

Stability of Growth Models with Generalised Lag Structures

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Abstract

This paper considers the lag structures of dynamic models in economics, arguing that the standard approach is too simple to capture the complexity of actual lag structures arising, for example, from production and investment decisions. It is argued that recent (1990s) developments in the theory of functional differential equations provide a means to analyse models with generalised lag structures. The stability and asymptotic stability of two growth models with generalised lag structures are analysed. The paper concludes with some speculative discussion of time-varying parameters.

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1 Introduction

Economic models often generate dynamic processes in which the current rate of change of a variable depends on the current and lagged levels of that variable. For simplicity the lag is usually taken to be constant, but this radically oversimplifies most economic problems. In this paper attention is focussed on dynamic processes with a generalised lag structure: such a process can be summarised by the following equation:

$$\dot{X}(t) = A(t)X(t) + B(t)X(t - \lambda(t)) + C(t)\dot{X}(t - \lambda(t)) \text{ with } X(0) = X_0 \text{ given} \quad (1)$$

The function $\lambda(\cdot)$ is called the *lag function*: it is assumed to be continuous and to satisfy:

$$0 \leq \lambda(t) \leq t \quad (2)$$

Moreover the functions $A(t)$, $B(t)$ and $C(t)$ are assumed continuous with:

$$A(t) \leq 0 \text{ and } |C(t)| \leq C^* < 1 \quad (3)$$

Note that this specification allows for three generalisations of the standard dynamic processes which arise in economics:

- (i) It allows time-varying coefficients ($A(t)$, $B(t)$, $C(t)$)
- (ii) It allows variable lags (via the function $\lambda(t)$)
- (iii) It allows dependence on lagged rates of change as well as lagged levels of variables.

Equation (1) may be generalised to cover many variables by allowing X to be a vector and A , B and C to be matrices. The particular case in which $t - \lambda(t) = \mu t$ where μ is constant and $0 < \mu < 1$ is analysed by Iserles (1993) [5]. Feldstein et. al. (1995) [6] consider a further generalisation of (1) and develop a numerical approach.

Examples of this kind of problem can be found in the theory of economic growth and in investment theory. Bar-Ilan and Strange (1996) [1] analyse models of irreversible investment which allow for the fact that most investments take time. Majd and Pindyck (1987) [2] report that this problem is particularly pronounced in the aircraft industry where a new model requires several stages of activity, including prototype production, testing and final tooling. Together these can introduce production lags of eight to ten years.

Sequeira (2008) [3] develops an endogenous growth model with an erosion effect, paying particular attention to its transitional dynamics. Li (2002) [4] explores the empirical validity of AK-type growth models, introducing complex lag structures. He introduces these complex lag structures in a more or less ad hoc way, aiming at a good empirical fit, concluding that the long-run relationship between growth and investment is consistent with the AK model. In this paper we modify the AK model in such a way as to provide theoretical underpinnings for a complex lag structure. The resulting dynamics are analysed using the theory of functional differential equations.

2 Stability, Asymptotic Stability and Convergence

In this paper we analyse the *stability* and *asymptotic stability* of two growth models. These two properties are defined below:

Definition 1 *The function $X(t)$ will be called stable if $|X(t)|$ is bounded.*

Definition 2 *The function $X(t)$ will be called asymptotically stable if it is stable and $\lim_{t \rightarrow \infty} X(t) = 0$.*

Note that it may be useful to think of $X(t)$ as the divergence of some economic variable (such as the capital/labour ratio $k(t)$) from its long-run equilibrium value, k^* , so that $X(t) = k(t) - k^*$.

Clearly asymptotic stability is stronger than stability and is the convergence concept typically employed in economic growth theory. It is generally considered a desirable property of growth models, largely to ensure that the method of comparative dynamics can be validly employed. This method involves analysing the effects of shocks (e.g. tax changes, productivity increases etc.) by comparing the new (post-shock) equilibrium with the old (pre-shock) equilibrium. This procedure only makes sense if the economy can be relied upon to converge to its new equilibrium when displaced from its old one. The weaker property of stability only guarantees that the variable of interest stays within a neighborhood of its equilibrium value, but nonetheless may still be useful in model construction, econometric analysis or simulation. The theorist may be satisfied that his model cannot diverge to infinity and the econometrician, with limited data available, may be satisfied with boundedness.

