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A valuation model for firms with stochastic earnings*

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ABSTRACT

In this paper, a model is set up for valuing a firm with stochastic earnings. It is assumed that the earnings of the considered firm follow a time-varying mean reverting stochastic process. It is shown that the value of the firm satisfies a boundary value problem of a second-order partial differential equation, which can be solved numerically. Special cases are discussed. Analytic solution is found for one special case. Moreover it is shown that the analytic solution is consistent with a previous result obtained by other researchers.

Keywords: stochastic earnings, firm valuation, debt valuation

I. INTRODUCTION

Firm valuation has always been an important research topic in finance and accounting. It is also one of the most fundamental problems faced by practitioners in the finance industry. Thus a large amount of work has been done on firm valuation.

The most popular valuation model is the dividend discount model (DDM), see e.g. Brealey and Meyers (1996), Dixit and Pindyck (1994), Copeland et al. (1990). This model describes equity valuation in terms of expected future dividends, i.e. $P_t = \sum_i \frac{E(d_{t+i})}{(1+r)^i}$ where P_t is the price of the equity at time t , $E(d_{t+i})$ is the expected dividend paid at time $t+i$ given the information at t and r is the discount rate (cost of equity). The DDM targets the actual distribution to shareholders. The formula requires the prediction of dividends to infinity for

* I would like to thank Prof. David Allen for bringing a few references into my attention.

going concerns, but the Miller and Modigliani (1961) dividend irrelevance proposition states that price is unrelated to the timing of expected payout prior or after any finite horizon. Therefore forecasted dividends to a finite horizon are uninformative about price and do not necessarily relate to value. This problem calls for forecasting something more fundamental than dividends.

Discounted cash flow (DCF) analysis substitutes free cash flows for dividends as the target for analysis, see e.g. Copeland et al. (1990). Free cash flow is a company's true operating cash flow. It is the total after-tax cash flow generated by the company that is available to all providers of the company's capital, both creditors and shareholders. It can be thought as the after-tax cash flow that would be available to the company's shareholders if the company had no debt. Free cash flow is generally not affected by the company's financial structure, even though the financial structure may affect the company's weighted average cost of capital and therefore its value. The DCF methodology gives the firm value as the present value of the future free cash flows.

Firm valuation has also been an active research topic in the accounting literature. For example, Ohlson (1995) and Feltham and Ohlson (1995) base their theory of valuation on the residual income valuation model and they show that under certain conditions share price can be expressed as a weighted average of book value and earnings. The Ohlson and Feltham-Ohlson models have spawned much empirical research examining the comparative valuation relevance of the balance sheet and income statement.

The roles of dividends, earnings and book value in equity valuation are being debated enormously in the accounting research see e.g. Penman and Sougiannis (1998). These accounting approaches aim to capture value-creating activities, rather than the value-irrelevant payout activities.

All the above techniques are discrete in time and often applied by financial analysts in forecasting equity prices. They are essentially variations of the NPV techniques. Consequently, they need estimates of the future benefits and an appropriate discount rate. The forecasting of these factors are often influenced by many uncertainties and can be sometimes subjective. These

techniques have the advantages of being easy to use and to gain acceptance. Thus they are popular in practice, though the precision of forecasted value is often questionable. No doubt, more theoretical and empirical research is required regarding these techniques.

Another different type of approach is based on continuous time models. These approaches are heavily influenced by the option pricing approach. Black and Scholes (1973) points out that a firm's liabilities can be priced as options. They regard the stock of a firm as being a call option on the value released when the firm shuts down and a 'final dividend' is paid out. The strike price is the value of the firm's bonds outstanding, the level which the value of the firm must clear if the shareholders are to receive anything at the termination. Models pricing a firm's liabilities as contingent claims on its underlying value include Bensoussan, Crouchy and Galai (1994), Black and Cox (1976), Cooper and Mello (1994), and Merton (1974,1990). All these models of firm valuation rest on two crucial assumptions. Firstly, it is clear that the behavior of any such model of a derivative claim is inextricably linked to the choice of process for the quantity underlying the value of the claim. These models generally assume that some reasonably high level process follows a simple geometric random walk. Secondly these models typically rely on assumptions about hedging opportunities which are invalid and unrealistic.

