Behavioral biases and representative agent

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Abstract

In this note, we consider an economy with heterogeneous agents, differing by their time preference rate and by their beliefs. We show that at the Pareto optimum, the representative agent exhibits interesting behavioral properties. More precisely, starting from a standard model with expected utility maximizers and exponential discounting, but allowing for heterogeneity among agents' beliefs and time preference rates, we obtain at the representative agent level an inverse S-shaped probability distribution weighting function and hyperbolic discounting. We provide possible interpretation and applications for this result.

1. Introduction

In this note, we start from a standard model with von Neuman Morgenstern utility maximizing agents and exponential discounting. Agents are heterogeneous, in the sense that they might differ in their beliefs and in their time preference rates. We examine the belief as well as the time preference rate of the representative agent and we show that we retrieve at the aggregate level "behavioral" properties that have been proved to be true at the individual level in recent behavioral economics developments.

We first show that our heterogeneous agents can be aggregated into a representative agent and we determine the expression for her belief and her time preference

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rate. We find that the belief of the representative agent is essentially a mixture (or a power/Hölder average) of the individual beliefs. In particular in a Gaussian setting, since a mixture of Gaussian variables is not Gaussian, this implies that if all agents anticipate normal distributions, with the same variance parameter but differ in the anticipated mean, the anticipated distribution at the aggregate level is not Gaussian. In the unbiased setting (i.e. in a setting where the average belief coincides with the objective belief), the mean remains the same but the variance is increased and the distribution exhibits kurtosis.

As far as behavioral properties are concerned, we find that the belief of the representative agent and the probability weighting functions of Cumulative Prospect Theory (CPT) share common properties. We assume that beliefs are heterogeneous enough in order to allow for optimistic as well as pessimistic agents in the initial set of von Neuman Morgenstern utility maximizing agents. We obtain that the transformation of the objective distribution function into the distribution function of the representative agent is inverse S-shaped as in CPT. The representative agent can neither be everywhere optimistic nor everywhere pessimistic. She is optimistic for the good states of the world and pessimistic for the bad states of the world. As in the SP/A Theory of Lopes (1987), the representative agent behaves as if she had fear (need for security) for very bad events and hope (desire for potential) for very good events. Moreover, we show that we are able to fit relatively well standard weighting functions of the Prospect Theory literature (Tversky and Kahneman, 1992, Tversky and Fox, 1995, Prelec, 1998, among others).

As far as the consensus time preference rate is concerned, we obtain at the aggregate level a time preference rate that is lower than the average of the individual time preference rates. The time preference rate of the representative agent is decreasing, which is consistent with "hyperbolic discounting". It converges to the time preference rate of the most patient individual. These properties are similar to those obtained in a deterministic setting by Gollier-Zeckhauser (2005) and Lengwiller (2005).

The implications of our results are twofold. On the one hand, we obtain that a "behavioral" individual (i.e. an individual whose preferences are governed by hyperbolic discounting and Cumulative Prospect Theory) behaves as would, at a Pareto optimum, a group of standard heterogeneous vNM individuals with exponential discounting. Heterogeneity alone leads to these behavioral properties. Our results can be related to a recent strand of research, called Neuroeconomics, in which the brain is considered as an organization. The purpose of neuroeconomic theory is to take a look into the brain (that is considered as a black box in economic theory including behavioral economic theory) and produce a novel theory of individual decision-making based on experimental neurosciences and on models of agents interactions. Developments in brain imaging have helped identify different regions of the brain that are associated to different types of processes, different time preferences, different information processing, etc. If each region or each process is represented by an agent, Economic Theory provides then many useful tools to analyze the brain. As underlined by Carillo and Brocas "the (...) goal of this strand of research is to revisit the individual decision-making paradigm and provide micro-microfoundations for characteristics of human behavior that have been traditionally ignored or considered as exogeneously given". Examples include hyperbolic discounting, distorded beliefs, mental accounting, etc. For instance, Carillo and Brocas (2008) analyze interactions between different brain areas through principal-agent models. Our results may be interpreted as a general equilibrium model for mental processes interactions. Our conclusions can then be rephrased as follows, a model of the brain with a central planner (the cortex) who relies on evaluations provided by doers (mental processes with heterogeneous time preference rates and beliefs) in order to evaluate risky prospects leads to hyperbolic discounting and to probability weighting functions as in the Cumulative Prospect Theory.

On the other hand, our results imply that in order to analyze the properties of Pareto optima with a group of individuals endowed with “behavioral” beliefs and time preference rates, one only needs to consider the properties of Pareto optima within a larger group of individuals endowed with heterogeneous standard beliefs and time preference rates.

Note that we don’t pretend to retrieve all features of CPT on the aggregate belief nor all features of the time preference rate as in e.g. Loewenstein and Prelec (1992). We only retrieve one of the three main features of CPT: the inverse S-Shaped probability distribution weighting function (the other two being the presence of a reference point and the presence of loss aversion). This comes from the fact that we have introduced heterogeneity on the beliefs only, hence the ”behavioral” property that we retrieve deals with the belief only. We also only obtain the ”hyperbolic” property of the time preference rate and not other behavioral properties such as the different (discounting) treatment of gains and losses. In order to retrieve such properties it likely would be necessary to introduce utility functions heterogeneity.

The note is organised as follows. Section 2 presents the model. Section 3 analyses the properties of the belief of the representative agent, while Section 4 analyses the properties of the time preference rate of the representative agent. Section 5 provides possible interpretations as well as applications and concludes.

All proofs are in the Appendix.

2. The Model

Let us consider an economy with a single consumption good and a set I of agents. The total endowment in consumption good at horizon t is described by a random variable e_t^*. All agents have the same CRRA utility function $u(x) = \frac{x^{1-\frac{1}{\eta}}}{1-\frac{1}{\eta}}$. They have different time preference rates $(\rho_i)_{i\in I}$ and different subjective beliefs Q_i about the distribution of e_t^*. Each agent wants to maximize her Von Neumann Morgenstern utility for future consumption of the form $U_i(c) = E^{Q_i}[\exp(-\rho_i t) u(c_t)]$. We let M_t^i denote the density of Q_i with respect to the objective probability P, and $D_t^i \equiv \exp(-\rho_i t)$ the discount factor of agent i, hence agent i's utility for consumption can equivalently be written in the form $U_i(c) = E[M_t^i D_t^i u(c_t)]$.

In such an economy, we consider the aggregate utility function defined as the solution of the following maximization program

$$\arg \max_{\sum_{i\in I} y_t^i = e_t^*} \sum_{i\in I} \lambda_i E\left[M_t^i D_t^i u(y^i)\right].$$

The aggregate utility function corresponds to the value of the social welfare function at the Pareto optimum when agent i is granted a weight λ_i by the social planner. From the social planner point of view, it corresponds then to the the highest social utility level among all possible endowment distributions accross agents.

The number of agents can be finite or infinite. In the case of a finite number of agents, $|I|$ denotes the number of agents and in the case of an infinite number of agents, sums are replaced by integrals. We will say that the characteristics $(M_t^i, D_t^i, \lambda_i)_{i\in I}$ are independent if for almost all states of the world ω, $M_t^i(\omega), D_t^i$ and λ_i are independent[1] as random variables on I. This property will be, in particular, satisfied when I can be written in the form $I = J \times K \times L$ and when there exist characteristics $\left(\bar{M}_t^j\right)_{j\in J}, \left(\bar{D}_t^k\right)_{k\in K}$ and $\left(\bar{\lambda}_\ell\right)_{\ell\in L}$ such that for $i = (j, k, \ell)$ we have $(M_t^i, D_t^i, \lambda_i) = \left(\bar{M}_t^j, \bar{D}_t^k, \bar{\lambda}_\ell\right)$. Roughly speaking, this property means

[1] More precisely, for any real valued (measurable) functions f, g, h defined on the real line, we have $\sum_{i\in I} f(M_t^i) g(D_t^i) h(\lambda_i) = \sum_{i\in I} f(M_t^i) \sum_{i\in I} g(D_t^i) \sum_{i\in I} h(\lambda_i)$ a.e.

that there is no specific correlation between beliefs and time preferences and that the weights granted by the social planner to the individuals in the economy are independent of their time and belief characteristics. Specific examples are given by a situation where beliefs and time preferences are independent and the agents are uniformly weighted in the social welfare function or by a situation where the agents' weights are given by their relative wealth and where wealth, beliefs and time preferences are assumed to e independent.