The standard approach to ensuring convergence (asymptotic stability) in growth theory is to assume the economy is controlled by a single ("representative") agent maximising an infinite horizon Ramsey utility function such as:

$$\int_0^{\infty} U(c(t))e^{(n-\rho)t} dt \quad (4)$$

where $c(t)$ = consumption per head, n = population growth rate, ρ = discount rate and U is concave. Equation (4) can be modified to make U a function of more variables (perhaps including human capital for example) but it is usually necessary to assume $\rho > n$ to ensure existence of the integral. Maximising (4) subject to the relevant (differential equation) constraints typically leads to saddlepoint dynamics and the imposition of a transversality condition such as:

$$p(t).k(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \quad (5)$$

where $p(t)$ is the costate variable associated with the state variable $k(t)$ (capital per head). The costate variable usually has the interpretation of a discounted shadow price of the corresponding state variable, so that the transversality condition (5) requires that the discounted shadow value of the capital stock tends to zero as t tends to infinity. This approach has a number of significant difficulties:

1. It relies on assuming the existence of a controlling agent with a particular type (Ramsey) of utility function. Such an assumption is usually completely orthogonal to the rest of the model. For example, it may be a plausible working hypothesis to assume rational expectations but, in the absence of central planning (which, presumably, is not what the theorist has in mind), it is going much further to assume the existence of single controlling agent of the requisite type.
2. The transversality condition is not, in general, a necessary condition for a maximum of (4). (See Halkin, 1971 [14]) Moreover, it only guarantees convergence in the infinite horizon case. The finite horizon version of (5), $p(T).k(T) = 0$, could easily be satisfied on a divergent path.
3. Even when the transversality condition does guarantee convergence, it does not explain how convergence is re-established after a shock is

applied to the model. In practice this is achieved by invoking the implausible notion of "jump variables" which, as if by magic, adjust instantaneously to ensure that the model is always on a convergent path (i.e. one lying in the stable manifold). Though some variables, such as prices, might be thought to jump in reality, the jump variable invoked in growth theory, for example, are usually quantities such as consumption. Jumps in quantity variables such as consumption are clearly less plausible than jumps in prices.

4. In practice the transversality condition/jump variables approach is applied to a local linearisation of the model which holds only in a neighbourhood of an equilibrium. Such a linearisation obscures potentially important behaviour of the model away from the particular equilibrium under consideration. Moreover it has problematic implications for the jump variables mentioned above. For example the linearisation may call for an upward jump under exactly the same conditions that the original (non-linear) model requires a downward jump. This raises insuperable difficulties for empirical testing.

These issues are discussed in George, Oxley and Carlaw (2003) [10], George and Oxley (2005) [11] and Buiter (2009) [12]. We dispense with the Ramsey/Pontryagin framework in this paper and focus directly on the issue of convergence (stability and asymptotic stability).

3 Economic Growth with a Variable Production Lag

It is widely assumed in growth theory that investment projects contribute to output as soon as the investment decision is taken. In reality there is usually a significant delay between investment decision and increased output. Sometimes referred to as "construction lags" or "time to build", these delays can be long and/or highly variable. Wheaton (1987) [7] discovers that, for the US, there is a lag between receiving a construction permit and completing the building which varies between 18 and 24 months. Macrae (1989) [8] describes lags of 6 to 10 years in the construction of power stations and Pindyck (1991) [9] notes similar delays in the aerospace and pharmaceutical

sectors. The complexity of prototype production, testing and certification (e.g. in pharmaceuticals) all contribute to this problem. There are reasons to believe that actual lags in all these cases are even longer and/or more variable than these authors describe. They only consider a subset of the lags applicable to any particular investment project.

The standard AK growth model, with a Ramsey utility function, leads to the reduced form dynamical system:

$$\dot{k} = (A - n - \delta)k - c \quad (6)$$

$$\dot{c} = c \left[\frac{A - \delta - \rho}{\theta} \right] \quad (7)$$

where k = capital per head, c = consumption per head, A = productivity parameter, n = population growth rate, δ = depreciation rate, ρ = discount rate, $\frac{1}{\theta}$ = intertemporal elasticity of substitution. Establishing this reduced form requires use of the transversality condition. There are no lags in the model and its solution is simply exponential growth (of all variables) at a rate $g = \frac{A - \rho - \delta}{\theta}$. A condition for asymptotic stability is therefore $g < 0$.

In this section and the next, we develop simple AK type growth models with complex lag structures. Their dynamics are analysed by appeal to the theory of functional differential equations. In this section we focus on a model with a variable production lag.

