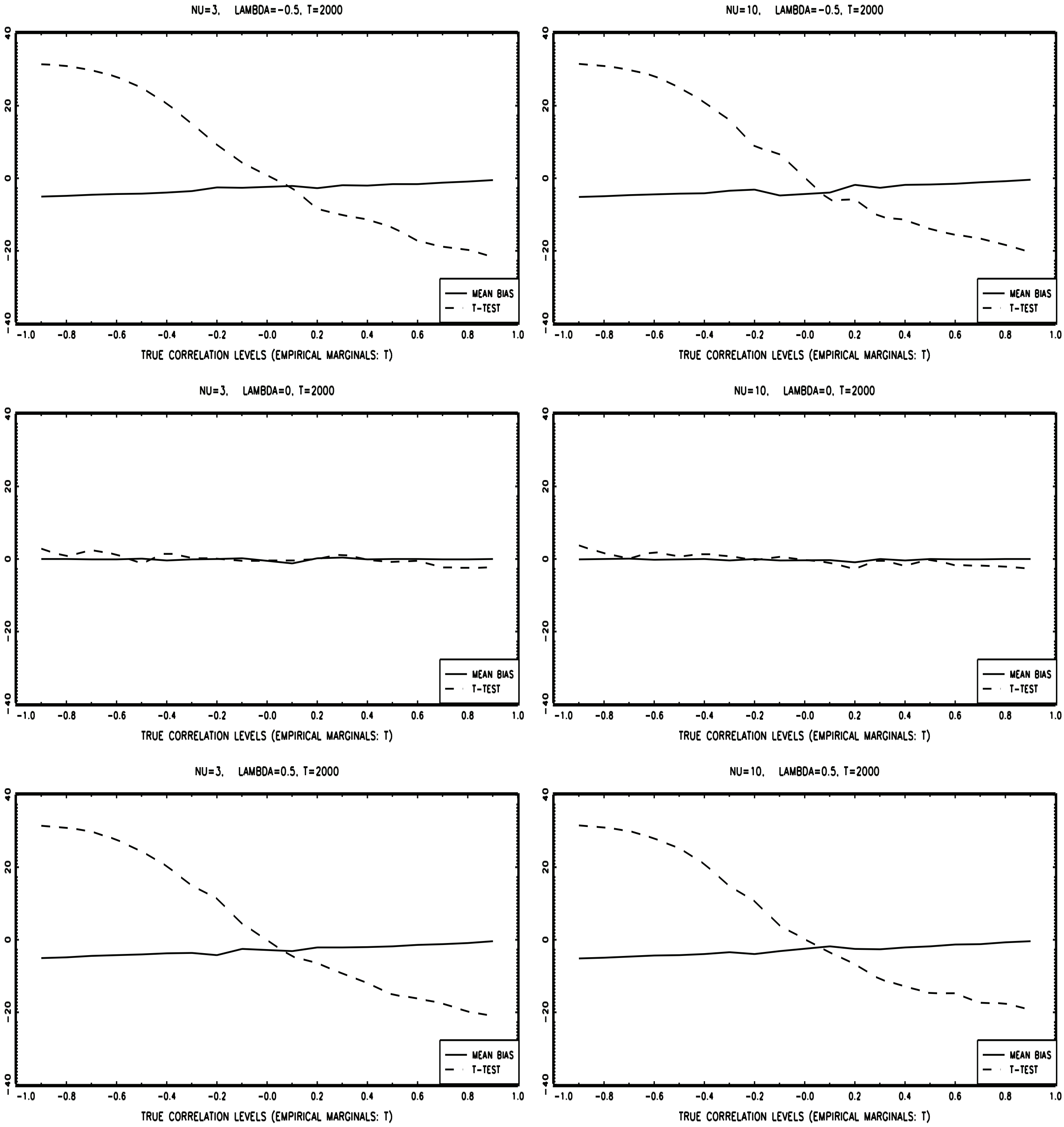

Figure 14: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the bivariate copula correlation when T=2000 and the empirical marginals are symmetric Student's t. The DGP values of ν and λ are reported in the top of the plots.

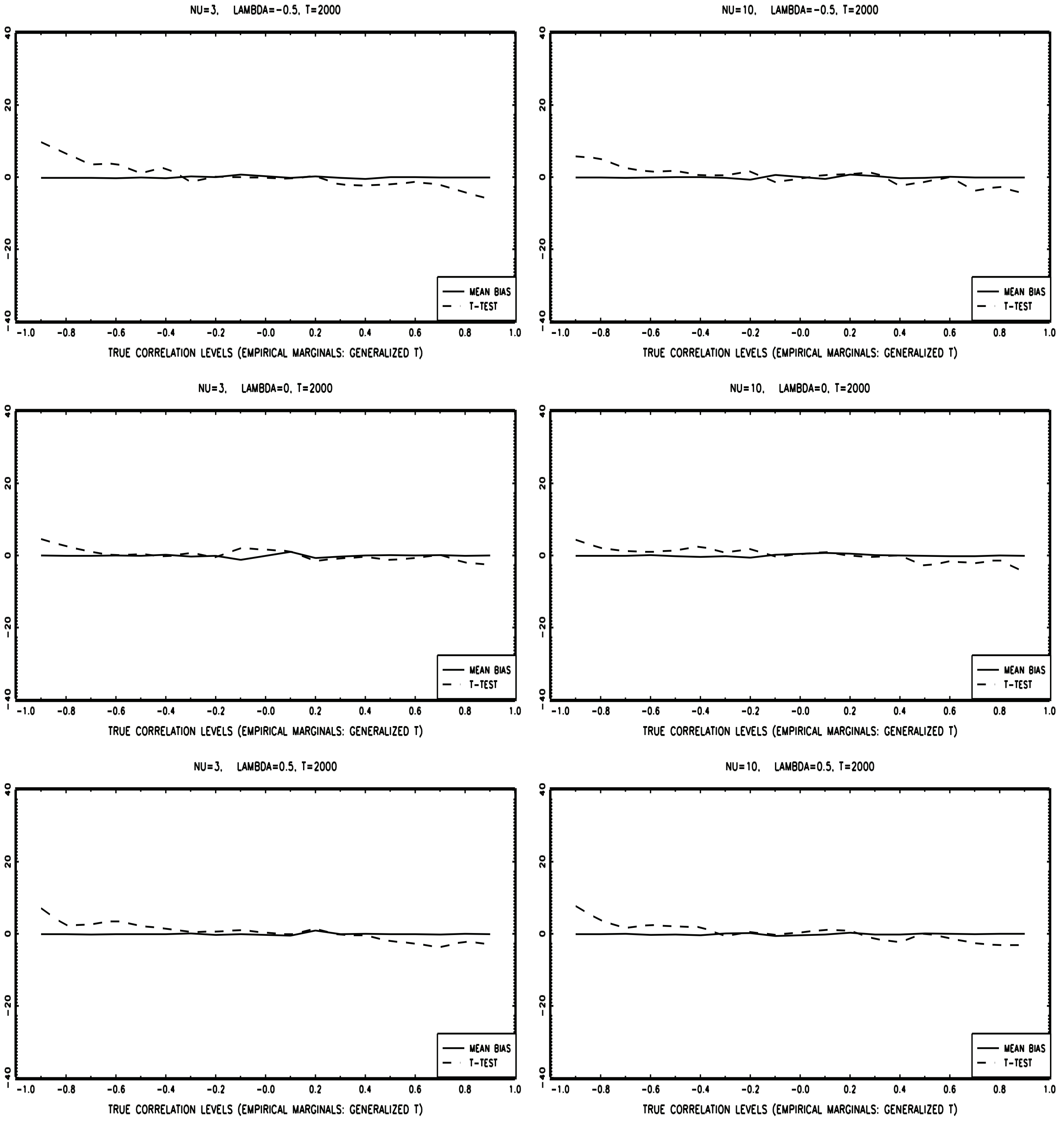

Figure 15: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the bivariate copula correlation when T=2000 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

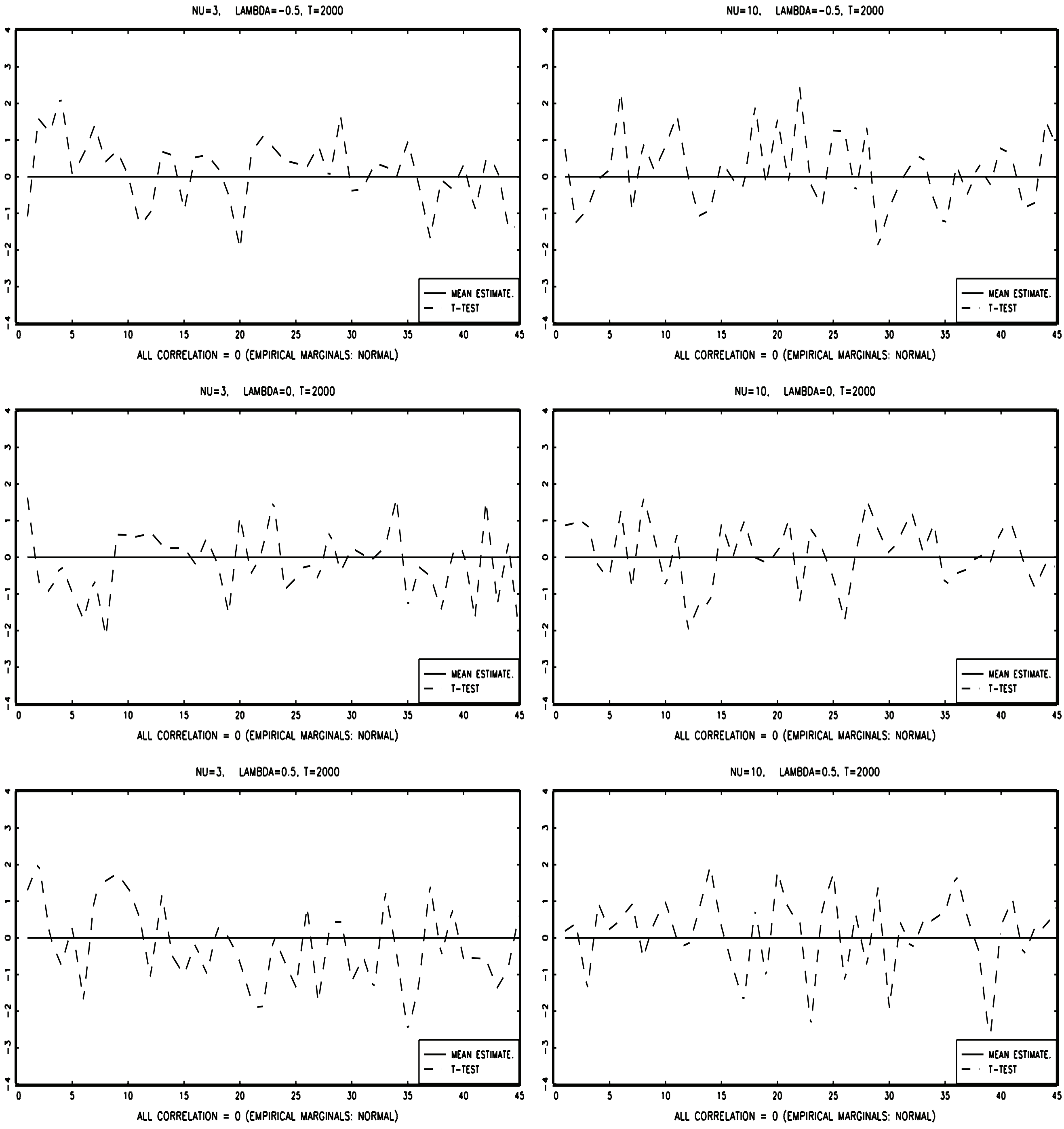

Figure 16: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 1, when T=2000 and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

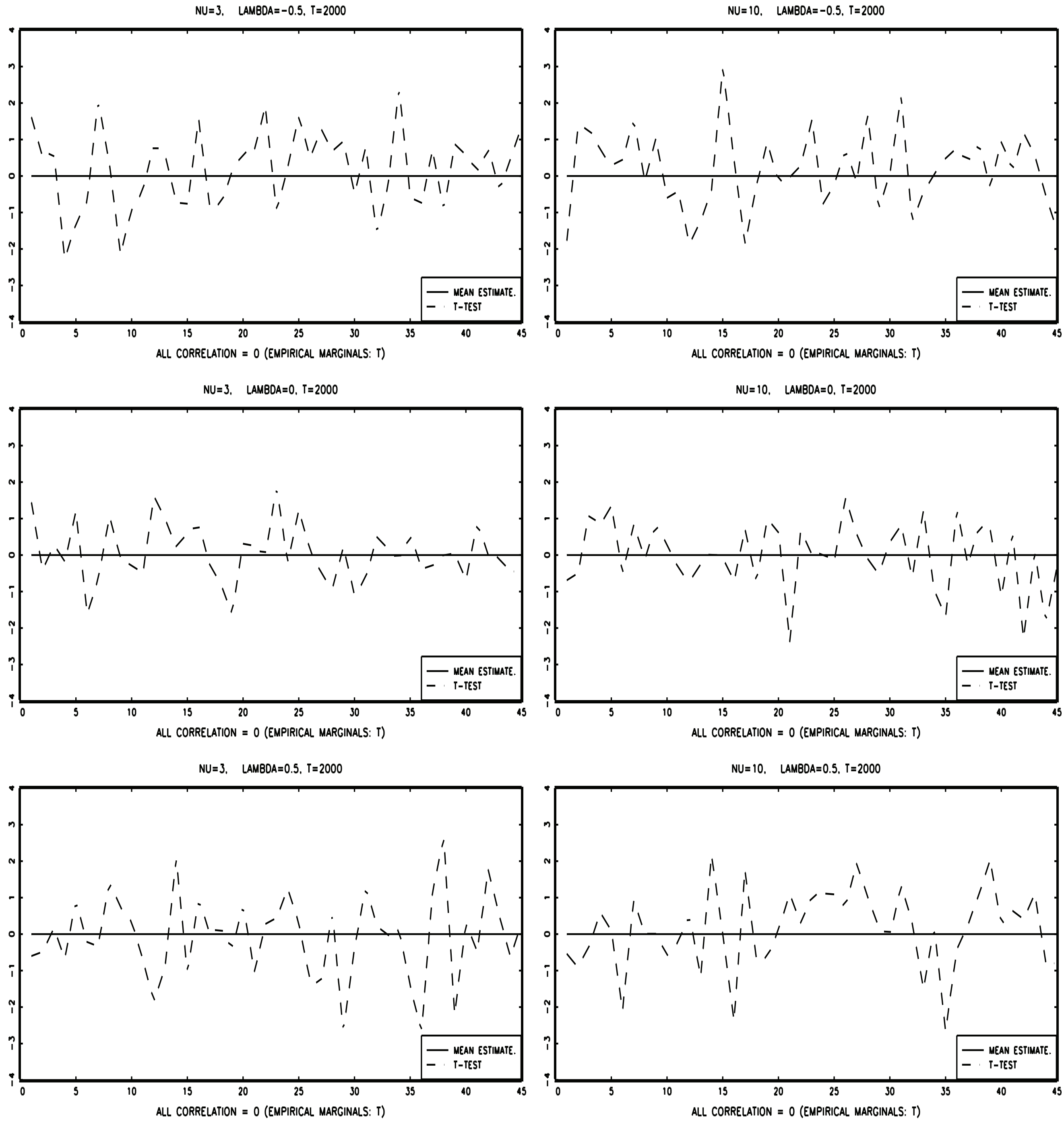

Figure 17: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 1, when T=2000 and the empirical marginals are symmetric Student's t. The DGP values of ν and λ are reported in the top of the plots.

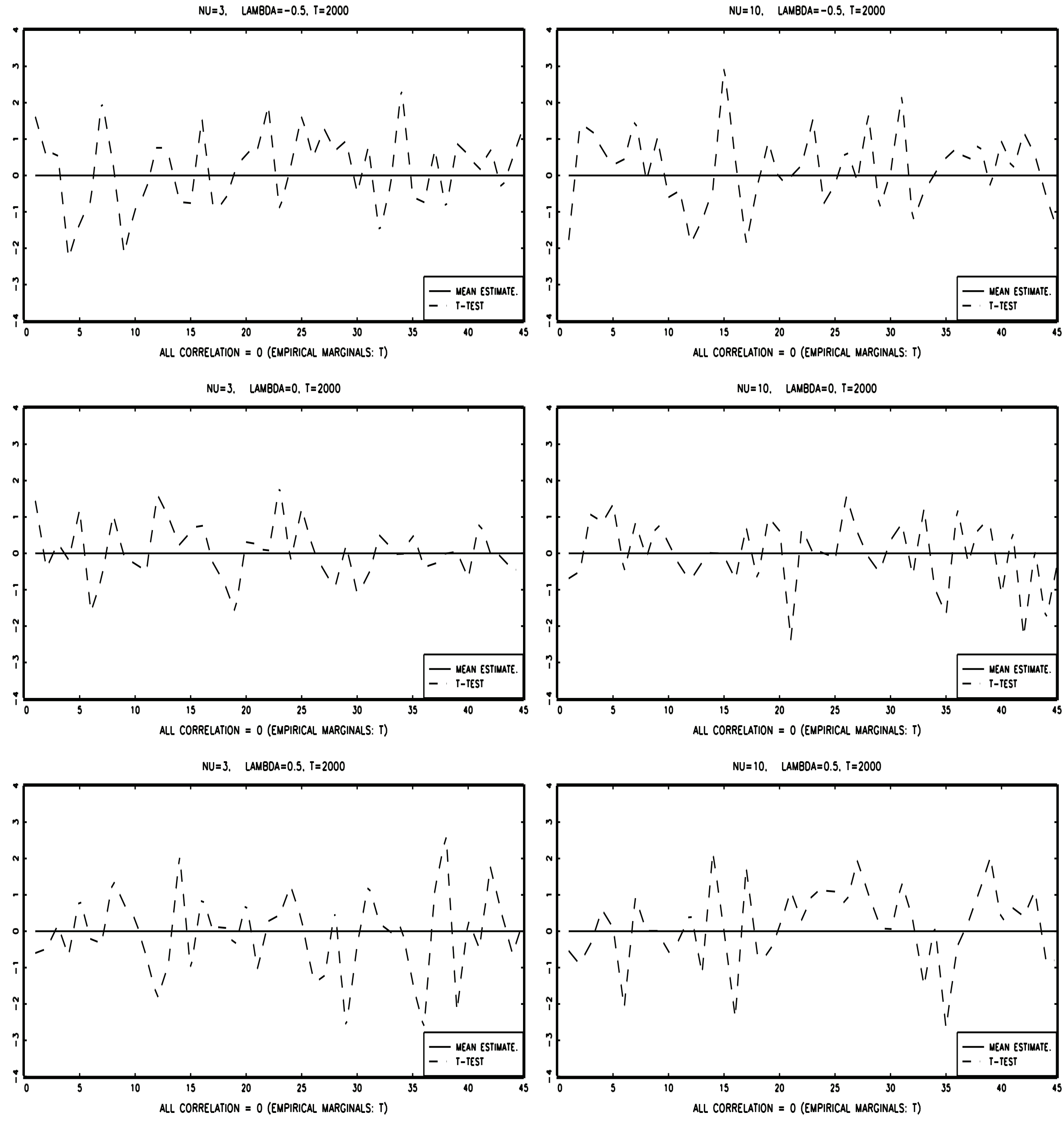

Figure 18: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 1, when T=2000 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