Proposition 1. *Representative Agent*

If the characteristics $(M_t^i, D_t^i, \lambda_i)_{i \in I}$ are independent, then

$$\max_{\sum_{i \in I} y_t^i = e_t^*} \sum_{i \in I} \lambda_i E\left[M_t^i D_t^i u(y^i)\right] = E\left[M_t D_t u(e_t^*)\right]$$

with

$$M_t = \left(\frac{1}{|I|} \sum_{i \in I} \left(M_t^i\right)^\eta\right)^{\frac{1}{\eta}} \text{ and } D_t = \left(\frac{1}{|I|} \sum_{i \in I} \left(D_t^i\right)^\eta\right)^{\frac{1}{\eta}}.$$

The representative agent belief is then given by $M_t = \left(\sum_{i \in I} (M_t^i)^\eta\right)^{\frac{1}{\eta}}$ and the representative agent time discount factor is given by $D_t = \left(\sum_{i \in I} (D_t^i)^\eta\right)^{\frac{1}{\eta}}$.

In other words, if the weights (λ_i) are chosen independently of the agents' characteristics then the social utility of a given prospect is given by its utility for a representative agent endowed with an average belief and an average discount factor.

We now turn to the analysis of the properties of the belief M_t and of the time discount factor D_t.

3. Representative Agent Belief

In this section, we focus on the representative agent belief at a fixed date t. For the ease of notation, we will omit the subscript t. As underlined by e.g., Jouini-Napp (2007), note that, except in the specific logarithmic utility setting, the weights of the representative agent belief do not add up to one, i.e. $E[M] \neq 1,$ and then M fails to be the density of a probability measure. In order to analyze the relative weights of the different states of the world from the representative agent point of view, we introduce the normalized belief $\widetilde{M} = \frac{M}{E[M]}$ and its associated probability measure Q defined by $\frac{dQ}{dP}=\widetilde{M}$. We also introduce the following terminology. We say that the distribution of a random variable $X\equiv\varphi\left(e^{*}\right)$ admits a "density f_X for the representative agent" if for all function h, we have $E\left[Mh\left(X\right)\right]=\int h\left(x\right)f_X\left(x\right)dx$.

We start by analysing the distribution of e^{*} (or $\log e^{*}$) for the representative agent. We suppose that for all $i\in I$, the distribution of e^{*} for agent i admits a density[2] (with respect to the Lebesgue measure on the real line), denoted by f^{i}. We also assume that the objective distribution of e^{*} (i.e. the distribution of e^{*} under the objective probability P) also admits a density and we denote it by f.

Proposition 1. *Distribution of aggregate endowment for the representative agent The distribution of e^{*} admits the following density for the representative agent*

$$f^{M}=\left(\frac{1}{|I|}\sum_{i\in I}\left(f^{i}\right)^{\eta}\right)^{1/\eta}$$

which is a power/Hölder average of the initial densities. In particular, for $\eta=1$, the distribution of e^{*} for the consensus investor is a mixture of the individual subjective distributions.

As an immediate consequence, we get that for any measurable real-valued function φ, the distribution of $\varphi\left(e^{*}\right)$ admits the density $f^{\varphi}=\left(\frac{1}{|I|}\sum_{i\in I}\left(f^{i,\varphi}\right)^{\eta}\right)^{1/\eta}$ for the representative agent where $f^{i,\varphi}$ denotes the density of the distribution of $\varphi\left(e^{*}\right)$ for agent i.

Consider first the implications in terms of mean and variance, in the setting with $\eta=1$. We have $E^{M}\left[e^{*}\right]=\frac{1}{|I|}\sum_{i\in I}E^{Q_i}\left[e^{*}\right]$, which means that in the unbiased setting (i.e. when $\frac{1}{|I|}\sum_{i\in I}E^{Q_i}\left[e^{*}\right]=E^{P}\left[e^{*}\right]$) the mean at the aggregate level coincides with the objective mean. For the variance, we have $Var^{M}\left[e^{*}\right]=E\left[M\left(e^{*}\right)^{2}\right]-E\left[Me^{*}\right]^{2}=\frac{1}{|I|}\sum_{i\in I}E^{Q_i}\left[\left(e^{*}\right)^{2}\right]-\left(\frac{1}{|I|}\sum_{i\in I}E^{Q_i}\left[e^{*}\right]\right)^{2}$. In the unbiased setting, and if we suppose that the agents agree on the (objective) variance, i.e. that $Var^{Q_i}\left[e^{*}\right]=Var^{P}\left[e^{*}\right]$, we get that $Var^{M}\left[e^{*}\right]=Var^{P}\left[e^{*}\right]+Var_{i}\left(E^{Q_i}\left[e^{*}\right]\right)$, where $Var_{i}\left(E^{Q_i}\left[e^{*}\right]\right)$ measures beliefs (on the mean) heterogeneity and is given by $Var_{i}\left(E^{Q_i}\left[e^{*}\right]\right)=\frac{1}{|I|}\sum_{i\in I}\left(E^{Q_i}\left[e^{*}\right]\right)^{2}-\left(\frac{1}{|I|}\sum_{i\in I}E^{Q_i}\left[e^{*}\right]\right)^{2}$.

[2] In other words, the distribution of e_t^{*} under Q_i is absolutely continuous with respect to the Lebesgue measure.

This means that in the unbiased setting and for $\eta = 1$, the mean is unchanged but there is more variance at the aggregate level than at the objective level even if all agents agree on the same level of variance, namely the objective variance: beliefs heterogeneity generates "doubt". Let us consider more precisely the case of lognormal distributions.

Example (E) We refer to as Example (E) the case where e^* follows a log normal distribution with $\log e^* \sim \mathcal{N}((\mu, \sigma^2)$. We assume that agents disagree about the mean of the random variable e^*, i.e., we assume that under Q^i the distribution of $\log e^*$ is given by[3] $\mathcal{N}(\mu_i, \sigma^2)$ and we suppose that half of the agents believe that the mean is given by μ_1 while the other half believe that it is given by μ_2. We then have $f_i^{\log}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(x-\mu_i)^2}{2\sigma^2}$ and $f^{\log} = \left(\frac{1}{2} \left(f_1^{\log} \right)^\eta + \frac{1}{2} \left(f_2^{\log} \right)^\eta \right)^{1/\eta}$. We will say that there is no bias when $\frac{\mu_1 + \mu_2}{2} = \mu$.