Definition 3 *Let $\mu(t) = t - \lambda(t)$. Note that $\mu(t)$ is continuous and $0 \leq \mu(t) \leq t$, from (2). No further restrictions are imposed on $\mu(t)$, allowing for a completely general lag structure.*

Now assume a variable production lag, so that:

$$Y(t) = AK(\mu(t)) \quad (8)$$

where $Y(t)$ = output, $K(t)$ = capital stock and A = constant. Assume growth of population ($N(t)$) is exponential:

$$\dot{L}(t) = nL(t) \quad (9)$$

where n = constant. Assume also that depreciation occurs at a constant rate (δ):

$$\text{depreciation} = -\delta K(t) \quad (10)$$

Then:

$$\dot{K}(t) = sY(t) - \delta K(t) = sAK(\mu(t)) - \delta K(t) \quad (11)$$

where s = savings rate (assumed constant).

We now derive a differential equation in the capital/labour ratio ($k(t) = \frac{K(t)}{L(t)}$). Logarithmic differentiation together with equation (9) yield:

$$\dot{k}(t) = \frac{\dot{K}(t)}{L(t)} - nk(t) \quad (12)$$

Equations (10), (11) and (12) yield:

$$\dot{k}(t) = \frac{sAK(\mu(t)) - \delta K(t)}{L(t)} - nk(t) = \frac{sAK(\mu(t))}{L(t)} - (n + \delta)k(t) \Rightarrow \quad (13)$$

$$\dot{k}(t) = \left[\frac{sAK(\mu(t))}{L(\mu(t))} \right] \cdot \left[\frac{L(\mu(t))}{L(t)} \right] - (n + \delta)k(t) \Rightarrow \quad (14)$$

$$\dot{k}(t) = sAk(\mu(t))e^{(\mu(t)-t)n} - (n + \delta)k(t) \quad (15)$$

Equation (15) relates $\dot{k}(t)$ to current and lagged values of $k(t)$. It has the form of equation (1) in its scalar form. Adopting lower case letters (to represent scalars) let:

$$a(t) = -(n + \delta), \quad b(t) = sAe^{(\mu(t)-t)n}, \quad c(t) = 0 \quad (16)$$

Now define the *aggregate coefficient* $\omega(t)$:

Definition 4 *The aggregate coefficient $\omega(t)$ is defined as $\omega(t) = b(t) + a(t - \lambda(t))c(t)$*

Standard approaches to determining stability fail because of the completely general form of the lag function $\lambda(t)$ (generating a completely general form for $\mu(t)$) but progress can be made by appeal to the theory of functional differential equations (see Azbelev, Maksimov, and Rakhmatullina (2007) [15] for an introduction). In particular Theorem 2 of Iserles and Terjeki

(1995) [13] provides a useful means to identify sufficient conditions for the stability and asymptotic stability (in the sense of section 2 above) of $k(t)$. It is relatively straightforward to confirm that the conditions of this theorem are satisfied by our growth model.

The coefficient $b(t)$ is depicted in figure 1.

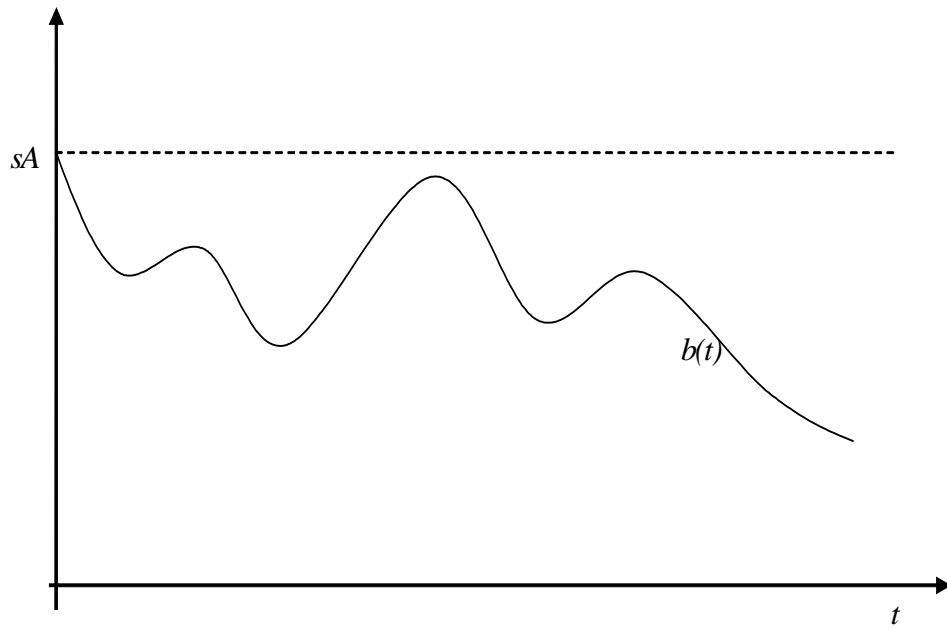


Figure 1. The coefficient $b(t)$

Clearly $\text{Re } a(t) < 0$ (note that $a(t)$ is real so $\text{Re } a(t) = a(t)$) and $b(t)$ is continuous. Moreover $|c(t)| \leq c^* < 1$, taking $c^* = 0$. Furthermore the aggregate coefficient $\omega(t)$ is given by:

