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Fairness Properties of Constrained Market Equilibria

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Abstract

This paper studies the notion of fairness in pure exchange economies involving uncertainty and asymmetric information. We propose a new concept of coalitional fair allocation in order to solve the tension that may exist between efficiency and envy-freeness when the equity of allocations is evaluated at the {\it interim} stage. Some characterizations of constrained market equilibria are derived extending the analysis to economies that have both an atomic and an atomless sector.

JEL Classification: C71, D51, D82

Keywords: Mixed markets, coalitional fairness, envy, efficiency, asymmetric information.

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1 Introduction

The problem of a fair distribution of resources among agents has been widely investigated and many notions of fair allocation have been adopted to evaluate equity. Since its introduction by Foley [6] and Varian [14], one of the most extensively studied concept is the one according to which an allocation is fair if it is envy free and efficient. An allocation is said to be envy free if each individual prefers to keep his bundle rather than to receive the bundle of some other agent. It is known that in a pure exchange economy a fair allocation always exists (see Theorem 2.3 in [14]). This is so because any competitive allocation that results from an equal sharing of the total initial endowment is fair. On the other hand, when production is allowed, envy-freeness may be incompatible with efficiency and therefore the set of fair allocations may be empty (see [12] and [14]). The same problem arises when agents are asymmetrically informed at the time of contracting. Recent papers by de Clippel [5] and Gajdos and Tallon [8] exhibit the tension between the concepts of envy-freeness and efficiency in models that explicitly encompasses uncertainty. The incompatibility may occur if the allocation is judged before or after the realization of uncertainty, as well as according to an interim stage evaluation.

Stronger notions of equity present in the fairness literature require that bundle comparisons are allowed between groups of agents. The pioneers of the concept of coalitional fair (c-fair) allocations are Varian [14] and Gabszewicz [7]. An allocation is c-fair if no coalition envies the aggregate bundle or the net trade of some other coalition. It comes out that c-fair allocations are Pareto optimal. They always exist in standard models of pure exchange economies. Moreover, despite of differences due to technical requirements about the measures of potentially envious coalitions, c-fair allocations introduced in both papers provide a complete characterization of competitive market equilibria (see also [17] and [19] for analogous results).

In de Clippel [5], the notion of (individual) fair allocation is extended to differential information economies, assuming that the true state of nature is commonly known at the time of implementing contracts. This requirement makes irrelevant incentive compatibility constraints and allows to cover situations, like payoff of options or insurance contracts, that are contingent on the realization of some observable event. An interim envy free allocation is a redistribution of the initial resources such that there is zero probability of an agent interim envying another. de Clippel shows that an interim fair allocation may not exist ([5, Example 1]), addressing the issue of the incompatibility between efficiency and envy freeness in pure exchange economies with informational asymmetries. A similar notion is introduced in [11] in a quite different context, to show that interim fair allocations are not implementable.

The purpose of this paper is to identify the coalitional fairness notion as a suitable criterion to evaluate allocations on an equitable basis when agents are asymmetrically informed. We analyze the general case of mixed markets in which the measure of agents may have atoms. As in [5], we assume that the true state of nature is commonly known at the time of implementing the contracts, focusing on interim fairness notions. We extend to differential information economies both c-fairness criteria introduced, respectively, by Gabszewicz [7] and Varian [14]. Our goal is to provide a notion of fairness which solves the conflict arising between efficiency and the absence of envy. To this end, we first extend to asymmetric information economies the notions of c-fair allocations: an allocation is *interim c-fair* if no coalition interim envies (in the sense of Varian or Gabszewicz) the bundle of any other coalition. In the light of the negative conclusions presented in [5], we obtain only partial existence results by using the notion of common knowledge event. Indeed, we prove that it is impossible to find two coalitions S_1 and S_2 for which it is common knowledge that S_1 interim envies S_2. Hence, the notions of interim coalitional fairness, that implicitly satisfy interim Pareto efficiency, do not seem to furnish a suitable criterion of equity when agents are asymmetrically informed. However, differently by individual fairness, they allow us to adapt, in the case of envy, the alternative blocking mechanism proposed by [4].