Recently, a new approach for valuing the entire worth of a firm has been proposed by Apahbai et al. (1996). They build a model based on the specification of a more fundamental process. Earnings are regarded as the key underlying variable. They assume that earnings follow a geometric Brownian motion. The free cash generated by earnings is paid into a bank account during the firm's lifetime. This cash is then allocated at the end of the firm's life. Current value, regarded as the expected present value of this terminal value, is modeled as a derivative claim on the processes followed by earnings and cash. No assumptions are made against fluctuations in a firm's value. This approach is also taken in Epstein et al. (1997), where a geometric mean reverting process for earnings is considered.

One important feature of the approach of Apahbai et al. (1996) is that they do not assume the no-arbitrage condition for the firm's value, which is often assumed in option pricing. Though arbitrages are possible theoretically, there are many situations in which the party interested in the value being modeled will not attempt to hedge, making any assumption to the contrary irrelevant.

In this paper, we follow the approach by Apabhai et al. (1996) to consider a very special type of business fulfilling the following assumptions:

- The firm is assumed to have a finite life;
- All earnings during the life of the firm are retained in a bank;
- The firm's earnings follow a time-varying mean reverting process as given in Chiang et al. (1997).

Since the company has a finite life, the cash balance, beside earnings, is also taken as a key variable. A few special cases are considered. For one special case, we show that our analytic solution is asymptotically consistent with the one obtained by Chiang et al. (1997).

The rest of the paper is organized as follows. In section 2, we illustrate our assumption on the earnings process. In Section 3, we set up the partial differential equation satisfied by the firm's value. In section 4, we consider the boundary conditions and final condition for some special cases. The problem of debt valuation for the firm is addressed in Section 5. We conclude in Section 6.

II. The choice of the earnings process

In recent times a number of studies have attempted to investigate the relations between share returns and different fundamental variables capable of explaining movements in returns. One of the approaches in examining this issue is to model the variables that predict movement in returns. For example, Fama and French (1988) propose dividends per share as being significant in explaining returns. Similarly, Campbell and Shiller (1988) propose dividends per share, dividend growth and long term earnings per share as being significant in explaining returns. They also propose that earnings have significant explanatory power in predicting share returns. We therefore propose that the firm value is the discounted value of expected future earnings. A similar approach is taken in Hodgson et al. (2000).

Valuation of firms with stochastic earnings has been studied by a number of authors. Apabhai et al. (1996) considered the following model:

$$dE = \mu E(t)dt + \sigma E dz(t) \quad (1)$$

where E is the gross earnings of the firm, μ, σ are constants and $dz(t)$ is the standard Wiener process. Using earnings and the cash balance as key variables, they set up a firm valuation model.

Later on, Epstein et al. (1997) consider the following geometric mean-reverting model

$$dE = \mu(\bar{E} - E)Edt + \sigma Edz(t) \quad (2)$$

where \bar{E} is a constant, E is the gross earnings of the firm, μ, σ are constants. They further value the firm in a similar way as in Apabhai et al.(1996) and consider the problem of optimal advertising.

In both Apabhai et al (1996) and Epstein et al. (1997), their assumptions on the earnings process are somehow arbitrary and artificial, to a large extent for the convenience in mathematical treatment. There is no empirical evidence for the stochastic processes (1) or (2).

In Chiang et al (1997), a time-varying mean reverting process for net earnings is proposed. Specifically, they assume that:

$$dE = (\alpha e^{kt} - \beta E(t))dt + \sigma dz(t) \quad (3)$$

where dE is the instantaneous change in earnings, $E(t)$ is the net earnings at time t , β is the speed of adjustment and $\frac{\alpha}{\beta} e^{kt}$ is the long term mean which grows or decays exponentially at the rate e^{kt} , σ is instantaneous standard deviation of the earnings. They use the nonlinear maximum likelihood techniques to estimate the parameters based on the annual observations of Standard & Poors Composite Stock Price index for the period 1871-1986. It is shown that earnings revert significantly towards the time dependent mean. The extent of mean reversion is approximately 32% per year and the growth of long term mean is 5.28% per year. This provides some evidence that the stochastic earnings process (3) matches the historical earnings to a certain degree.

Under the formulation (3) earnings can take on both positive and negative values and, by use the factor e^{kt} , the long term mean can be allowed to grow in times of inflation or decay in times of deflation. This formulation also allows earning to be mean reverting, which is reasonable given that there is evidence of mean reversion in the dividend series [Shiller (1981), Marsh and Merton (1987), and Fama and French (1988)] and there is generally a strong relationship between

earnings and dividends. For example, for $k=0$ and $\beta>0$ this represents mean reversion of earnings towards a stable long term mean. On the other hand, if $k>0$ and $\beta>0$, this represents mean reversion about a time varying long term mean. As earnings grow over time, one might expect this to be the more appropriate model to capture the behavior of earnings. In the case when $\alpha=0$, $\beta=0$, then the earnings are represented by a random walk.

