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Abstract

In this article we discuss the dynamics of organizational change when agents have heterogeneous initial conjectures and do learn. In this framework, conjectural equilibrium is defined as a steady state of the learning process, and all the adjustment occurs in disequilibrium. We discuss the properties of the system under different "rationality" assumptions, and using well-known learning algorithms. We prove analytically that multiplicity of equilibria, and failure of good organizational routines, cannot be ruled out: better, they are fairly probable. Stability is a crucial matter: it is shown to depend on initial conjectures. Finally, learning does not necessarily select the best.

JEL Classification: D21, D23, D84

Keywords: Heterogeneity, Learning, Theory of the firm

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1 Introduction

In this article we investigate the emerging of new organizational forms as a learning process on the part of the manager and of workers. We start from a conceptualization of technology that is familiar to evolutionary economics, namely a set of problem-solving routines that assign resources to specific nodes of the input-output graph; then we add a parallel view of the firm as a set of organizational routines that try to harmonize the conflicting interests at the shop floor level, given technological constraints. The study of the learning scheme is included as an essential feature of the setup, including initial conjectures as *primitives* of the problem.

The novelty of this approach is twofold. On the one hand, we put forth a setup in which the capability view and the incentive view of the firm can be reconciled, making a step forward in the agenda proposed by [Dosi et al.(2003)]. On the other hand, we introduce a discussion of the equilibrium process as genuinely based on disequilibrium (or *non-tâtonnement*) adjustments: equilibrium is a terminal state of the dynamical system, or to put it differently, a steady state of the learning algorithm.

From the methodological point of view, this approach is *open*, in that it requires a formalization of the learning algorithm: it can be completed by adding empirically grounded formalizations of the learning dynamics, which will further increase the robustness of existing evolutionary theorizing.[1] Moreover, the possibility of performing (at least) local stability analysis can help making concrete predictions.

Coming to the main results of the article, we show that even under rather general assumptions *lock-in* results are unavoidable: organizational innovation may fail because of what we term 'expectational bottlenecks'. In the second part, using a Bayesian formalization, we prove that there exists a continuum of conjectural equilibria, implying *path-dependency*, and we discuss some conditions for stability.

The article proceeds as follows: Section 2 presents the background on technology and firm theory in an evolutionary framework; Section 3 discusses the building blocks and a general characterization of equilibria; Section 4 presents an example of the general model under Bayesian conjectures and learning; finally, Section 5 concludes.

[1] In the second part of this article we will use Bayesian algorithm, without suggesting that this should be the only road to be followed.

2 Production theory: alternative views, and a conjectural equilibrium approach

In the mainstream framework a technology is a set of activities. An activity is a *complete* list of inputs and outputs. The emphasis on 'complete' is necessary: technically, such things as entropy, mental concentration, time should be included as well, and different qualities of the same input should be treated as different inputs. Moreover, for descriptive purpose the level of disaggregation of each production should be maximum. Once put this way, constant returns to scale are simply a tautological implication: duplication of the inputs should produce the same output. In the description of the system, only the activities that are feasible given the time time window considered in the analysis should be included. This is the approach to General Equilibrium Theory put forth by [Neumann(1945)].

In this framework: (1) no such a thing as a firm exists, and everyone can enter a market and produce, given the available technology (this is the way in which, [Walras(2003), Hicks(1939)] were addressing the problem, formalizing the free-entry concept of the Classics); and (2) a market must exists for every input and output, which implies zero transaction costs everywhere. The "firm" was introduced in the theory by [Arrow and Debreu(1954)], but this was done in a framework where entry and exit are not allowed: one lacks an explanation for the make-or-buy decision, and it is unclear what is controlled by command and what is left to market exchange. Put plainly, no answer to [Coase(1937)]'s question is provided.

The blank is filled, in partial equilibrium, by the theory of corporate governance (e.g. [Aghion and Bolton(1992), Holmstrom and Tirole(1997)] [La Porta et al.(2002)]), which is game-theoretical in spirit and focuses on incentive compatibility, and which can be included in general equilibrium along the lines provided by [Demichelis and Ritzberger(2011)].

In an alternative *capability based* perspective, a technology is a concrete knowledge base. Here, a set of problem solving procedures are available or can be developed, which match the system's agents with a set of input-output vectors [Winter(2006), Dosi and Grazzi(2006), Dosi and Grazzi(2010)]: a problem solving procedure assigns agents and resources to the nodes of the input output graph.[2]

[2]In the words of [Dosi(1988)] a technological paradigm is a set of pieces of knowledge, involving heuristics about "how to do things" and some basic templates of artifacts (i.e.

In this framework, there is radical uncertainty on the dimension and the characteristics of the production possibility set, which is the cartesian product of a *fuzzy* space of problem-solving procedures and a set of input-output vectors. Knowledge is dispersed, with various degrees of tacitness and significant costs of acquisition, exploration and replication. As stressed by [Dosi and Grazzi(2006)], firms know, possibly in a tacit way, their current technique, and tend to explore, in a local and cumulative way, some neighborhood of their location in the production set, departing from their present position.

If we incorporate organizational theory into the latter framework, we can conceive routines in a larger sense as including mechanisms of governance for conflicting interests. By exploring the role of institutional arrangements, this enlarged perspective tries to bridge the gap between the capability view and the incentive view of the firm [Dosi et al.(2003)].

In our view, organizational arrangements are tentative answers given by economic agents to the existence of conflicting interests *and* to the management of knowledge: this is, in our opinion, the way to approach [Coase(1937)]'s question. Indeed, the Coasian balance between market exchange and command aims also at efficiently running the way knowledge is coordinated, used, and modified in economic affairs: the very notion of "corporate culture" (e.g. [Kreps(1990), Crémer(1993)]) is grounded on the intuition that organizations can sometimes perform this task better than markets. This is obviously due to commonality of experiences-routines-languages, that tends to reinforce reciprocal understanding and expectations.

However, this reciprocal reinforcement is the source of a different difficulty: when an *organizational innovation* is conceived and proposed, by definition some parts of the exiting routines are called into question, and there is no guarantee that this innovation is well understood by all participants. In fact, what a corporate culture ensures is a sort of 'local' agreement/coordination on the *preexisting* set of routines, and nothing ensures that this agreement still holds when some elements of the corporate culture are displaced by the proposed innovation. What needs to be carefully studied is how *learning* takes place, in order to understand whether, and to which degree, the innovation will be successful: from this point of view, the configuration of accumulated knowledge and of individual initial conjectures is essential. Observe that what is crucial are not only the initial conjectures of

already known activities or production processes).

those *to whom* the innovation is proposed, but also the conjectures of those *by whom* the innovation is launched.

It is our contention that innovations, including organizational ones, can fail due precisely to the configuration of initial conjectures and to the way learning unfolds: we call *expectational bottleneck* this phenomenon. Many real-world examples display this feature, from major innovations (e.g. mergers, where the mixing of two corporate cultures is at work), to minor ones (e.g. the introduction of a single new routine).