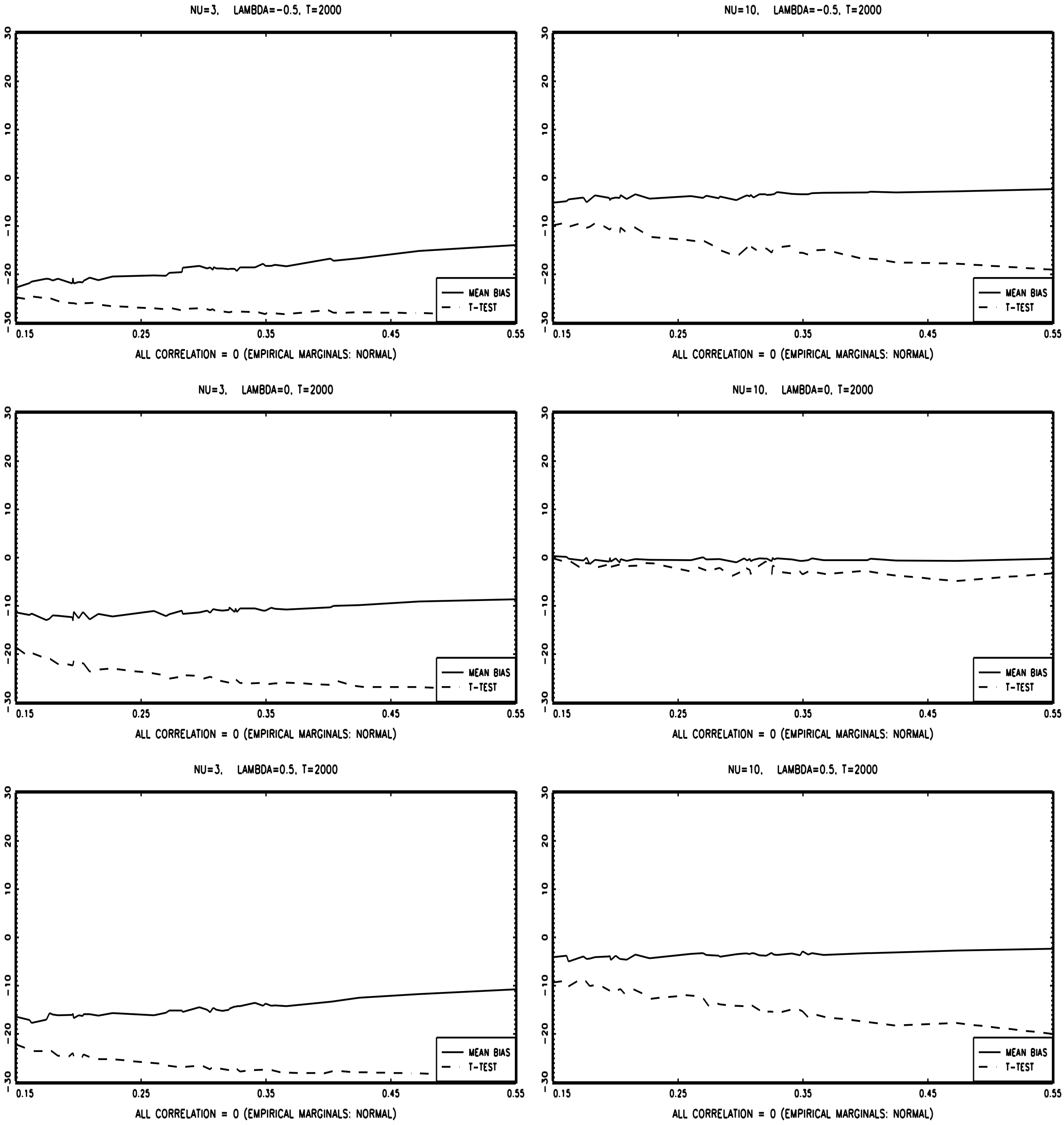

Figure 19: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 2, when T=2000 and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

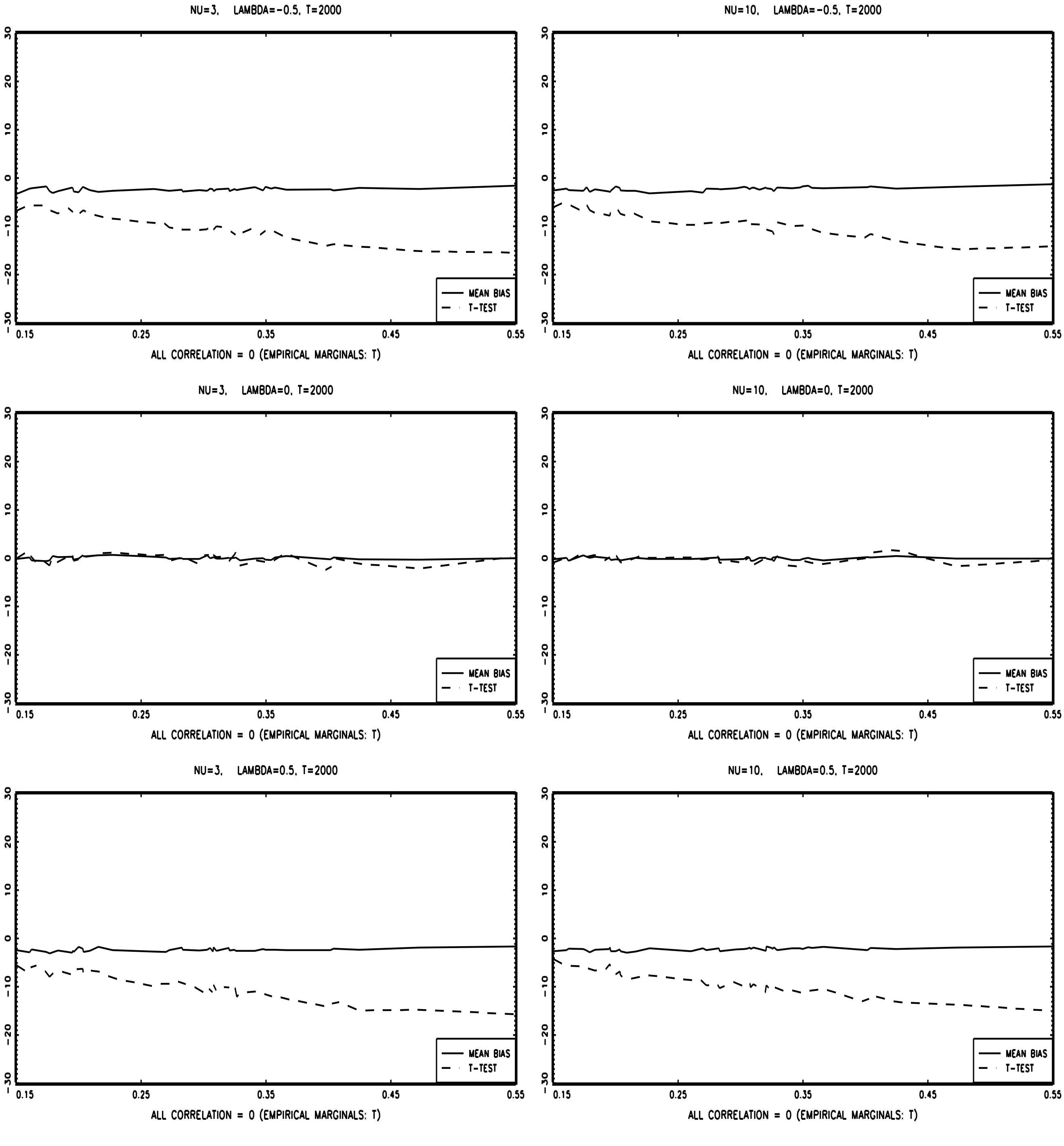

Figure 20: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 2, when T=2000 and the empirical marginals are symmetric Student's t. The DGP values of ν and λ are reported in the top of the plots.

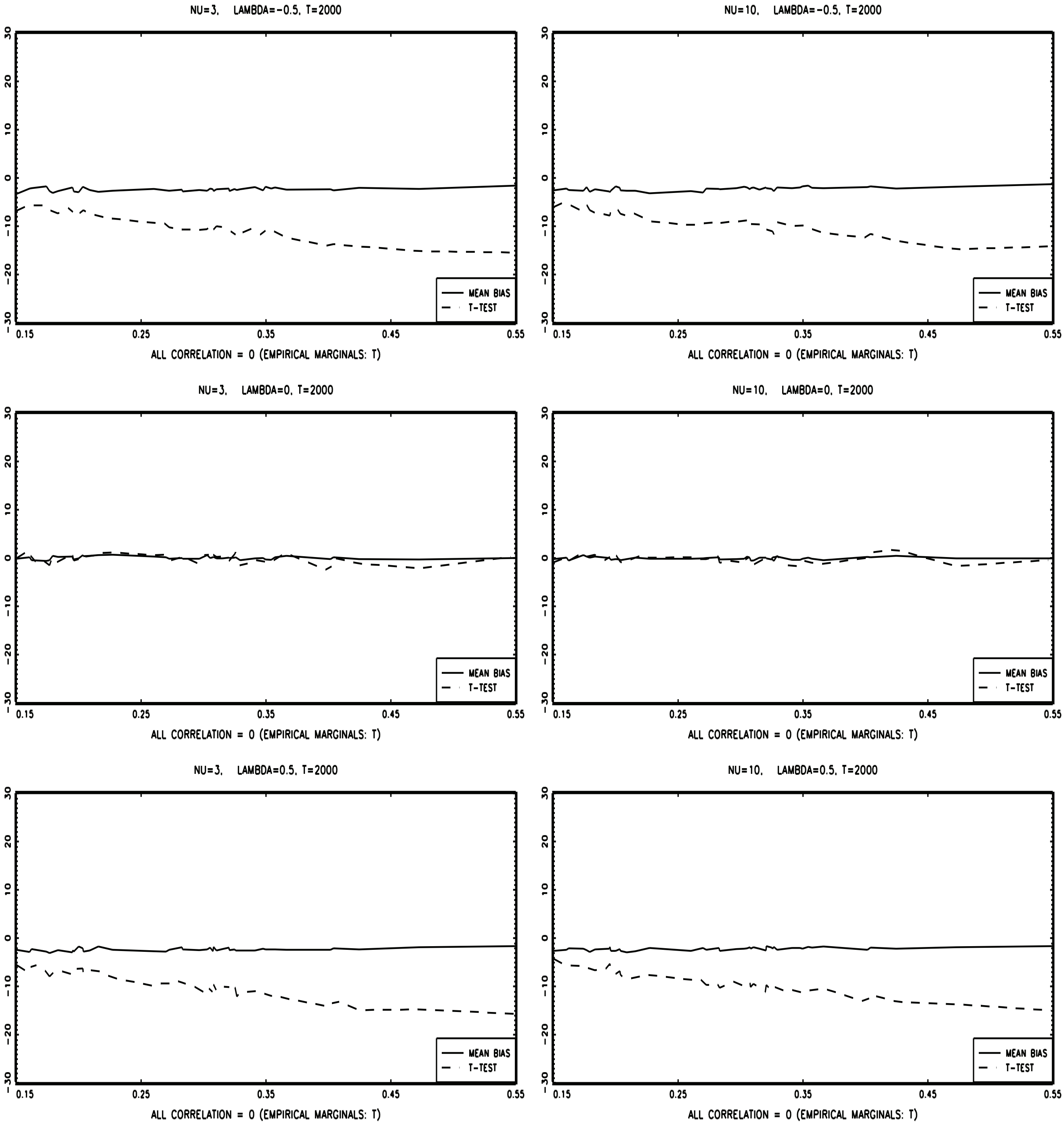

Figure 21: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 2, when T=2000 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

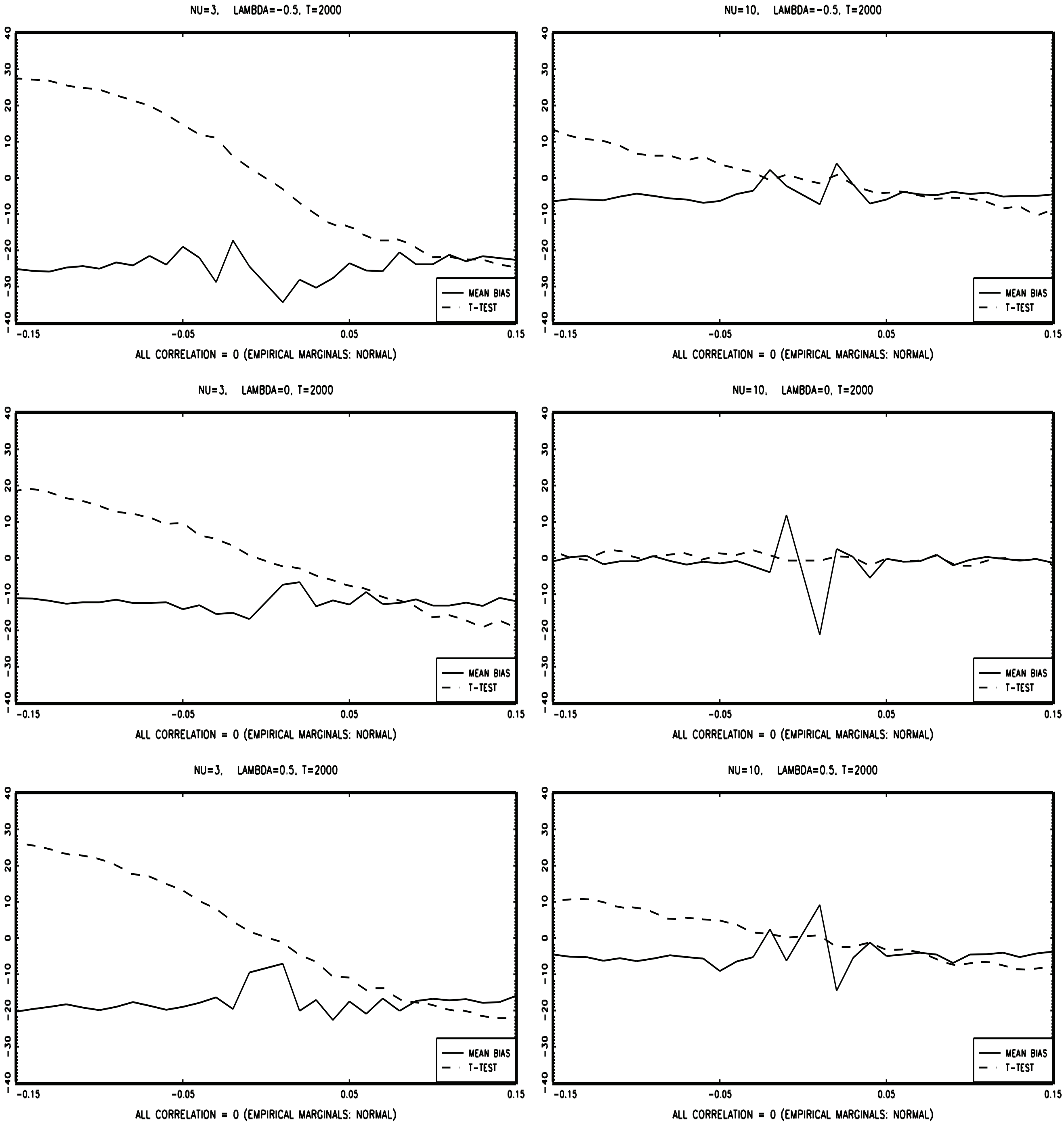

Figure 22: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 3, when T=2000 and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

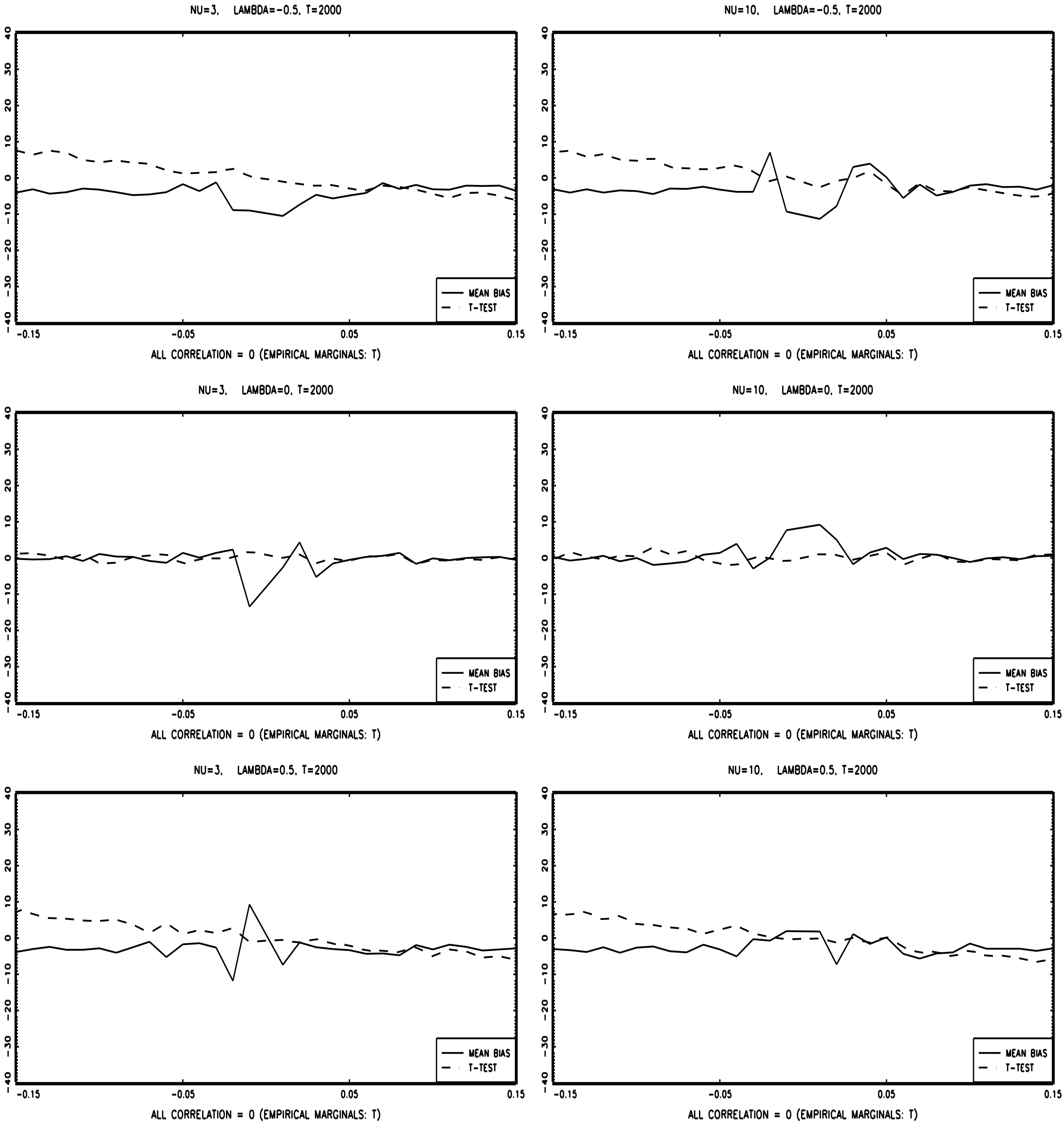

Figure 23: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 3, when T=2000 and the empirical marginals are symmetric Student's t. The DGP values of ν and λ are reported in the top of the plots.