Corollary 2. *Distribution of* $\log e^*$ *for the representative agent in the case of Example (E)*

1. *The distribution of* $\log e^*$ *for the representative agent is not Gaussian. Moreover, when agents' beliefs are heterogeneous enough, the distribution of* $\log e^*$ *is bimodal.*

2. *For* $\eta = 1$, *the distribution of* $\log e_t^*$ *for the representative agent has the following moments*

$$
\begin{aligned}
E^Q [\log e^*] &= E^P [\log e^*] + E_i (\mu_i - \mu) = \frac{\mu_1 + \mu_2}{2} \\
Var^Q [\log e^*] &= Var^P [\log e^*] + Var_i (\mu_i) = \sigma^2 + \left[\frac{\mu_1 - \mu_2}{2} \right]^2 .
\end{aligned}
$$

3. *For* $\eta = 1$, *and when there is no bias, i.e.* $\frac{\mu_1 + \mu_2}{2} = \mu$, *the distribution of* $\log e^*$ *for the consensus belief exhibits kurtosis excess. Indeed,*

$$
\beta_2 \equiv \frac{E^Q \left[\left(\log e^* - E^Q (\log e^*) \right)^4 \right]}{E^{Q_t} \left[\left(\log e^* - E^{Q_t} (\log e^*) \right)^2 \right]^2} = 3 - \frac{2\Delta^4}{\Delta^4 + 2\Delta^2 \sigma^2 + \sigma^4} < 3
$$

where $\Delta = |\mu_i - \mu|$, $i = 1, 2$.

[3] We whall also consider divergence on the volatility parameter σ.

4. *When there is no bias and for general* η*, the distribution of* $\log e^*$ *under* Q *is Portfolio Dominated by the distribution of* $\log e^*$ *under* P*. In particular, we have* $E^Q[\log e^*] = E^P[\log e^*]$ *and* $Var^Q[\log e^*] > Var^P[\log e^*]$.

5. *For* $\eta > \eta'$ *and associated representative agent probability measures* Q^η *and* $Q^{\eta'}$*, the distribution of* $\log e^*$ *under* Q^η *is Portfolio Dominated by the distribution of* $\log e^*$ *under* $Q^{\eta'}$*. In particular,* $Var^Q[\log e^*]$ *increases with* η.

This means, in particular that in the case with no bias, the distribution of $\log e_t^*$ for the representative agent has the same mean, more variance and is more flat (platikurtic, smaller peak around the mean) than the objective distribution. It also has more variance and more kurtosis than each of the individual subjective distributions.

Let us know analyze how mean and variance evolve in a dynamic setting and let us consider a wealth process e_t^* that follows a geometric Brownian motion with drift $\mu + \frac{1}{2}\sigma^2$ and volatility σ. The distribution of e_t^* is then lognormal and we have $\log e_t^* \sim_P \mathcal{N}(\mu_t, \sigma_t^2)$ with $\mu_t = \mu t$ and $\sigma_t^2 = \sigma^2 t$. By Girsanov Theorem, the subjective distribution of e_t^* from agent i point of view is necessarily of the form $\log e_t^* \sim_{Q^i} \mathcal{N}(\mu_t^i, \sigma_t^2)$ and μ_t^i can be written in the form $\mu_t^i = \mu_t + \delta_t^i \sigma_t$. As far as the excess variance is concerned, notice that if δ_t^i is constant, then $Var_i(\mu_t^i) = \sigma^2 t^2 \delta^2$ which dominates $Var^P[\log e_t^*] = \sigma_t^2 = \sigma^2 t$, for t large enough. The variance of $\log e_t^*$ from the representative agent point of view is then arbitrarily large when the horizon becomes longer. If δ_t is of the form $\frac{\delta}{\sqrt{t}}$ for some constant δ, or in other words if $\mu_1(t) = \mu t + \delta\sigma\sqrt{t}$ and $\mu_2(t) = \mu t - \delta\sigma\sqrt{t}$(agent's i deviation from the objective mean μt is equal to δ times the objective standard deviation on $\log e_t^*$) we have $Var^M[\log e_t^*] = \left(\sigma^2 + \delta^2\right) t$. In both cases, we get that the kurtosis coefficient $\beta_2(t)$ converges to 1 when t becomes large. Note that excess kurtosis is to be compared with ambiguity. Roughly speaking the previous proposition means that, in the unbiased setting, difference in beliefs induces, at the representative agent level, more variance and more ambiguity while leaving the mean unchanged.

In Example (E), all agents believe that e^* has a lognormal distribution and they all agree about its variance. Hence, there is a natural order on the set of possible distribution functions induced by the natural order on the μ_i s. Agents with a larger μ_i can be referred to as more optimistic. In a more general setting, we introduce the following notion of optimism/pessimism.

Definition 1. *An agent is said to be (everywhere) optimistic (resp. pessimistic) if* $\frac{f_i}{f}$ *is nondecreasing (resp. nonincreasing). Agent* i *is said to be more optimistic* *than agent j and we denote by $f_i \succcurlyeq f_j$ if and only if $\frac{f_i}{f_j}$ is nondecreasing. The relation $\succcurlyeq$ is an order on the set $(f_i)_{i\in I}$.*

This definition can be rephrased in terms of Monotone Likelihood ratio Dominance (MLR)[4] : agent i is more optimistic than agent j if the distribution of e_t^* for agent i (under Q_i) dominates the distribution of e_t^* for agent j (under Q_j) in the sense of the MLR. For a given agent i, the probability weighting function g_i transforms the objective distribution function F into the agent's subjective distribution function F_i, i.e. $F_i = g_i \circ F$. It is easy to check that $\frac{f_i}{f}$ is nondecreasing (resp. nonincreasing) if and only if g_i is convex (resp. concave). This means that our concept of optimism/pessimism is the analog, in the expected utility framework, of the concept of optimism/pessimism introduced by Diecidue and Wakker (2001) in a RDEU framework. Other concepts of optimism/pessimism have been proposed in the literature. In particular, Abel (2002) and Chateauneuf et al XXX, propose a definition based on First Stochastic Dominance[5]. Note that MLR dominance is stronger than FSD.

A MLR dominated shift for a given distribution reduces the mean and if agent i is more pessimistic than agent j we have $E^{Q_i}[e_t^*] \leq E^{Q_j}[e_t^*]$. In the framework of Example (E), this last condition characterizes the MLR dominance and the we retrieve then that agent i is more optimistic than agent j if and only if $\mu_i(t) > \mu_j(t)$. Optimistic agents (resp. pessimistic) are then characterized by $\mu_i(t) > \mu$ (resp. $\mu_i(t) < \mu$) as in Shefrin (2005).

Let us now assume that there is at least one pessimistic and one optimistic agent in our economy and let us analyze more in detail the belief of the representative agent. Proposition (1) and Corollary (2) suggest that the belief of the representative agent shares interesting properties with the probability weighting functions (who distorts the objective distribution function) introduced in the recent behavioral economics literature (Cumulative Prospect Theory of Kahneman and Tversky, 1992, or SP/A Theory of Lopes, 1987). The next proposition analyses these similarities more precisely.

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[4]This concept is widely used in the statistical literature and was first introduced in the context of portfolio problems by Landsberger and Meilijson (1990). More precisely, Landsberger and Meilijson (1990) showed that in the standard portfolio problem a MLR shift in the distribution of returns of the risky asset leads to an increase in demand for the risky asset for all agents with nondecreasing utilities.

[5]More precisely, in an expected utility framework Abel (2002) defines pessimism by the condition $F_i \geq F$ (First Stochastic Dominance) that corresponds to the condition $g_i \geq Id$ introduced by Chateauneuf et al.XXXX in a RDEU setting .

Proposition 3. *Behavioral properties of the representative agent belief. We suppose that the set I is made of both optimistic and pessimistic agents.*

1. *The representative agents can neither be (everywhere) optimistic, nor (everywhere) pessimistic, i.e.* $\frac{f_M}{f}$ *is non monotone*

2. *The representative agent is optimistic for "good states of the world" (high values of e_t^*) and pessimistic for "bad states of the world" (low values of e_t^*), i.e.* $\frac{f_M}{f}(x) \searrow$ *for* $x \leq \underline{x}$ *and* $\frac{f_M}{f}(x) \nearrow$ *for* $x \geq \overline{x}$.