The aim of the theory should be, thus, to describe the properties of the terminal states of the the system, i.e. the steady states of the learning process of participants. In this theory, first of all, agents must be construed as being *heterogeneous* and as acting sequentially: the "policy" functions are in general time-dependent, which is the definition of learning. Secondly, the system must be *self-referential*, in the sense that the outcomes of individual policies feed the learning of all participants at each date. Thirdly, in the face of the huge difficulty often implied by the choice under strategic interaction and uncertainty, one must be open to accept *bounded rationality* in the theory. Fourthly, any steady state of the learning process must have the property that no participant is induced to modify her conjecture given what she can observe. To the best of our knowledge, this is the *conjectural* equilibrium concept described by [Hahn(1974)], but one should not overlook [Hayek(1937)]'s arguments. We accept this equilibrium notion, with the additional requirement that it be an attracting state of the dynamical system, i.e. it is somehow robust with respect to shocks: in other terms, (at least local) *stability* is a fundamental part of the theory to be studied, given the multiplicity of equilibria.

Finally, we should point out that the theory is indeterminate unless we specify a precise learning algorithm. In other words, it asks for a meta-theory of learning algorithms. An investigation of this issue is beyond the scope of this article, and we think that further insights can be brought by psychological studies and behavioral economics.

3 Conjectural equilibria: a preliminary characterization

3.1 Preliminaries

We start by discussing the concept of conjectural equilibria in a setup which is demanding in terms of cognitive ability of the agents. We consider the problem of the implementation of a new organizational form, or of a new production arrangement required by the introduction of an innovation. The result depends on the effort performed by workers, and a manager is in charge of defining the compensation scheme and supervising the workers. The essential point is that agents, both the workers and the manager, are initially uncertain about the effort that will be exerted by everyone.

There are M workers: each of them maximizes a utility function that is separable in money wage, W, and the utility cost of the effort, $g(X_i)$, of the following type:

$$U(W, X_i) = W - g(X_i). \tag{1}$$

Individual efforts are not perfectly observable, neither by the other workers nor by the firm. The only signal is aggregate output[3] that, for simplicity, is the sum of the individual efforts:

$$Y = \sum_{i=1}^{M} X_i.$$

The manager is the residual claimant of the production unit: she will get the output minus the payments made to workers.[4] As a result, she would like to implement the effort that maximizes surplus. In fact workers will not enter the transaction unless their cost of providing the effort is covered (individual rationality constraint): thus the only way to increase the residual claim is by increasing the surplus, and getting more output for more effort will also increase the payment to cover the cost. Given the effort cost for

[3] We could add a zero-mean random noise term, independent across workers: this would not change our results but add increase notational complexity.

[4] We keep the setup sufficiently general: it can capture either the functioning of a unit of the firm, or the implementation of a single project, or the whole firm (in which case the manager simply gets the profits).

workers, the manager would like to implement:

$$X_i^* \in \arg\max_{X_i} \sum_{i=1}^{M} (X_i - g(X_i)).$$

In this setup the manager cannot implement the optimum through a simple scheme that links proportionally the pay to the result, since this cannot guarantee optimality in terms of efficiency *and* incentive compatibility [Holmstrom(1982)]. The standard solution is to fix a threshold $\bar{\pi}$ for aggregate output, above which a bonus payment B is granted. We depart from [Holmstrom(1982)] in assuming that agents have *subjective* prior conjectures on the effort that is going to be exerted. Conjectures are as follows. Each worker i maintains a certain distribution over the aggregate effort of the others, with an unknown mean parameter over which she has a prior conjecture with parameter μ_{it} collecting all the available information up to date t. The manager maintains a distribution over the way in which workers assume the effort by the others, again with an unknown mean over which she has a prior conjecture with parameter μ_{ft}, again using all the information up to time t.[5] The $\mu_{it}(i = 1, ..., M, f)$ can be *vectors* of parameters.

The bonus is to be interpreted as a wage *premium* for participating in the new project. In practice, we imagine workers as receiving a basic fixed wage that is equal to the average wage outside the new unit, or to some other outside option: the bonus is then added if the threshold is reached.

The timing is as follows: at every time t the firm announces an output threshold $\bar{\pi}_t$ and a bonus B_t (with perfect commitment); after this, each worker decides her effort level; production takes then place, and only now aggregate output Y_t can be observed; the firm and the workers update their individual priors, using some algorithm and the available information, viz. Y_t. At $t + 1$ the process restarts. Notice, then, that choices at time t can only be made on the basis of conjectures formulated at date $t - 1$.

Clearly the following relation holds from each worker's point of view:

$$\text{Prob}(Y_t \leq \bar{\pi}) = \text{Prob}(\sum_{j \neq i} X_j \leq \bar{\pi} - X_i)$$

[5] This last element possibly captures the perception, on the part of the manager, of the shop-floor relationship between the workforce.

All this given, the manager's problem can be stated as:

$$\begin{array}{c} \text{choose } \{B_t\ , \bar{\pi}_t\} \\ \textit{such that:} \\ \bar{\pi}_t = \sum_{i=1}^{M} X_i \\ X_{ft} \in \arg\max_{X_{it}} \sum_{i=1}^{M} (X_{it} - g(X_{it})) \\ X_{ft} \in \arg\max_{X_{it}} B_t(1 - F(\bar{\pi}_t - X_{it}|\mu_{f,t-1})) - g(X_{it})) \end{array} \tag{2}$$

where we call X_{ft} the level of effort that the manager wants to implement. The first constraint is always binding because of the definition of the technology, and because of the role of the individual rationality constraint discussed above. For each worker i the objective is:

$$\max_{X_{it}} B_t(1 - F(\bar{\pi}_t - X_{it}|\mu_{i,t-1})) - g(X_{it}). \tag{3}$$

We take $F(\cdot|\ \mu_{it})$ to be the Cumulative Density Function (CDF), and $f(\cdot|\ \mu_{it})$ to be the PDF, given the individual prior of i at date t (with $i = f, 1, ..., M$). Observe that the functional form of F and f is taken to be the same across agents, as it is common in standard literature (see the next Section for a different hypothesis): only the μ_{it}'s can be different.

The learning algorithm used by workers and the manager is a mapping

$$[\mu_{f,t+1},\ \mu_{1,t+1}, \dots\ , \mu_{M,t+1}]' = [v_f(\mu_{f,t}),\ v_1(\mu_{1,t}),\ \dots\ , v_M(\mu_{M,t})]') \tag{4}$$

We make the following assumptions.

Assumption 1.

$$g : \mathbb{R} \to \mathbb{R},\ g \in C^2,\ g(0) = 0,\ g' > 0\ \lim_{X \to 0} g'(X) = 0,\ g'' > 0$$

Assumption 2. The PDFs $f(\cdot|\ \mu_{it})$ are continuous. They admit a sufficient statistic that is a continuous function of the aggregate output. The v_j's ($j = 1,\ \dots\ M, f$) defined in (4) are continuous functions of the sufficient statistic.

Assumption 3. The manager is risk neutral and is not cash constrained.

Remark 1. Assumption 1 is fairly reasonable. Assumption 2 is standard for most conjugate families in Bayesian analysis [De Groot(1970)]. Of course it fits any non-Bayesian updating algorithm, for which continuity holds. Since the only available signal is aggregate output, the sufficient statistic must be some continuous transformation of Y_t. Assumption 3 can be weakened (see the comments at the end of this Section).

