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Abstract

A necessary and sufficient condition for an input to be inferior is that, taking into account the input adjustment, an increase of its price raises the marginal productivity of all inputs. Contrary to a widespread opinion, it is not necessary that (some) inputs are "rivals" (i.e., that some marginal productivity cross derivative is negative). We discuss these facts and illustrate them by introducing a few simple functional forms for the production function. Our results suggest that the existence of inferior inputs is naturally associate to the presence of increasing returns, and possibly make the case for inferiority considerably stronger.

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1. INTRODUCTION

An “inferior” input is one the demand for which decreases with output, at given prices. Clearly, this feature is a property of the cost-minimizing “conditional” demand system, $\boldsymbol{x}(\boldsymbol{w}, y)$, where $\boldsymbol{x}$ is a vector of n inputs whose positive prices are given by $\boldsymbol{w}$, and y indicates the output level.[1] We will discuss the case for an inferior input by assuming that the production function $y = f(\boldsymbol{x})$ is twice differentiable, strictly increasing and (locally) strongly quasi-concave. Accordingly (see e.g. Avriel et alii, 1988: paragraph 4.3), at any interior solution $\boldsymbol{x}(\cdot)$ is differentiable, and input i is (locally) inferior if and only if $x_{iy} = \partial x_i/\partial y < 0$.

In spite of its simple definition, the case for an inferior input has not yet (as far as we know) received a convincing interpretation in terms of the underlying technology. For a given level of output, at an interior solution the optimal input mix will equate the Marginal Rates of Technical Substitution (which are given by the ratios of marginal productivities) to the corresponding price ratios. Accordingly, the question of the existence of an inferior input concerns the way these rates change across isoquants (i.e., for changes in the output level). It is easy to make a graphical argument for inferiority in the two-input case (see e.g. Katz and Rosen, 1998: chapter 10, Figure 10.16), but surprisingly difficult to relate it to properties of the production function. However, it has been known for a long time (see Hicks, 1946: chapter VII, Samuelson, 1947, chapter IV, and Puu, 1971) that, under (strong) concavity of the production function, an input is inferior if and only if it is “regressive”, i.e., if a raise of its price increases the profit-maximizing level of output $y(p, \boldsymbol{w})$, where p is the output price. The simple reason is that an input is inferior if and only if a raise of its price decreases the marginal cost. This fact is easily established by noting that, by Shephard’s Lemma, the derivative of the cost function $c(\boldsymbol{w}, y)$ with respect to input prices is equal to the demand system, i.e., in matrix terms,

$$\boldsymbol{D}_w c(\boldsymbol{w}, y) = \boldsymbol{x}(\boldsymbol{w}, y) \tag{1}$$

(the operator $\boldsymbol{D}$ stands for the set of first derivatives), and thus it must be the case that

$$\boldsymbol{D}_w c_y(\boldsymbol{w}, y) = \boldsymbol{D}_y \boldsymbol{x}(\boldsymbol{w}, y), \tag{2}$$

where $c_y = \partial c/\partial y$ is marginal cost.

[1] Formally, the case of an inferior consumption commodity, whose characteristic depends on the Hicksian “compensated” demand system, $\boldsymbol{h}(\boldsymbol{p}, u)$, where $\boldsymbol{h}$ is a vector of goods whose prices are indicated by $\boldsymbol{p}$ and u is a utility index, is completely analogous: see e.g. Fisher (1990).

The result given in (2) is a nontrivial implication of cost minimization. Its simple economic intuition is that an increase in the price of an input will actually raise the marginal cost if and only if that input will not be substituted away if output increases. As a further consequence, under (strong) concavity of the production function all inputs must be "normal" (that is, their demand must increase with respect to output) if they are "cooperant", i.e. if *all* the cross derivatives of the production function are non-negative.[2] This comes from the fact that the Jacobian of the profit-maximizing demand system, $\tilde{\boldsymbol{x}}(p, \boldsymbol{w})$, with respect to input prices is given by:

$$\boldsymbol{D}_w\tilde{\boldsymbol{x}}(p,w) = \frac{1}{p}\boldsymbol{D}^2 f(\tilde{\boldsymbol{x}}(p,w))^{-1}, \tag{3}$$

where $\boldsymbol{D}^2 f(\boldsymbol{x})$ is the Hessian of the production function and p the output price. Now, if all the off-diagonal elements of $\boldsymbol{D}^2 f(\cdot)$ are non-negative, a clear-cut conclusion concerning the substitutability properties of $\tilde{\boldsymbol{x}}(\cdot)$ follows. In fact, it is well known that in that case its inverse $\boldsymbol{D}^2 f(\cdot)^{-1}$ must be a non-positive matrix (see e.g. Takayama, 1985: chapter 4, and in particular Theorem 4.D.3, p. 393). That is, according to a terminology introduced by Hicks (1956), under concavity all inputs must be *gross p*-complements (i.e., $\partial\tilde{x}_i/\partial w_j < 0$, $i,j = 1, \ldots, n$) if they are all *gross q*-complements ($f_{ij} = \frac{\partial^2 f}{\partial x_i x_j} \geq 0$, $i,j = 1, \ldots, n$, $i \neq j$, where f_{ij} is the cross derivative of the production function with respect to inputs i and j): see e.g. Bertoletti (2005). Since