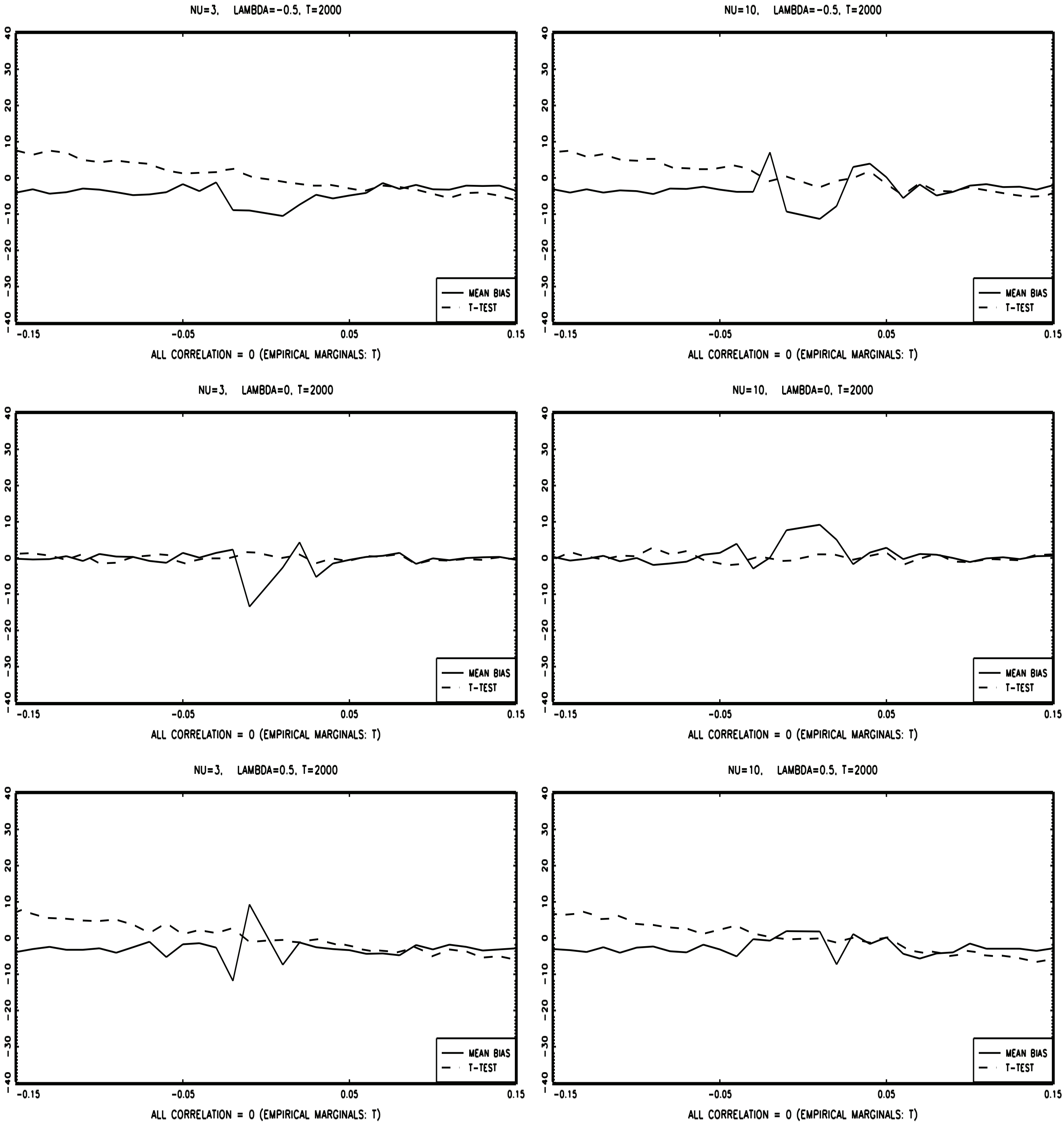

Figure 24: Mean bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for 10-variate Normal copula with correlation matrix reported in Table 3, when T=2000 and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

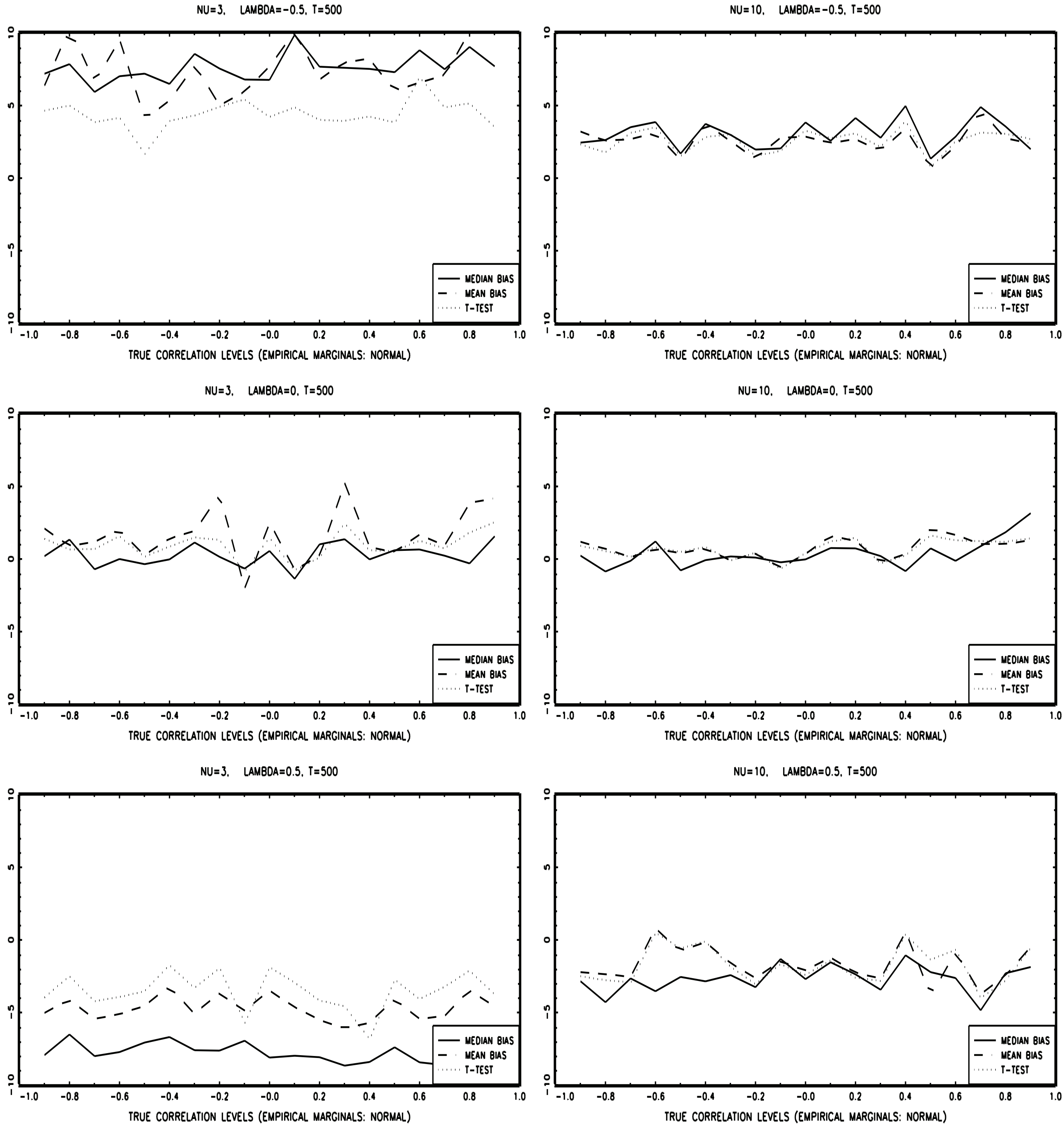

Figure 25: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient μ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

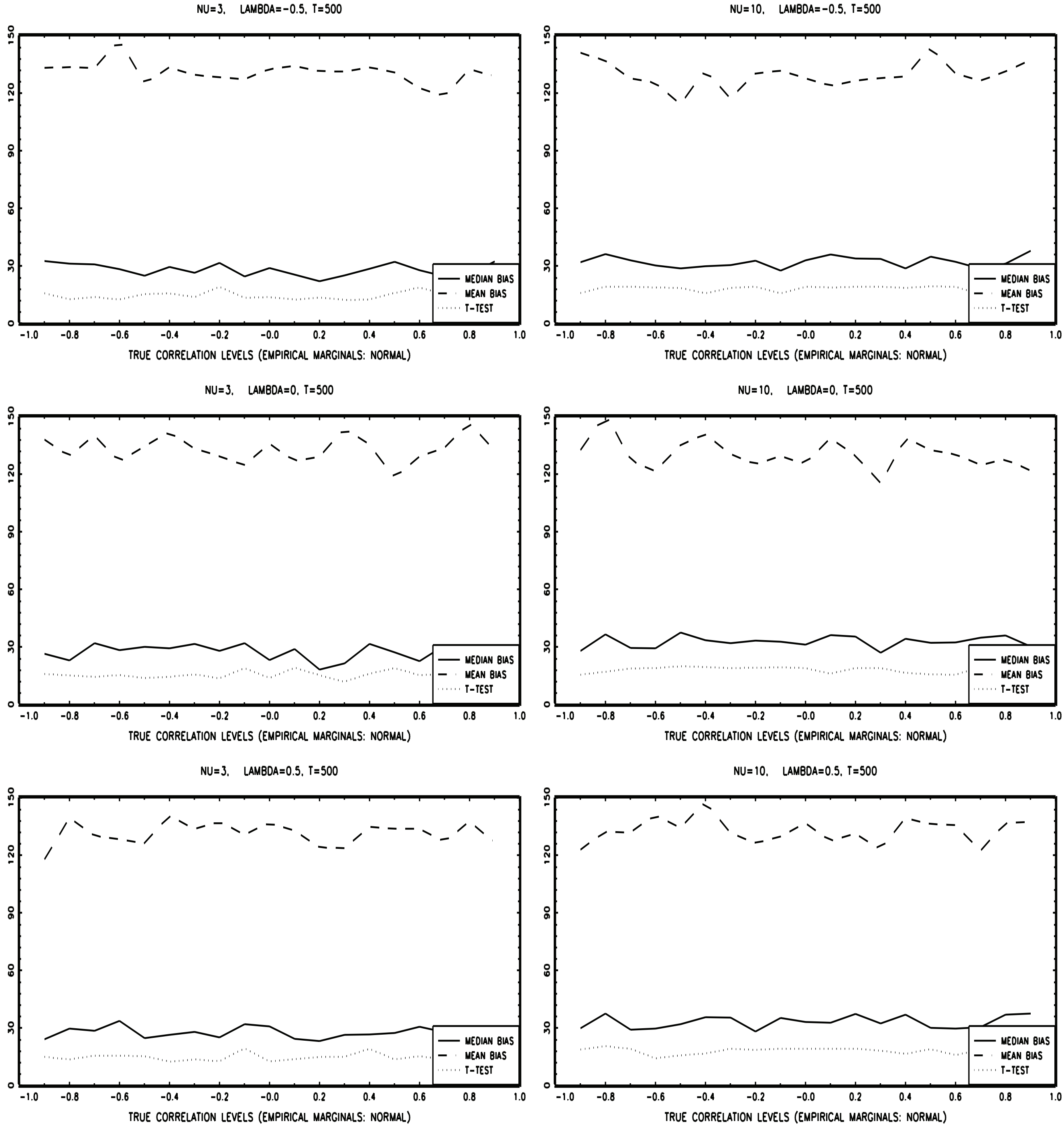

Figure 26: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ω when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

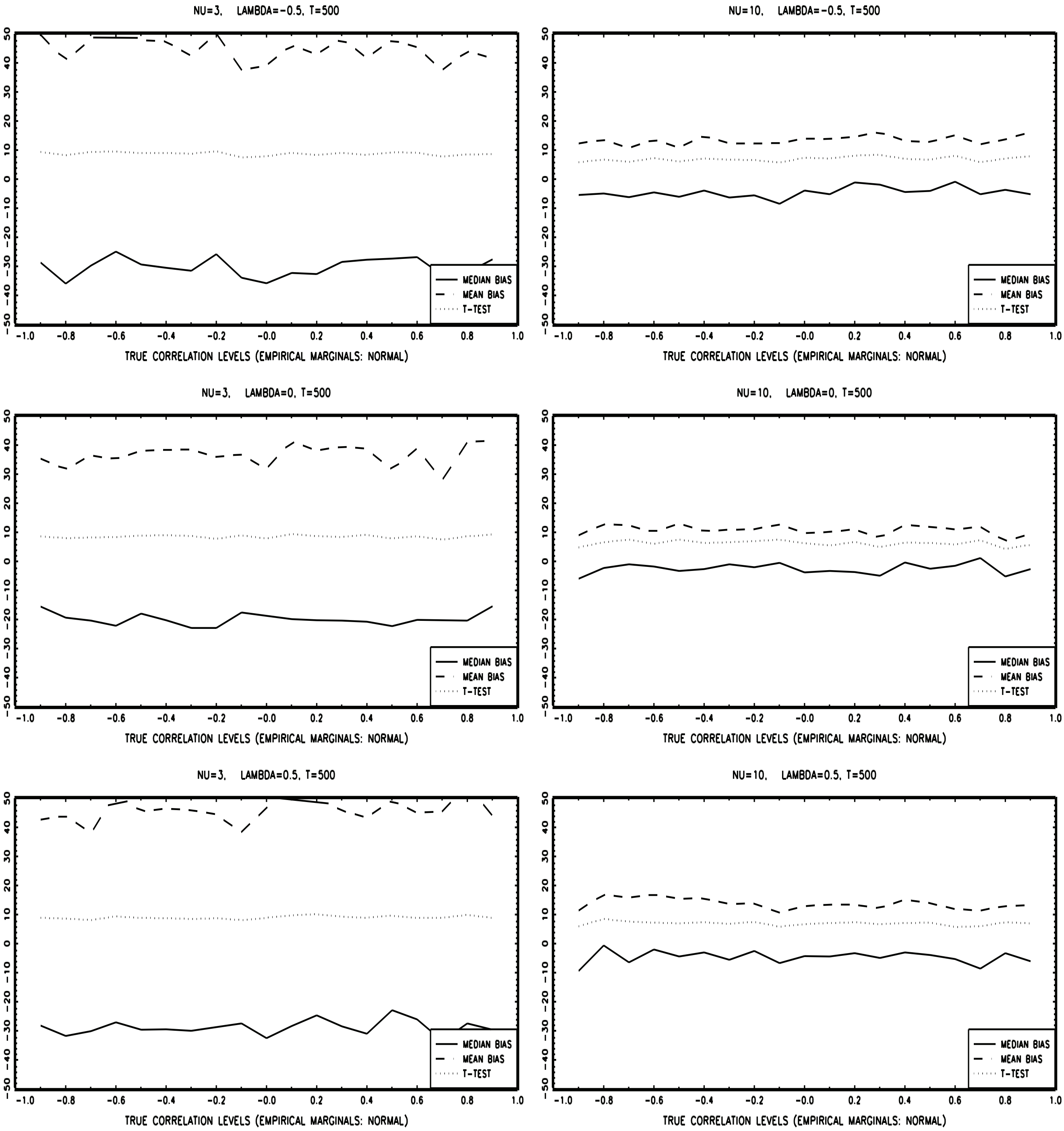

Figure 27: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient α when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

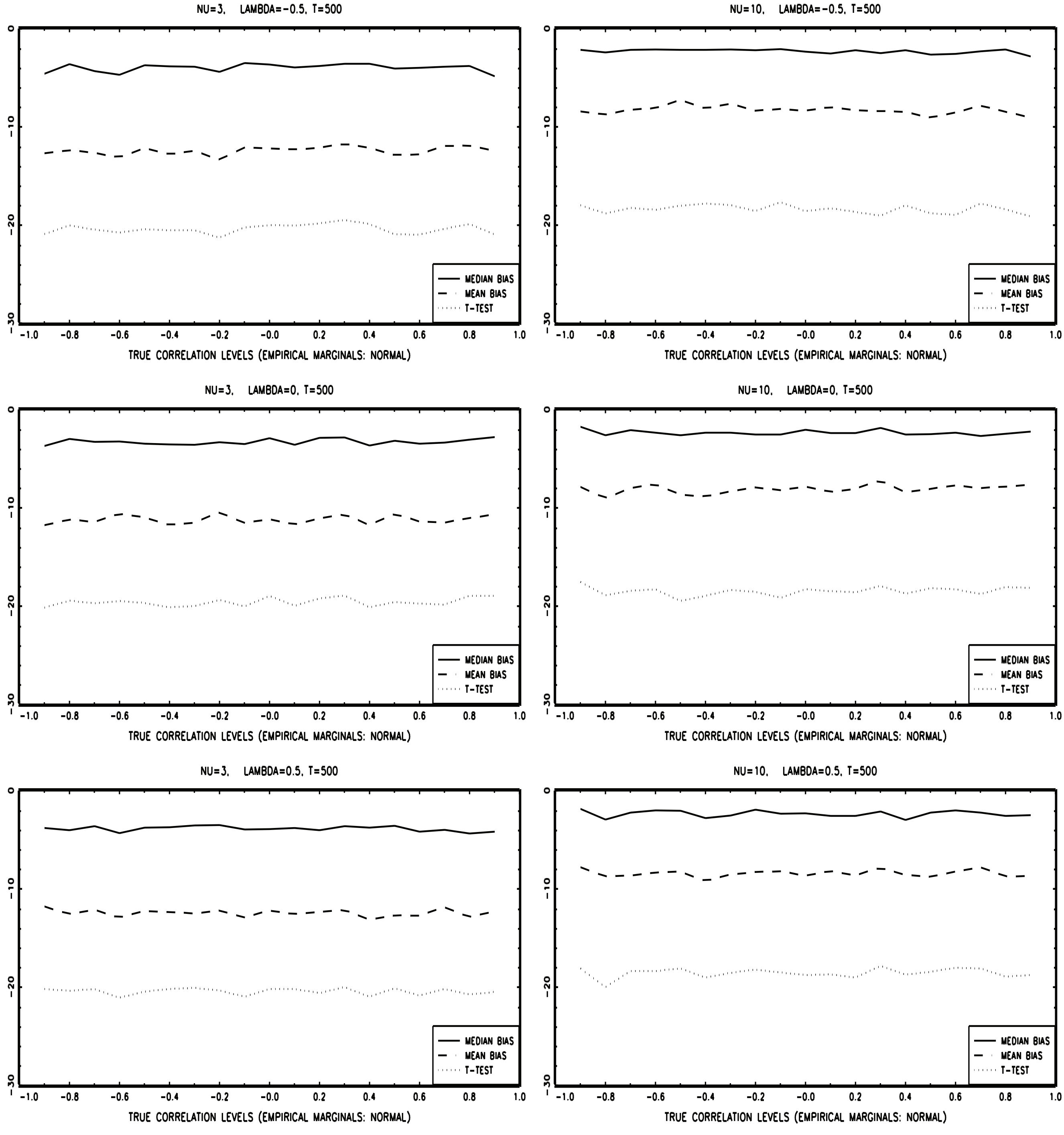

Figure 28: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient β when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