$$\omega(t) = sAe^{(\mu(t)-t)n} \quad (17)$$

Hence:

$$\omega^*(t) = \max_{\tau \in [0, t]} b(\tau) = sA \quad (18)$$

because $e^{(\mu(t)-t)n} \leq 1$ and $e^{(\mu(t)-t)n} = 1$ at $t = 0$. The theorem now provides a sufficient condition for the stability (in the sense of section 2 above) of $k(t)$. That condition is:

$$\omega^*(t) + \operatorname{Re} a(t)(1 - c^*) \leq 0 \text{ for } \forall t \geq 0 \quad (19)$$

this yields (from equations (16) and (18)):

$$sA - (n + \delta) \leq 0 \quad (20)$$

Note that this a sufficient condition for stability in the sense of section 2. That is it guarantees the uniform boundedness of $k(t)$. It is *not* sufficient for the asymptotic stability of $k(t)$, but, surprisingly given the general form of the lag structure, it is very similar to a necessary and sufficient condition for asymptotic stability in the unlagged case (with exogenous savings ratio s), namely:

$$sA - (n + \delta) < 0 \quad (21)$$

However, neither (20) nor (21) provide sufficient conditions for the stronger property of asymptotic stability in the sense of section 2 above. Theorem 3 of Iserles and Terjeki (1995) [13] does provide such conditions. Consider conditions (iv), (v) and (vi) of that theorem.

Condition (iv) The condition: $\omega^*(t) + \operatorname{Re} a(t)(1 - c^*)\kappa \leq 0$ ($0 < \kappa < 1$) becomes $sA - (n + \delta)\kappa \leq 0$ ($0 < \kappa < 1$). Since s, A, n and δ are constants, this is equivalent to:

$$sA - (n + \delta) < 0 \quad (22)$$

which is slightly stronger than equation (20).

Condition (v) The condition: $\int_0^\infty \operatorname{Re} a(s) ds = -\infty$ becomes:

$$\int_0^\infty -(\delta + n) ds = -\infty \quad (23)$$

in terms of our model, and is clearly satisfied.

Condition (vi) In terms of our model this becomes:

$$\mu(t) \rightarrow \infty \text{ as } t \rightarrow \infty \quad (24)$$

Imposing this last condition (equation 24), in addition to the other assumptions, will therefore guarantee asymptotic stability of $k(t)$. This condition admits a wide variety of lag structures (fig 2) but rules out others. For example, figure 3 depicts a function $\mu(t)$ which tends to some finite limit as $t \rightarrow \infty$. Another example is given by the function:

$$\mu(t) = t \sin^2 t \quad (25)$$

depicted in figure 4. This function does not tend to a finite limit, but continues to oscillate as $t \rightarrow \infty$, violating condition 24.