3.2 Main Result

Definition 1. $[B_t,\ \bar{\pi}_t,\ X_{1,t}\ X_{M,t}] = T([B_{t-1},\ \bar{\pi}_{t-1},\ X_{1,t-1}\ X_{M,t-1}])$ is the mapping governing the dynamics of the system under (2)-(4).

This mapping is the composition of the solutions to (2) and (3) and the updating (4), given Assumption 2. We define now a *conjectural equilibrium* as a steady state of the updating process:

Definition 2. A conjectural equilibrium is a $(M+2)$-tuple of individual choices $[B^*,\ \bar{\pi}^*,\ \{X_i^*\}_{i=1,\ ...M}]$ such that

$$[B^*,\ \bar{\pi}^*,\ \{X_i^*\}_{i=1,\ ...M}] = T([B^*,\ \bar{\pi}^*,\ \{X_i^*\}_{i=1,\ ...M}])$$

This equilibrium concept is equivalent to the one discussed in Section 2 above: it describes a termination of the dynamical system, once we have described the behavioral rules of agents. Notice that we do not attach to it any particular normative label, i.e. we do not state that the equilibrium satisfies a particular efficiency criterion.

Remark 2. A CE is defined as a fixed point of the updating of choices: as it will be apparent from the proof of Theorem 1 below, it is also a fixed point of the updating of priors. If (4) is a Bayesian smooth updating process, e.g. a (log)normal-normal couple, then $T(\cdot)$ depends also on individual *precisions.* In this case a conjectural equilibrium should be defined asymptotically, i.e. for precisions tending to infinity, as we shall see in Section 4, and as in [Bogliacino and Rampa(2010)] and [Bogliacino and Rampa(2011)].

This given, the following Theorem holds.

Theorem 1. *Under Assumptions* 1-3*, system* (2)-(3) *admits a conjectural equilibrium.*

Proof. See A □

Theorem 1 cannot rule out a *multiplicity* of conjectural equilibria, as it happens with all fixed-point arguments: indeed the Jacobian of the $Z(\cdot)$ mapping is generically non-zero everywhere off the diagonal, and the conditions that would rule out multiplicity are not easy to be put forth, since they involve the properties of both the conjectures and the updating algorithm.

From the economic point of view, multiplicity derives form the fact that ours is a standard coordination problem, where high (low) effort drives high (low) effort. As a result, it is almost impossible to discard the possibility of *lock-in*, i.e. failure, or low effectiveness, of innovations in raising surplus, regardless of their *objective* characteristics. In addition, since the model allows for vectorial conjectures μ_i, it is likely that the equilibria form a *continuum* (as we shall see in Section 4 below).

The existence of conjectural equilibria with a lock-in property is a pervasive feature of learning processes involving heterogeneous agents, as shown by [Bogliacino and Rampa(2011)] and [Bogliacino and Rampa(2010)]. It captures our basic insight about the role of *expectational bottlenecks*. Even at the shop-floor, the existence of uncertainty -a traditional property of the technological domain- transforms the problem of cooperation into a beauty contest, in a [Keynes(2007)] fashion.

The present general setup allows for some natural extensions. First, the cost of the effort is usually *private information* of the worker, or it can even be unknown to the worker: indeed, since the new organizational form is implemented for the first time in the firm, there is no reason to imagine a perfect knowledge of the required effort. This corresponds to a slight modification of the second constraint in the manager's problem (2). The proof of Theorem 1 could be easily adapted to account for this generalization, and would use the same analytical arguments as those put forth in A below. Second, one can generalize the assumption on technology: as long as output is a continuous function of efforts, the basic properties remain unchanged.

Third, one can relax Assumption 3, e.g. assuming that agents are cash-constrained and/or bear some fixed costs of participation in the new organizational form.[6] In this case the manager can run out of money before the new organizational form is functioning, which will eventually lock-in the system into a failure. Alternatively, pessimism on the part of workers might convince them that it is not worth investing in the necessary human capital:

[6]E.g. investment in physical capital on the part of the manager, and/or investment in human capital on the part of workers

in this case the innovation does not even take off.

As we discussed in the previous Section, the model is 'open': a concrete learning algorithm should be specified. In the next Section we discuss an example with Bayesian learning, a mechanism that is widely used in both orthodox and heterodox approaches.

4 Learning under "bounded" rationality

In this Section we study a setup where the standard assumption that the manager is fully informed about the workers' preferences is dropped. In addition, some specific functional forms are adopted, in order to obtain more definite results as regards (a) the learning process, (b) the characterization of conjectural equilibria, and (c) their robustness under the learning dynamics.

4.1 Technology and the bonus scheme

Given individual efforts $X_{i=1,\ldots,M}$, total output is $Y = M \cdot \prod_{i=1}^{M} X_i^{1/M}$: as anticipated at the end of the previous Section, this is coherent with the general setup developed there. Using lower-case letters to denote logarithms, we have the following expression for the log of per-capita output:

$$y - \ln M = \frac{1}{M} \sum_{i=1}^{M} x_i \tag{5}$$

The manager announces the following bonus scheme:

$$B = \beta Y^2 \tag{6}$$

with $\beta > 0$. Being the bonus a non-decreasing function of Y, this scheme belongs to the same general family already treated in part 3.1. Both (5) and (6) are assumed to be known to all workers.

4.2 Workers' conjectures, choices and learning

Each worker is endowed with the utility function (1). As regards the effort cost for each worker, we posit

$$g_i(X_i) = \frac{2}{\alpha} X_i^{\alpha}, \qquad \alpha > 2. \tag{7}$$

This assumption, that is coherent with Assumption 1, is necessary for having a solution to the worker's problem written below, given the quadratic bonus scheme. We take α to be equal across workers: this is due to the quest for simplicity. Of course, agent *heterogeneity* is an essential feature of our setup: it is modeled by assuming workers to be heterogenous from the point of view of their *conjectures*, as we are going to see.

Worker i is uncertain about the effort that is going to be exerted by all other workers, denoted by X_{-i}. We assume that worker i takes X_{-i} to be log-normally distributed, meaning that $x_{-i} \doteq \ln X_{-i}$ is normally distributed with mean μ, and precision assumed to be equal to 1.[7] Being uncertain on μ, worker i models this uncertainty by means of the following *normal* prior conjecture, that is coherent with our Assumption 2:

Assumption 4. $\mu \sim_{i,t} N(\mu_{i,t}, \tau_{i,t}^{-1})$ *where* $\sim_{i,t}$ *means "is distributed, according to i at date t, as".* $\tau_{i,t}$ *is i's subjective precision.*

Each worker's objective is to maximize her expected utility at each date. As regards the *timing* of events, we assume that workers choose their utility maximizing efforts simultaneously at date t: hence, each worker does not know y_t when choosing her effort, and she must utilize her conjecture as of date $t-1$. This given, it easy to see that, under the assumption on technology, worker i's optimization problem can be written as

$$\max_{X_{i,t}} \beta_t M^2 E_{i,t-1}(X_{-i,t}^{2/M}) \cdot X_{i,t}^{2/M} - \frac{2}{\alpha} X_{i,t}^{\alpha}$$

where $E_{i,t-1}(\cdot)$ denotes expectation taken with respect to i's conjecture formulated at date $t-1$. It can be shown that the first order condition for the solution to this problem is[8]

$$X_{i,t} = (\beta_t M)^{\frac{M}{\alpha M - 2}} \cdot \exp\left[\frac{2}{\alpha M - 2}(\mu_{i,t-1} + \frac{1}{M} + \frac{1}{M\tau_{i,t-1}})\right] \tag{8}$$

Taking logarithms, we obtain

$$x_{i,t} = \frac{M}{\alpha M - 2} \cdot \ln(\beta_t M) + \frac{2}{\alpha M - 2}(\mu_{i,t-1} + \frac{1}{M} + \frac{1}{M\tau_{i,t-1}}) \tag{9}$$

[7] The 'objective' precision of x_{-i} is taken to be equal across workers, equal to 1, and maintained with certainty, for simplicity.