In this paper, we follow the approach in Apabhai et al (1996), but we use the earnings process specified by Chiang et al. (1997) which is supported by some empirical evidence. Furthermore, we assume that α , β and σ are positive constants throughout this paper.

III. THE VALUATION MODEL

As in Apabhai (1996), we assume that the firm under consideration has a finite life T . Earnings of the firm follow the stochastic process given by (3). The earnings are paid into a bank account during the firm's life time. This cash is allocated at the end of the firm's life to the owners of the firm. The cash balance at time t is defined as

$$C(t) = \int_0^t e^{r(t-s)} E(s) ds \quad (4)$$

where r is the risk-free interest rate, E is the net earning. Given the above assumptions, the worth of the entire firm value V at any time t is the present value of the current account balance at time T , i.e.

$$V = e^{r(t-T)} E(C(T)). \quad (5)$$

This is reasonable provided that the salvage value at time T of the firm is negligible or the firm is nearly a going concern. For simplicity, we ignore the salvage value of the firm at time T .

The use of the cash balance $C(t)$ is an important feature of this approach. At any time t , the firm value V can be regarded as a function of time t , cash balance $C(t)$, and earnings $E(t)$, i.e. $V = V(E, C, t)$. This enables us to use partial differential equations to model the firm value. The key variables $C(t)$ and $E(t)$ are both observable at time t . This is in contrast to the traditional approaches where the expected future dividends or free cash flows are discounted to obtain firm value.

Below we derive the partial differential equation for the firm value. Differentiating (4) yields

$$dC(t) = (E(t) + rC(t))dt. \quad (6)$$

Using Ito's Lemma we have

$$dV = \left(\frac{\partial V}{\partial t} + (E + rC) \frac{\partial V}{\partial C} + (\alpha e^{kt} - \beta E) \frac{\partial V}{\partial E} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial E^2} \right) dt + \sigma \frac{\partial V}{\partial E} dz. \quad (7)$$

On the other hand, rational expectation implies

$$E(dV) = rVdt. \quad (8)$$

Combining (7) and (8) yields

$$\frac{\partial V}{\partial t} + (E + rC) \frac{\partial V}{\partial C} + (\alpha e^{kt} - \beta E) \frac{\partial V}{\partial E} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial E^2} - rV = 0. \quad (9)$$

Note that the drift term $(\alpha e^{kt} - \beta E)$ appears in the above equation explicitly. This is in contrast to the Black-Scholes equation for the value of an option and is due to the absence of a traded instrument with which to hedge the risk (the randomness) in the firm's value.

Equation (9) describes the firm value in terms of E , C and t . To complete the specification of this problem, we need impose appropriate boundary and final conditions. These conditions depend on the operating procedure and ownership structure etc. In the next section, we consider a few special cases.

IV. SOME SPECIAL CASES

In this section we consider some special cases of the general business model established in the previous section. We use these cases to illustrate how to formulate a well-defined boundary value problem, which can be solved numerically or analytically. But the cases that can be considered are not limited to these ones.

4.1 Partnership, no restriction on borrowing

In this case, partners have unlimited liability and are responsible for any loss when the firm ceases to exist. Hence the firm value must be equal to the cash balance at time T . Since there is no restriction on the borrowing, the business continues running regardless of its success

or otherwise. In other words, if there is a negative amount in the bank at any time then the business continues to run. The problem for V is specified for all C , both positive and negative. The boundary and final conditions are:

$$\frac{\partial^2 V}{\partial C^2} \rightarrow 0, \quad \text{as} \quad |C| \rightarrow \infty,$$

$$\frac{\partial^2 V}{\partial E^2} \rightarrow 0, \quad \text{as} \quad |E| \rightarrow \infty,$$

$$V(E, C, T) = C.$$

This problem need to be solve in

$$C \in (-\infty, \infty), \quad E \in (-\infty, \infty).$$

Note that this boundary value problem can be simplified by using the transformation:

$$U = V - C. \quad (10)$$

The partial differential equation for U is

$$\frac{\partial U}{\partial t} + (\alpha e^{kt} - \beta E) \frac{\partial U}{\partial E} + \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial E^2} - rU + E = 0. \quad (11)$$