3. *The representative agent behaves like the more pessimistic individual for low values of e_t^* and behaves like the more optimistic investor for high values of e_t^*, i.e.* $f_M \sim_{\infty} f_{i_{opt}}$ and $f_M \sim_{-\infty} f_{i_{pess}}$

4. *The representative agent acts as if he had fear (need for security) for very bad events and hope (desire for potential) for very good events, i.e. for* $Q(x \leq \underline{x}) \geq P(x \leq \underline{x})$ *for* $\underline{x} \leq x_{\inf}$ *and* $Q(x \geq \overline{x}) \geq P(x \geq \overline{x})$ *for* $\overline{x} \geq x_{\sup}$. *In other words, the representative agent puts more weight on small probability events with large consequences.*

5. *The transformation of the objective distribution function into the distribution function of the consensus investor is inverse S-shaped: concave for small probabilities, and convex for moderate and high probabilities.*

It appears from this Proposition that the representative agent in a standard expected utility framework with heterogeneous beliefs behaves like the individual agents considered in the behavioral economics and/or psychology literature. Indeeed, she puts more weight on small probability events with large consequences as in the Cumulative Prospect Theory of Kahneman and Tversky, 1992,, has fear (need for security) for very bad events and hope (desire for potential) for very good events as in the SP/A Theory of Lopes, 1987.

As far as the transformation of the objective distribution function is concerned, a variety of methods have been used to determine the shape of the probability weighting function. Tversky and Kahneman (1992), Fox and Tversky (1995) and Prelec (1998) among others specify parametric forms (respectively $\omega(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{1/\gamma}}$, $\omega(p) = \frac{\delta p^\gamma}{[\delta p^\gamma + (1-p)^\gamma]^{1/\gamma}}$ and $\omega(p) = \exp - (-\log p^\gamma))$ and estimate them through standard techniques. Wu and Gonzalez (1996, 1998) and Abdellaoui (2000) avoid the potential problems of parametric estimation and directly derive from experimental studies the shape of the probability weighting function at the aggregate or individual level. The results of all these studies are (mostly) consistent with an inverse S-shaped weighting function, concave for small probabilities, and convex for moderate and high probabilities. As it is illustrated on Figure 1, these properties are retrieved at the representative agent level in a standard expected utility model with heterogeneous beliefs. More precisely, Figure 1 represents the representative agent probability weighting function in a model with two logarithmic utility agents without aggregate bias.

Gonzalez and Wu (1999) exhibit two main features for the shape of the probability function: diminishing sensitivity and attractiveness. Diminishing sensitivity corresponds to the fact that people become less sensitive to changes in probability as they move away from a reference point. In the probability domain, the two endpoints 0 (certainly will not happen) and 1 (certainly will happen) serve as reference points and under this principle, increments near the end points of probability loom larger than increments near the middle of the scale. This concept is related to the concept of discriminability in psychophysics literature and can be illustrated by two extreme cases: a function that approaches a step function and a function that is almost linear. In the setting of Example (E) without aggregate bias, Figure XX shows that discriminability decreases with the level of disagreement among the two agents. When both agents agree on the objective distribution, the probability weighting function is linear. When the agents disagree, one of them overestimating the average payoff by twice the standard deviation and the other one underestimating it by twice the standard deviation, we obtain a function that approaches a step function.

Attractiveness characterizes the absolute level of the probability weighting function. Indeed, an inverse S-shaped function can be completely below the identity line, can cross the identity line at some point or can be completely above the identity line. If an agent has a probability weighting function graph more "elevated" than the probability weighting function graph of another agent, then this means that the first agent finds betting on the chance domain more attractive than the second agent. In the setting of Example (E), Figure XX shows that attractiveness increases when the optimistic agent becomes more optimistic (the level of divergence among agents being unchanged). Another way to increase attractiveness consists in increasing the weight attached to the optimistic agent (relatively to the weight of the pessimsitic agent) and this is illustrated by Figure XX.

The definition of attractiveness can be rephrased in terms of First Stochastic

Dominance (FSD)[6]. Indeed, if we denote by g_i the probability weighting function of agent i (i.e. the function that transforms the objective distribution function F into the agent's subjective distribution function F_i) then agent i finds betting on the chance domain more attractive than agent j if $g_i \geq g_j$ or, in other words, if $F_i \geq F_j$ which characterizes the First Stochastic Dominance. In the next we will say that the density function f_i is more attractive than the density function f_j. We can then generalize the results illustrated by Figure XX and XX as follows. We denote by γ_i the proportion of agents having the same density function f_i and we assume that the set $(f_i)_{i \in I}$ of agents' density functions is totally ordered (nondecreasing) with respect to the FSD order, i.e. $F_i \leq F_j$ for $i \leq j$. Let us consider (γ'_i) another possible distribution of agents' characteristics; as usual, we will say that the distribution (γ'_i) dominates the distribution (γ_i) in the sense of the FSD if for any increasing family (x_i), we have $\sum \gamma_i x_i \leq \sum \gamma'_i x_i$. In other words, the distribution (γ'_i) puts more weight on more attractive distributions.

Proposition 4. *If all the agents have logarithmic utility functions and if the set $(f_i)_{i \in I}$ of agent's density functions is totally ordered with respect to the FSD order then a FSD dominated shift in agents' density functions distribution leads to a less attractive density function for the representative agent.*

When all agents have logarithmic utility functions, attractiveness at the representative agent level increases then when the weight granted to the more attractive density functions increases. Since FSD is weaker than MLR, attractiveness at the representative agent level increases when the weight granted to the more optimistic agents increases. As shown in the next proposition, this last property can be extended to power utility functions if we replace FSD shifts on the distribution of agents' characteristics by MLR shifts. More precisely, we assume that the set $(f_i)_{i \in I}$ of agent's density functions is totally ordered with respect to the MLR order, i.e. for all (i, j) we have either $f_i \succeq f_j$ or $f_j \succeq f_i$ and we will say that the distribution (γ'_i) dominates the distribution (γ_i) in the sense of the MLR if whenever $f_i \succ f_j$ we have $\frac{\gamma'_i}{\gamma_i} \geq \frac{\gamma'_j}{\gamma_j}$. In other words the ratio between the two densities (γ'_i) and (γ_i) increases with agents' optimism and, in particular, the distribution (γ'_i) puts more weight on more optimistic agents.

[6]

Proposition 5. *If the set $(f_i)_{i \in I}$ of agent's density functions is totally ordered with respect to the MLR order then a MLR dominated shift in agents' density functions distribution leads to a more pessimistic representative agent.*

4. Representative Agent Time Preference Rate

The properties of the representative agent time preference rate are easy to obtain. Note that the properties of a "consensus" time preference rate when there is heterogeneity on individual time preference rates has already been studied in varying contexts. Indeed, the problem of the aggregation of the utility discount rates has been studied by Reinschmidt (2002) through a certainty equivalent approach, by Gollier-Zeckhauser (2005) and Nocetti and al. (2008) through a Benthamite/Pareto optimal approach, and by Lengwiler (2005) through an equilibrium approach. All these papers adopt a deterministic setting with no divergence on the beliefs of the agents. On the contrary our aim here is to derive the properties at the aggregate level simultaneously on the beliefs and on the time preference rate (and in a quite general stochastic setting).

We know that the representative agent time discount factor is given by $D_t = \left(\sum_{i \in I} (D_t^i)^\eta\right)^{\frac{1}{\eta}}$ where $D_t^i \equiv \exp(-\rho_i t)$. We introduce the representative agent marginal time preference rate ρ_m as well as the representative agent average time preference rate ρ_a, respectively defined as

$$\begin{aligned} \rho_m^D(t) &= -\frac{D_t'}{D_t} \\ \rho_a^D(t) &= -\frac{1}{t}\log D_t \end{aligned}$$

The average discount rate corresponds to the rate which, if applied constantly for all intervening years, would yield the discount factor D_t, whereas the marginal discount rate is the rate of change of the discount factor. It is easy to recover the average discount rate from the marginal discount rate since $\rho_a(t) = \frac{1}{t}\int_0^t \rho_m(s)\,ds$.