$$\boldsymbol{D}_w y(p,\boldsymbol{w}) = \boldsymbol{D}f(\tilde{\boldsymbol{x}}(p,w))'\boldsymbol{D}_w\tilde{\boldsymbol{x}}(p,\boldsymbol{w}), \tag{4}$$

it follows that if no cross derivative of the marginal products is negative then $\boldsymbol{D}_{\boldsymbol{w}}y(\cdot) < \boldsymbol{0}$ and no input can be regressive. In other words, any regressive input j must have at least a gross p-substitute (i.e., there must exist an input i such that $\partial\tilde{x}_i/\partial w_j > 0$), otherwise the profit-maximizing level of output could not increase. This result is often, and to some extent misleadingly, stated by asserting that a necessary but not sufficient condition for an input to be inferior is that (some) inputs are rivals (see e.g. Epstein and Spiegel, 2000: Proposition 1, p. 505), and this has apparently shaped the search for technologies exhibiting inferior inputs: see Epstein and Spiegel (2000) and Weber (2001).

In next section we will discuss the conditions for getting an inferior input under bare (strong) quasi-concavity of the production function (the standard assumption for analyzing cost-minimizing

[2] Notice that, in order to establish the case for all consumption goods to be normal, Leroux (1987) gave conditions on the preferences sufficient to represent them by a concave utility function with positive cross derivatives.

behavior). Intuitively, an input is inferior if at a larger productive scale it can be conveniently substituted for. From (2), this can be interpreted as requiring that the marginal productivities of *all* inputs are raised by an increase of the inferior input price. We will illustrate the case of inferiority without "rivalry" among inputs (meaning negative cross derivatives of marginal productivities) by introducing a simple additive functional form for the production function in the case of two inputs, with the normal input (there must be at least one) exhibiting increasing marginal returns. The point we make also applies to the case of many inputs if the input with increasing returns enters additively the production function, and under certain restrictions to any sign of the cross derivative in the two-input case. These results provide an economically meaningful rationale for the existence of inferior inputs, namely their association to the existence of increasing returns with respect to another input, and suggest that the case for inferiority could be stronger than what it is usually thought.

2. Marginal productivities and inferiority

Our starting point is the well-known identity:[3]

$$c_y(\boldsymbol{w}, y) \equiv \frac{w_i}{f_i(\boldsymbol{x}(\boldsymbol{w}, y))} \tag{5}$$

$i = 1, \ldots, n$, where $f_i = \partial f / \partial x_i$ is the marginal productivity of input i. Assume that input j is (locally) inferior and that its price w_j increases: the conditional demand system has to vary in a way to increase *all* the marginal productivities. That is, the following necessary and sufficient condition must hold:

$$\boldsymbol{D}^2 f(\boldsymbol{x}(\boldsymbol{w}, y))\boldsymbol{d}_j \boldsymbol{x}(\boldsymbol{w}, y) = \boldsymbol{a} > \boldsymbol{0}, \tag{6}$$

where $\boldsymbol{d}_j\boldsymbol{x}(\boldsymbol{w}, y) = \boldsymbol{D_w x}(\boldsymbol{w}, y)\boldsymbol{e}_j dw_j$ is the change in demand induced by an increase in w_j, and $\boldsymbol{e}_j$ is the *jth* natural unit vector.

(6) provides a simple alternative explanation of why, in the case of (strong) concavity of the production function, if *all* cross derivatives of the production function are non-negative no inferior input can exist. In fact in such a case (6) would be equivalent to:

$$\boldsymbol{d}_j \boldsymbol{x}(\boldsymbol{w}, y) = \boldsymbol{D}^2 f(\boldsymbol{x}(\boldsymbol{w}, y))^{-1} \boldsymbol{a} < \boldsymbol{0}, \tag{7}$$

[3] For the sake of simplicity we assume an interior solution, i.e., $\boldsymbol{x} > \boldsymbol{0}$.

which says that all the changes dx_i should be negative. But this is impossible since the output has to stay constant, i.e., $\boldsymbol{Df}'\boldsymbol{d}_j\boldsymbol{x} = 0$. In fact, a *net p*-substitute for input j ought to exist; that is, there must be an input i such that $x_{ij} = \partial x_i/\partial w_j > 0$ (again, see e.g. Bertoletti, 2005 for this terminology). An intuition for this result can be grasped by considering the two-input case. Clearly, in such a setting, under decreasing marginal returns (an implication of concavity) the productivity of the normal input substitute (whose use increases after the rise in the inferior input price) cannot increase unless the production function cross derivative is negative.