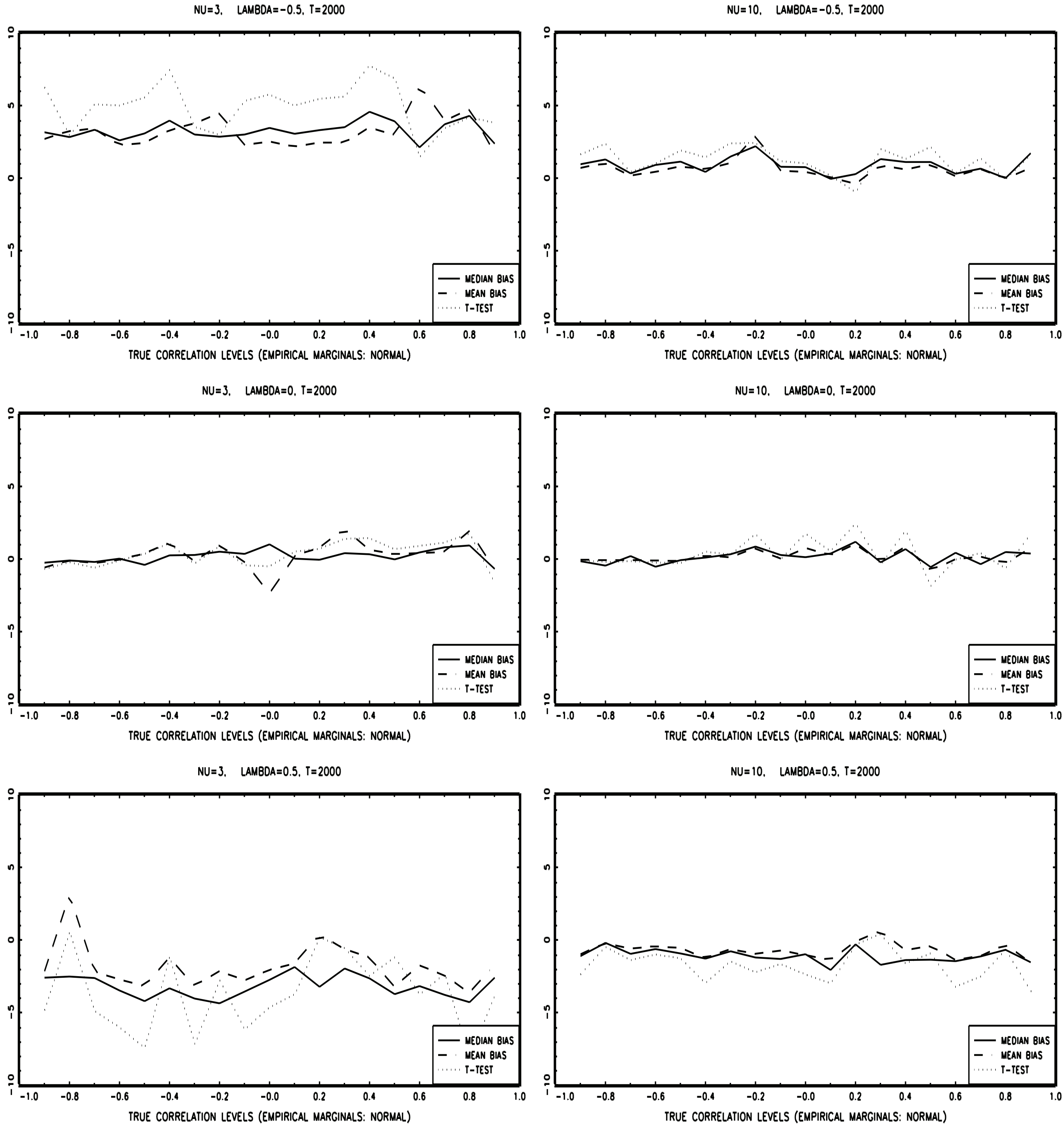

Figure 29: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient μ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

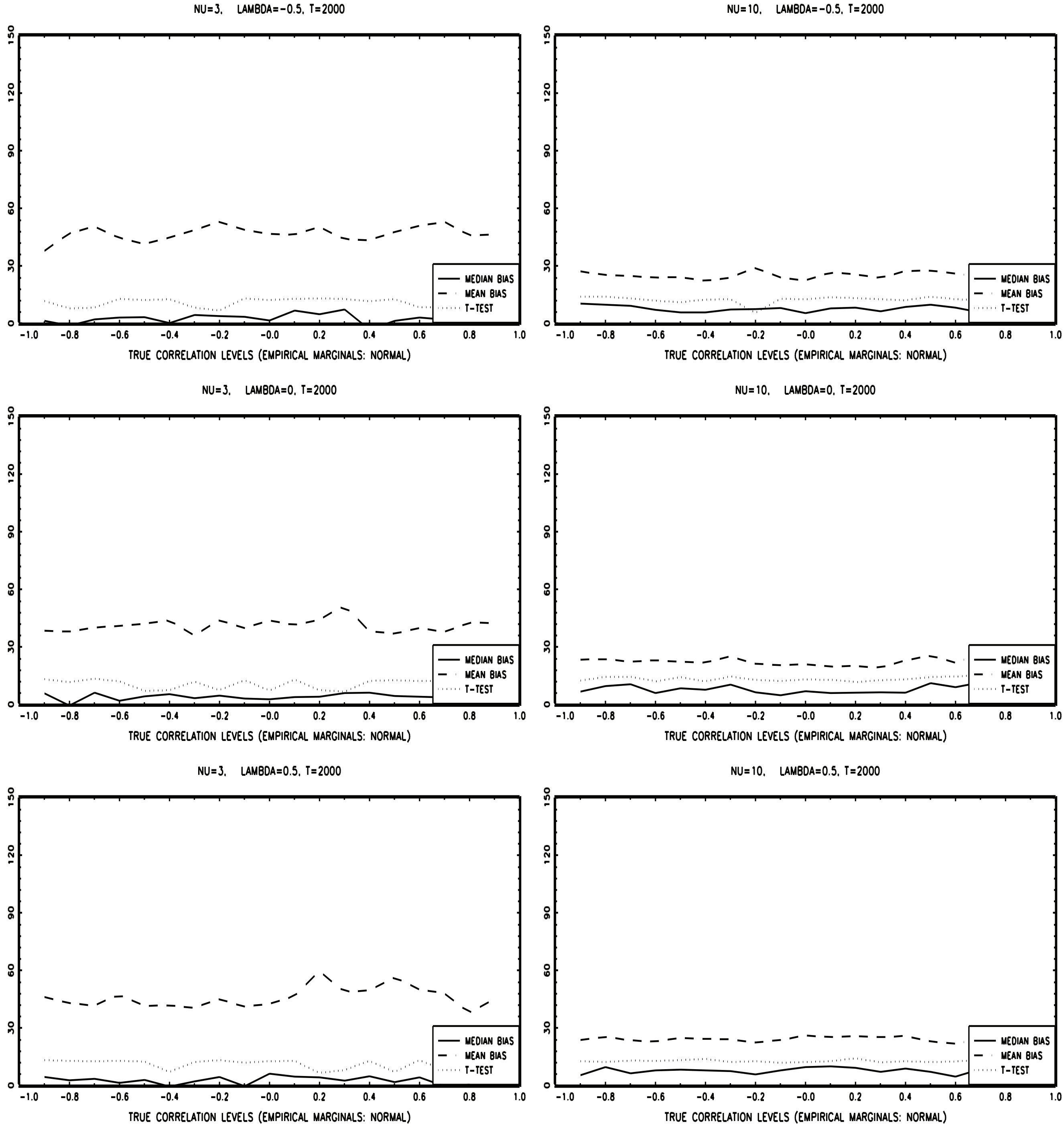

Figure 30: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ω when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

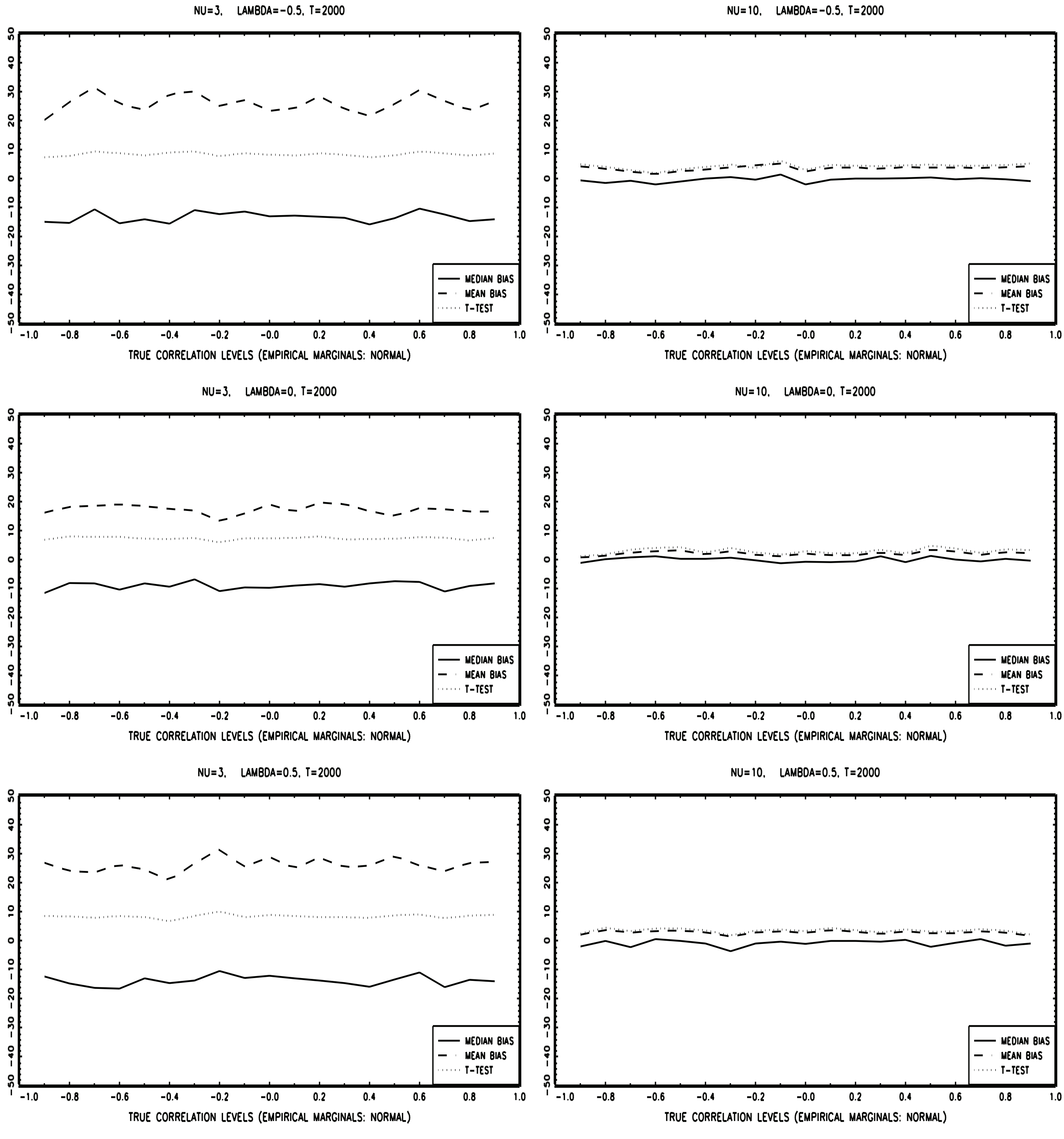

Figure 31: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient α when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

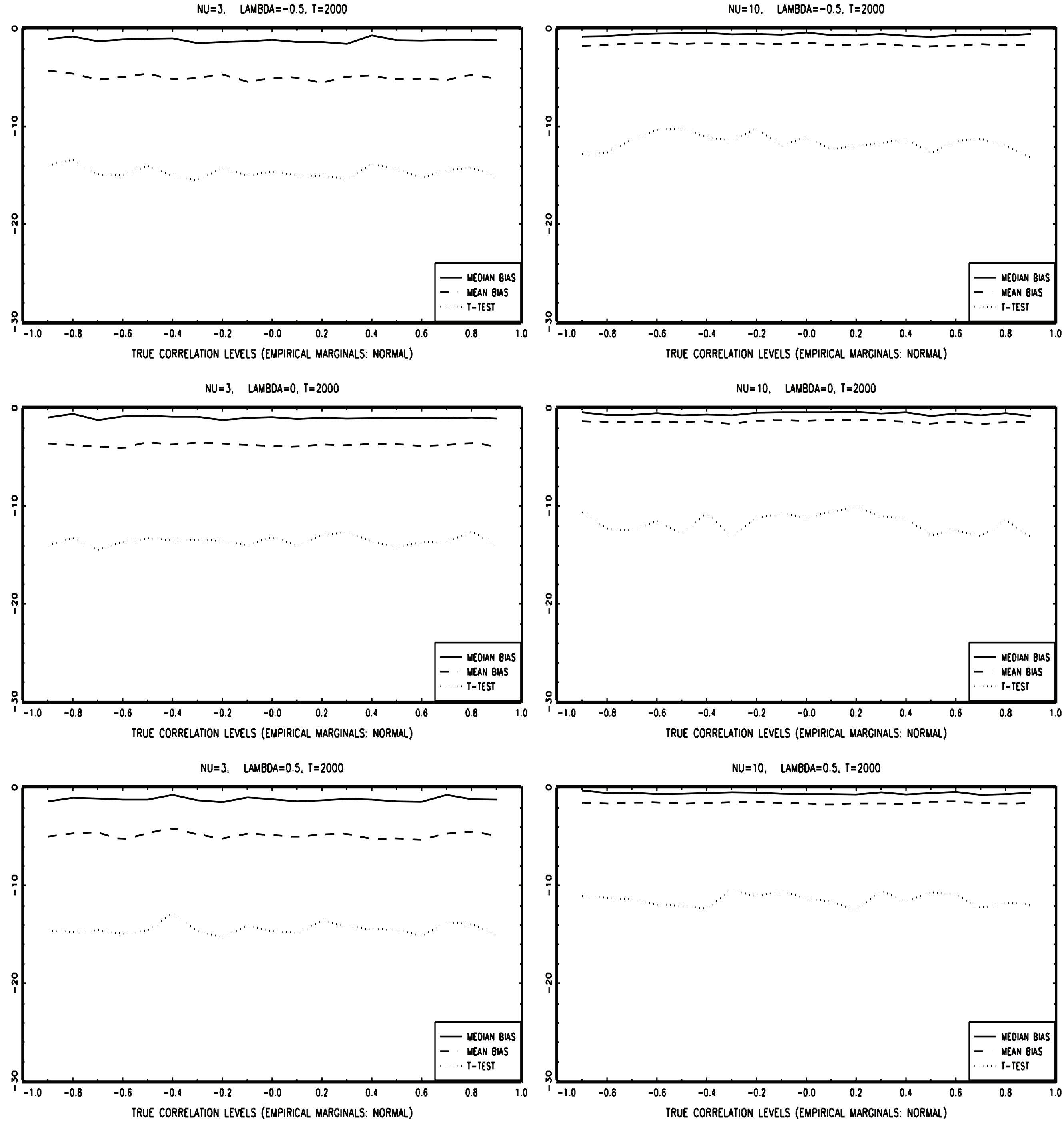

Figure 32: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient β when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Normal. The DGP values of ν and λ are reported in the top of the plots.

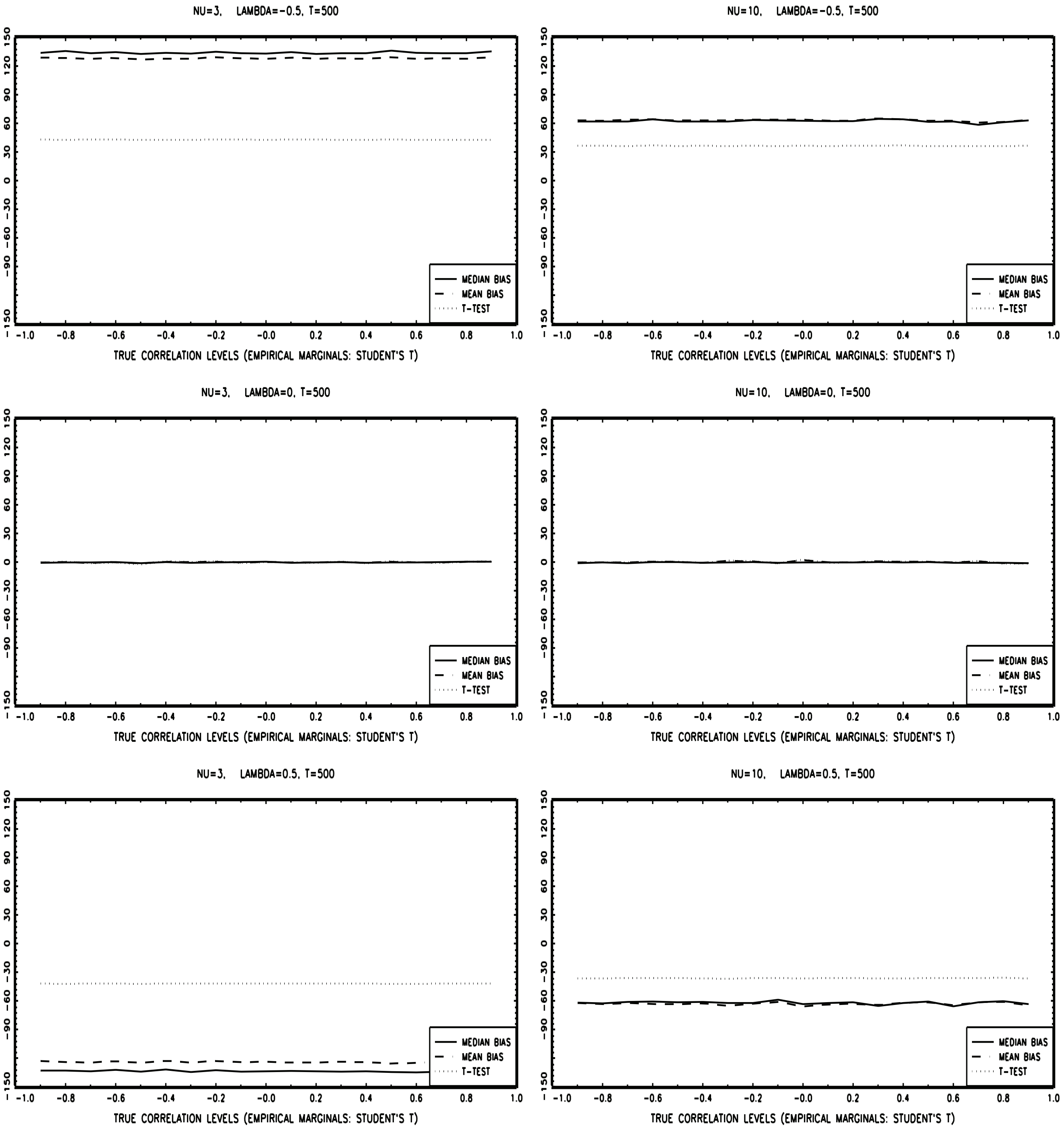

Figure 33: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient μ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