[8] To obtain the result, one must take account of the following fact: if $\ln(Z)$ is believed to be normally distributed with mean μ and precision 1, and if the prior over μ is normal with mean μ_i and precision τ_i, then the subjectively expected value of Z is $\exp(\mu_i + 1 + \frac{1}{\tau_i})$. Concavity is guaranteed by the hypothesis $\alpha > 2$.

From (5) and (9) one derives the resulting log of per-capita output:

$$y_t - \ln M = \frac{M}{\alpha M - 2} \ln(\beta_t M) + \frac{2/M}{\alpha M - 2} \left(1 + \sum_i (\mu_{i,t-1} + \frac{1}{M \tau_{i,t-1}}) \right) \quad (10)$$

We come now to workers' learning. After individual efforts have been chosen, production takes place and workers are informed about the resulting output at date t, Y_t. Each worker i, starting from (5) and knowing her effort $X_{i,t}$, can compute $\ln(X_{-i,t}) \doteq x_{-i,t} = My_t - M \ln M - x_{i,t}$. Hence, worker i updates her conjecture using standard Bayesian techniques (see [De Groot(1970)], chap. 10):

$$\mu_{i,t} = \frac{\mu_{i,t-1}\tau_{i,t-1} + x_{-i,t}}{\tau_{i,t-1} + 1} \qquad \text{and} \qquad \tau_{i,t} = \tau_{i,t-1} + 1. \quad (11)$$

4.3 The manager's conjecture, choice and learning

The manager knows neither workers' preference, that is $g_i(X_i)$, nor their conjectures, that is their stochastic assumptions about the environment, nor the shape of their heterogeneity. In order to grasp how individual efforts, and hence output, respond to the bonus scheme, the manager assumes first of all that workers are all alike for simplicity. Second, she conjectures the following relation between workers' effort and the bonus: $X_i = C\beta^A$, where it is reasonable to assume $A, C > 0$. Notice that this conjecture is coherent with the general functional form of (8).[9] Taking account of (5), the managers' conjecture on total output is then $Y = MC\beta^A$, implying the following log-linear relation between β and per-capita output:

$$y - \ln M = A \ln \beta + c \quad (12)$$

where $c \doteq \ln C$. The manager believes that, for any A, c and β, the log of per-capita output is a normal variable with mean described by (12) and with a given precision, assumed equal to 1 for simplicity.

The manager wishes to maximize expected profit with respect to β (risk neutrality), under the assumption that the output price is 1. However, the bonus must be chosen at date t before Y_t is known: as a consequence, the manager chooses β_t using her conjecture formulated at date $t-1$. So, the output expected by the manager is $E_{m,t-1}(Y_t|\beta_t) = MC_{t-1}\beta_t^{A_{t-1}}$.

[9]Hence, the manager's rationality is only "boundedly" bounded!

This given, the manager's problem is as follows:

$$\max_{\beta_t} E_{m,t-1}(Y_t|\beta_t) - M\beta_t[E_{m,t-1}(Y_t|\beta_t)]^2.$$

It can be shown that the solution, expressed in logarithms, is

$$\ln\beta_t = \frac{1}{A_{t-1}+1}\left[\ln(\frac{A_{t-1}}{2A_{t-1}+1}) - c_{t-1} - 2\ln M\right]. \qquad (13)$$

We come to learning on the part of the manager. Being uncertain about the parameters (A, c), the manager maintains a *normal* prior conjecture, that is coherent with our Assumption 2, with the following characteristics

Assumption 5. The manager's hyper-parameters at date t are the following: the mean of the prior is the couple (A_t, c_t), and the symmetric variance-covariance matrix is $\mathbf{\Gamma}_t$, defined as

$$\mathbf{\Gamma}_t = \begin{bmatrix} \gamma_{1,t} & \gamma_{12,t} \\ \gamma_{12,t} & \gamma_{2,t} \end{bmatrix}.$$

where the suffixes 1 and 2 refer, respectively, to A and c. We assume $\gamma_{12,0} = 0$, and define $\gamma_{1,0} = \gamma_1$ and $\gamma_{2,0} = \gamma_2$

Define now the vectors $\mathbf{m}_t = [A_t \ c_t]^T$ and $\mathbf{x}_t = [\ln(\beta_t) \ 1]^T$, i.e. the conjectured parameters and the 'regressors' of conjecture (12), respectively: conjecture (12) can thus be written compactly as $\upsilon_t^m = \mathbf{x}_t^T\mathbf{m}_{t-1}$, where υ_t^m is the *conjectured* log of per-capita output at date t. Define, instead, the *actual* log of per-capita output at date t as $\upsilon_t \doteq y_t - \ln M$.

The manager, after having chosen β_t, observes the resulting output of date t, and hence computes υ_t. On the basis of this, she can update her conjecture along standard Bayesian lines ([De Groot(1970)], ch. 10):

$$\mathbf{m}_t = [\mathbf{\Gamma}_{t-1} + \mathbf{x}_t\mathbf{x}_t^T]^{-1}[\mathbf{\Gamma}_{t-1}\mathbf{m}_{t-1} + \upsilon_t\mathbf{x}_t] \qquad (14)$$

and

$$\mathbf{\Gamma}_t = \mathbf{\Gamma}_{t-1} + \mathbf{x}_t\mathbf{x}_t^T. \qquad (15)$$

Notice that, after some passages, expression (14) becomes

$$\mathbf{m}_t = \mathbf{m}_{t-1} + \mathbf{\Gamma}_t^{-1}[\mathbf{x}_t(\upsilon_t - \upsilon_t^m)] \qquad (16)$$

4.4 Characterization of conjectural equilibria

Define the row vector of workers' parameters: $\mathbf{h}_t^T = [\mu_1, ..., \mu_M, \tau_1, ..., \tau_M]$; recall the definition of vector $\mathbf{m}_t$; and define the row vector of the manager's subjective variances-covariances $\mathbf{g}_t^T = [\gamma_{1,t}, \gamma_{12,t}, \gamma_{2,t}]$. Define finally the $(2M + 5)$-row-vector $\mathbf{p}_t^T = [\mathbf{h}_t^T, \mathbf{m}_t^T, \mathbf{g}_t^T]$.

A careful inspection of (9), (10), (11), (13), (14) and (15) reveals that one can write the following system of $2M + 5$ first-order difference equations in all hyper-parameters

$$\mathbf{p}_t = H(\mathbf{p}_{t-1}), \tag{17}$$

that completely describes the learning dynamics.