Before proceeding, let us briefly discuss the case for an inferior input under the perspective we are considering. Is there any reason why we should expect the marginal productivities to decrease monotonically with respect to prices (along the path of the conditional demand system)? A cost-minimizing behavior implies that the cost has to rise after an input price increase (or that the output that can be produced at a given cost should decrease).[4] But there seem to be no general argument for expecting a rise in *marginal* cost too. When the price of a factor rises its demand decreases, and this is compatible with either an increase or a decrease of its so-called "weighted marginal productivity" (the reciprocal of the right-hand-side of (5)). According to (6), what happens to the marginal productivities of the *other* inputs depends on the second order derivatives of the production function and (endogenously) on their net *p*-substitutability relationships with the input whose price increased. However, since the price ratios of these inputs (and thus the corresponding Marginal Rates of Technical Substitution) remain unchanged, their productivities must move together. Notice that at least one net *p*-substitute of the input whose price augmented ought to exist, and to some extent it would be natural to expect a decrease in the productivity of this input. Though, as we have seen above, even under concavity its productivity might on the contrary increase unless the cross derivatives of the production function are all non negative. Besides this case, there is actually a class of well-known technologies with the previously alleged property. If the technology is homothetic, it is easily seen that the marginal cost is proportional to the cost, and in particular that the vector $\boldsymbol{D}_w c_y$ and $\boldsymbol{D}_w c$ are related by a positive scalar multiplication. However, as Tönu Puu (1971: p. 243) wrote almost forty years ago: "[homotheticity][5] is assumed for mathematical simplicity in exemplifications and in econometric applications." and "I cannot see anything to make the case [of an inferior input][6] unlikely".

[4] This is one property of the so-called "indirect production function": see e.g. Cornes (1992: section 5.1).
[5] *Homogeneity* in the original text.
[6] Added to the original text.

Let us consider the special case in which there are only two inputs, and let us assume that the inferior input is 1 (so that 2 is a normal input). We can then uniquely characterize the differential $\boldsymbol{d}_1\boldsymbol{x}$, since:

$$d_1x_2(\boldsymbol{w},y) = -\frac{f_1(x(\boldsymbol{w},y))}{f_2(x(\boldsymbol{w},y))}d_1x_1(\boldsymbol{w},y) > 0\,. \tag{7}$$

Accordingly, condition (6) is equivalent to the system:

$$\begin{aligned}&\left[f_{11}(x(\boldsymbol{w},y)) - f_{12}(x(\boldsymbol{w},y))\frac{f_1(x(\boldsymbol{w},y))}{f_2(x(\boldsymbol{w},y))}\right] < 0,\\ &\left[f_{21}(x(\boldsymbol{w},y)) - f_{22}(x(\boldsymbol{w},y))\frac{f_1(x(\boldsymbol{w},y))}{f_2(x(\boldsymbol{w},y))}\right] < 0,\end{aligned} \tag{8}$$

and it is easily interpreted as requiring that the movement along the relevant isoquant increases the productivity of input i, either "directly" through d_1x_i, or "indirectly" through d_1x_j ($i, j = 1, 2, i \neq j$). It is equivalent to the condition that the elasticities of the marginal products, $\varepsilon_{ij} = f_{ij}x_j/f_i$, are ordered in such a way that $\varepsilon_{21} > \varepsilon_{11}$ <u>and</u> $\varepsilon_{22} > \varepsilon_{12}$: in other words, both inputs 1 and 2 have a larger proportional impact on f_2 than on f_1. Notice that homotheticity requires on the contrary that the sums of the elasticities of each marginal product with respect to all inputs should be equal (i.e., $\Sigma_j\varepsilon_{ij}$ should be independent from i), to keep constant the Marginal Rates of Technical Substitution with respect to any proportional input change.

Condition (8) can be written compactly as:

$$f_{22}(x(\boldsymbol{w},y))\frac{f_1(x(\boldsymbol{w},y))}{f_2(x(\boldsymbol{w},y))} > f_{12}(x(\boldsymbol{w},y)) > \frac{f_2(x(\boldsymbol{w},y))}{f_1(x(\boldsymbol{w},y))}f_{11}(x(\boldsymbol{w},y))\,. \tag{8'}$$