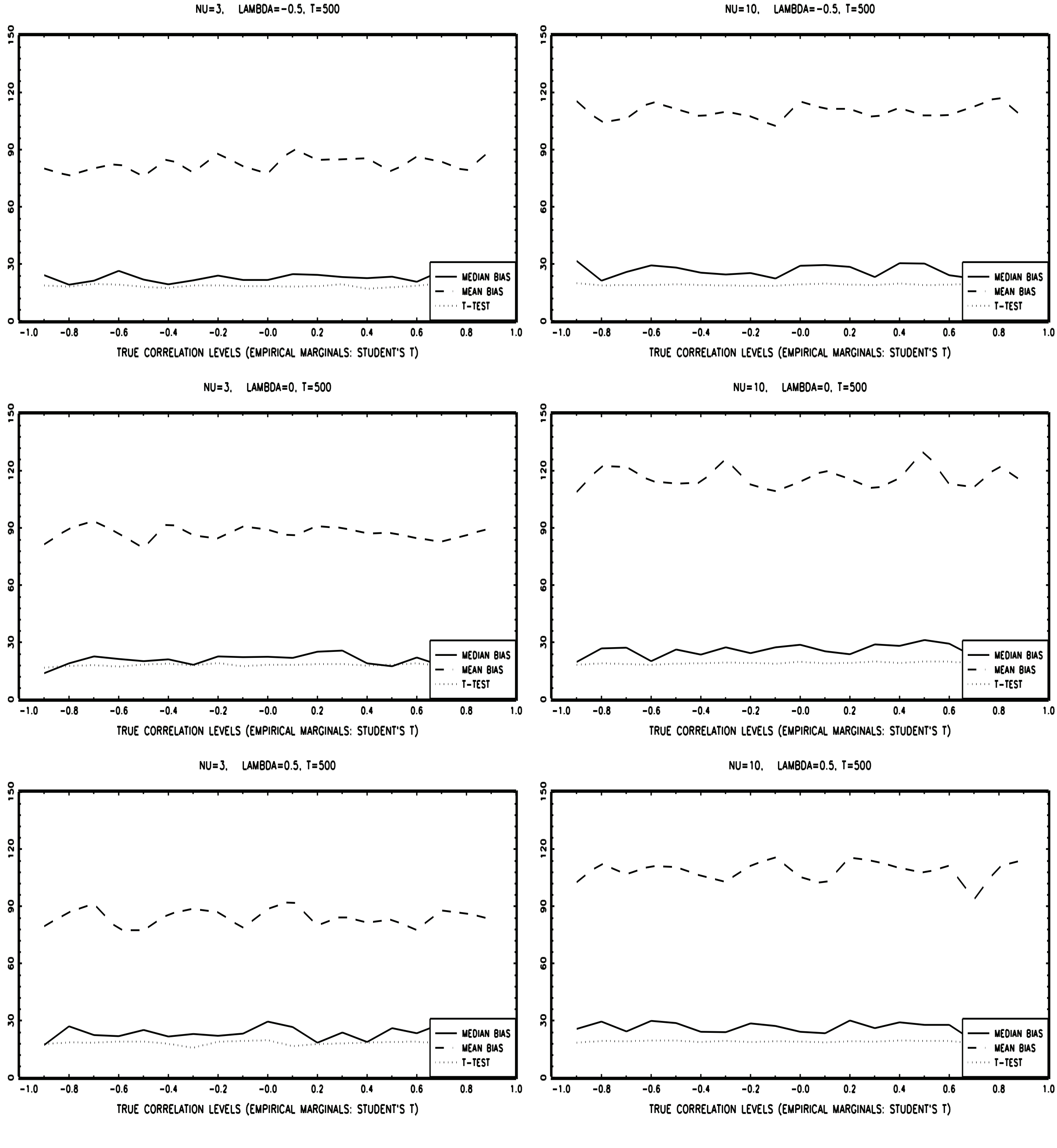

Figure 34: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ω when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

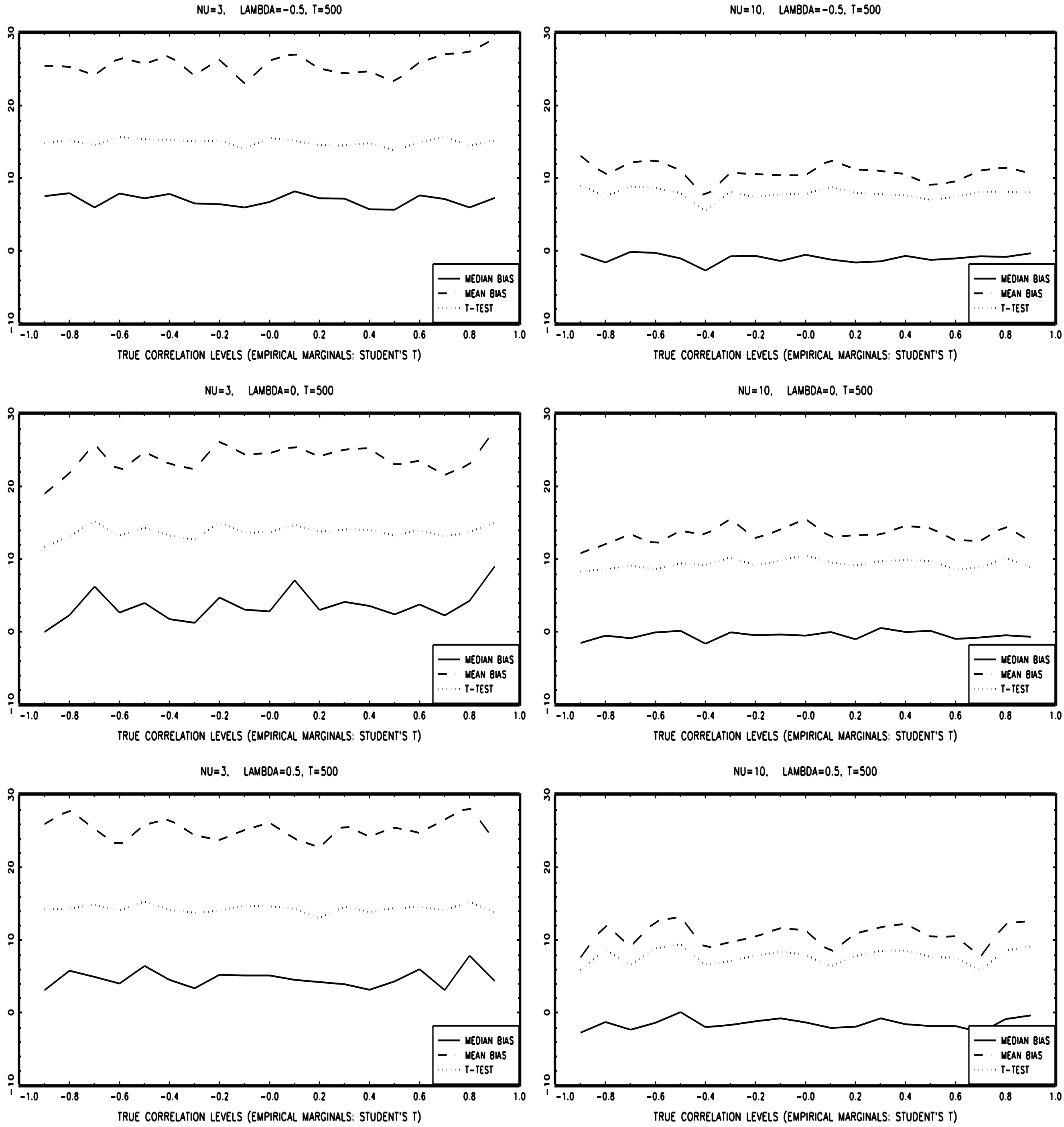

Figure 35: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient α when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

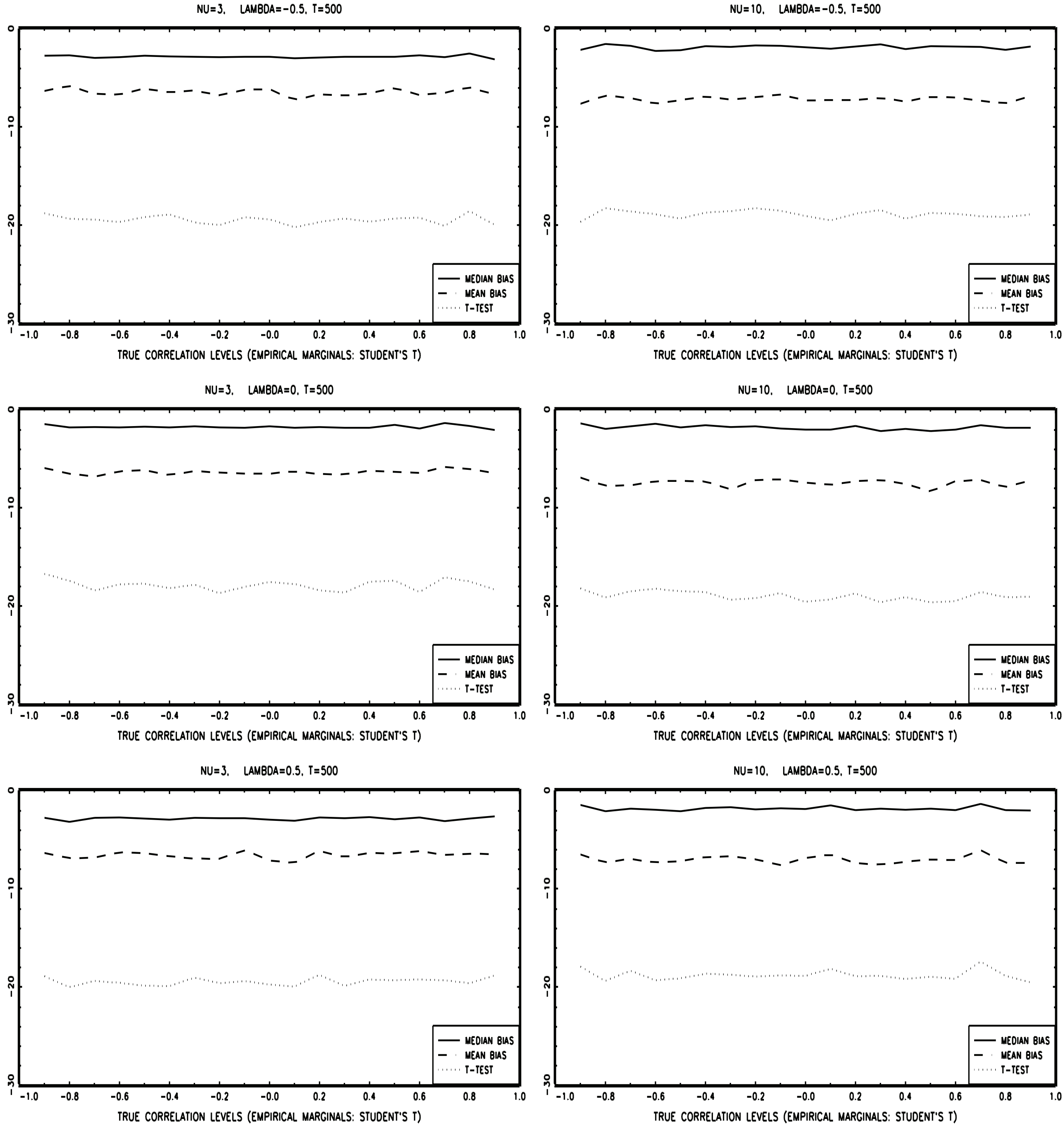

Figure 36: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient β when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

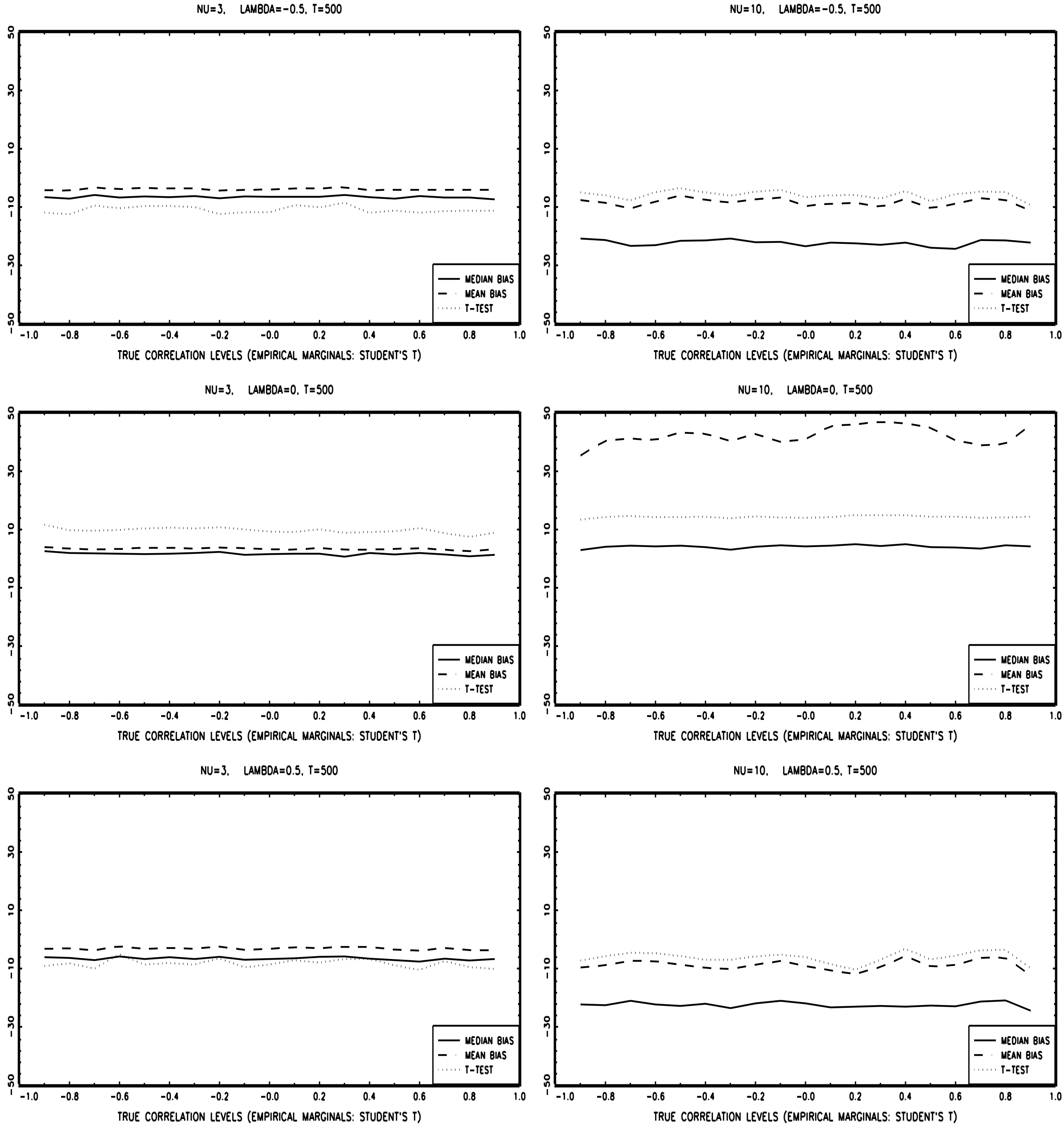

Figure 37: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ν when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

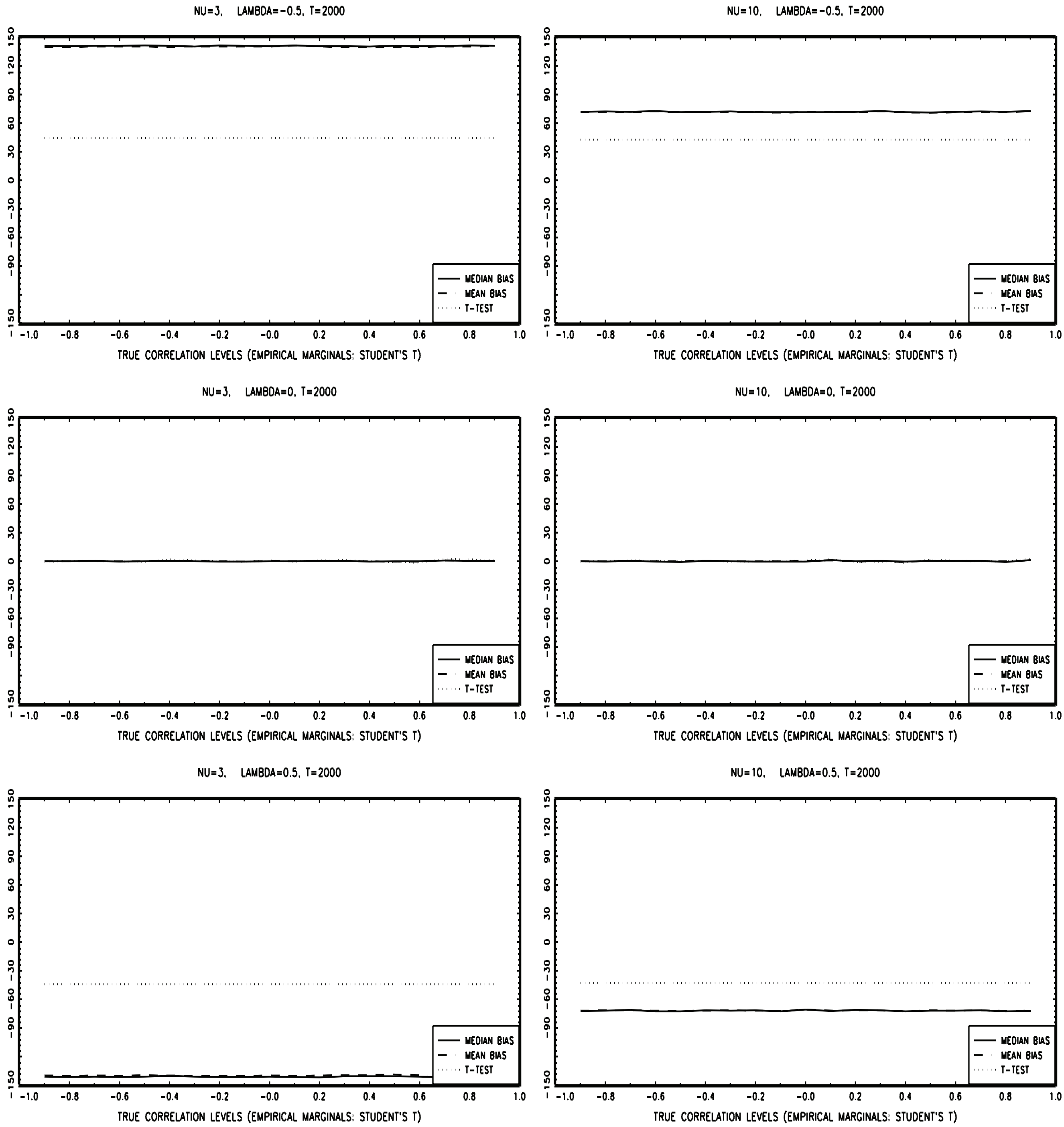

Figure 38: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient μ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

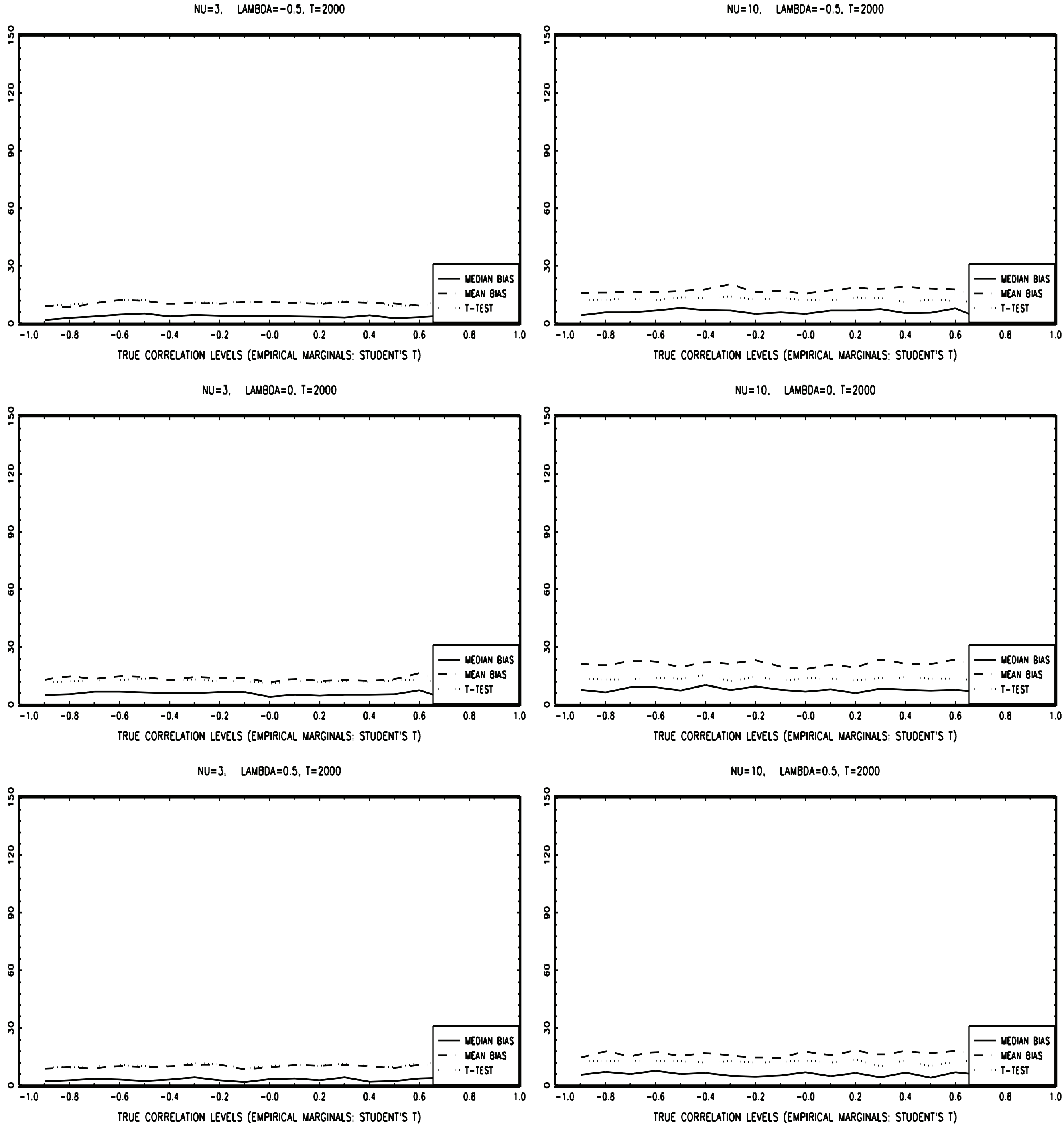

Figure 39: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ω when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