In [4], objections to a given allocation of resources emerge from coalitions depending on the state of nature. Unblocked allocations are interpreted as sub game-perfect equilibrium outcomes of competitive screening games. Under this approach, uninformed intermediaries anticipate the coalitions of agents that are going to form to buy net trade vectors. Since agent's decisions depend on their private information, coalitions depend on the future state of nature. With this interpretation in mind, coalitions that are going to be formed emerge endogenously as a function of the state of nature.

Given an allocation a and an alternative allocation a', the set of deviators in a given state is the set of agents that prefer to receive, given their private information, the allocation a' instead of keeping a. We say that there is coalitional envy in a state under the allocation a, if there exists an alternative allocation a' for which the set of deviators could benefit from achieving the net trade of some other disjoint coalition. Adapting the Gabszewicz's notion ([7]) to the asymmetric information framework, an allocation is qualified *c-type fair* if there is zero probability of a set of deviators interim envying the net trade of any other coalition. Under such distribution of resources, there does not exist an alternative allocation and a state at which, potentially, deviators are treated in a discriminatory way by the market. According to the c-type fair criterion, agents in a potentially envious coalition make comparisons given their private information and not after the observation of the state. Our main result shows that the set of c-type fair allocations is nonempty.

We prove that c-type fair allocations correspond to the c-fair allocations of an auxiliary Arrow-Debreu exchange economy with uncertainty and symmetric information. Agents of the auxiliary economy are defined adapting the idea used by Harsanyi [10] to define Bayesian games. A type-agent is a couple $(t, \mathcal{E})$, where t is an agent and $\mathcal{E}$ is an atom of his information partition. The future state of the fictitious economy is uncertain, but each type-agent has no private information. Moreover, since contracts are contingent on the future state of the economy, standard Arrow-Debreu equilibrium notions can be applied. We adapt this representation to the case of mixed markets, namely large exchange economies in which some agents are endowed with an exceptional initial endowment of resources. Mixed market is the natural framework to deal with coalitional fairness. We prove that there is a natural correspondence between the original mixed market and the auxiliary economy. A correspondence that preserves core, c-fair allocations and constrained market equilibria. This allows us to conclude that the set of c-type fair allocations is non empty, since it contains the set of constrained market equilibria (see [16]). Moreover, c-type fair allocations form a subset of the type-core defined in [4]. Though inclusions relating the above equilibrium notions are strict, additional assumptions allow to show their convergence towards the set of constrained market equilibria.

The paper is organized as follows. In Section 2 we present the model describing the general framework of mixed markets. In Section 3 an extension to differential information economies of the notion of c-fairness due to Varian is introduced and a first partial existence theorem is proved. In Section 4 we introduce c-type fairness. Hence we define the fictitious economy associated to the original differential information one. Section 5 deals with the one-to-one correspondence between both economies in terms of core, c-type fair allocations and constrained market equilibria. We conclude proving the non emptiness of the set of c-type fair allocations and convergence results.

2 The model

2.1 The setting

We consider a Radner like exchange economy E with uncertainty and differential information. We refer to this economy as a mixed market since it exhibits both an atomic and an atomless sector (see [13] and the references given there). The economy E is modeled by the following collection:

$$E = \left\{ (\Omega, \mathcal{F}, \pi);\ (T, \mathcal{T}, \mu);\ I\!R^{\ell}_{+};\ (\mathcal{F}_t, u_t, e_t)_{t \in T} \right\}$$

where:

1. $(\Omega, \mathcal{F}, \pi)$ is a probability space describing the exogenous uncertainty. The set Ω is finite: $\Omega = \{\omega_1, \dots, \omega_k\}$ represents the possible states of nature and $\mathcal{F}$ is the field representing the set of all the events. The common prior π describes the relative probability of the states. We assume, without loss of generality, that $\pi(\omega) > 0$ for each $\omega \in \Omega$.

2. $(T, \mathcal{T}, \mu)$ is a complete, finite measure space representing the space of agents, where T is the set of agents, $\mathcal{T}$ is the σ-field of all eligible coalitions, whose economic weight on the market is given by the measure μ. An arbitrary finite measure space of agents makes us deal simultaneously with the case of discrete economies, non-atomic economies as well as economies that may have atoms. Indeed, discrete economies are covered by a finite set T with a counting measure μ. Atomless economies are analyzed by assuming that $(T, \mathcal{T}, \mu)$ is the Lebesgue measure space with $T = [0, 1]$. Finally, mixed markets are those for which is composed by two sets: T_0 and T_1, where T_0 is the atomless sector that describes the ocean of small agents and T_1 the set of atoms. We will refer to T_0 as the set of "small" traders and to T_1 as the set of "large" traders.[1]

3. $I\!R^{\ell}_{+}$ is the commodity space.