We exploit now our Definition 2, and say that a *conjectural equilibrium* (CE) is a fixed point of (17). From our previous analysis, it is apparent that in a CE not only all hyper-parameters, but also all individual choices are stabilized. As observed in Remark 2 above, in the present setup a state of CE can hold only 'asymptotically', namely if $\tau_i \to \infty, i = 1, ..., M$, meaning that workers have become certain of their conjectures. Indeed, from expressions (9) and (11) (second part) it is clear that workers' choices keep changing if their precisions take on finite values, and this cannot be a CE state. On the contrary, a CE does *not* require that the manager has become certain of her conjecture: in fact (13) says that her choice does not depend on her subjective variance-covariance parameters.

Our Theorem 1, properly adapted to the present setting, ensures that a CE does exist. However, we can be more precise than this, and go further to characterize CE's under the assumption of the present Section. First of all, we give the following

Definition 3. A symmetric CE (SCE) is a CE in which all workers conjecture the same mean parameter μ with certainty (i.e. $\tau_i \to \infty, i = 1, ..., M$), and hence exert the same (log) effort x.

A first set of results are included in the following Theorem.

Theorem 2. *Under* (5)-(7) *and Assumptions* 4-5 *the following holds:*
(a) *any* CE *of the system* (17) *is a* SCE*;*
(b) *there exists a continuum of* SCE*'s, corresponding to a well defined one-dimensional manifold in the* (A,c) *space;*
(c) *output is increasing in* A *along the above manifold;*
(d) *there exists one profit-maximizing* SCE*; however, there exists no optimal*

-i.e. surplus-maximizing- SCE*, and surplus is strictly increasing in* A *along the* SCE*-manifold.*

Proof. See B □

Part (a) characterizes the set of equilibria as symmetric. Part (b) indicates that the new organizational form can remain trapped in infinitely many different states, depending obviously on the state of agents' mutual expectations (*indeterminacy*). Indeed, as the proof shows, the manager's conjectured mean parameter, hence the bonus announced, increases with workers' conjectured μ along the SCE manifold: in other words, conjectures positively feed each other. As a further consequence, the system displays $path - dependency$: to different initial conditions of conjectures there corresponds a different final state, supposing that the latter is stable (see below). As remarked at the end of Section 3, the evolution could display $lock{-}in$ phenomena: if the *subjectively* expected gross return from participation is lower than some fixed cost, and if agents are cash-constrained, even an *objectively* high-return organizational form can fail to take-off.

Parts (c) and (d) characterize further SEC's: the activity level depends positively on the manager's conjecture, and in addition SCE's can be Pareto-ranked along the SCE manifold[10]. Hence there is room for welfare-improving policies. However, there does not exist a surplus-maximizing policy, and increasing surplus may harm profits (past the profit-maximizing CE), thus requiring more articulated policies.

4.5 Robustness under learning

One wonders whether indeterminacy can be somehow 'refined' by means of stability arguments: in fact, one might hope that a large part of CE's, hopefully the Pareto-inferior ones, are actually uninteresting being unstable. As regards this question, we put forward two major points. First, expressions (11), (15) and (16) reveal that as time elapses agents' mean hyper-parameters become more and more stabilized: indeed, all τ_i's and Γ_t grow in time, hence the contribution of new information becomes less and less important in modifying the conjectured mean parameters. As a consequence, *all* SCE's become stable sooner or later.

[10]Surplus is measured as total output minus total effort cost borne by workers, as we did in Section 3.

Second, by undertaking a *local* stability analysis we are able to offer the following

Proposition 1. *Under* (5)-(7) *and Assumptions* 4-5 *the following holds:*
(a) *Local stability/instability is completely determined by one of the eigenvalues relating to the manager's learning scheme;*
(b) *if the elasticity of output to* β *that is conjectured by the manager,* A, *is sufficiently lower than the true elasticity,* $\frac{M}{\alpha M-2}$, *and if the manager is initially highly uncertain uncertain about her conjectured elasticity, meaning* $\gamma_1 \to 0$, *then* SCE*'s are locally unstable in the first phases of learning*[11]*;*
(c) *under the conditions of part (b),* SCE*'s become more stable if, ceteris paribus, the number of workers,* M, *and the effort elasticity of utility,* α, *increase;*
(d) *Pareto-superior* SCE*'s are more stable*[12].

Proof. See C □

Parts (a)-(c) show that stability depends not only on the 'fundamental' parameters, but also on the manager's conjecture. Given parts (b)-(d), the answer to the question raised in this subsection is *only partially* 'yes'. In the present setup it is true that Pareto-superior outcomes are relatively more stable at the outset: however, it is not true 'learning selects the best', since, when time elapses and/or the manager becomes more self-confident, *all* CE's become stable.

5 Discussion and Concluding Remarks

In this article we have presented a Conjectural Equilibrium framework (*à la* [Bogliacino and Rampa(2010), Bogliacino and Rampa(2011)]) to discuss some aspects of the theory of production, in particular the implementation of an organizational change.

We have studied some properties of the steady states of the learning dynamics, viz. their number, stability, and welfare properties. The main results pertain to the *multiplicity* of conjectural equilibria: multiplicity turns into

[11]The proof shows that in this case instability is of the *ocillatory* type (a negative eigenvalue).

[12]Here "greater stability" means a lower modulus of the relevant eigenvalue.

indeterminacy if one weakens the assumption of perfect rationality. This implies that lock-in and path-dependency are typical properties of any process of organizational innovation: since these phenomena are driven by conjectures, we speak of ‘expectational bottlenecks’.

We deem that future research should address the role of alternative learning algorithms, and the question of how the results can be generalized to the case of multiple conflicting interests.

A Proof of Theorem 1

We need to prove two Claims. We remind that μ_{it-1} is the relevant conjecture at time t because it incorporates all the information before taking any action.

Claim 1. *At every time t, for given B_t and $\bar{\pi}_t$ and any positive and finite $\mu_{i,t-1}$, each worker chooses an optimal level of effort belonging to the set $[0, \bar{\pi}_t]$. For any $t \in \boldsymbol{N}$, (X^*_{it}) is a non empty, upper hemi-continuous, compact value and closed graph correspondence of $(X^*_{j,t-1})$ $j \neq i$ and of B_t*

Proof. Rationality implies $X_{it} \leq \bar{\pi}_t$. The objective function is continuous in both $[0, \bar{\pi}_t]$ and the domain of $\mu_{i,t-1}$ and B_t. Thus, by the Maximum Theorem ([Berge(1997)], Maximum Theorem, p. 116), we can define $(X^*_{it}) = C(\mu_{i,t-1}, B_t)$, where $C(\cdot)$ is non empty, upper hemi-continuous, compact value and closed graph. Finally, using the updating rule and the last part of Assumption 2, we can define $\mu_{i,t-1} = u_i(X^*_{j,t-1}),\ i \neq j$, where $u(\cdot)$ is *continuous*. Composing $C(\cdot)$ and $u_i(\cdot)$ completes the proof ([Berge(1997)], Theorems VI.1' and IV.5). □

Claim 2. *For any given threshold $\bar{\pi}$, the optimal bonus B_t is bounded and is a continuous function of $(X^*_{i,t-1})$ $i = 1, ...M$. $B_t \in [0, \pi]$.*

Proof. The FOC for Equation (2) implies:

$$B_t = \frac{g'(X_{ft})}{f(\bar{\pi} - X_{ft}|\mu_{f,t-1}))} \tag{18}$$

where X_{ft} is the level of effort that the firm wants to implement (the one that maximizes surplus). Given the RHS in expression (18), B_t it is unique and continuous in $\mu_{t,t-1}$. In addition, according to the last part of Assumption 2 we can define $\mu_{f,t-1} = u_f((X^*_{j,t-1})_t), j = 1, ..., M$, where $u_f(\cdot)$ ic *continuous*. As a consequence, the composition of continuous function is again continuous.