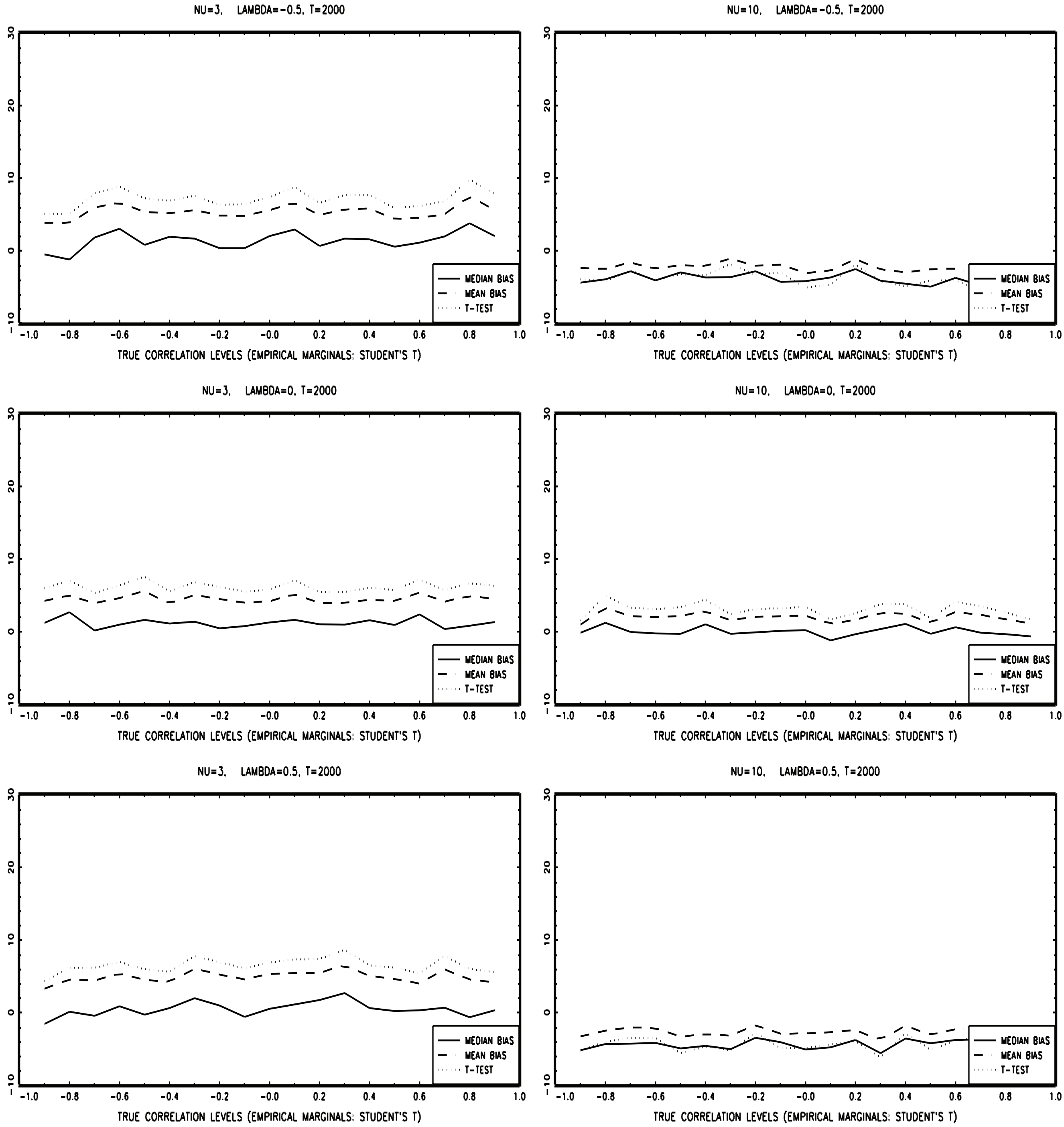

Figure 40: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient α when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

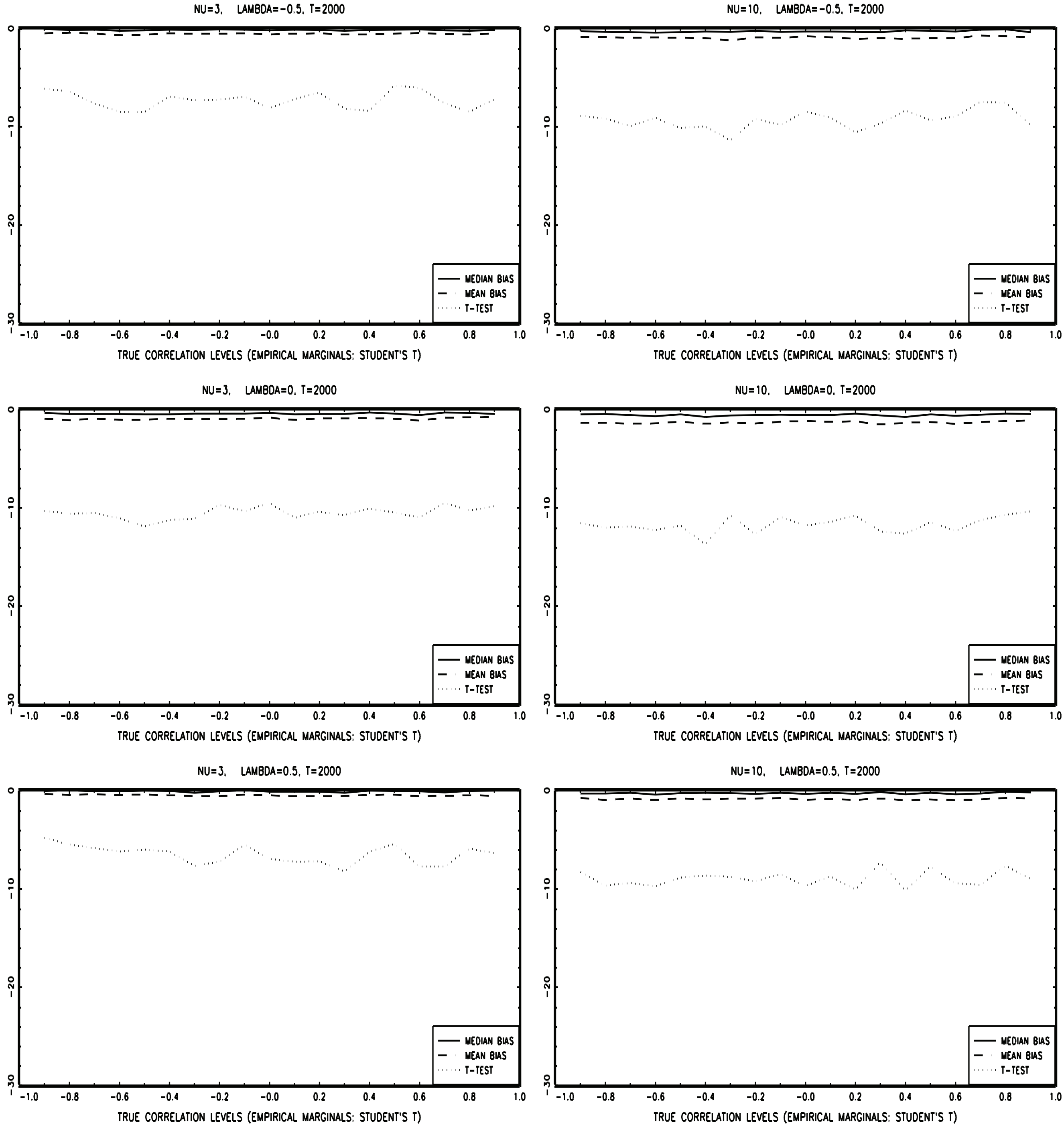

Figure 41: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient β when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

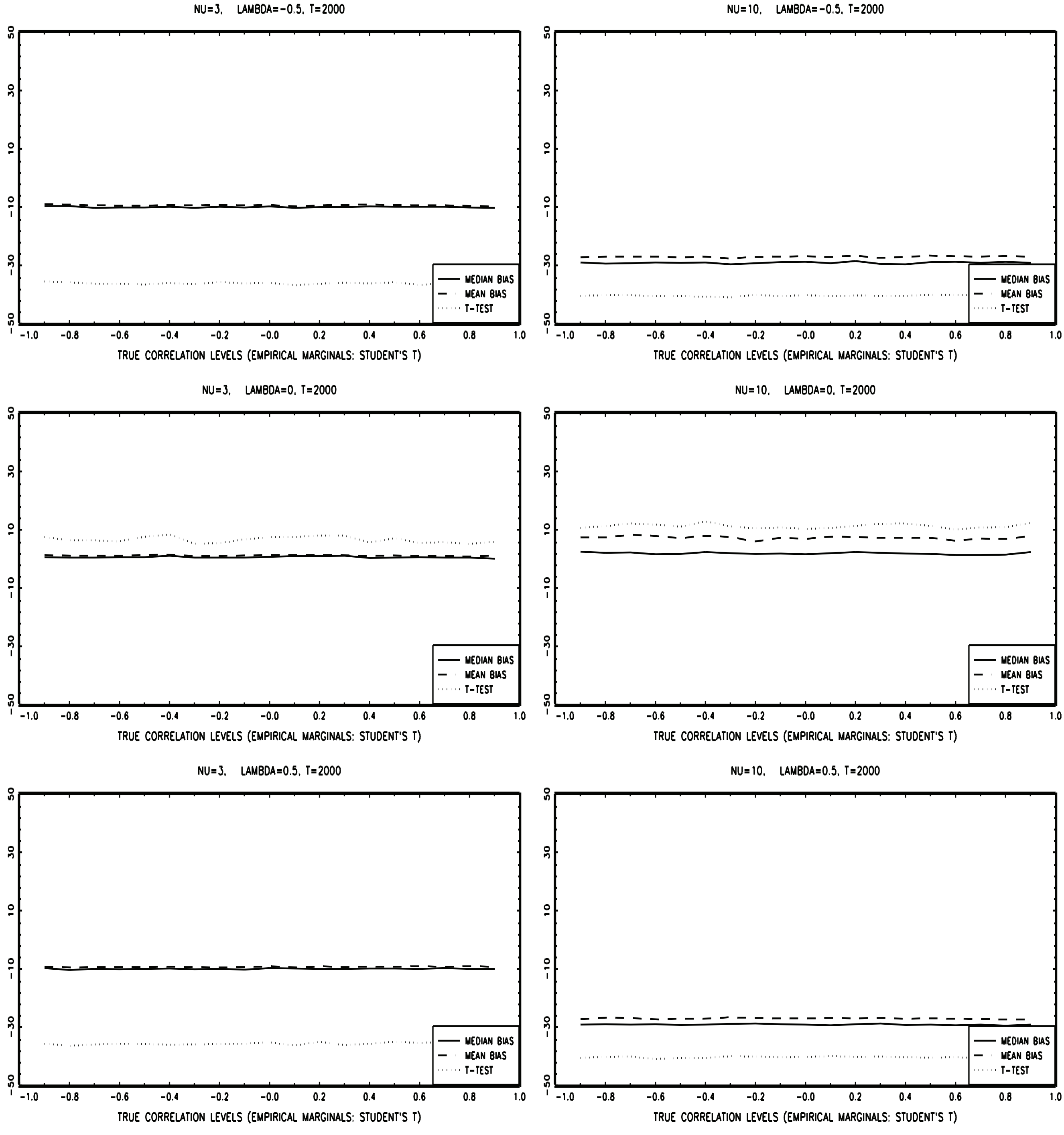

Figure 42: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ν when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Student's t. The DGP values of ν and λ are reported in the top of the plots.

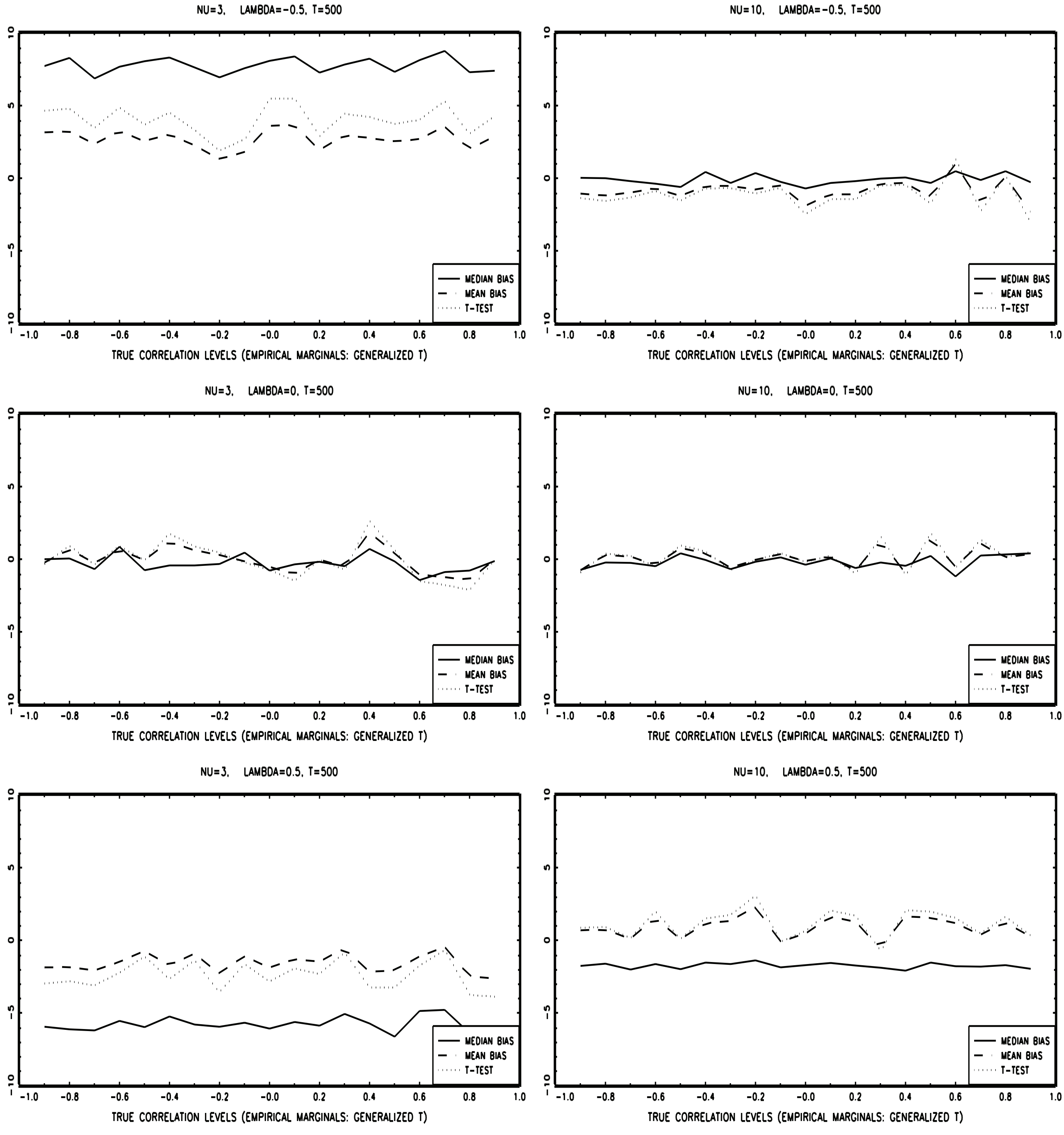

Figure 43: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient μ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

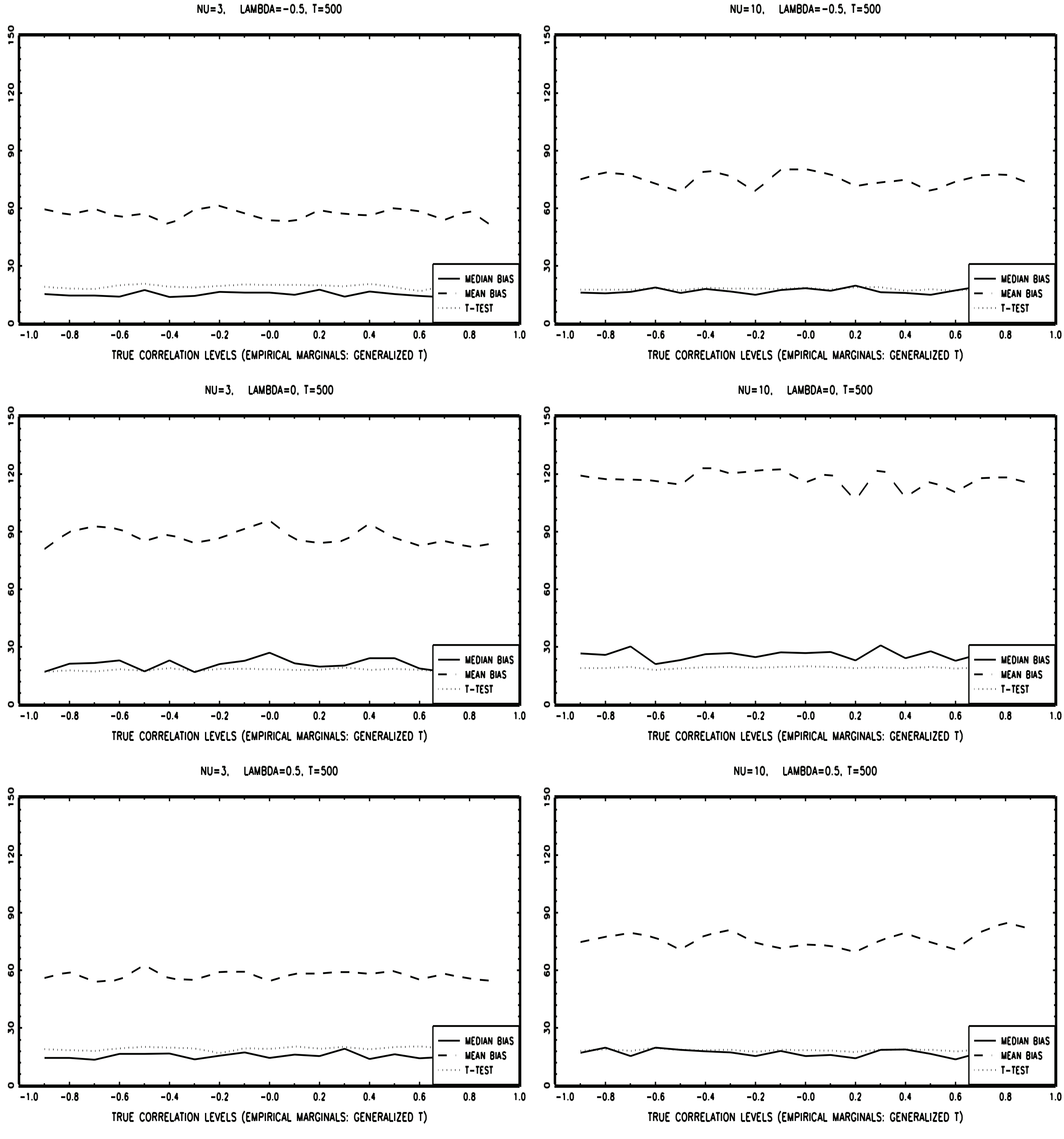

Figure 44: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ω when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