Geometrically, (8') says that the relevant "iso-marginal-productivity curves" f_i = constant (i = 1,2) are (locally) "steeper" than the isoquant if f_{12} is negative (see Puu, 1971: p. 247, Figure 2.c, for the case in which both f_{11} and f_{22} are negative), and (locally) "flatter" if f_{12} is positive (see Figure 1 below for the case in which $f_{22} > 0 > f_{11}$). Notice that the iso-marginal-productivity curve of input i is increasing if f_{ii} and f_{ij} do not agree in sign, and that they are orthogonal if $f_{12} = 0$ (this requires $f_{22} > 0 > f_{11}$ to satisfy (8')). Also note that (8') cannot hold under concavity unless f_{12} is negative. But even if f_{12} is non-negative the existence of an inferior input cannot actually be ruled out if the other input exhibits increasing marginal returns. In fact, local (strong) quasi-concavity only requires that:

$$f_{21}(\boldsymbol{x}(\boldsymbol{w},y)) - f_{22}(\boldsymbol{x}(\boldsymbol{w},y))\frac{f_1(\boldsymbol{x}(\boldsymbol{w},y))}{f_2(\boldsymbol{x}(\boldsymbol{w},y))} + f_{12}(\boldsymbol{x}(\boldsymbol{w},y)) - f_{11}(\boldsymbol{x}(\boldsymbol{w},y))\frac{f_2(\boldsymbol{x}(\boldsymbol{w},y))}{f_1(\boldsymbol{x}(\boldsymbol{w},y))} > 0 . \quad (9)$$

We conclude that increasing (marginal) returns in the other input naturally satisfy the requirement of having a price increase to raise marginal productivities if quasi-concavity can be guaranteed. In particular, note that if the production function satisfies (9) globally and one has $f_{22} > 0$ everywhere, then $f_{12} \leq 0$ is *sufficient* to guarantee that (8') holds (at any interior solution). The intuition is again simple: in a two-input setting, if one input exhibits increasing returns, a rise in the price of the other input must raise its productivity unless the production function cross derivative is positive.

To illustrate this possibility, consider the following functional form for the production function:[7]

$$g(\boldsymbol{x}) = \ln(x_1 + 1) + e^{x_2} - 1 . \quad (10)$$

$g(\cdot)$ is a strictly increasing, additive, (at least) twice differentiable function which is also strongly quasi-concave (but not concave) for positive input quantities, with $g(\mathbf{0}) = 0$. Notice that the Marginal Rate of Technical Substitution is given by:

$$\frac{g_1(\boldsymbol{x})}{g_2(\boldsymbol{x})} = \frac{1}{(x_1 + 1)e^{x_2}} , \quad (11)$$

and thus the isoclines are always decreasing. Also notice that the strictly decreasing isoquants intercept the horizontal axis at $x_1 = e^y - 1$, where their slope is e^{-y}, and the vertical axis at $x_2 = \ln(y + 1)$, with slope $y + 1$. A typical isoquant is depicted in Figure 2, together with an interior solution and the relevant isocline. At any interior solution[8] the conditional demand system moves continuously along the relevant isoquant for changes in the input price ratio, and along the relevant isocline for changes in the output level, confirming that input 1 is inferior. Of course, the marginal cost will be decreasing with respect to output.

Let us now return to the case of $n > 2$. Again, let us assume additivity of the production function, and suppose that at an interior solution at which the production function is (locally) quasi-concave there is one[9] inputs with increasing marginal returns. Now suppose that the price of this factor, say input 2, rises, decreasing its marginal productivity and raising the marginal cost. Call input 1 a net

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[7] It is not difficult to find other production functions with properties similar to those of $g(\cdot)$: $r(\boldsymbol{x}) = x_1^{\alpha} + x_2^{1/\alpha}$, $1 > \alpha > 0$, is one instance, and another it is considered in the Appendix.

[8] This requires $1/(y + 1) > w_1/w_2 > e^{-y}$.

[9] At an interior solution, to satisfy local quasi-concavity of the production function under additivity there can be but one input exhibiting increasing returns.

p-substitute of input 2: the former input must be inferior, since when its price increases it symmetrically increases the demand for input 2 and its productivity. Thus for sure at least an inferior input will exist in such a case, generalizing the result of our example. Moreover, for an additive production function it must be the case that ($i \neq j$):

$$x_{iy}(\mathbf{w}, y) = -\frac{c_{yy}(\mathbf{w}, y) f_i(x_i(\mathbf{w}, y))}{c_y(\mathbf{w}, y) f_{ii}(x_i(\mathbf{w}, y))},$$
$$x_{ij}(\mathbf{w}, y) = \frac{x_{iy}(\mathbf{w}, y) x_{jy}(\mathbf{w}, y)}{c_{yy}(\mathbf{w}, y)}, \tag{12}$$

where $c_{yy} = \partial c_y / \partial y$, as it can be easily proved by differentiating the identity (5). It follows that actually all the inputs with decreasing marginal returns will be inferior, net p-substitutes with respect to 2 (i.e., $x_{2j} > 0, j \neq 2$) and net p-complements among them (i.e., $x_{ij} < 0$, $i, j \neq 2$). Notice that the marginal cost must be decreasing with respect to output, and that conditions (6) are a fortiori satisfied for any $dw_j > 0, j \neq 2$.