Proposition 1. *Properties of the representative agent time preference rate*

1. *The representative agent average and marginal time preference rates are*

given by

$$\rho_a^D(t) = -\frac{1}{t}\log\left[\frac{1}{N}\sum_{i=1}^{N}\exp(-\eta\rho_i t)\right]^{1/\eta}$$

$$\rho_m^D(t) = \sum_{i=1}^{N}\frac{\exp(-\eta\rho_i t)}{\sum_{i=1}^{N}\exp(-\eta\rho_i t)}\rho_i$$

2. *The representative agent time preference rates are lower than the average of the time preference rates, i.e.*

$$\rho_m^D(t) < \frac{1}{N}\sum_{i=1}^{N}\rho_i \text{ and } \rho_a^D(t) < \frac{1}{N}\sum_{i=1}^{N}\rho_i$$

3. *"Behavioral Properties" : The representative agent time preference rates are decreasing with time. Moreover, the asymptotic discount rates are given by the lowest time preference rate, i.e.* $\lim_{t\to+\infty}\rho_a^D(t) = \lim_{t\to+\infty}\rho_m^D(t) = \inf_i(\rho_i)$. *The representative agent behaves for* t *large enough like the more patient agent.*

Let us denote by f_ρ the distribution of the ρ_is on the real line, the representative agent average and marginal time preference rates are then given by

$$\rho_a^D(t) = -\frac{1}{t}\log\left[\int\exp(-\eta xt)\,f_\rho(x)dx\right]^{1/\eta},$$

$$\rho_m^D(t) = \frac{\int x\exp(-\eta xt)\,f_\rho(x)dx}{\int\exp(-\eta xt)\,f_\rho(x)dx}.$$

These formulas permit explicit computations for specific distributions of the individual time preference rates. For instance, if we assume a Gamma[7] distribution $\gamma(\alpha,\beta)$ for the ρ_is we obtain

$$\rho_m^D(t) = \frac{m^2}{m+\eta v^2 t}$$

[7] Recall that the density function of a gamma distribution $\gamma(\alpha,\beta)$ is given by $\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta x)$. Its mean m and its variance v^2 are respectively given by $m = \frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$.

where m and v^2 respectively denote the mean and the variance of the considered distribution. It is immediate on this simple example that the marginal discount rate decreases with time and is hyperbolic as in Weitzman (). Furthermore, the speed of the decrease increases with the level of heterogeneity v^2 as well as with the level of risk tolerance.

Let us now analyze more in detail the impact of the choice of the distribution f_ρ on the average and marginal discount rates. More precisely, the next proposition provides comparative statics results for shifts of the distribution f_ρ.

Proposition 2. *1. A FSD (resp. SSD) dominated shift on the distribution f_ρ of individual marginal time preference rates decreases the representative agent average time preference rate ρ_a^D.*

2. A MLR (resp. PD) dominated shift on the distribution f_ρ of individual marginal time preference rates decreases the representative agent average time preference rate ρ_m^D.

It is easy to verify that the "hyperbolic" property as well as the asymptotic property remain valid for non constant time preference rates as long as these rates are nonincreasing.

5. Interpretation, Applications and Discussion

We have seen that starting from a standard model with heterogeneous agents endowed with Von Neuman Morgenstern preferences and exponential discouting, we obtain at the representative agent level properties such as an S-shaped distribution transformation function and hyperbolic discounting, that are in line with recent empirical and experimental results.

A possible interpretation of such a result is to consider that each individual subject to experiments behaves as a group of individuals at the equilibrium.

A possible application of this result consists in the study of equilibrium models with preferences and time discounting that are consistent with recent empirical results. Indeed, it has been argued that recent developments in decision theory better reflect than the standard ones the true behavior of individuals but that they are difficult to handle. Our results imply that we can interpret a model with n agents with behavioral preferences and preference for the present as a model with for instance $2n$ agents with heterogeneous standard VNM preferences and exponential discounting. These models have been studied for instance by Jouini and Napp (2007) and lead to tractable results.

BIBTEX

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APPENDIX

Proof of Proposition 1

At the Pareto optimum, we have

$$\lambda_i M_t^i D_t^i u'(y_t^i) = q_t$$

for some random variable q_t. It follows that

$$y_t^i = \left[\frac{q_t}{\lambda_i M_t^i D_t^i}\right]^{-\eta}$$

hence

$$e_t^* = \sum_{i \in I} \left[\frac{q_t}{\lambda_i M_t^i D_t^i}\right]^{-\eta} = q_t^{-\eta} \sum_{i \in I} \left[\frac{1}{\lambda_i M_t^i D_t^i}\right]^{-\eta}$$

and

$$y_t^i = e_t^* \frac{[\lambda_i M_t^i D_t^i]^\eta}{\sum_{i \in I} [\lambda_i M_t^i D_t^i]^\eta}.$$

We have then

$$\begin{aligned}
\sum_{i\in I} \lambda_i E\left[M_t^i D_t^i u(y_t^i)\right] &= \sum_{i\in I} \lambda_i E\left[M_t^i D_t^i \frac{\left[\lambda_i M_t^i D_t^i\right]^{\eta-1}}{\left(\sum_{i\in I}\left[\lambda_i M_t^i D_t^i\right]^{\eta}\right)^{1-\frac{1}{\eta}}} u(e_t^*)\right] \\
&= E\left[\frac{\sum_{i\in I}\left[\lambda_i M_t^i D_t^i\right]^{\eta}}{\left(\sum_{i\in I}\left[\lambda_i M_t^i D_t^i\right]^{\eta}\right)^{1-\frac{1}{\eta}}} u(e_t^*)\right] \\
&= E\left[\left[\sum_{i\in I}\left[\lambda_i M_t^i D_t^i\right]^{\eta}\right]^{1/\eta} u(e_t^*)\right]
\end{aligned}$$

Now, if the characteristics $(\lambda_i, M_t^i, D_t^i)$ are independent, then

$$\left[\sum_{i\in I}\left[\lambda_i M_t^i D_t^i\right]^{\eta}\right]^{1/\eta} = \left[\left(\frac{1}{|I|}\sum_{i\in I}\left(M_t^i\right)^{\eta}\right)\right]^{1/\eta}\left[\left(\frac{1}{|I|}\sum_{i\in I}\left(D_t^i\right)^{\eta}\right)\right]^{1/\eta}$$

and

$$\sum_{i\in I} \lambda_i E\left[M_t^i D_t^i u(y_t^i)\right] = E\left[\left(\frac{1}{|I|}\sum_{i\in I}\left(M_t^i\right)^{\eta}\right)^{1/\eta}\left(\frac{1}{|I|}\sum_{i\in I}\left(D_t^i\right)^{\eta}\right)^{1/\eta} u(e_t^*)\right]$$