4. $\left(\mathcal{F}_t, u_t, e_t\right)_{t \in T}$ is the set of agent's characteristics. Each agent $t \in T$ is characterized by:

   - A private information. This is described by the field $\mathcal{F}_t$ generated by a partition Π_t of the set Ω. The interpretation is as usual: if $\omega \in \Omega$ is the state of nature that is going to be realized, agent t observes the event $P_t(\omega)$ of Π_t which contains ω, that is agent t knows and only knows that it will be an element of $P_t(\omega)$. Moreover agent's beliefs are described from π by Bayesian updating. Given an event $\mathcal{E} \subseteq \Omega$, the probability $\pi(\omega|\mathcal{E})$ of any state ω given the event $\mathcal{E}$ is $\dfrac{\pi(\{\omega\} \cap \mathcal{E})}{\pi(\mathcal{E})}$. The true state of the economy is common knowledge among the agents at some future date.
     Let $\Pi_1, \dots \Pi_N$ be the partitions of the set Ω. For each $i \in \{1, \dots, N\}$, denote by T_i the information type set defined by $T_i = \{t \in$

[1] This terminology is, in particular, motivated when T is a separable metric space. Indeed, in this case, T_0 is the set of traders $t \in T$ for which $\mu(t) = 0$, while T_1 is the set of traders such that $\mu(t) > 0$.

$T: \Pi_t = \Pi_i\}$. Throughout the paper we assume that, for every $1 \leq i \leq N$, the set T_i is measurable with positive measure[2].

- A state-dependent utility function representing preferences:

$$\begin{array}{rcl} u_t: \Omega \times I\!R^\ell_+ & \to & I\!R \\ (\omega, x) & \to & u_t(\omega, x). \end{array}$$

  The utility function of each agent is strictly increasing, continuous and concave in each state of the economy, moreover for all ω in Ω the mapping $(t, x) \mapsto u_t(\omega, x)$ is $\mathcal{T} \times \mathcal{B}$-measurable, where $\mathcal{B}$ is the σ-field of Borel subsets of $I\!R^\ell_+$.

- An initial endowment of physical resources represented by the function

$$e_t: \ \Omega \to I\!R^\ell_+.$$

Decisions are taken today about the way to redistribute the endowments when the state will be common knowledge. Therefore, incentive and measurability constraints are irrelevant.
An *allocation* is a function $a: \Omega \times T \to I\!R^\ell_+$ and, for each $t \in T$, $a_t = a(\cdot, t)$ represents what agent t obtains by a. An allocation a is *feasible* if for each $\omega \in \Omega$

$$\int_T a_t(\omega) d\mu \leqslant \int_T e_t(\omega) d\mu.$$

For any allocation a we denote by $V_t(a_t)$ the ex ante expected utility from a of trader t, that is

$$V_t(a_t) = \sum_{\omega \in \Omega} \pi(\omega) u_t(\omega, a_t(\omega)),$$

while given an event $\mathcal{E}$ the interim expected utility of agent t for some allocation a conditional on the event $\mathcal{E}$ is

$$V_t(a_t|\mathcal{E}) = \sum_{\omega \in \Omega} \pi(\omega|\mathcal{E}) u_t(\omega, a_t(\omega)) = V_t(a_t|\mathcal{E}) = \sum_{\omega \in \mathcal{E}} u_t(\omega, a_t(\omega)) \frac{\pi(\omega)}{\pi(\mathcal{E})}.$$

Definition 2.1. *A feasible allocation a' interim Pareto dominates an allocation a if almost all agents, given their own private information, prefer a' over a in each state, i.e.,*

$$V_t(a'_t|P_t(\omega)) > V_t(a_t|P_t(\omega))$$

for almost all $t \in T$ and each $\omega \in \Omega$.