The threshold is the sum of the individual efforts X_{ft}, with $X_{ft} \in \arg\max_{X_i} \sum_{i=1}^{M}(X_i - g(X_i))$, which is unique by Assumption 1. □

We are ready to prove Theorem 1:

Proof. Take the compact $K = [0, \bar{\pi}]^{M+2}$. The dynamical system is

$$[B_t, \bar{\pi}, (X_{jt})_{j=1,...M}] = T(B_{t-1}, \bar{\pi}, (X_{j,t-1})_{j=1,...M}) \quad T(\cdot) \quad T : K \to K$$

$T(\cdot)$ is non empty, upper hemi-continuous, compact value and closed graph correspondence ([Berge(1997)], Theorems VI.4'): existence follows from the Kakutani Fixed Point Theorem ([Berge(1997)], Kakutani's Theorem, pag 117). □

B Proof of Theorem 2

In this Appendix we drop the time suffixes of variables wherever we are interested in stationary states.

Part (a)

Proof. From (11), a necessary condition for a CE is $\mu_i = x_{-i} = \sum_{j \neq i} x_j$, together with $\tau_i \to \infty$, $\forall i$. Coupling this with (5), after some passages one gets the following alternative CE condition: $y - \ln M = \frac{1}{M}(\mu_i + x_i), \forall i$. Now, from (9) we see that x_i depends only on μ_i, plus other terms that are common to all workers (provided that $\tau_i \to \infty$, as we require for a CE); moreover, x_i is an increasing function of μ_i. So, for any given y the above CE condition admits a single solution in μ_i, which is necessarily the same for all i's. □

Part (b)

Proof. Call μ the common value of the μ_i's in a SCE; recall that we require also $\tau_i \to \infty$. From (9) one computes the log of individual effort in a SCE:

$$x_i = \frac{M}{\alpha M - 2} \ln(\beta M) + \frac{2\mu}{\alpha M - 2} + \frac{2}{M(\alpha M - 2)} \tag{19}$$

and from (10) one derives the log of per-capita output in a SCE:

$$y - \ln M = \frac{M}{\alpha M - 2} \ln(\beta M) + \frac{2\mu}{\alpha M - 2} + \frac{2}{M(\alpha M - 2)}. \tag{20}$$

multiplying (19) by $(M-1)$, we obtain the effort exerted in a SCE by all workers different from a given one. From the argument of the proof of Part (a), this must be equal to μ in a SCE. Hence we require $\mu = \frac{M(M-1)}{\alpha M-2} \ln(\beta M) + \frac{2\mu(M-1)}{\alpha M-2} + \frac{2(M-1)}{M(\alpha M-2)}$, that boils down to

$$\mu = \frac{M-1}{\alpha - 2} \ln(\beta M) + \frac{2(M-1)}{M^2(\alpha - 2)}. \tag{21}$$

Substitute now (21) in (20); some manipulation will lead to the following condition:

$$y - \ln M = \frac{1}{\alpha - 2} \ln(\beta M) + \frac{2}{M^2(\alpha - 2)}. \tag{22}$$

We have now to take account of the manager's conjecture (12). In order that this is a CE conjecture, it must agree with the log output defined by (22); hence, after substitution and by rearranging terms, the following condition is to be satisfied:

$$c = \frac{1 - A(\alpha - 2)}{\alpha - 2} \ln(\beta) + \frac{1}{\alpha - 2} \ln(M) + \frac{2}{M^2(\alpha - 2)}. \tag{23}$$

The term $\ln(\beta)$, in turn, must agree with profit maximization. Thus, we substitute (13) in (23), obtaining

$$c = \frac{1 - A(\alpha - 2)}{\alpha - 2} \frac{1}{A+1} \left[\ln(\frac{A}{2A+1}) - c - 2\ln M \right] + \frac{1}{\alpha - 2} \ln(M) + \frac{2}{M^2(\alpha - 2)}.$$

It is easy to show that the last condition is equivalent to

$$c = \frac{1 - A(\alpha - 2)}{\alpha - 1} \ln(\frac{A}{2A+1}) + \left[\frac{(2\alpha - 3)(A+1)}{\alpha - 1} - 2\right] \ln M + \frac{2(A+1)}{M^2(\alpha - 1)}. \tag{24}$$

Expression (24) defines precisely the one-dimensional SCE-manifold, expressing the *continuum* of SCE's. The economically sound branch of (24) is where $A > 0$. □

Part (c)

Proof. Substitute (24) in expression (13). By doing so, after some passages we get

$$\ln \beta = \frac{\alpha - 2}{\alpha - 1} \ln(\frac{A}{2A+1}) - \frac{2\alpha - 3}{\alpha - 1} \ln M - \frac{2}{M^2(\alpha - 1)} < 0. \tag{25}$$

Expression (25) defines the value that $\ln \beta$ must take on, as a function of A, in order that the system is in a SCE. Notice that $\ln \beta$ is *negative* in any SCE. In addition, it is easy to show that, under our assumption $A > 0$ and $\alpha > 2$, $\ln \beta$ is an *increasing* function of A. Going back to (22), it turns out finally that the log of per-capita output is an *increasing* function of A along the SCE manifold. □

Part (d)

Proof. We prove first that there exists indeed a profit-maximizing SCE. Given the bonus scheme, and having assumed that the output price is 1, profit is $Y - M\beta Y^2$. From (22), the level of output in a SCE is $Y = M(\beta M)^{\frac{1}{\alpha-2}} \exp\left(\frac{2}{M^2(\alpha-2)}\right)$. Hence, one must maximize $(\beta M)^{\frac{1}{\alpha-2}} \exp\left(\frac{2}{M^2(\alpha-2)}\right) - M\beta M(\beta M)^{\frac{2}{\alpha-2}} \left[\exp\left(\frac{2}{M^2(\alpha-2)}\right)\right]^2$ with respect to β: in fact, from (25) we know that β, a well-defined function of A, parametrizes completely all SCE's. This is equivalent to maximizing $(\beta M)^{\frac{1}{\alpha-2}} - M(\beta M)^{\frac{\alpha}{\alpha-2}} \exp\left(\frac{2}{M^2(\alpha-2)}\right)$.

The FOC for this problem is $\frac{1}{\alpha-2}(\beta M)^{\frac{3-\alpha}{\alpha-2}} = M\frac{\alpha}{\alpha-2}(\beta M)^{\frac{2}{\alpha-2}} \exp\left(\frac{2}{M^2(\alpha-2)}\right)$, that after some passages reduces to $(\beta M)^{\frac{1-\alpha}{\alpha-2}} = \alpha M^{\frac{2\alpha-3}{\alpha-2}} \exp\left(\frac{2}{M^2(\alpha-2)}\right)$. Passing to logarithms, and rearranging terms, one obtains $\ln(\beta) = \frac{2-\alpha}{\alpha-1}\ln(\alpha) - \frac{2\alpha-3}{\alpha-1}\ln(M) - \frac{2}{M^2(\alpha-1)}$.