It is easy to see that the boundary and final conditions in terms of U . In summary, we have the following boundary value problem for U:

$$\begin{cases} \frac{\partial U}{\partial t} + (\alpha e^{kt} - \beta E) \frac{\partial U}{\partial E} + \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial E^2} - rU + E = 0, \\ \frac{\partial^2 U}{\partial E^2} \rightarrow 0, \quad \text{as} \quad |E| \rightarrow \infty, \quad U(E, T) = 0. \end{cases}$$

Note that this boundary value problem is independent of C . Hence U is also independent of C . This boundary value problem can be solved analytically. The solution is given by:

$$U = \frac{E}{(\beta + r)} [1 - e^{(\beta+r)(t-T)}] + \frac{\alpha}{(\beta + k)(k - r)} e^{rt+(k-r)T} + \frac{\alpha}{(\beta + r)(\beta + k)} e^{(\beta+k+r)t-(\beta+r)T} + \frac{\alpha}{(\beta + r)(r - k)} e^{kt} \quad (12)$$

for $k \neq r$ and

$$U = \frac{E}{(\beta + r)} [1 - e^{(\beta+r)(t-T)}] + \frac{\alpha}{(\beta + r)^2} [(\beta + r)T - 1]e^{rt} + \frac{\alpha}{(\beta + r)^2} e^{(\beta+2r)t - (\beta+r)T} - \frac{\alpha}{(\beta + r)} te^{rt} \quad (13)$$

for $k = r$.

For large T , the firm can be regarded as a going concern. Equation (10) implies that U is equivalent to the equity value obtained by discounting future earnings in the usual DDM approach. Thus it is of great interest to see the behavior of the solution given by (12) and (13) for large T . Two cases need to be considered: $k \geq r$, and $k < r$.

Consider first the case $k \geq r$. Using Equations (12) and (13), we get immediately that

$$U = O(e^{(k-r)T}) \quad \text{as } T \rightarrow \infty$$

for $k > r$ and

$$U = O(T) \quad \text{as } T \rightarrow \infty$$

for $k = r$. Thus the solution blows up for large T . This is in accordance with our intuition. As the earnings growth exceeds the discount rate, the present value of futures earnings accumulates to infinity.

Now let us consider the case $k < r$. Letting $T \rightarrow \infty$ in (12) gives

$$U = \frac{E}{(\beta+r)} + \frac{\alpha}{(\beta+r)(r-k)} e^{kt}. \quad (14)$$

This is the same as obtained by Chiang et al. (1997, Footnote 13). This consistency reveals that our result can be regarded as a generalization of the one obtained in Chiang et al. (1997). Based on (3) and (14), a simulated path for earnings and values is plotted in Figure 1, where we use the empirical data obtained by Chiang et al. (1997): $\alpha=0.012$, $\beta=0.324$, $k=0.0528$, $\sigma=0.205$ and assume that initial earning is \$1 per share and interest rate is 7% p.a. Figure 1 shows that our model appears to be consistent with realities.