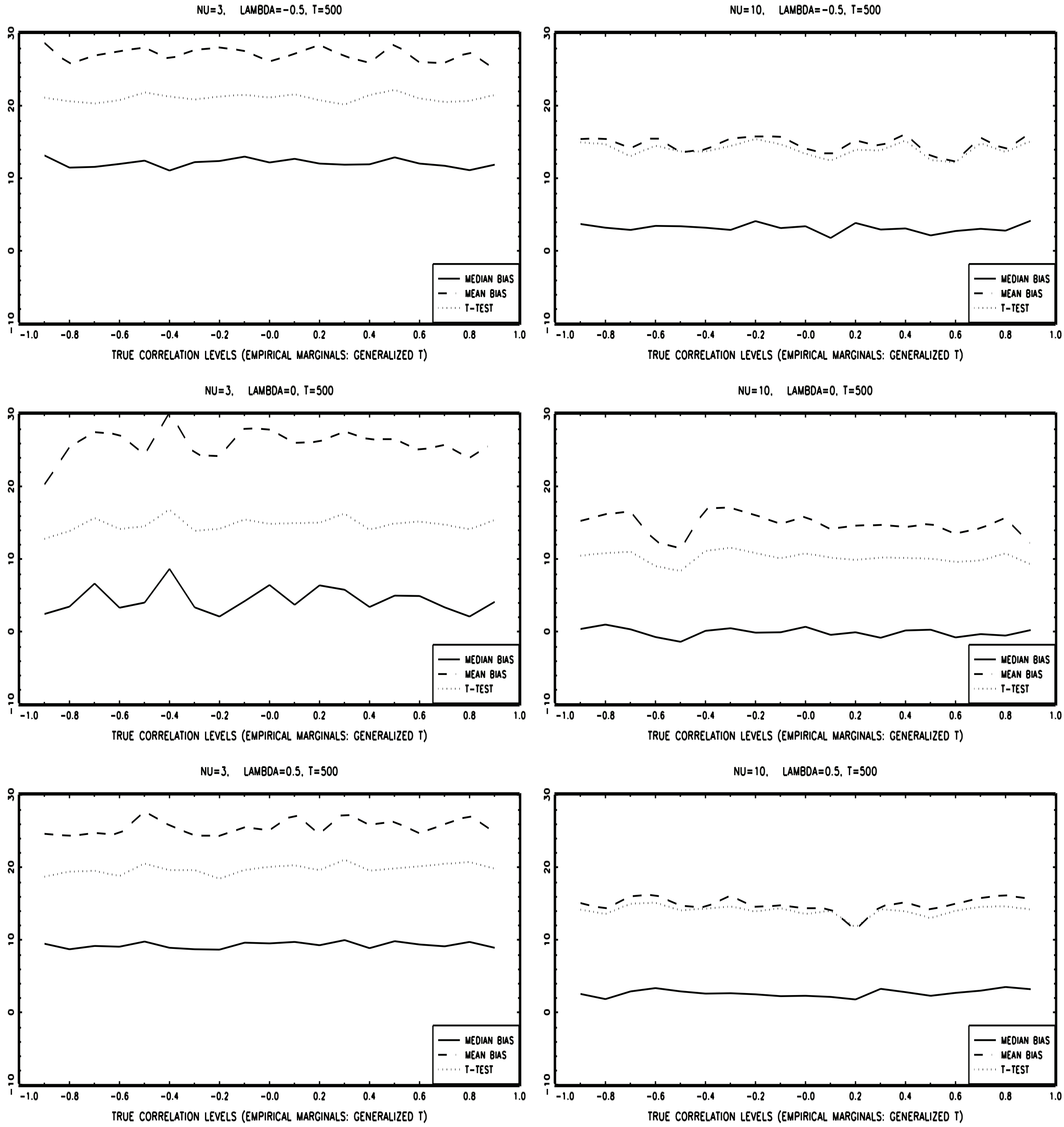

Figure 45: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient α when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Generalized-t.

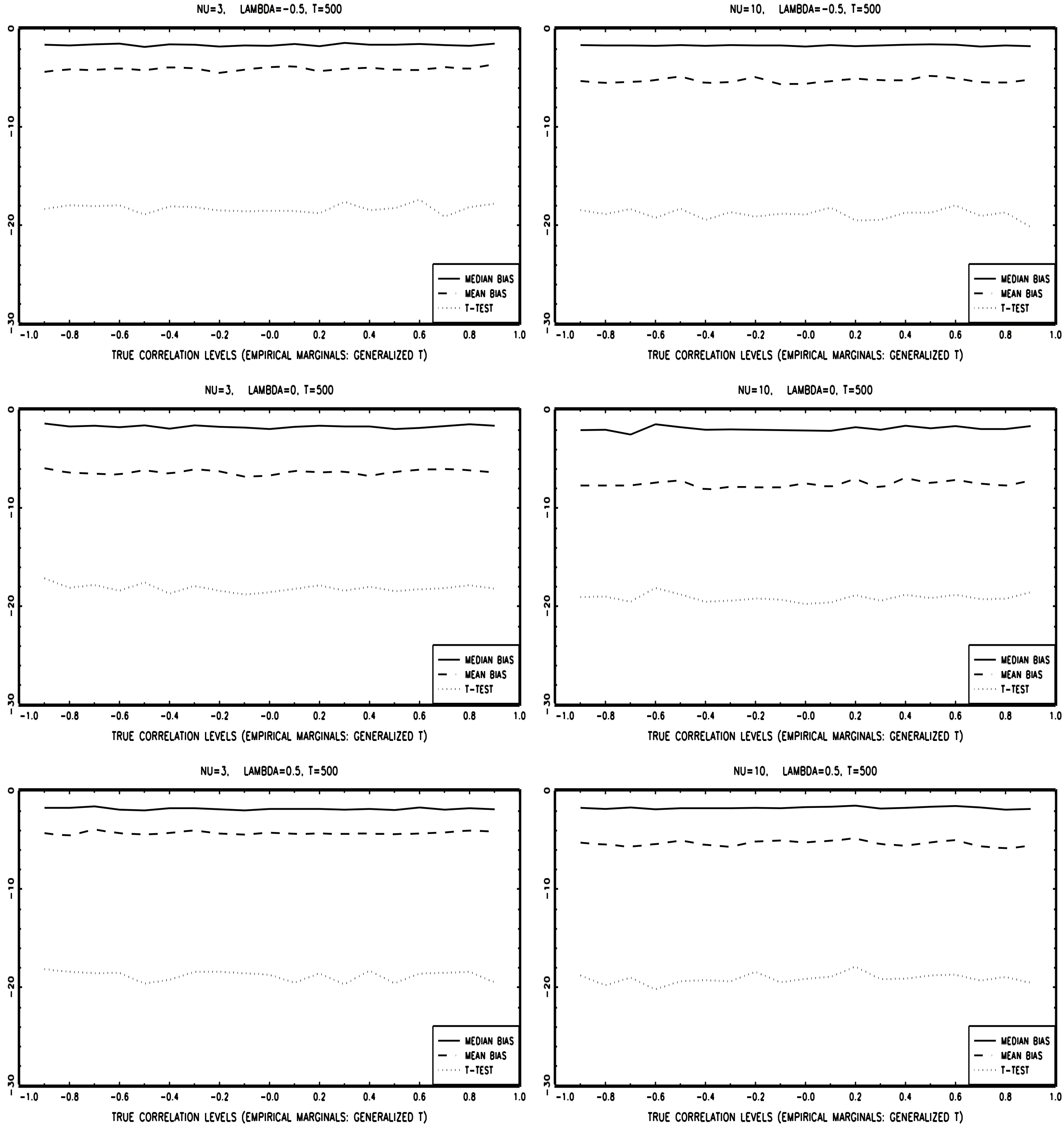

Figure 46: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient β when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

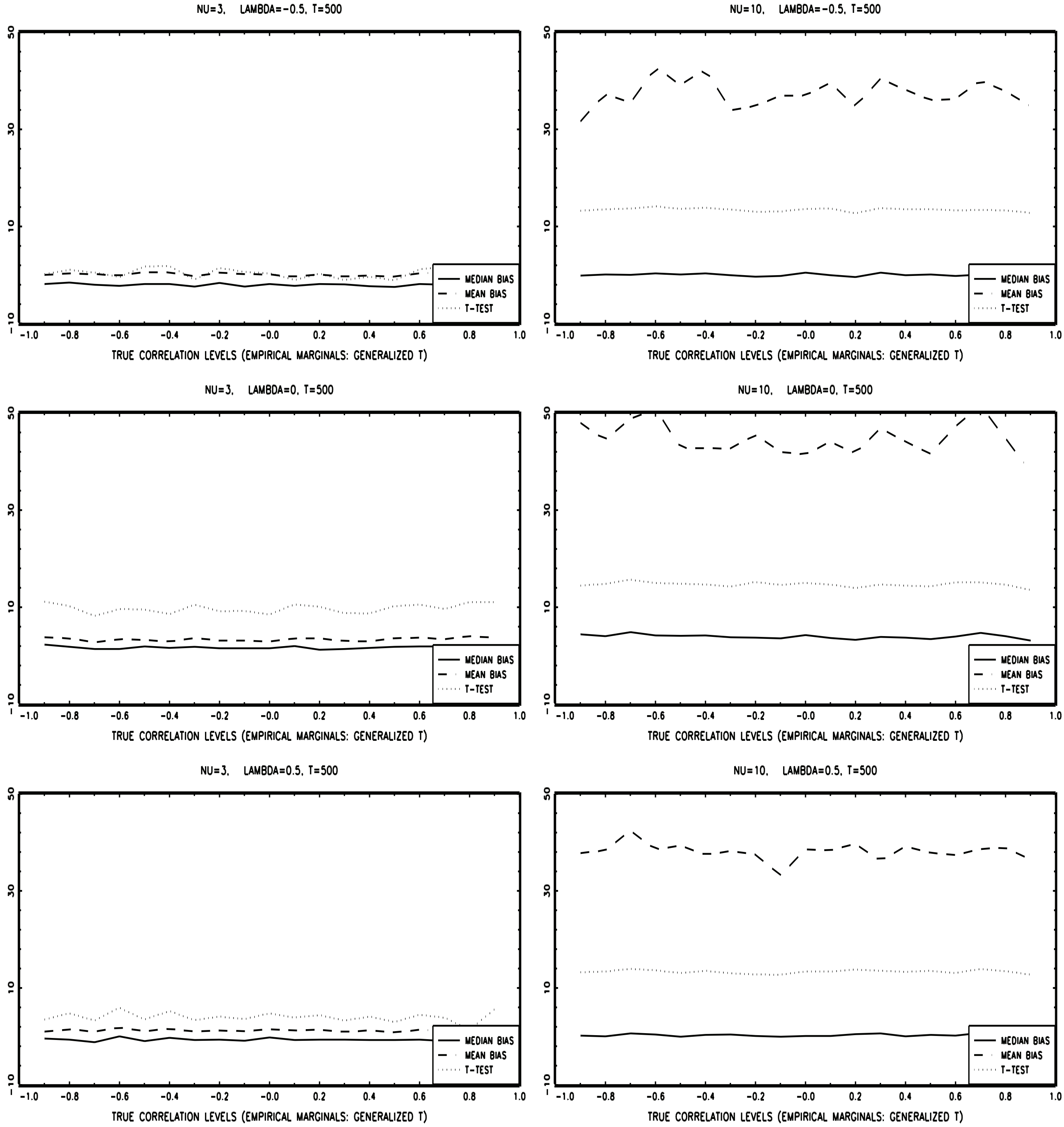

Figure 47: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ν when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

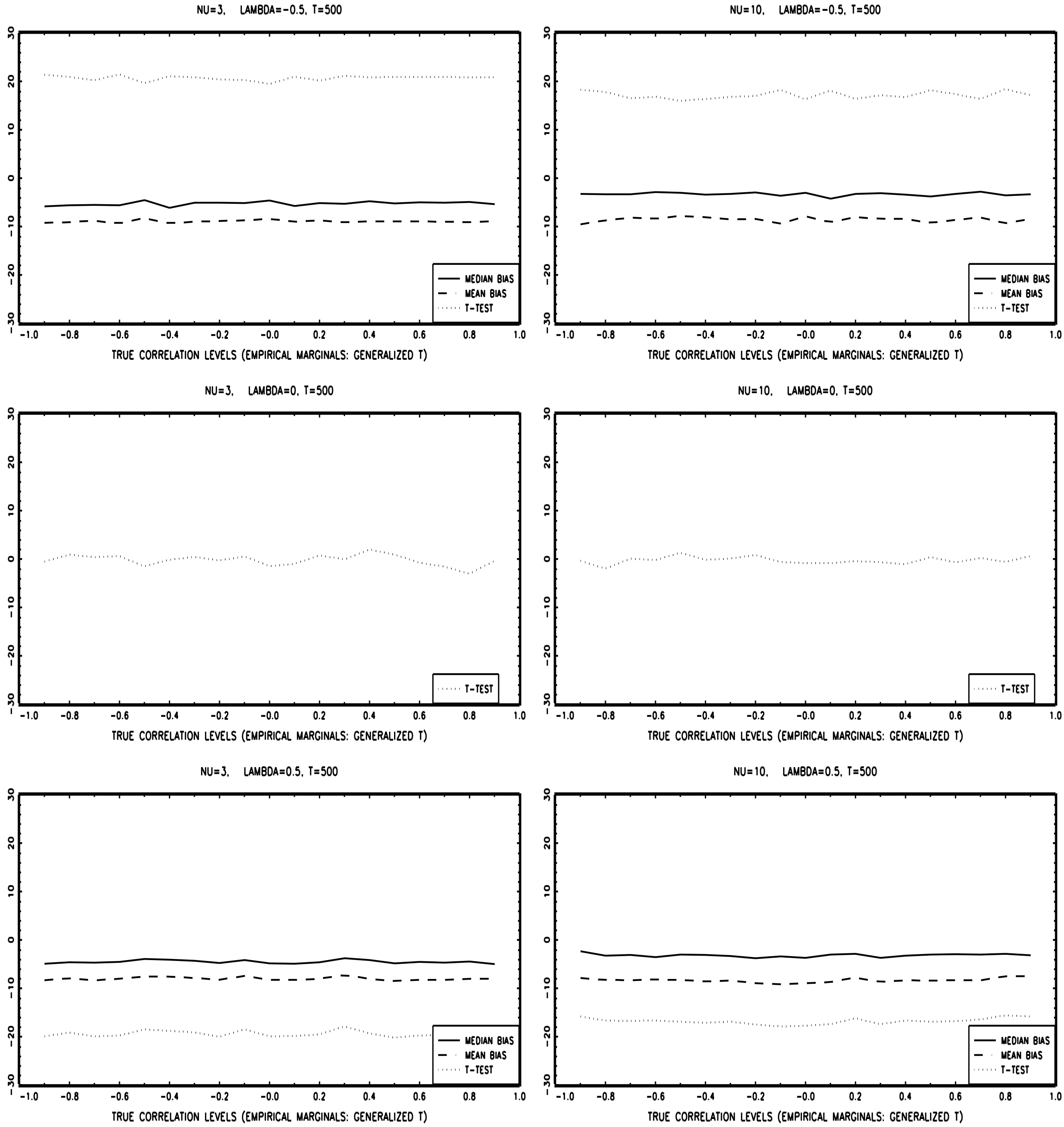

Figure 48: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient λ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 500$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

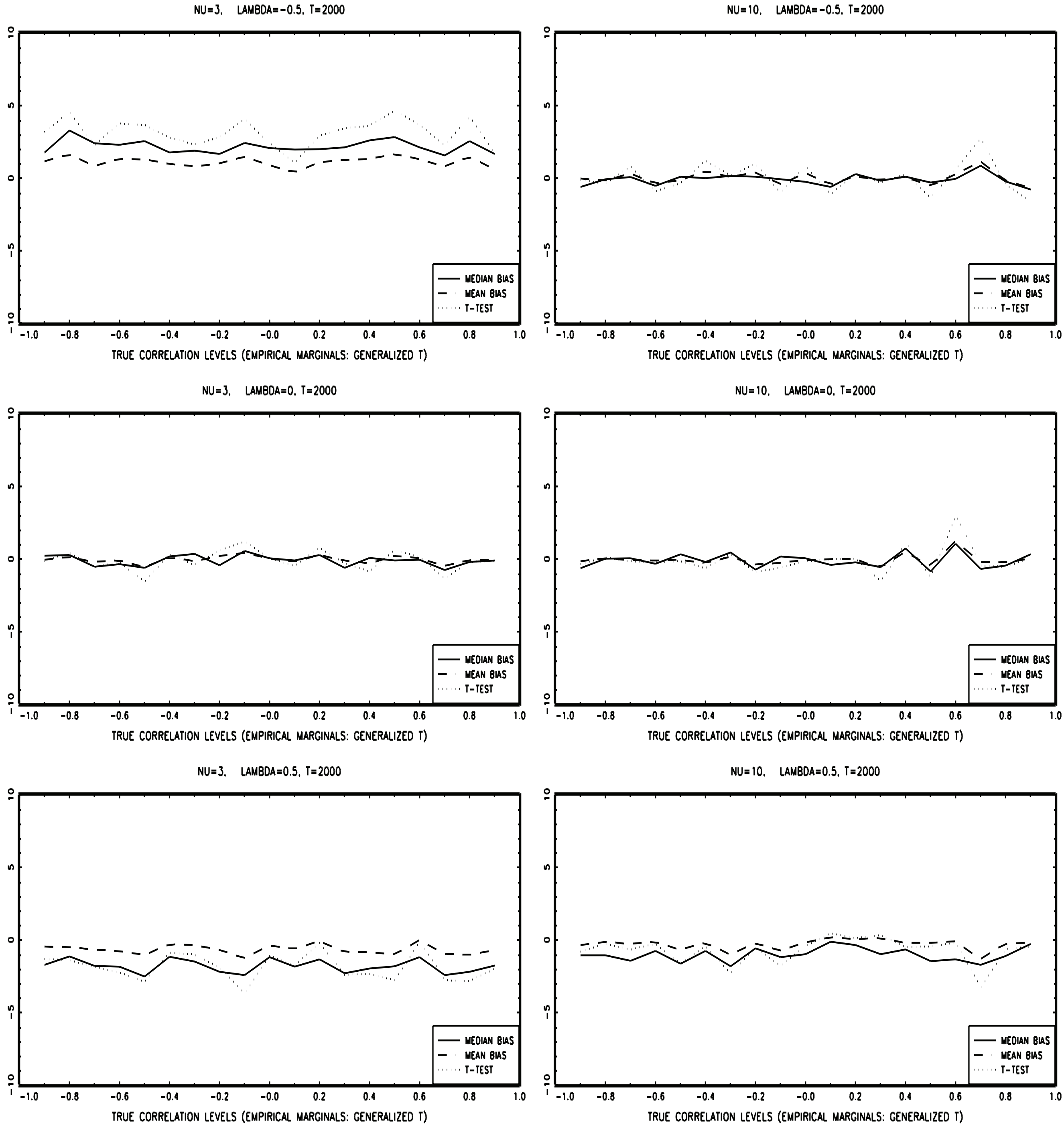

Figure 49: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient μ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