We know that $\ln(\beta)$ must also obey (25) in order that a SCE obtains: substituting (25) in our last expression, simple algebra leads us to the condition $A = \frac{1}{\alpha-2}$. Having assumed $\alpha > 2$, the profit-maximizing SCE is economically meaningful ($A > 0$).

We prove now that there does *not* exist a surplus-maximizing SCE. Surplus is defined as $Y - M\frac{2}{\alpha}X^\alpha$, where X is the level (*not* log) of individual efforts in a SCE. Given technology, any SCE-output satisfies $Y = MX$: thus maximizing surplus is equivalent to maximizing $X - \frac{2}{\alpha}X^\alpha$. Consider again (22), saying that in a SCE output must satisfy $Y = M(\beta M)^{\frac{1}{\alpha-2}} \exp\left(\frac{2}{M^2(\alpha-2)}\right)$. Some algebra shows that surplus is maximized if and only if one maximizes $(\beta M)^{\frac{1}{\alpha-2}} M \left[\exp\left(\frac{2}{M^2(\alpha-2)}\right)\right]^{1-\alpha} - \frac{2}{\alpha}(\beta M)^{\frac{\alpha}{\alpha-2}}$: maximization is with respect to β, for the same reason discussed beforehand in this proof.

The FOC for the problem is $\frac{1}{\alpha-2}(\beta M)^{\frac{3-\alpha}{\alpha-2}}\left[\exp\left(\frac{2}{M^2(\alpha-2)}\right)\right]^{1-\alpha} = \frac{2}{\alpha-2}(\beta M)^{\frac{2}{\alpha-2}}M$. After tedious algebra, we get the condition $\ln(\beta) = \frac{2-\alpha}{\alpha-1}\ln(2) - \ln(M) - \frac{2}{M^2}$. Substitute this in (25): some passages lead to the condition $A = \frac{M}{2\exp(\frac{2}{M^2})-2M}$.

However, one sees that the denominator of last expression is *negative* for $M \geq 2$: hence, no economically sound SCE, meaning $A > 0$, can be surplus-maximizing. Finally, since $A > \frac{M}{2\exp(\frac{2}{M^2})-2M}$ in any economically sound SCE, if one reasons backwards one sees that the derivative of surplus with respect to β is always positive along the SCE-manifold. In addition, from the proof of part (a) we know that β is strictly increasing in A. As a consequence, surplus is strictly increasing in A along that manifold. □

C Proof of Proposition 1

Part (a)

Proof. The Jacobian of system (17) is

$$\mathbf{J}_{H,t} = \left[\begin{array}{cc|cc} \left[\frac{\partial\mu_{i,t}}{\partial\mu_{j,t-1}}\right] & \left[\frac{\partial\mu_{i,t}}{\partial\tau_{j,t-1}}\right] & \left[\frac{\partial\mu_{i,t}}{\partial m_{k,t-1}}\right] & \left[\frac{\partial\mu_{i,t}}{\partial\gamma_{p,t-1}}\right] \\ \mathbf{0}_{M,M} & \mathbf{I}_M & \mathbf{0}_{M,2} & \mathbf{0}_{M,3} \\ \hline \left[\frac{\partial m_{k,t}}{\partial\mu_{j,t-1}}\right] & \left[\frac{\partial m_{k,t}}{\partial\tau_{j,t-1}}\right] & \left[\frac{\partial m_{k,t}}{\partial m_{k,t-1}}\right] & \left[\frac{\partial m_{k,t}}{\partial\gamma_{p,t-1}}\right] \\ \mathbf{0}_{3,M} & \mathbf{0}_{3,M} & \left[\frac{\partial\gamma_{p,t}}{\partial m_{k,t-1}}\right] & \mathbf{I}_3 \end{array}\right]$$

where: $i, j = 1, \ldots, M$; $k = 1, 2$ and $m_1 \doteq A$ and $m_2 \doteq c$; $p = 1, \ldots, 3$; $\mathbf{I}_m$ is the identity matrix of size m; and $\mathbf{0}_{m,n}$ is a m by n null matrix.

For a CE we require $\tau_{i,t} \to \infty$: given (11), this implies $\frac{\partial\mu_{i,t}}{\partial\mu_{j,t-1}} = 1$ and $\frac{\partial\mu_{i,t}}{\partial\tau_{j,t-1}} = \frac{\partial\mu_{i,t}}{\partial m_{k,t-1}} = \frac{\partial\mu_{i,t}}{\partial\gamma_{p,t-1}} = 0$. A further condition is $\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1} = 0$, since in this case expected and realized output coincide, terminating the manager's learning. From this condition, and considering (16), we get $\frac{\partial m_{k,t}}{\partial\gamma_{p,t-1}} = \left[\frac{\partial}{\partial\gamma_{p,t-1}}[\Gamma_t]^{-1}\right][\mathbf{x}_t(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})] = \mathbf{0}_{2,3}$.

It is easy to see, then, that $\mathbf{J}_{H,t}$ reduces to a decomposable matrix: the north-western block is the $2M$-identity matrix; and the south-eastern block decomposes into $\left[\frac{\partial m_{k,t}}{\partial m_{j,t-1}}\right]$ and $\mathbf{I}_3$. Hence, $2M+3$ eigenvalues of $\mathbf{J}_{H,t}$ are equal to 1, meaning that in any CE the μ's are are stabilized, while the τ_i's and the γ_i's are free to diverge to infinity. What remains to be studied are the last two eigenvalues of $\mathbf{J}_{H,t}$, i.e. the eigenvalues of $\left[\frac{\partial m_{k,t}}{\partial m_{j,t-1}}\right]$.

From (16) we see that $\left[\frac{\partial m_{k,t}}{\partial m_{j,t-1}}\right] = \mathbf{I}_2 + \frac{\partial}{\partial m_{j,t-1}}\Big(\mathbf{\Gamma}_t^{-1}[\mathbf{x}_t(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})]\Big)$, so its eigenvalues are equal to 1 plus the the eigenvalues of the second addend. Consider the latter term, and call it $\mathbf{K}_t$.

By the product rule of derivatives, and by the CE condition $\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1} = 0$, one sees that $\mathbf{K}_t = \mathbf{\Gamma}_t^{-1}\frac{\partial}{\partial m_{j,t-1}}[\mathbf{x}_t(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})]$. Using again the product rule and the CE condition, we get $\mathbf{K}_t = \mathbf{\Gamma}_t^{-1}\mathbf{x}_t\frac{\partial}{\partial m_{j,t-1}}(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})$.