Now note that the previous arguments for the existence of inferior inputs generalize to the case in which the production function is just additive with respect to the input exhibiting increasing returns, i.e., to the case in which $f(\mathbf{x}) = f^{-2}(\mathbf{x}_{-2}) + f^2(x_2)$, where $\mathbf{x}_{-2}$ is the vector of all inputs but 2 and $f^{2}{}''(\cdot) > 0$ (in such a case, the results given by (12) hold for $i = 2$). While things become more involved if the cross derivatives of the marginal productivities are not null, by a continuity argument sufficiently small cross derivatives *of any sign* would not change the previous results. Moreover, it is easy to see that the functional form:

$$\tilde{g}(\mathbf{x}) = \ln(x_1 + 1) + e^{x_2}(x_1 + 1) - 1, \tag{13}$$

which generalizes (10) to the case of a strictly positive cross derivative, does satisfy both conditions (8') and (9).

In summary, the net p-substitute of an input exhibiting increasing marginal returns *tends* (it depends on the cross derivatives of the production function) to be an inferior input. Accordingly, our results have uncovered an association between the existence of inferior inputs and the presence of increasing returns. In addition, notice that it would be natural to think of an additive technology as referring to the use of many different plants by the firm. Indeed, our results apply to the case in which a single firm owns n plants, and each quantity x_i is actually internally produced at plant i by using m_i inputs $\mathbf{z}^i$ into a sub-production function $x_i = h^i(\mathbf{z}^i)$, where each $h^i(\cdot)$ is monotonically increasing, concave and linearly homogenous (accordingly, an appropriate version of "two-stage budgeting" applies, with the "price" of input i being computable as a well-defined index of the z^i prices: see e.g. Deaton and Muellbauer, 1980: section 5.2). Following such an interpretation, let us suppose that there is a single plant i where the output y is produced by using x_i with increasing (marginal) returns, while all the others exhibit decreasing returns at the plant level. An increase of total output will then be associated to a decrease of the production in the latter plants, whose underlying inputs are inferior if the ones used in the former plant are specific to it. Thus, in our examples it is the presence of increasing returns which creates an opportunity for input substitution as the output increases.

We conclude this section by reminding the careful reader that any twice-differentiable, strong quasi-concave function $f(\cdot)$ is so-called "transconcave", that is it can be transformed into a concave function by means of a monotonically increasing function of one variable $G(\cdot)$: see e.g. Avriel et alii (1988: Theorem 8.25, p. 278). This implies that our production functions (12) and (13) are concavifiable, and that their concavized versions could then be used to describe profit-maximizing behavior exhibiting regressive inputs (of course, the process of concavification would generate negative cross derivatives for the production function $F(\boldsymbol{x}) = G(f(\boldsymbol{x}))$. What matters more is that our "increasing returns story" would still apply to the inner productive stage described by $f(\cdot)$, while $G(\cdot)$ could then be interpreted as an outer stage of production exhibiting decreasing (marginal) returns. Notice that, conversely, starting from a (two-input) concave technology exhibiting an inferior input, one should always be able to de-concavize it by taking a monotonically increasing convex transformation of the associate production function. While preserving both its quasi-concavity and the satisfaction of (8'), this operation will leave increasing marginal returns with respect to the normal input to emerge once the second order cross derivative of the resulting production function is turned from negative into positive.

3. CONCLUDING REMARKS

In this note we have revisited the case for the existence of inferior inputs. We have argued that to assume concavity of the underlying technology, as it is usually done in the literature, is restrictive and possibly misleading.[10] In particular, by assuming bare (strong) quasi-concavity of the

[10] In a fine paper that anticipated some of our arguments, Puu (1971: pp. 243-4) was apparently leaded by the assumption of concavity to suggest that inferiority could be expected by inputs used at a plant exhibiting *increasing* returns: "As to the presence of factor inferiority in reality, the phenomenon probably may be encountered when a firm operates several plants simultaneously. […] An increase of total output will in such a case be combined with a decrease

underlying technology, we have shown that inferior inputs ought to exist if the underlying technology is additive with respect to *another* input exhibiting increasing marginal returns (a result which admits an interpretation in terms of returns to scale at the plant level). In the two-input case, we have similarly shown that a negative cross derivative of the production function is sufficient but not necessary to make an input inferior if there are increasing marginal returns with respect to the other (see the functional form (13) above). As a corollary to these results, rivalry among inputs is not needed to deliver input inferiority. Thus, in addition to present some (simple) functional forms exhibiting inferior inputs (according to Weber, 2001, only a few examples were already known), we have uncovered a novel (as far as we know) and economically meaningful reason for their existence, namely their association with the presence of increasing returns. We believe that this should considerably strengthen the case for inferiority, which is widely held to be dubious: see e.g. Cowell (2005: p. 32).