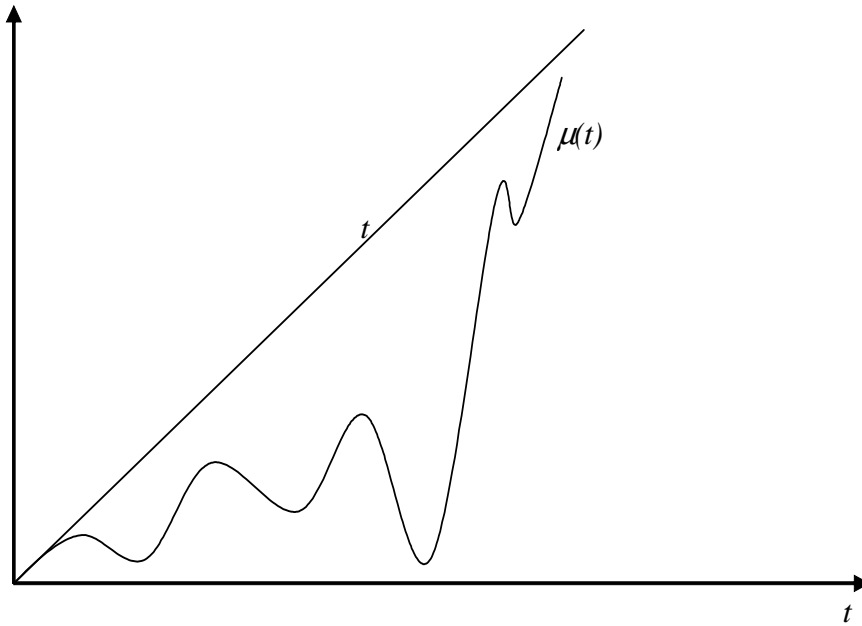


Figure 2. Lag structure satisfying equation 24

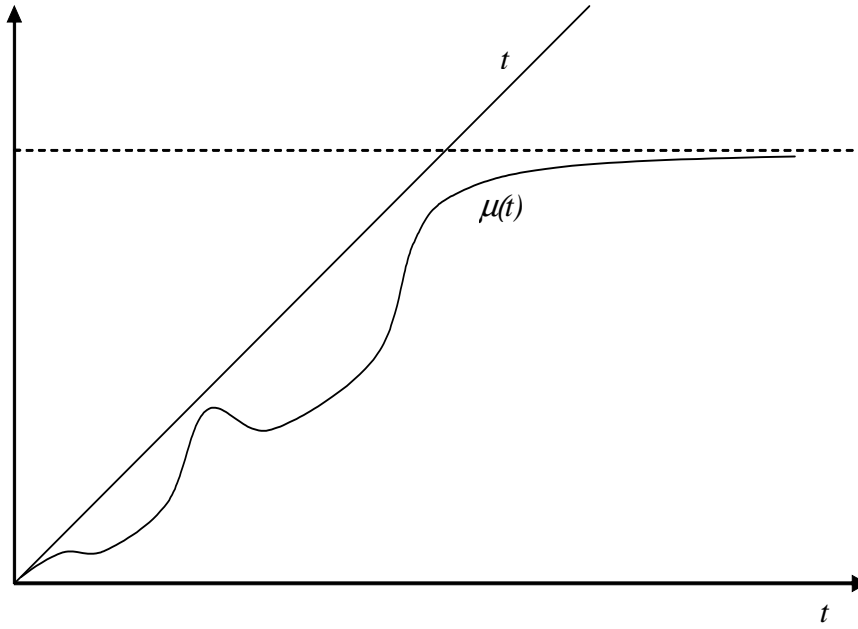


Figure 3. Lag structure violating equation 24

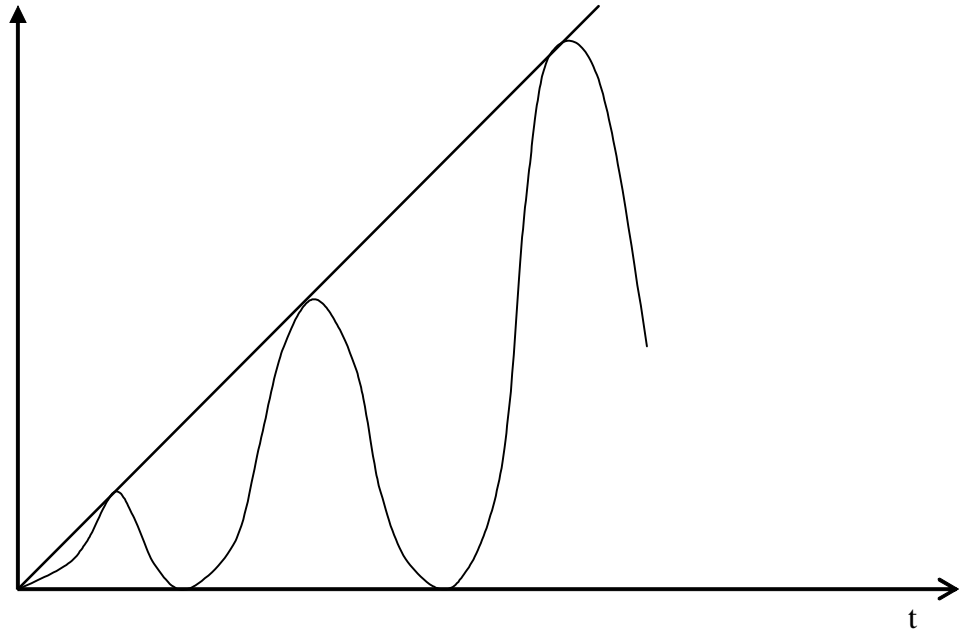


Figure 4. The function $\mu(t) = t \sin^2 t$. This lag structure violates equation 24.