■

Proof of Proposition 1

We have

$$
\begin{aligned}
E\left[M_t h\left(e_t^*\right)\right] &= E\left[\left(\frac{1}{|I|}\sum_{i\in I}\left(M^i\right)^\eta\right)^{1/\eta} h\left(e_t^*\right)\right]\\
&= E\left[\left(\frac{1}{|I|}\sum_{i\in I}\left(\frac{f^i}{f}\left(e_t^*\right)\right)^\eta\right)^{1/\eta} h\left(e_t^*\right)\right]\\
&= E\left[\frac{\left(\frac{1}{|I|}\sum_{i\in I}\left(f^i\left(e_t^*\right)\right)^\eta\right)^{1/\eta}}{f\left(e_t^*\right)} h\left(e_t^*\right)\right]\\
&= \int \frac{\left(\frac{1}{|I|}\sum_{i\in I}\left(f^i\left(x\right)\right)^\eta\right)^{1/\eta}}{f\left(x\right)} h\left(x\right) f\left(x\right) dx\\
&= \int \left(\frac{1}{|I|}\sum_{i\in I}\left(f^i\left(x\right)\right)^\eta\right)^{1/\eta} h\left(x\right) dx
\end{aligned}
$$

hence $f^M = \left(\frac{1}{|I|}\sum_{i\in I}\left(f_i\right)^\eta\right)^{1/\eta}$. ■

Proof of Proposition 2

1. Since a mixture of Gaussian distributions is not Gaussian, the first part is immediate. For the second part, we have

$$
\left(f^{\log}\right)^\eta = \frac{1}{2}\left(f_1^{\log}\right)^\eta + \frac{1}{2}\left(f_2^{\log}\right)^\eta = \frac{1}{2\sqrt{2\pi}\sigma}\exp\left(-\frac{\eta\left(x-\mu_1\right)^2}{2\sigma^2}\right) + \frac{1}{2\sqrt{2\pi}\sigma}\exp\left(-\frac{\eta\left(x-\mu_2\right)^2}{2\sigma^2}\right).
$$

This function has either two maxima that are symmetric with respect to $\frac{\mu_1+\mu_2}{2}$ or only one maximum at $\frac{\mu_1+\mu_2}{2}$. In the first case $\frac{\mu_1+\mu_2}{2}$ would be a local minimum. It suffices then to analyze the sign of the second derivative of $\left(f^{\log}\right)^\eta$ at $\frac{\mu_1+\mu_2}{2}$. We obtain that the distribution is bimodal for $\mu_1 - \mu_2 > 2\sigma/\sqrt{\eta}$ and unimodal for $\mu_1 - \mu_2 \leq 2\sigma/\sqrt{\eta}$.

2. For $\eta = 1$, we have $E^Q\left[\log e^*\right] = \frac{1}{2}E^{Q^1}\left[\log e^*\right] + \frac{1}{2}E^{Q^2}\left[\log e^*\right] = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$.

We have

$$
\begin{aligned}
Var^Q[\log e^*] &= E^Q\left[(\log e^*)^2\right] - E^Q[\log e^*]^2 \\
&= \frac{1}{2}\left(Var^{Q^1}[\log e_t^*] + E^{Q^1}[\log e_t^*]^2\right) + \frac{1}{2}\left(Var^{Q^2}[\log e_t^*] + E^{Q^2}[\log e_t^*]^2\right) - \left(\frac{1}{2}\mu_1 + \frac{1}{2}\right. \\
&= Var^P[\log e^*] + \frac{1}{4}[\mu_1 - \mu_2]^2 \\
&= \sigma^2 + Var_i(\mu_i)
\end{aligned}
$$

3. Suppose that $\mu = 0$. We have

$$
\begin{aligned}
E^Q\left[\left(\log e^* - E^Q(\log e^*)\right)^4\right] &= E^Q\left[(\log e^*)^4\right] - 4E^Q\left[(\log e^*)^3\right]E^Q[\log e^*] + 6E^Q\left[(\log e^*)^2\right]E^M[\log \\
&\quad -4E^Q[\log e^*]E^Q[\log e^*]^3 + E^Q[\log e^*]^4 .
\end{aligned}
$$

We easily get that $E^{Q^i}\left[(\log e^*)^2\right] = \sigma^2 + \mu_i^2$, $E^{Q^i}\left[(\log e^*)^3\right] = \mu_i\left[\mu_i^2 + \sigma^2\right]$ and $E^{Q^i}\left[(\log e^*)^4\right] = \mu_i^4 + 6\mu_i^2\sigma^2 + 3\sigma^4$.

If we set $\mu_1 + \mu_2 = 0$, we get that $E^Q\left[\left(\log e^* - E^Q(\log e^*)\right)^4\right] = \frac{1}{2}\left(\mu_1^4 + \mu_2^4\right) + 3\left(\mu_1^2 + \mu_2^2\right)\sigma^2 + 3\sigma^4$ and $E^Q\left[\left(\log e^* - E^Q(\log e^*)\right)^2\right] = \sigma^2 + \frac{1}{2}\left(\mu_1^2 + \mu_2^2\right)$ hence

$$
\frac{E^Q\left[\left(\log e^* - E^Q(\log e^*)\right)^4\right]}{E^Q\left[(\log e^* - E^Q(\log e^*))^2\right]^2} = \frac{\mu_i^4 + 6\mu_i^2\sigma^2 + 3\sigma^4}{\mu_i^4 + 2\mu_i^2\sigma^2 + \sigma^4} = 3 - \frac{2\mu_i^4}{\mu_i^4 + 2\mu_i^2\sigma^2 + \sigma^4} < 3.
$$

For general μ, it suffices to translate uniformly all the considered distributions to obtain the result.

4. The ratio between the density of $\log e^*$ under Q and the density of $\log e^*$ under P is given by $\frac{f^{M\log}}{f^{\log}}(x) = \left(\frac{1}{2}\exp\left(\eta\frac{-2(x-\mu)(\mu-\mu_1)+\mu^2-\mu_1^2}{2\sigma^2}\right) + \frac{1}{2}\exp\left(\eta\frac{-2x(\mu-\mu_2)+\mu^2-\mu_2^2}{2\sigma^2}\right)\right)^{\frac{1}{\eta}}$ which is clearly symmetric with respect to μ, decreasing before μ and increasing after μ. Following Jouini and Napp (2008), this is a sufficient condition for Portfolio Dominance and gives that the distribution of $\log e^*$ under Q is Portfolio Dominated by the distribution of $\log e^*$ under P. Moreover, since the distributions of $\log e^*$ under Q and under P are both symmetric with respect to μ, we have $E^Q[\log e^*] = E^P[\log e^*] = \mu$. This last property with the Portfolio Dominance property give $Var^Q[\log e^*] = Var^P[\log e^*]$ (see Jouini and Napp, 2008).

5. For two different values η and η' of the risk tolerance parameter, it suffices to consider $\frac{f_{\eta'}^{M\log}}{f_{\eta}^{M\log}}$ and to apply the same reasoning as in 4.

Proof of Proposition (4) Let us consider a distribution (γ_i') and a FSD dominated shift (γ_i). We want to prove that $\sum \gamma_i' F_i \geq \sum \gamma_i F_i$. For a given x, letting x_i denote the quantity $F_i(x)$, it suffices to prove that $\sum \gamma_i' x_i \geq \sum \gamma_i x_i$ for a non-decreasing family $(x_i)_{i \in I}$ which is true since (γ_i') dominates (γ_i) in the sense of the FSD.