It is also worth concluding by returning to the correspondence between the inferiority of inputs and of consumption goods (see footnotes 1 and 2 above). It is an interesting paradox that while they are formally identical, the latter seem to be much more popular (see any microeconomic textbook, in which the case of inferior inputs is usually not even mentioned).[11] Moreover, the paradox deepens if one considers that to provide an intuitive economic explanation of the existence of a *normal* commodity, it is necessary to refer to the somehow exotic result that a raise of its price decreases the marginal utility of income, *with utility held constant*: see Fisher (1990).[12] The point being, of course, that in production theory the reciprocal of latter quantity is known as marginal cost. In particular, notice that in consumption theory inferior commodities are usually but informally interpreted as "low-quality goods" (see e.g. Varian, 1996: p. 96: "examples might include gruel […], or nearly any kind of low-quality good."). Our results, which relate the input substitution associated to inferiority to the existence of increasing returns, appear to provide only a limited support to the extension of the previous interpretation (inferior inputs as "poor inputs", e.g. some kind of unskilled work) to production theory.

Finally, we have to mention that, as we discovered after having completed the first draft of this paper, the property of the additive technology we have exploited in the previous section is already

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of production in the plant with decreasing marginal cost. If there is some factor which is employed especially intensively in this plant, it is reasonable to expect that total demand factor will decrease as total production is increased."

[11] For example, Varian (1990) does not refer to input inferiority, while Varian (1992: chapter 5, exercise 5.12) considers it in an exercise.

[12] Fisher (1990: p. 433): "Having said this, I confess that I can give no intuitive explanation for the fact that the Corollary speaks in terms of the effects of price changes on the marginal utility of income with *utility* rather than *income* held constant.". Italic in the original.

known in consumer theory but considered "very peculiar" (Deaton and Muellbauer, 1980: section 5.3) or even "clearly pathological" (Barten and Bohn, 1982: section 15), apparently because strictly speaking it implies that only one commodity will be normal. However, we cannot see any special difficulty in our production story of plants with different returns to scale. In particular, while it corresponds to economic commonsense that at any interior solution only a single plant with increasing returns is operated, notice that there can actually be many normal inputs (all those uniquely associated to that plant).[13] Yet another instance of the aforementioned paradox?

APPENDIX

Consider the following functional form alternative to (10):

$$p(\boldsymbol{x}) = \ln x_1 + \frac{x_2^2}{2}. \tag{A1}$$

$p(\cdot)$ is (strongly) quasi-concave for $x_2 > 1$, and accordingly behaves well for $x_1, x_2 \geq 1$. The Marginal Rate of Technical Substitution is given by $p_1/p_2 = 1/(x_1 x_2)$, and thus the isoclines are rectangular hyperbolas. The isoquants are asymptotic to the vertical axis, and intercept the horizontal one at $x_1 = e^y$, where they have a vertical tangent. Their concavity turns into convexity at $x_2 = 1$, where the slope of the isoquants is $1/x_1$. A typical isoquant is depicted in Figure 3, together with an interior solution and the relevant isocline.

The conditional demand that can be derived from (A1) is not everywhere continuous. For a small enough input price ratio w_1/w_2 the cost-minimizing way of producing uses only the decreasing-marginal-return input 1. But when w_1 gets large enough there will be a "jump"[14] from such a corner solution to an interior choice in which also the increasing-marginal-return input 2 will be put at work. And for even higher levels of w_1 (or larger levels of output) the conditional demand system will move continuously, confirming that input 1 is inferior. In particular, a bit of algebra shows that for *any* level y of output the minimum positive amount of input 2 to be used, $\underline{x}_2$, is the unique root to the expression:

[13] It seems worth to quote the concluding remark by Green (1961: p. 136): "the implication of the alternative assumption [one good exhibiting increasing marginal utility] seems scarcely credible. But any tests of the hypothesis of a utility function additive in terms of groups of commodities must, of course, also be greatly influenced by recent work on 'utility trees'.". Parenthesis added to the original text.

[14] There is a threshold $(\underline{\frac{w_1}{w_2}})(y)$ for the price ratio such that at that value there will be two optimal activites $\boldsymbol{x}(y)$, a corner solution in which $x_1 = e^y$ and $x_2 = 0$, and an interior solution $\underline{\boldsymbol{x}}$ with associated the same cost: see below.

$$\frac{x_2^2}{2} = \ln(1 + x_2^2)\,. \tag{A2}$$

Let us indicates with $\underline{x}_2 > 1$ this root. Then $\underline{x}_1(y)$ is given by:

$$\underline{x}_1(y) = e^{y - \frac{\underline{x}_2^2}{2}}\,, \tag{A3}$$

and there will exist an interior solution (and 1 will be locally an inferior input) if:

$$\frac{w_1}{w_2} > \underline{(\frac{w_1}{w_2})}(y) = \frac{\underline{x}_2}{e^y - \underline{x}_1(y)} \tag{A4}$$

(notice that $\underline{(\frac{w_1}{w_2})}$ is decreasing with respect to y).