[2]Notice that this assumption implies that the correspondence $\Pi: T \to 2^{\mathcal{F}}$ defined by $\Pi(t) = \Pi_t$, has measurable graph. It means that the set $G_\Pi = \{(t, \mathcal{E}) : \mathcal{E} \in \Pi_t\}$ belongs to $\mathcal{T} \otimes \mathcal{B}(2^{\mathcal{F}})$, where $\mathcal{B}(2^{\mathcal{F}})$ denotes the Borel σ-algebra on $2^{\mathcal{F}}$ and $\otimes$ denotes the product σ-algebra.

A feasible allocation is *interim efficient* (or *interim Pareto optimal*) if it is not interim Pareto dominated by any other feasible allocation (for a similar definition see [18]).

The following notion of constrained market equilibrium represents a natural generalization of Arrow-Debreu equilibria to markets with asymmetric information ([16], [4]).

Definition 2.2. *An allocation a is a constrained market equilibrium if it is feasible and there exists a price system $p : \Omega \to I\!R^{\ell}_{+}$ such that, for each $\omega \in \Omega$ and $t \in T$*

$$a_t \in \arg\max_{a'_t \in B_t(p,\omega)} V_t(a'_t | P_t(\omega))$$

where

$$B_t(p,\omega) = \left\{ a'_t \in I\!R^{\ell k}_{+} | \sum_{\bar{\omega} \in P_t(\omega)} p(\bar{\omega}) \cdot a'_t(\bar{\omega}) \leq \sum_{\bar{\omega} \in P_t(\omega)} p(\bar{\omega}) \cdot e_t(\bar{\omega}) \right\}$$

is the budget set of agent t at ω.

We observe that a constrained market equilibrium is a feasible allocation a for which there exists a price system such that, for each ω and for each agent t, a_t maximizes the interim expected utility function on the budget set of t at ω. The budget set of agent t depends also on the state ω, because it takes in account only states belonging to the atom of t's partition containing ω. Intuitively, each agent, using only his own information, maximizes his expected utility function under the additional constraint that he may not sell contingent commodities associated to states that he knows are not going to occur.

Whenever we need to require the equal distribution of initial resources, we define the *equal income budget set* of agent t in the state ω as

$$\widetilde{B}_t(p,\omega) = \left\{ a'_t \in I\!R^{\ell k}_{+} | \sum_{\bar{\omega} \in P_t(\omega)} p(\bar{\omega}) \cdot a'_t(\bar{\omega}) \leq \sum_{\bar{\omega} \in P_t(\omega)} p(\bar{\omega}) \cdot \frac{e(\bar{\omega})}{\mu(T)} \right\}$$

where e denotes the aggregate initial endowment.

The existence of a constrained market equilibrium is proved by [16] in the case of discrete economies (see also [4, Theorem 4]). We will prove that the existence of constrained market equilibria holds true in the general case of mixed markets (compare Remark 5.5).

3 Motivation

The problem of a fair division of resources among agents has been widely studied and many notions of fair allocation have been developed in different economic contexts (see among the others, [3], [6], [14], [17]). One of the concepts that has played a central role is due to Foley [6] and Varian [14] (see also [15]). According to [6] and [14], an allocation is fair if it is envy free and efficient. Equitable or envy free allocations (in the sense of Varian)are those for which each individual prefers to keep his bundle rather than to receive the bundle of any other agent. It is proved that in a pure exchange economy, under monotonicity and convexity assumptions on preferences, a fair allocation always exists ([14, Theorem 2.3]). On the other hand, when agents are asymmetrically informed, envy-freeness may be incompatible with efficiency and therefore the set of fair allocations may be empty. De Clippel defines an interim fair allocation as an (interim) efficient redistribution of initial resources such that there is zero probability of an agent interim envying another.

Definition 3.1. ([5]) *An agent t interim envies an agent t' at the allocation a in the state ω, if*

$$V_t(a_{t'}|P_t(\omega)) > V_t(a_t|P_t(\omega)).$$

An allocation a is: *interim envy-free* if there does not exists a state at which an agent interim envies another; *interim fair* if it is interim envy-free and interim efficient. This concept generalizes the usual (individual) fairness when uncertainty and asymmetric information are encompassed by the model. It is possible to show that an interim fair allocation may not exist (see [5, Example 1] and Example 3.7 below). Our goal is to provide a notion of fairness which solves the conflict. To this end, we extend to asymmetric information economies the stronger notions of equity according to which groups of agents are allowed to make utility comparisons, as we argue that an appropriate notion of fairness is the coalitional one. Coalitional notions of fair were introduced by Varian [14] and Gabszewicz [7]. The two notions differ for what follow: Varian requires that a c-fair allocation must be such that no coalition envies the aggregate bundle of any other coalition of the same or smaller size. According to Gabszewicz's definition, different groups of agents compare their net trades without any requirement on the measure of the potentially envious coalition. In both cases, differently from individual fairness, the efficiency is implicitly satisfied.