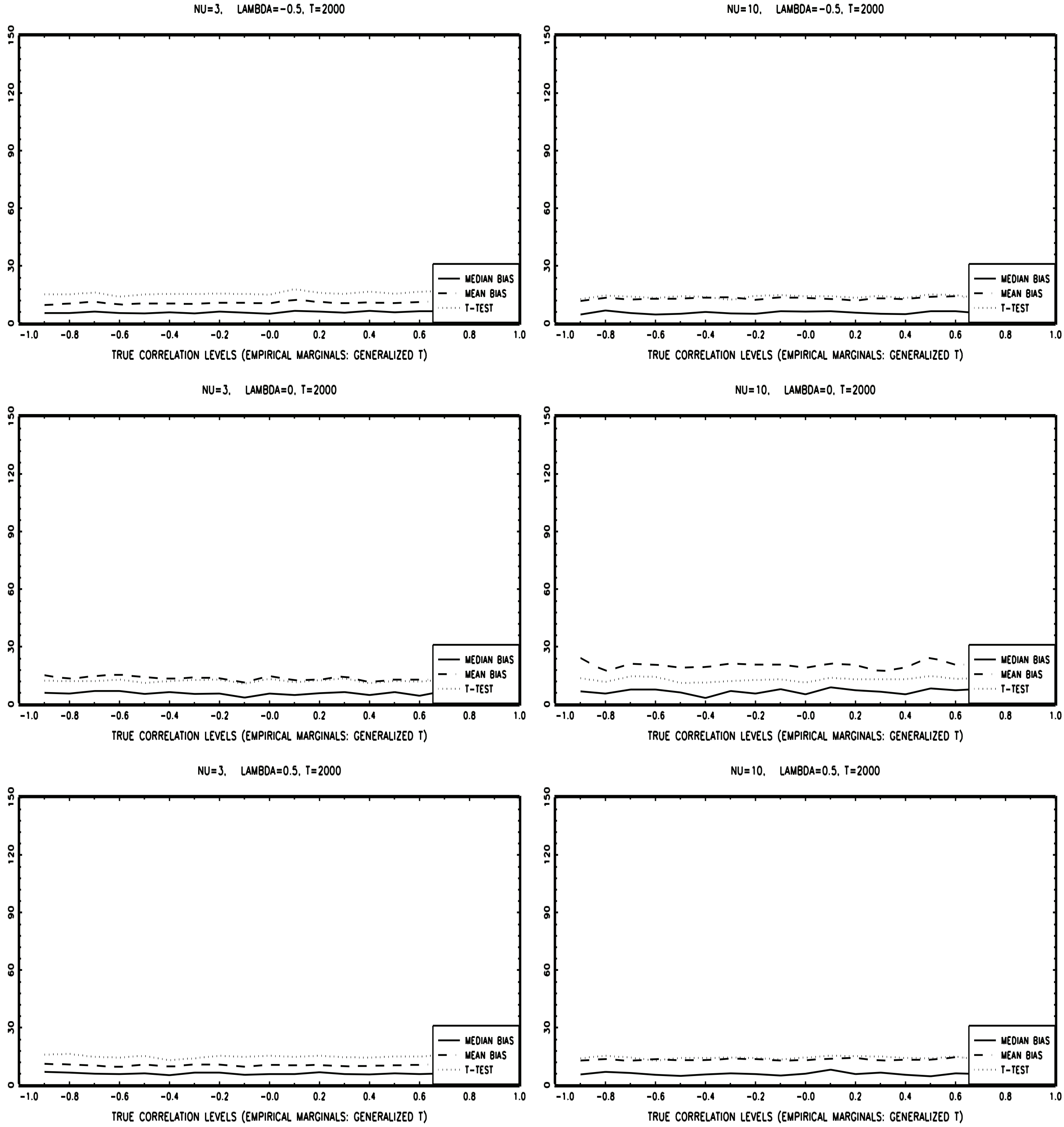

Figure 50: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ω when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

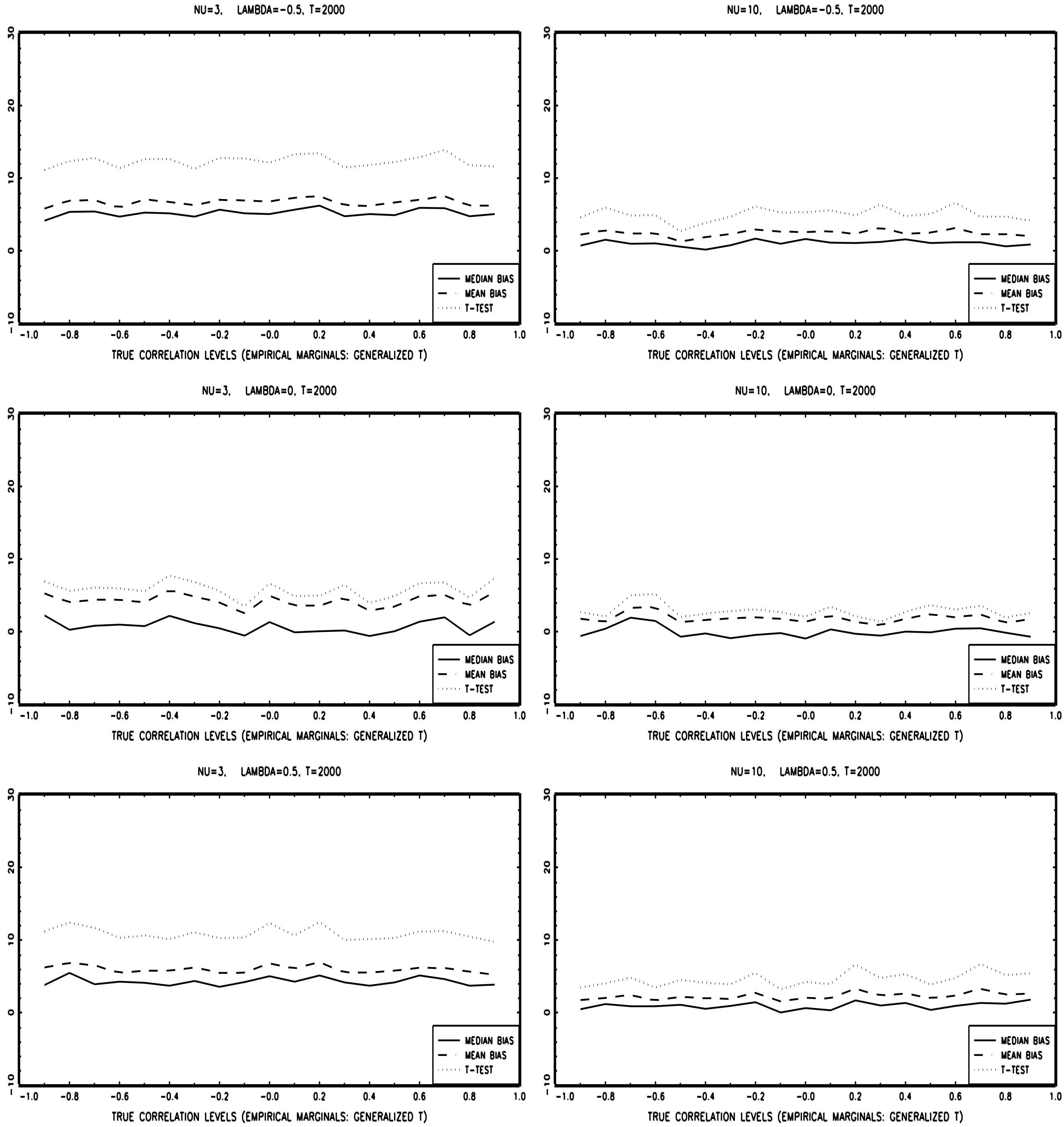

Figure 51: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient α when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

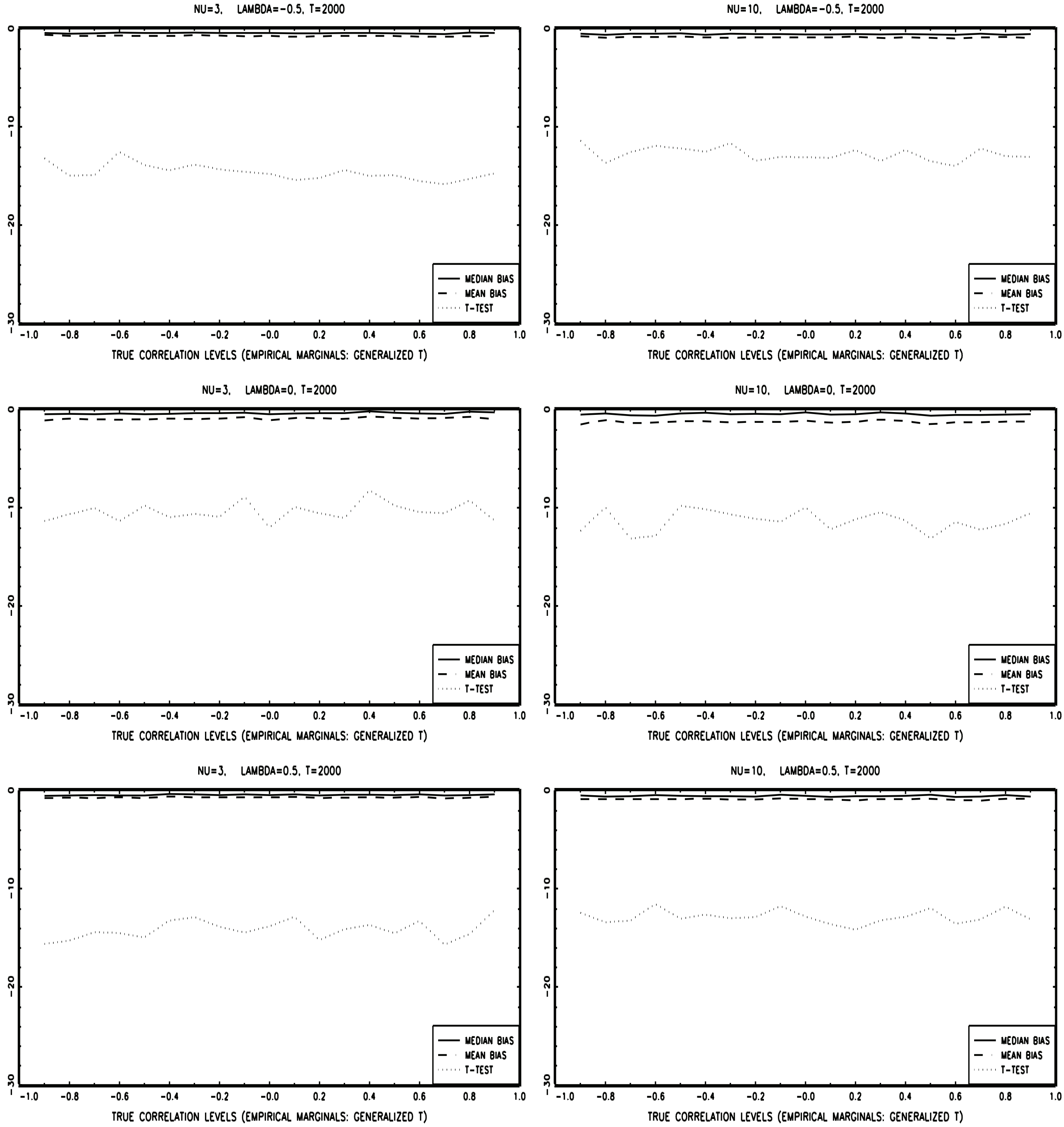

Figure 52: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient β when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

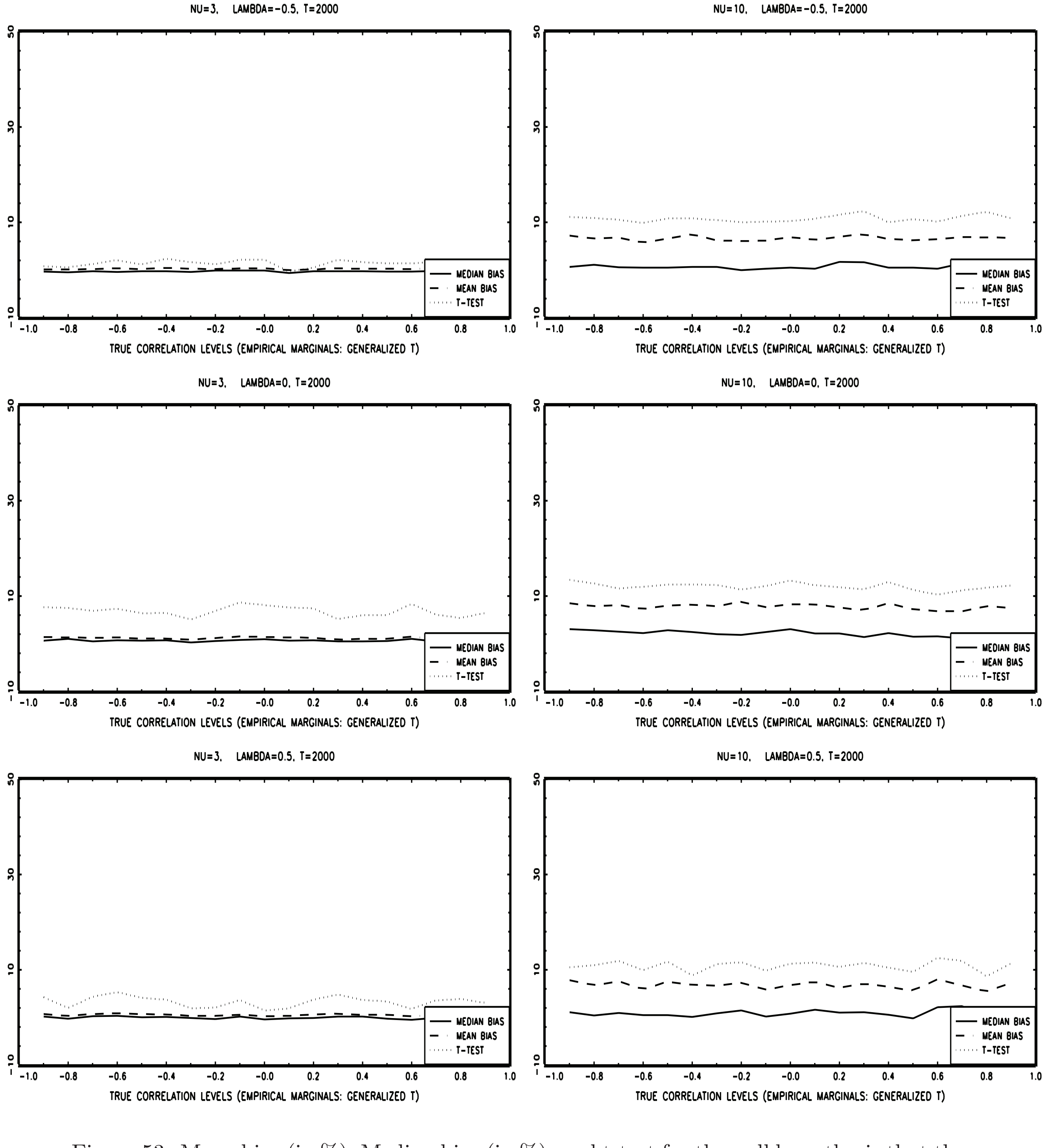

Figure 53: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient ν when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.

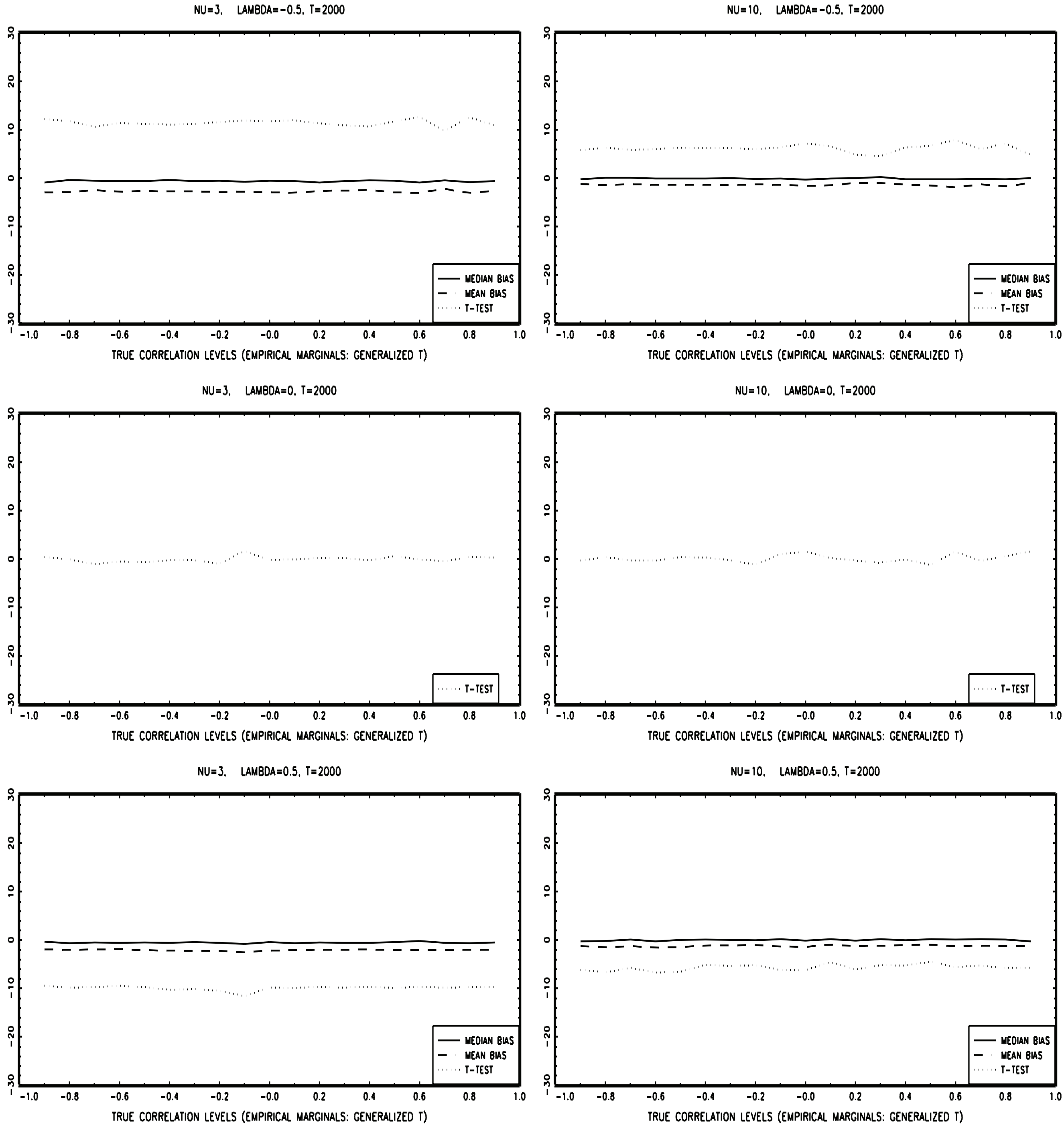

Figure 54: Mean bias (in %), Median bias (in %), and t-test for the null hypothesis that the empirical mean across simulations is equal to the true value. Monte Carlo results for the marginal coefficient λ when the True DGP is a Generalized-t with parameters $\nu = [3, 10]$, $\lambda = [-0.5, 0, -0.5]$, the bivariate copula correlation ranges in $\rho \in [-0.9, 0.9]$, $T = 2000$ and the empirical marginals are Generalized-t. The DGP values of ν and λ are reported in the top of the plots.