4 Economic Growth with an Accelerator Investment Function

It has been widely assumed that aggregate investment depends on the rate of change of output not its level, leading to an *accelerator* investment function. For similar reasons to those discussed in section 3 above, it is reasonable to suppose that this relationship operates with a variable lag so that:

$$I(t) = s\dot{Y}(\mu(t)) \quad (26)$$

where $I(t)$ = gross investment, $Y(t)$ = output, s = constant and $\mu(t)$ is the lag function, as before. As before, take depreciation = $-\delta K(t)$, an AK production $Y(t) = AK(t)$, and exponential population growth $\dot{L}(t) = nL(t)$. Then:

$$\dot{K}(t) = I(t) - \delta K(t) = s\dot{Y}(\mu(t)) - \delta K(t) = sA\dot{K}(\mu(t)) - \delta K(t) \quad (27)$$

As before, we seek a differential equation in $k(t)$, the capital labour ratio. Logarithmic differentiation yields:

$$\dot{k}(t) = \frac{\dot{K}(t)}{L(t)} - nk(t) \quad (28)$$

Using equation (26) this yields:

$$\dot{k}(t) = \frac{sA\dot{K}(\mu(t))}{L(t)} - (n + \delta)k(t) \quad (29)$$

Noting that equation (9) holds for the lagged variable $\mu(t)$, i.e. $\dot{L}(\mu(t)) = nL(\mu(t))$, equation (29) yields:

$$\dot{k}(t) = sAe^{n(\mu(t)-t)}\dot{k}(\mu(t)) + sAne^{n(\mu(t)-t)}k(\mu(t)) - (\delta + n)k(t) \quad (30)$$

As before, we seek sufficient conditions for the stability and asymptotic stability of equation (30) by appeal to Iserles and Terjeki (1995) [13] theorem 2. Let:

$$a(t) = -(\delta + n), b(t) = sAne^{n(\mu(t)-t)}, c(t) = sAe^{n(\mu(t)-t)} \quad (31)$$

Clearly the function $a(t)$ satisfies the conditions of Iserles and Terjeki (1995) [13] theorem 2 because $a(t) < 0$. To satisfy the conditions of the theorem we require that $|c(t)| \leq c^* < 1$ but, in contrast to the model of section (3) above, in this case $c(t) \neq 0$. In this model $c(t) = sAe^{n(\mu(t)-t)}$, so by taking $c^* = sA$ and assuming:

$$sA < 1 \quad (32)$$

the condition on $c(t)$ is satisfied. Figure (5) depicts $c(t)$ with condition (32) satisfied.

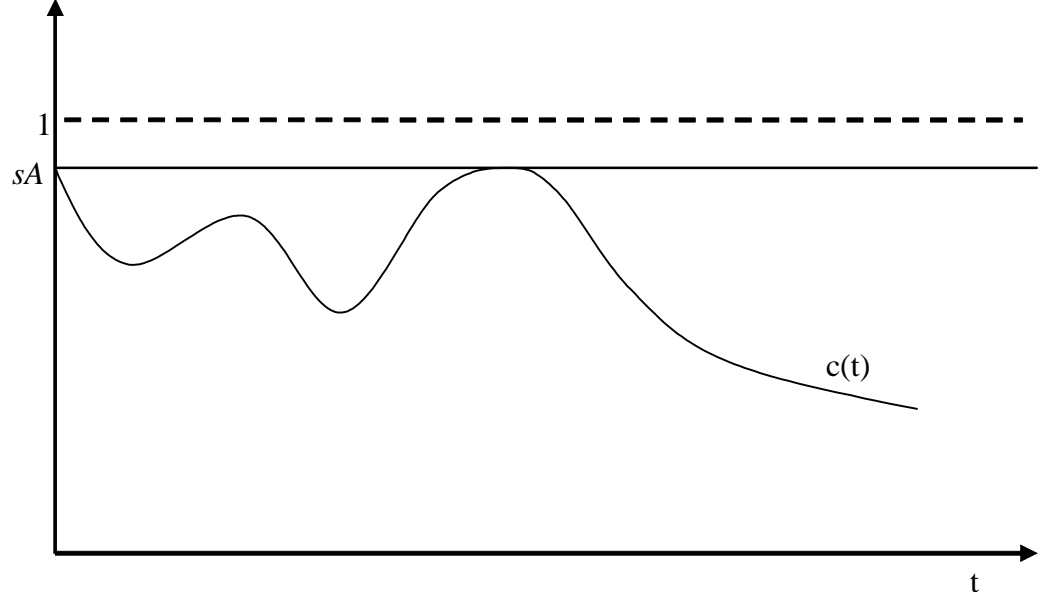


Figure 5. The function $c(t)$ shown satisfying condition (32).