Proof of Proposition (5) Let us consider a distribution (γ_i') and a MLR dominated shift (γ_i). It suffices to prove that $\frac{\left(\sum \gamma_i' f_i^\eta\right)^{\frac{1}{\eta}}}{\left(\sum \gamma_i f_i^\eta\right)^{\frac{1}{\eta}}}$ is increasing or that $\frac{\sum \gamma_i' F_i}{\sum \gamma_i F_i}$ is increasing with $F_i = f_i^\eta$. Without any loss of generality, we may assume that all the considered functions are differentiable and let us consider the derivative of $\frac{\sum \gamma_i' F_i}{\sum \gamma_i F_i}$

$$\begin{aligned}\left(\frac{\sum \gamma_i' F_i}{\sum \gamma_i F_i}\right)' &= \frac{\left(\sum \gamma_i' F_i'\right)\left(\sum \gamma_i F_i\right) - \left(\sum \gamma_i' F_i\right)\left(\sum \gamma_i F_i'\right)}{\left(\sum \gamma_i F_i\right)^2} \\ &= \frac{\sum_{f_i \succeq f_j} \gamma_i \gamma_j \left(\frac{\gamma_i'}{\gamma_i} - \frac{\gamma_j'}{\gamma_j}\right)\left(F_i' F_j - F_i F_j'\right)}{\left(\sum \gamma_i F_i\right)^2}.\end{aligned}$$

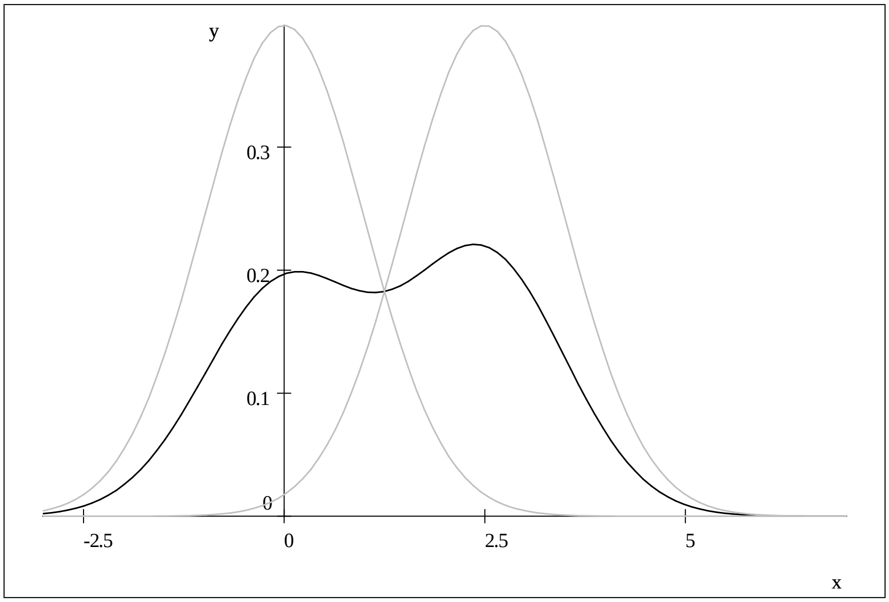
Remark that for $f_i \succeq f_j$ we have $F_i \succeq F_j$ and then $F_i' F_j - F_i F_j' \geq 0$. Furthermore, for $f_i \succeq f_j$ we also have $\frac{\gamma_i'}{\gamma_i} - \frac{\gamma_j'}{\gamma_j} \geq 0$ which leads then to the conclusion.

Proof of Proposition 3

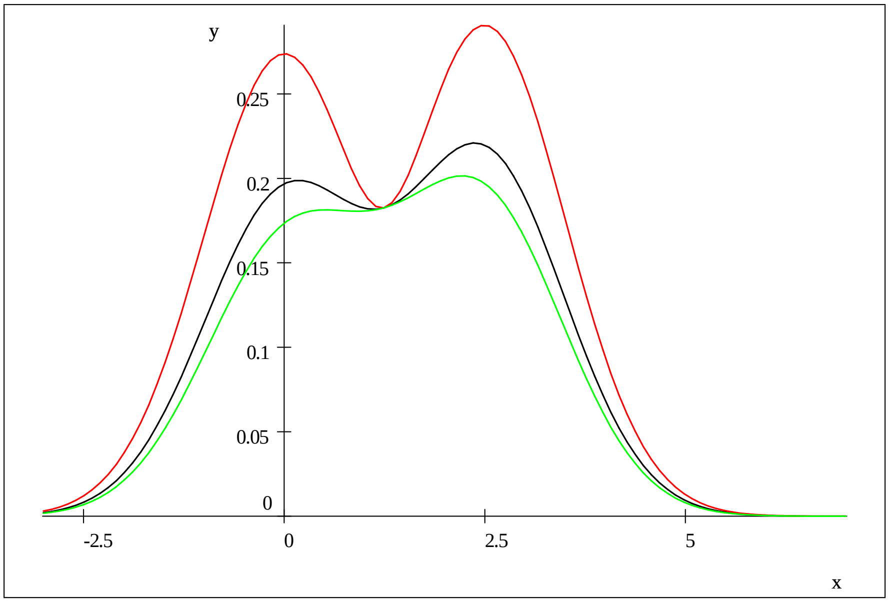
1. 2. and 3. We let i_{opt} (resp. i_{pess}) denote the most optimistic (resp. pessimistic) agent. Since $f^M(u) = \left(\frac{1}{N}\sum_{i=1}^N (f_i(u))^\eta\right)^{1/\eta}$, we have for all j, $f^M(u) = f_j(u)\left(\frac{1}{N} + \frac{1}{N}\sum_{\substack{i=1,\ldots,N \\ i \neq j}} \left(\frac{f_i(u)}{f_j(u)}\right)^\eta\right)^{1/\eta}$. For u large enough, we know that for all $i \neq i_{opt}$, $\frac{f_i(u)}{f_{i_{opt}}(u)}$ we have $f^M(u) = f_{i_{opt}}(u)\left(\frac{1}{N} + \frac{1}{N}\sum_{\substack{i=1,\ldots,N \\ i \neq i_{opt}}} \left(\frac{f_i(u)}{f_{i_{opt}}(u)}\right)^\eta\right)^{1/\eta} \frac{f^M(u)}{f(u)} = \left(\frac{1}{N}\sum_{i=1}^N \left(\frac{f_i(u)}{f(u)}\right)^\eta\right)^{1/\eta}$

5. Let g be given by $F^M(u) = \int_{-\infty}^u \left(\frac{1}{N}\sum_{i=1}^N (f_i(x))^\eta\right)^{1/\eta} dx = g\left[F(u)\right]$. We have $\left(\frac{1}{N}\sum_{i=1}^N (f_i(u))^\eta\right)^{1/\eta} = g'\left[F(u)\right] f(u)$ and $g_t'\left[F(u)\right] = \left(\frac{1}{N}\sum_{i=1}^N \left(\frac{f_i}{f}(u)\right)^\eta\right)^{1/\eta}$. As in the previous proof, it is easy to verify that for large u, $\left(\frac{1}{N}\sum_{i=1}^N \left(\frac{f_i}{f}(u)\right)^\eta\right)^{1/\eta}$ is nondecreasing and that for small u , $\left(\frac{1}{N}\sum_{i=1}^N \left(\frac{f_i}{f}(u)\right)^\eta\right)^{1/\eta}$ is nonincreasing.

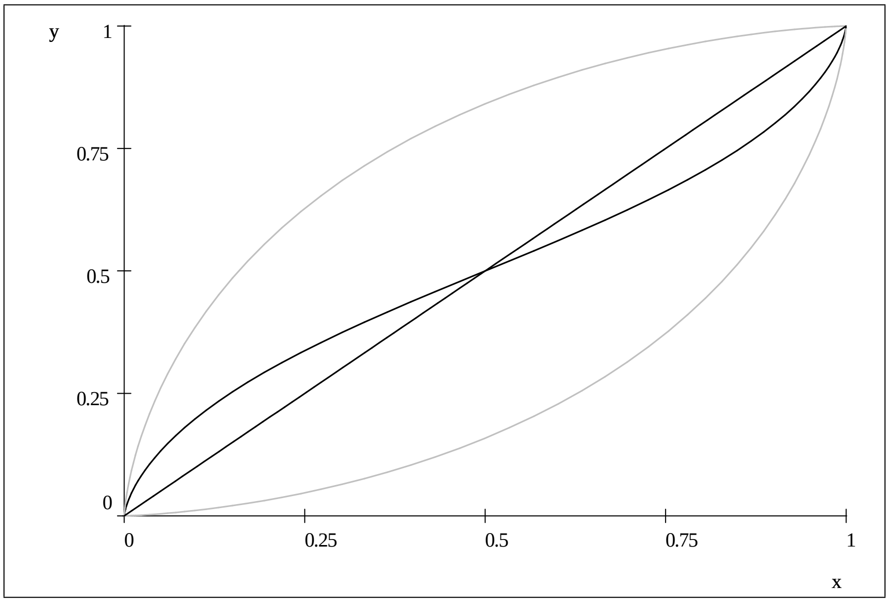
Since F is an increasing function, with $F(0) = 0$ and $F(\infty) = 1$, we get that g is concave for small probabilities, and convex for high probabilities. The inverse S-shape is immediate.