We first extend to asymmetric information economies the notion of c-fair allocations due to Varian [14][3]. We say that an allocation is *interim c-fair* if it is interim efficient and if no coalition interim envies the bundle of any other coalition with the same or smaller measure. Each member of a potentially envious coalition makes bundle comparisons given his private

[3]Differently from the perfect information notions, we will explicitly require interim Pareto efficiency due to the free disposal condition imposed on allocations. However, this is not a restriction for the aim of this Section since the motivation relying on Example 3.7 remains valid removing the free disposal requirement.

information. For the aim of this Section, it will enough to consider the case of a finite exchange economy E.

Definition 3.2. *A non-empty coalition S_1 interim envies (in the sense of Varian) a coalition S_2 at the allocation a in the state ω, if there exists an allocation a' such that*

$$\begin{array}{lll}
(1) & \mu(S_1) \geq \mu(S_2) & \\
(2) & V_t(a'_t|P_t(\omega)) > V_t(a_t|P_t(\omega)), & \textit{for all } t \in S_1 \\
(3) & \displaystyle\sum_{t\in S_1} a'_t(\omega) \leq \sum_{t\in S_2} a_t(\omega), & \textit{for all } \omega \in \Omega.
\end{array}$$

The interpretation goes as follows: We say that a coalition S_1 interim envies another one with smaller or equal measure at an allocation a in a state ω, if each member of S_1 prefers, given his private information, the bundle he receives from the allocation a' and the commodity bundle received by S_2 would allow S_1 to achieve this improvement.

Definition 3.3. *An allocation a is interim c-fair (in the sense of Varian) for the economy E if it is interim efficient and if there does not exist a state at which some coalition interim envies at a another one.*

Notice that in the case of perfect information, the above definition reduces to the one given by Varian [14]. Moreover, we show in the next prpoposition that an interim c-fair allocation (in the sense of Varian) is (individual) interim fair.

Proposition 3.4. *Any interim c-fair allocation (in the sense of Varian) is interim fair.*

PROOF: Let a be an interim c-fair allocation and assume that a is not interim fair. Then, there exist two agents t and t' and a state of nature $\bar{\omega}$ such that t interim envies t' at $\bar{\omega}$, that is $V(u_t(a_{t'})|P_t(\bar{\omega})) > V(u_t(a_t)|P_t(\bar{\omega}))$.
Define $S_1 = \{t\}$, $S_2 = \{t'\}$ and

$$a'_r(\cdot) = \begin{cases} a_{t'}(\cdot) & \text{if } r = t \\ a_t(\cdot) & \text{if } r = t' \\ a_r(\cdot) & \text{if } r \in T \setminus \{t, t'\}. \end{cases}$$

Notice that

$$V(u_t(a'_t)|P_t(\bar{\omega})) = V(u_t(a_{t'})|P_t(\bar{\omega})) > V(u_t(a_t)|P_t(\bar{\omega}))$$

and for all ω in Ω

$$\sum_{r\in S_1} a'_r(\omega) = a'_t(\omega) = a_{t'}(\omega) = \sum_{r\in S_2} a_r(\omega).$$

This contradicts the assumption that a is an interim c-fair allocation. $\square$

Since the set of (individual) interim fair allocations may be empty (see [5, Example 1]), we can conclude that Varian's notion does not totally help to solve the conflict between efficiency and the absence of envy. The same problem occurs extending to asymmetric information economies the notion of coalitional fairness due to Gabszewicz [7].