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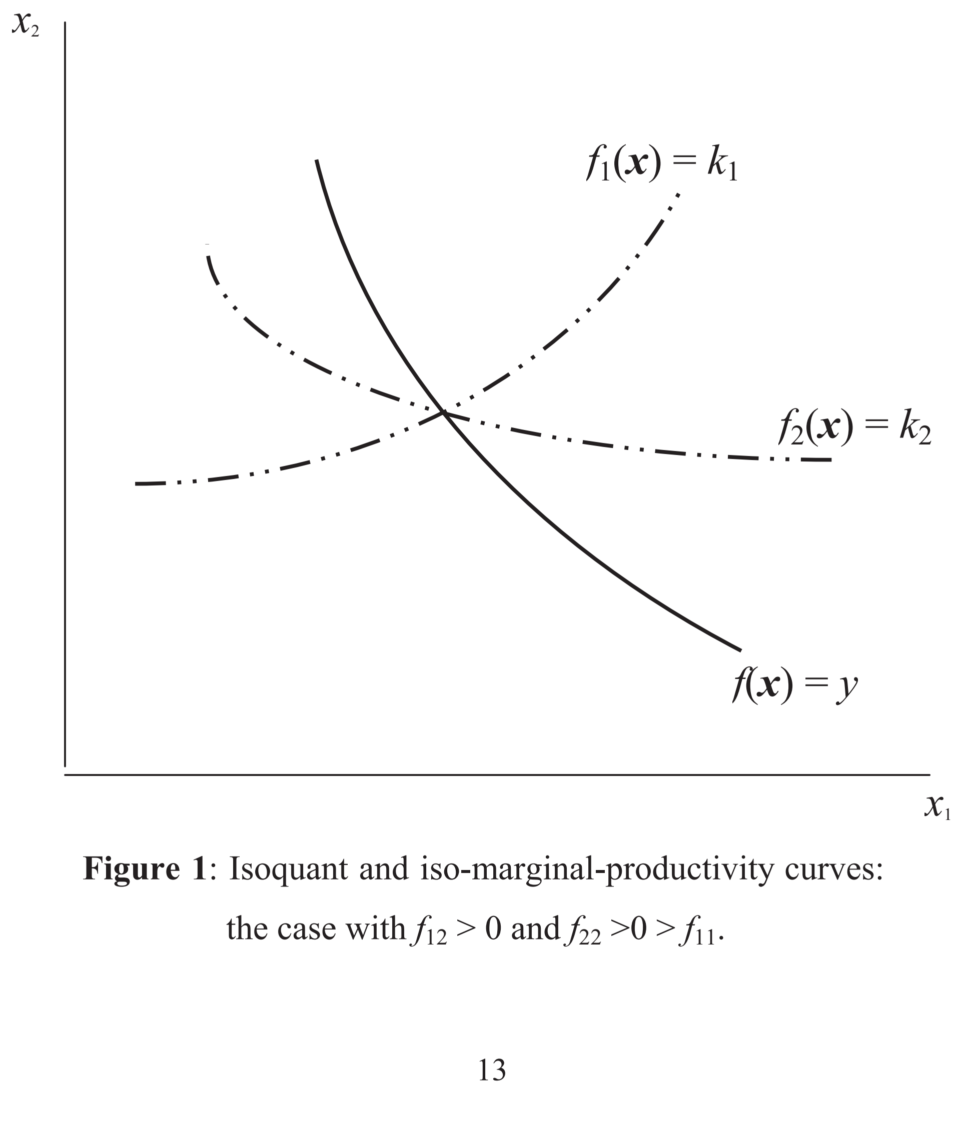

Figure 1: Isoquant and iso-marginal-productivity curves: the case with $f_{12} > 0$ and $f_{22} > 0 > f_{11}$.