In this model the aggregate coefficient $\omega(t)$ is given by: $\omega(t) = -\delta sA e^{n(\mu(t)-t)}$ so that $\omega^*(t) = \max_{\tau \in [0, t]} \delta sA e^{n(\mu(\tau)-\tau)} = \delta sA$. So the additional condition for stability is:

$$\omega^*(t) + \alpha(t)(1 - c^*) \leq 0 \quad (33)$$

which becomes (noting that $c^* = sA$ in this model) $\delta sA - (\delta + n)(1 - sA) \leq 0$. Re-arranging yields:

$$(2\delta + n)sA - \delta - n \leq 0 \quad (34)$$

Turning now to the conditions for asymptotic stability:

Condition (iv) The condition $\omega^*(t) + \alpha(t)(1 - c^*)\kappa \leq 0$ (where $0 < \kappa < 1$) is a stricter version of equation (33) and may therefore be treated as technical. It yields a stricter version of (34) namely:

$$\delta sA - (\delta + n)(1 - sA)\kappa \leq 0 \quad (\text{where } 0 < \kappa < 1) \quad (35)$$

Condition (v) As in the production lag model of section (3) above, the condition: $\int_0^{\infty} \operatorname{Re} a(s) ds = -\infty$ becomes:

$$\int_0^{\infty} -(\delta + n) ds = -\infty \quad (36)$$

in this model, and is clearly satisfied.

Condition (vi) As in the production lag model this becomes:

$$\mu(t) \rightarrow \infty \text{ as } t \rightarrow \infty \quad (37)$$

Once again this is the crucial condition for asymptotic stability. Figure 2 shows a lag structure satisfying this condition and figures 3 and 4 show lag structures violating it.

5 Time-varying parameters

The approach adopted above is general enough to admit time-varying parameters. As an example consider the production lag model of section 3 above with a time-varying productivity parameter $A(t)$ (perhaps arising from exogenous technical progress) and depreciation rate $\delta(t)$. For simplicity we keep population growth (n) and the savings ratio (s) constant. Now $\omega^*(t)$ takes a slightly more complicated form. The production function is now $Y(t) = A(\mu(t))K(\mu(t))$ and $\omega(t) = b(t) = sA(\mu(t))e^{(\mu(t)-t)n}$ so that:

$$\omega^*(t) = \max_{\tau \in [0, t]} sA(\mu(\tau))e^{(\mu(\tau)-\tau)n} \quad (38)$$

Because $A(\mu(t))$ is a now function of time, the relationship between $\omega^*(t)$ and $b(t)$ is now more complicated than in the model of section 3 above. This relationship is illustrated in figure 6.

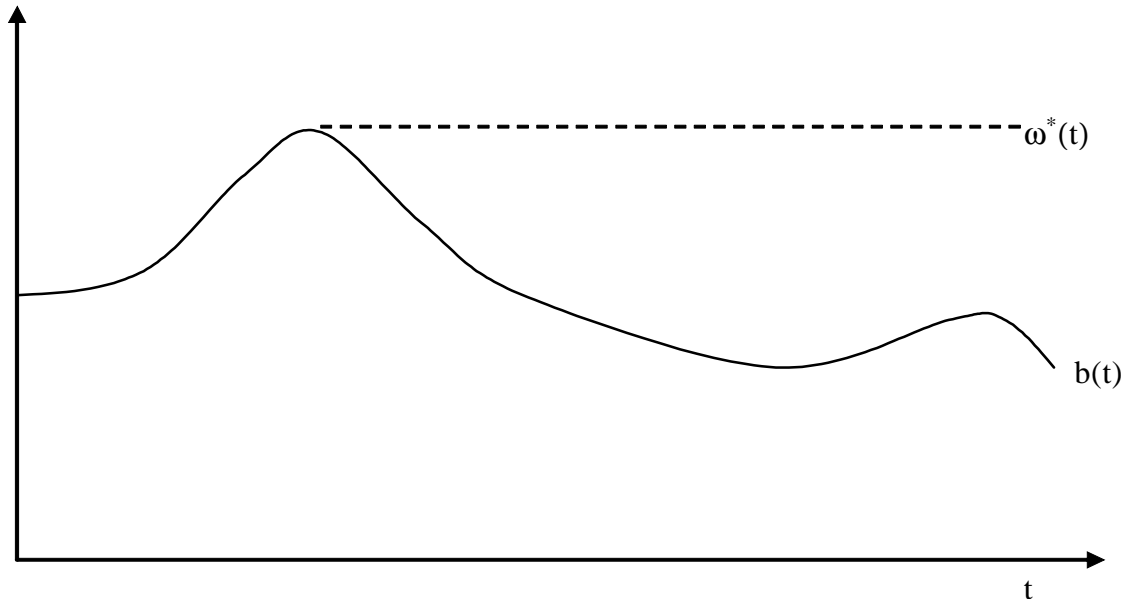


Fig 6. The relationship between $b(t)$ and $\omega^*(t)$ with a time-varying productivity parameter