Definition 3.5. *A non empty coalition S_1 interim envies (in the sense of Gabszewicz) a coalition S_2 at the allocation a in the state ω, if there exists an allocation a' such that*

$$
\begin{array}{ll}
(1) & S_1 \cap S_2 = \emptyset \\
(2) & V_t(a'_t|P_t(\omega)) > V_t(a_t|P_t(\omega)), \qquad \text{for all } t \in S_1 \\
(3) & \displaystyle\sum_{t\in S_1}(a'_t - e_t)(\omega) \leq \sum_{t\in S_2}(a_t - e_t)(\omega), \qquad \text{for all } \omega \in \Omega.
\end{array}
$$

Definition 3.6. *An allocation is interim c-fair (in the sense of Gabszewicz) for the economy E if there does not exist a state at which some coalition interim envies another one.*

Notice that this definition requires that interim c-fair allocations are interim efficient, since for $S_1 = T$ and $S_2 = \emptyset$ there is no way to rearrange an allocation in such a way that each agent is better off. Moreover it reduces to the one given in [7] in the case of perfect information. Assuming that agents have the same initial endowment in each state, we find that any interim c-fair allocation in the sense of Gabszewicz is interim fair (the argument of Proposition 3.4 works). Both notions of coalitional fairness in models with perfect information include competitive equilibria. More precisely, it was proved by [14, Theorem 4.1] that equal income competitive equilibria are c-fair in the sense of Varian and by [7, Proposition 1], that each competitive allocation is c-fair in the sense of Gabszewicz. This desirable property of c-fairness also fails to be true in the presence of asymmetric information. The example below exhibits the non existence of c-fair allocations in markets with asymmetric information as well as the fact that a constrained market equilibrium may not be c-fair.[4]

Example 3.7. Consider an economy composed by three agents, one good (money) and two different states of nature. More precisely, the future state may be low ($\omega = L$) or high ($\omega = H$) with equal probability. Agent 3 is able to distinguish between the two states, while agents 1 and 2 are not. So $\Pi_1 = \Pi_2 = \{\{L, H\}\}$ and $\Pi_3\{\{L\}, \{H\}\}$. Agent 1 is risk neutral, while agent 2 is risk averse. The total endowment is 1200 dollars when the state is low,

[4] Later we will prove that a constrained market equilibrium is c-type fair.

it is 1800 dollars when the state is high and in each state it is equal shared among the agents, i.e. $e_t(L) = 400$ and $e_t(H) = 600$ for each $t = 1, 2, 3$. The utility functions are $u_t(\cdot, x) = x$ for $t = 1, 3$ and $u_2(\cdot, x) = \sqrt{x}$.

Then, solving the maximization problems, we get that the allocation a defined as $a_1(L) = 300$, $a_2(L) = 500$, $a_3(L) = 400$, $a_1(H) = 700$, $a_2(H) = 500$ and $a_3(H) = 600$ is a constrained market equilibrium with p such that $p(L) = p(H)$.

Moreover, a is not a fair allocation, and therefore it is not c-fair (whatever definition is used). Indeed, imposing the condition for interim envy-freeness, we get that the only interim envy-free allocation is a' defined as $a'_t(L) = 400$ and $a'_t(H) = 600$ for each t. But a' is fair, since it is Pareto dominated by a'' such that $a''_t(L) = 301$, for $t = 1, 3$, $a''_2(L) = 498$, $a''_1(H) = 701$, $a''_2(H) = 498$ and $a''_3(H) = 601$. Notice that in both states L and H, there is an equal distribution of initial resources; thus even an equal income constrained market equilibrium may not be a c-fair allocation. Therefore the incompatibilities presented by [5] in the case of individual fairness, persist, when agents are asymmetrically informed, with reference to fairness notions of coalitional nature.

Following [5], a partial positive result can be obtained using the notion of common knowledge event. We recall that an event $\mathcal{E}$ is common knowledge if it can be written as a union of pairwise elements of Π_t for each $t \in T$.
The following Proposition extends Varian's positive results under the weaker condition of common knowledge.