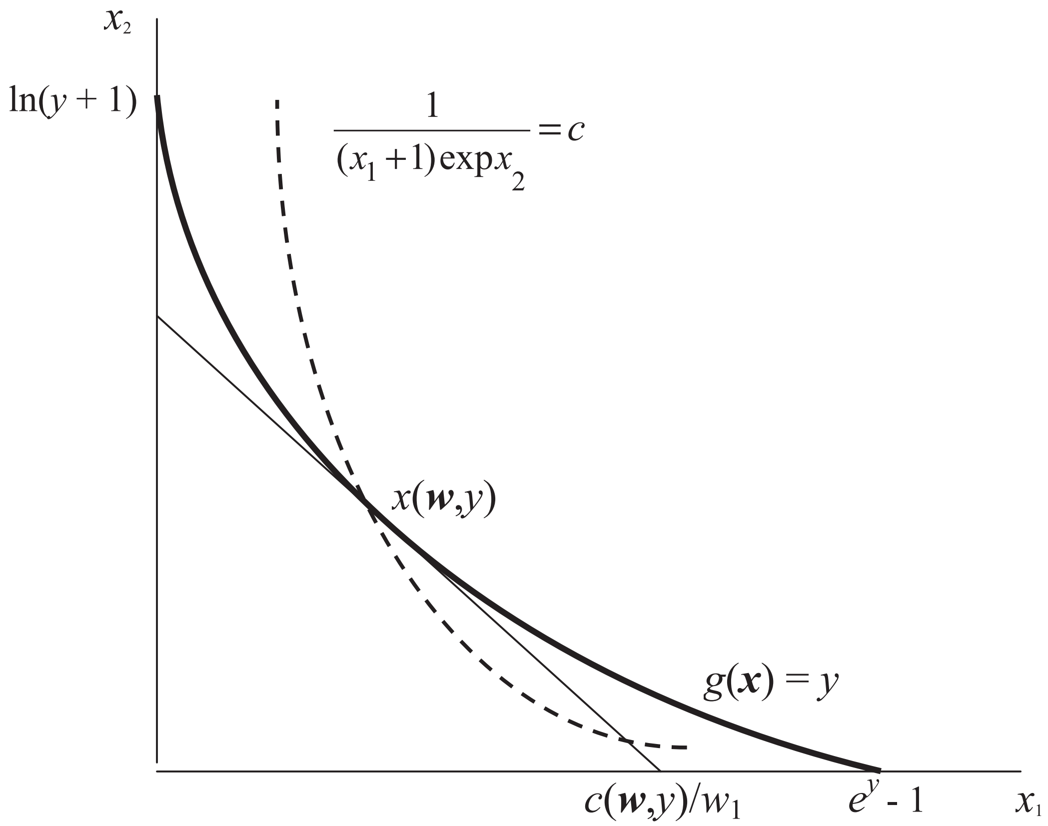

Figure 2: Isoquant, isocline and optimal input choice for the $g(\cdot)$ p.f.

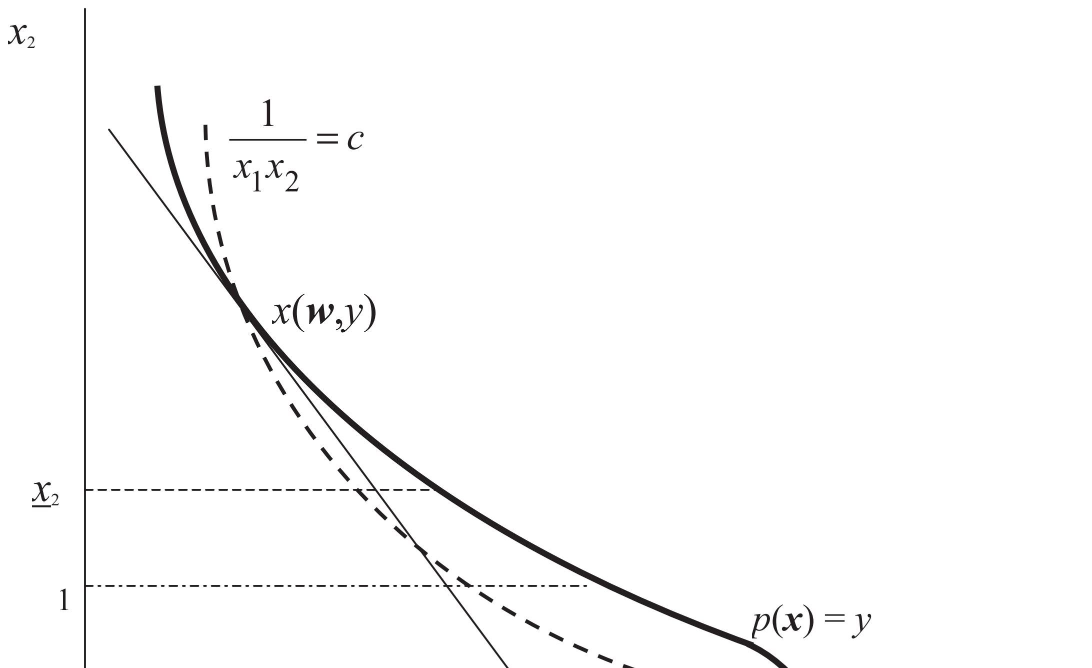

Figure 3: Isoquant, isocline and optimal input choice for the $p(\cdot)$ p.f.