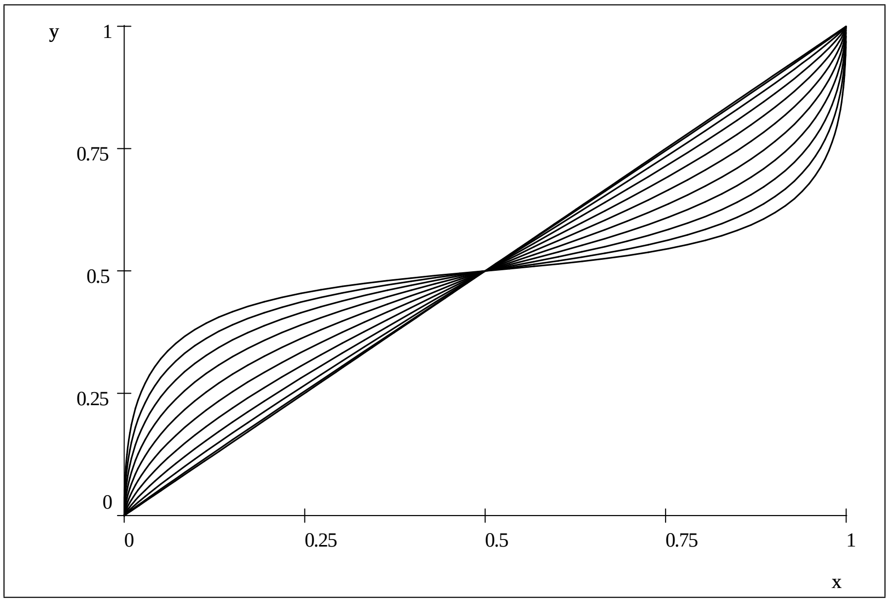
In this figure, we have represented in black the consensus belief in a log-utility agents setting. A proportion of 47% of the agents believe that $\log e_t \sim \mathcal{N}(0,1)$ and the remaining 53% believe that $\log e_t \sim \mathcal{N}(2.5,1)$.The beliefs of these two categories of agents are represented in grey.



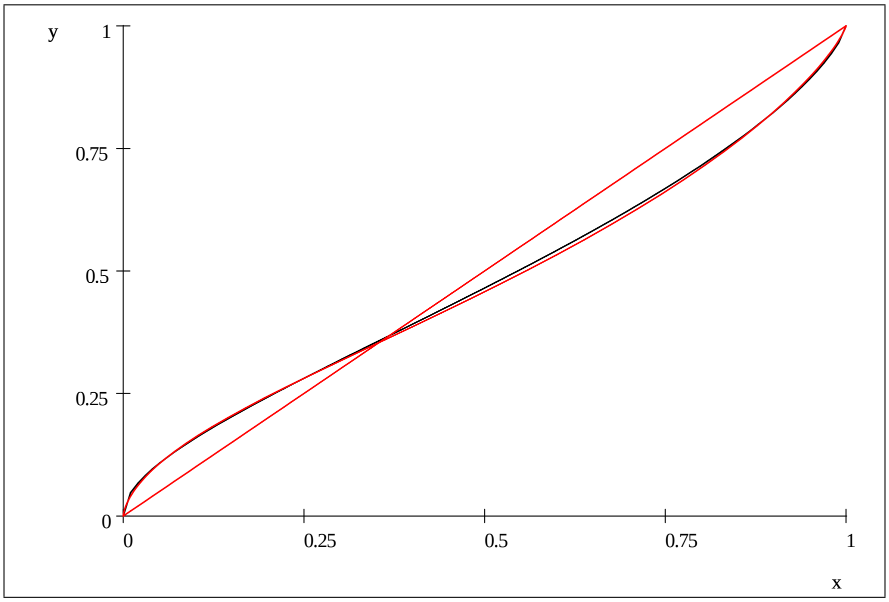
In this figure, we represent the consensus belief for three different levels of risk aversion. We assume that a proportion of 47% of the agents believe that $\log e_t \sim \mathcal{N}(0,1)$ and the remaining 53% believe that $\log e_t \sim \mathcal{N}(2.5,1)$. The upper curve corresponds to $\eta = 2$, the lower curve to $\eta = 0.8$ and the middle curve to $\eta = 1$. An increase of η increases the distance between the peaks and their size.



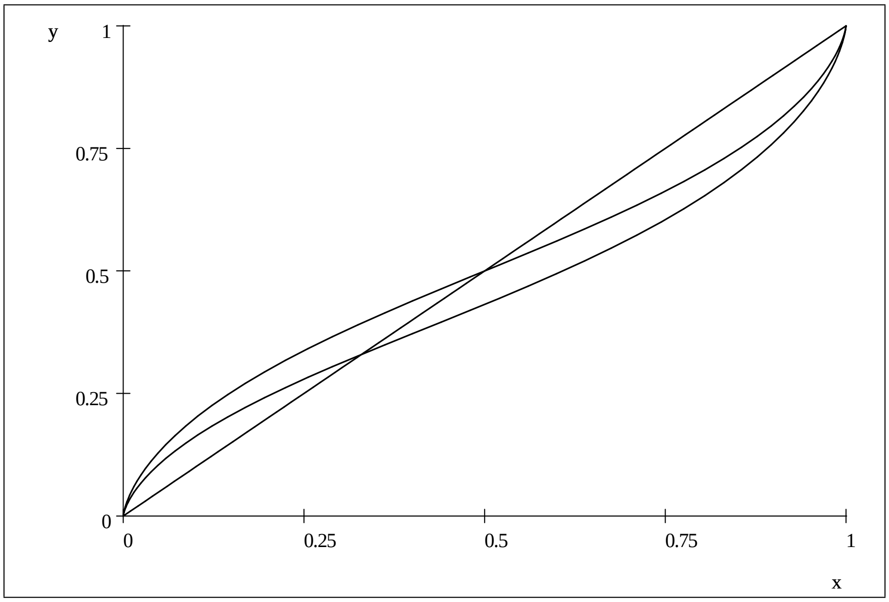
In this figure, we represent the probability weighting function of the optimistic agent (upper curve), of the pessimistic agent (lower curve) and of the representative agent (intermediary curve)



The probability weighting function for different levels of divergence of opinion. Both agents agree on a normal distribution in the form $\mathcal{N}(\mu, 1)$ but one agent is optimistic and the other one is pessimistic and there is no bias at the aggregate level, i.e. $\delta_1 = -\delta_2 = \delta$. The value of δ varies from 0 to 2 (twice the standard deviation). The discriminability decreases with δ (in other words the curvature increases with δ).



In this figure we represent Prelec's probability weighting function as well as the probability weighting function of our representative agent



In this Figure we represent the probability weighting function of the representative agent in a model with logarithmic utility agents. In the upper curve the optimistic and the pessimistic agents are equally weighted. In the lower curve, the pessimistic agents have a 60% weight and the optimistic ones have a 40% weight. The weight granted to the pessimistic agents decreases attractiveness.