Condition (19) of section 3 now becomes:

$$\omega^*(t) - (n + \delta(t)) \leq 0 \quad (39)$$

This need not be satisfied even if $s.A(0) \leq 0$. Figure 7 shows both cases (satisfying (39) and violating it).

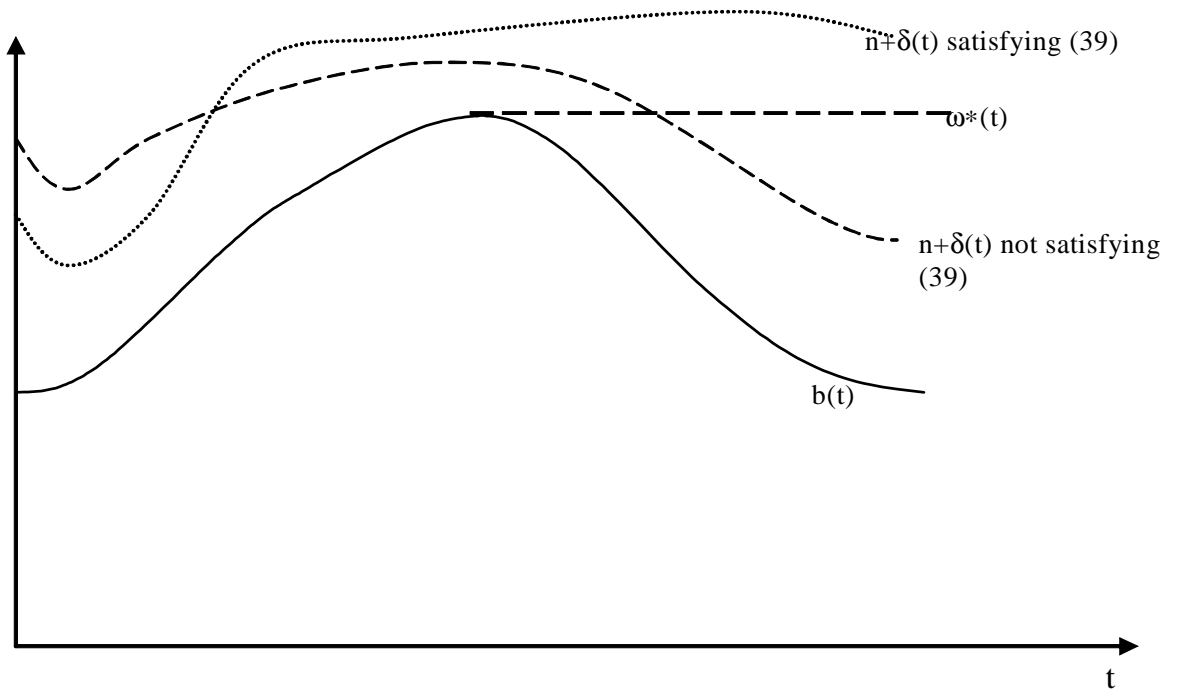


Fig. 7. Convergence condition with time-varying parameters

6 Conclusions

Economic models often generate complicated lag structures which cannot be captured with the standard modelling techniques. This particularly true in the theory of economic growth. Recent (1990s) developments in the theory of functional differential equations however provide a useful approach to this problem. Two growth models with generalised lag structures are analysed, involving production lags and an accelerator investment function respectively. By applying the theory of functional differential equations, sufficient conditions for stability and asymptotic stability can be obtained. In both cases the important extra condition required to guarantee asymptotic stability (convergence) is that $\mu(t) \rightarrow \infty$ as $t \rightarrow \infty$, where $\mu(t) = 1 - \lambda(t)$ for any lag function $\lambda(t)$.

The approach adopted can be extended to cover time-varying parameters and vector-valued equations (using matrix algebra). These extensions should encompass most nonlinearities arising in economic modelling. However, the approach via functional differential equations can be extended to full nonlinearity at the expense of increased mathematical complexity.

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