Now, $\frac{\partial}{\partial m_{j,t-1}}(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})$ is a *row* vector, hence $\mathbf{x}_t\frac{\partial}{\partial m_{j,t-1}}(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})$ is a $rank-$ *one* matrix, and same holds for the product $\mathbf{K}_t = \mathbf{\Gamma}_t^{-1}\mathbf{x}_t\frac{\partial}{\partial m_{j,t-1}}(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})$. So one of the eigenvalues of $\mathbf{K}_t$ is zero, and the other one, call it λ, is equal to the trace of $\mathbf{K}_t$. As a result, the last two eigenvalues of $\mathbf{J}_{H,t}$ are, respectively, 1 and $1+\lambda$, where $\lambda = trace(\mathbf{K}_t)$.

The unitary eigenvalue is the expression of the continuum of CE, implying *Lyapunov*-stability. The last eigenvalue, $1+\lambda$, rules stability/instability of any CE. □

Part (b)

Proof. Being in a CE at date t means having been there since date 1, with a constant value of $\mathbf{x}_t$. Use Assumption 5 and the definition of $\mathbf{x}$: dropping the time suffix from $\ln\beta$, you get $\mathbf{\Gamma}_t = \begin{bmatrix} \gamma_1 + t(\ln\beta)^2 & t\ln\beta \\ t\ln\beta & \gamma_2 + t \end{bmatrix}$. Define now $\frac{\partial}{\partial m_{j,t-1}}(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1}) \doteq \mathbf{d}_t^T = [d_{1t} \quad d_{2t}]$. One checks that we can write $\mathbf{x}\mathbf{d}_t^T = \begin{bmatrix} \ln\beta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_{1t} & d_{2t} \\ d_{1t} & d_{2t} \end{bmatrix}$.

As a consequence, after some passages we can write matrix $\mathbf{K}_t$ in the following way $\mathbf{K}_t = \begin{bmatrix} \frac{\gamma_1}{\ln\beta} + t\ln\beta & t \\ t\ln\beta & \gamma_2 + t \end{bmatrix}^{-1} \begin{bmatrix} d_{1t} & d_{2t} \\ d_{1t} & d_{2t} \end{bmatrix}$. Consider now the first factor appearing in the last expression: the determinant of the matrix to be inverted can be written as $D = \frac{\gamma_1\gamma_2 + t\gamma_1 + t\gamma_2(\ln\beta)^2}{\ln\beta}$, hence the inverse is equal to $D^{-1}\begin{bmatrix} \gamma_2 + t & -t \\ -t\ln\beta & \frac{\gamma_1}{\ln\beta} + t\ln\beta \end{bmatrix}$.

Collecting the above material, some algebra leads to $\mathbf{K}_t = D^{-1}\begin{bmatrix} \gamma_2 d_{1t} & \gamma_2 d_{2t} \\ \frac{\gamma_1}{\ln\beta} d_{1t} & \frac{\gamma_1}{\ln\beta} d_{2t} \end{bmatrix}$. Thus, the trace of $\mathbf{K}_t$, i.e. its non-null eigenvalue λ by Part (a), turns out to be

$$\lambda = trace(\mathbf{K}_t) = \frac{\gamma_2 \ln\beta d_{1t} + \gamma_1 d_{2t}}{\gamma_1\gamma_2 + t\gamma_1 + t\gamma_2(\ln\beta)^2}. \tag{26}$$

Pass now to the derivatives $d_{1t} = \frac{\partial}{\partial A_{t-1}}(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})$ and $d_{2t} = \frac{\partial}{\partial c_{t-1}}(\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1})$. Recall that $\upsilon_t - \mathbf{x}_t^T\mathbf{m}_{t-1} = y_t - \ln M - \ln\beta A_{t-1} - c_{t-1}$; from (10) we know that y_t depends on $\ln\beta_t$; the latter term, in turn, depends on A_{t-1} and c_{t-1}, given (13). Hence, we can write $d_{1t} = \frac{\partial \ln\beta_t}{\partial A_{t-1}}(\frac{M}{\alpha M - 2} - A_{t-1}) - \ln\beta_t$. As regards the term $\frac{\partial\ln\beta_t}{\partial A_{t-1}}$, starting from (13) and after some passages, we get $\frac{\partial\ln\beta_t}{\partial A_{t-1}} = \frac{1}{A_{t-1}+1}(\frac{1}{A_{t-1}(2A_{t-1}+1)} - \ln\beta_t)$. We have thus:

$$d_1 = \frac{1}{A+1}\left[\frac{1}{A(2A+1)} - \ln\beta\right] \cdot \left[\frac{M}{\alpha M - 2} - A\right] - \ln\beta \tag{27}$$

where we dropped all time suffixes, as this derivative must be evaluated in a CE.

Consider now the derivative d_{2t}. Using again (10) and (13), in a handful of passages one obtains

$$d_2 = \frac{-1}{A+1}\left[\frac{M}{\alpha M - 2} + 1\right]. \tag{28}$$

having dropped again the time suffixes. Observe that, under our Assumptions, $-1 < d_2 < 0$ for $M > 1$.

Now, go back to (26). It is apparent that its absolute value decreases as t increases: so, in order to detect a possible *in*stability we put ourselves in the worst possible position, choosing $t = 1$. It is also apparent that what matters in (26) is the value of γ_1 *relative* to γ_2. Thus, we consider the two limiting cases $\gamma_1 = 0$ and $\gamma_2 = 0$.

If one takes $\gamma_2 = 0$, one gets $\lambda = d_2$, and hence the last eigenvalue of our dynamical system is equal to $1 + d_2$: from what we said after expression (28), we deduce that this eigenvalue is positive and lower than 1 (*stability*).

Moving to the case $\gamma_1 = 0$, one obtains $\lambda = d_1 / \ln \beta$: from (27), the last eigenvalue of our dynamical system is thus

$$1 + \lambda = \frac{1}{A+1} \left[\frac{1}{\ln \beta A(2A+1)} - 1 \right] \cdot \left[\frac{M}{\alpha M - 2} - A \right]. \tag{29}$$

The product of the first two factors in (29) is negative: in fact, from (25) we know that $\ln \beta < 0$ in a CE. As a consequence, the sign of the last eigenvalue is opposite to the sign of $(\frac{M}{\alpha M - 2} - A)$. Now, from (10), the term $\frac{M}{\alpha M-2}$ is the *true* elasticity of output to β; while, from (12), A is the elasticity *conjectured* by the manager. It follows that the last eigenvalue is positive for high values of the conjectured elasticity; however this very condition implies a low absolute of the first two factors of $1 + \lambda$, and hence we expect that CE's are stable in this case. On the contrary, if the conjectured elasticity A is sufficiently low with respect to the true one the last factor of $1 + \lambda$ is positive, implying a *negative* eigenvalue, and in addition the first two factors are high in absolute value: hence instability is likely in this case. Of course, looking at (26) one sees that, when t increases, and/or γ_1 increases in relation to γ_2, instability disappears[13]. □

Part (*c*)

Proof. Looking at (25) it is apparent that the absolute value of $\ln \beta$ increases when α and/or M increase: hence, the absolute value of the second factor of (29) decreases, lowering the absolute value of the last eigenvalue of the system. At the same time, the true elasticity appearing in (29) decreases when α and/or M increase, and this lowers the absolute value of the last factor of (29), that is positive under the present conditions. □

Part (*d*)

Proof. This derives from coupling the last sentence of Theorem 2 with (the proof of) part (*b*) of the present Proposition. □

[13] A numerical simulation confirmed the properties argued in this paragraph

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