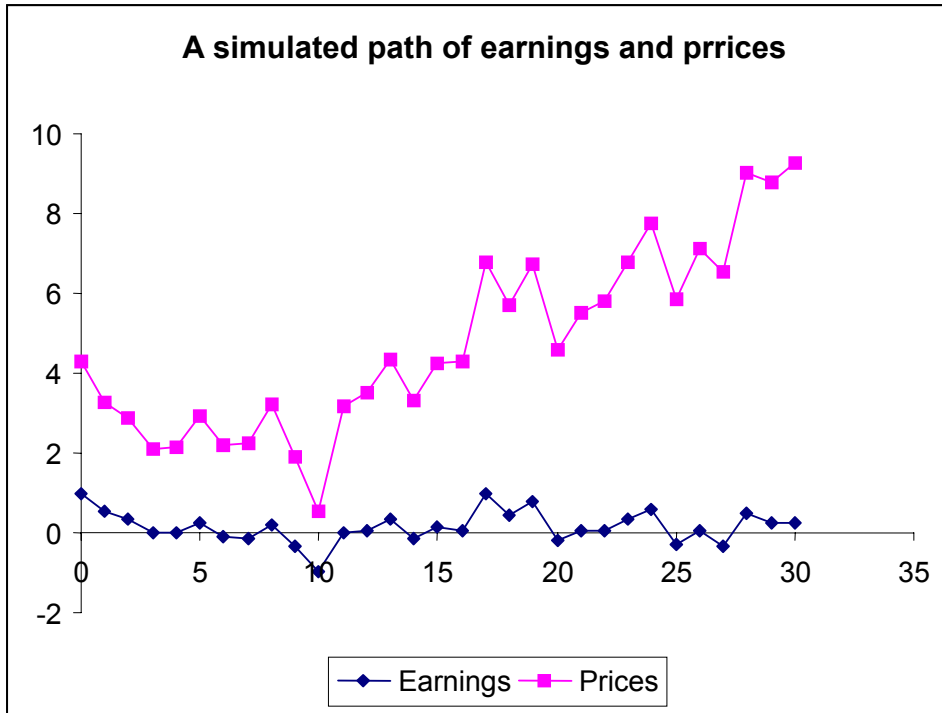


Figure 1.

4.2 Partnership, but business ceases if account balance reaches zero

In this case, the cash balance is constrained to be positive. Other conditions are the same as the former case. Hence the boundary and final conditions are as follows:

$$\frac{\partial^2 V}{\partial C^2} \rightarrow 0, \quad \text{as} \quad C \rightarrow \infty,$$

$$\frac{\partial^2 V}{\partial E^2} \rightarrow 0, \quad \text{as} \quad |E| \rightarrow \infty,$$

$$V(E, 0, t) = 0,$$

$$V(E, C, T) = C.$$

This problem is to be solved in

$$C \in [0, \infty), \quad E \in (-\infty, \infty).$$

Note that in this case, it will not be helpful to use the transformation (10). This boundary value problem can be solved numerically by finite difference method.

4.3 Limited Liability

Assume that the company is run regardless of the state of its current account, but the partners take only limited liability. In this case, the boundary and final condition are:

$$\frac{\partial^2 V}{\partial C^2} \rightarrow 0, \quad \text{as} \quad |C| \rightarrow \infty,$$

$$\frac{\partial^2 V}{\partial E^2} \rightarrow 0, \quad \text{as} \quad |E| \rightarrow \infty,$$

$$V(E, C, T) = \max(C, 0).$$

This problem is to be solved in $C \in (-\infty, \infty)$, $E \in (-\infty, \infty)$

This boundary value problem can also be solved by standard numerical techniques such as finite difference method.

V. VALUING DEBT

In this section, we consider the problem of valuing debt for the present model. Assume that the firm has limited liability. Suppose that the business must repay an amount D at time T . For simplicity, we assume that if the firm has more than D in the bank at the time T , then it is fully paid. If it has less then it pays as much as it has. If it has a negative amount in the bank at the time T , then it pays nothing. This implies that the payment at the time T is: $\text{Max}(\min(C, D), 0)$.

Now assume that the value of loan at time t is the present value of the expected repayment. Thus given the earnings process (3), the value of debt $W(E, C, t)$ can be similarly modeled as a derivative claim. W satisfies

$$\frac{\partial W}{\partial t} + (E + rC) \frac{\partial W}{\partial C} + (\alpha e^{kt} - \beta E) \frac{\partial W}{\partial E} + \frac{1}{2} \sigma^2 \frac{\partial^2 W}{\partial E^2} - rW = 0$$

with

$$W(E, C, T) = \text{Max}(\min(C, D), 0).$$

Other boundary conditions can be specified similarly as before. For example, if there is no limit on borrowing, then we require

$$\frac{\partial^2 W}{\partial C^2} \rightarrow 0, \quad \text{as} \quad |C| \rightarrow \infty,$$

$$\frac{\partial^2 W}{\partial E^2} \rightarrow 0, \quad \text{as} \quad |E| \rightarrow \infty.$$

By solving the above boundary value problem, we can obtain the debt value.

VI. CONCLUSIONS

The use of stochastic differential equations in finance is a common practice. These equations are used to value derivative securities, and have also been used to analyze and value various opportunities and decisions faced by a firm.

In this paper, we have presented a model for the value of a firm using observable quantities as the variables and parameters. The firm value is modeled as a boundary value problem of a partial differential equation similar to the Black-Scholes equation. A number of special cases are addressed and one analytic solution is presented. It is shown that the model can be also applied to the debt valuation of the firm.

This paper has illustrated the advantages of using a more fundamental process on which the value of the firm depends. As long as the earnings process is specified, the firm valuation can be obtained by solving a boundary value problem for the partial differential equation satisfied by the firm value.

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