Theorem 3.8. *There exists an interim efficient allocation such that it is impossible to find two coalitions S_1 and S_2 for which it is common knowledge that S_1 interim envies S_2 (in the sense of Varian).*

PROOF: As observed in [16] the set of constrained market equilibria is non-empty and every constrained market allocation is interim efficient (see [4]). Let a be a constrained market equilibrium with equal income. We mean that a is a constrained market equilibrium resulting from an equal sharing of the aggregate endowment in each state. More precisely, each agent in each state ω maximizes his interim expected utility function subject to his equal income budget set at ω.
Suppose that there exist a coalition S_1 of positive measure and a coalition S_2 for which it is common knowledge that S_1 interim envies S_2, that is there exists a common knowledge event $\mathcal{E}$ such that S_1 interim envies S_2 at each state $\omega \in \mathcal{E}$.
Thus, there exists a function $a' : T \times \Omega \to I\!R^\ell_+$ such that for all $\bar{\omega} \in \mathcal{E}$

$$
\begin{array}{ll}
(1) & \mu(S_1) \geq \mu(S_2) \\
(2) & V_t(a'_t|P_t(\bar{\omega})) > V_t(a_t|P_t(\bar{\omega})) \quad \text{for all } t \in S_1 \\
(3) & \displaystyle\sum_{t \in S_1} a'_t(\omega) \leq \sum_{t \in S_2} a_t(\omega) \quad \text{for all } \omega \in \Omega.
\end{array}
$$

Since a is a constrained market equilibrium with equal income, from (2) it follows that $a'_t(\omega) \notin \widetilde{B}_t(p,\omega)$ for all $\omega \in \mathcal{E}$ and for all $t \in S_1$. Thus, for all $t \in S_1$ and for all $t' \in S_2$,

$$\begin{aligned}
\sum_{\omega\in\mathcal{E}} p(\omega)\cdot a'_t(\omega) &= \sum_{P_t\in\Pi_t P_t\subseteq\mathcal{E}} \sum_{\omega\in P_t(\bar\omega)} p(\omega)\cdot a'_t(\omega) \\
> \sum_{P_t\in\Pi_t P_t\subseteq\mathcal{E}} \sum_{\omega\in P_t(\bar\omega)} p(\omega)\cdot \frac{e(\omega)}{\mu(T)} &= \sum_{\omega\in\mathcal{E}} p(\omega)\cdot \frac{e(\omega)}{\mu(T)} \\
= \sum_{P_{t'}\in\Pi_{t'} P_{t'}\subseteq\mathcal{E}} \sum_{\omega\in P_{t'}(\bar\omega)} p(\omega)\cdot \frac{e(\omega)}{\mu(T)} &\geq \sum_{P_{t'}\in\Pi_{t'} P_{t'}\subseteq\mathcal{E}} \sum_{\omega\in P_{t'}(\bar\omega)} p(\omega)\cdot a_{t'}(\omega) \\
&= \sum_{\omega\in\mathcal{E}} p(\omega)\cdot a_{t'}(\omega).
\end{aligned}$$

Thus, for all $t \in S_1$ and for all $t' \in S_2$,

$$\sum_{\omega\in\mathcal{E}} p(\omega)\cdot a'_t(\omega) > \sum_{\omega\in\mathcal{E}} p(\omega)\cdot a_{t'}(\omega).$$

Therefore, from (1), it follows that

$$\sum_{\omega\in\mathcal{E}} p(\omega)\cdot \left[\sum_{t\in S_1} a'_t(\omega) - \sum_{t'\in S_2} a_{t'}(\omega)\right] > 0.$$

This contradicts the condition (2). $\square$

The previous theorem can be reread in the following way: any constrained market equilibrium with equal income is (interim) efficient and such that it is impossible to find two coalitions S_1 and S_2, with $\mu(S_1) \geq \mu(S_2)$, for which it is common knowledge that S_1 interim envies S_2.

Contrary to coalitional notions introduced in this Section, we are going to prove that extending to fairness properties the blocking mechanism proposed by [4], the set of *c-type fair* allocations is non empty.

4 Main Results

4.1 Main Definitions

The notion of coalitional fairness that we introduce in this Section needs the extension to the mixed market framework of the blocking mechanism proposed by de Clippel for discrete economies with asymmetric information (see [4]).

In [4], objections to a given allocation of resources emerge from coalitions depending on the state of nature, as unblocked allocations are interpreted like sub game-perfect equilibrium outcomes of competitive screening games. Under this interpretation, uninformed intermediaries anticipate the coalitions of agents that are going to form to buy net trade vectors they propose. Since agent's decisions depend on their private information, coalitions that are going to be formed emerge endogenously as depending on the state of nature.

Let a be an allocation that one can see as a potential outcome and let a' be an alternative allocation. Denote by $D(a, a', \omega)$ the set of deviators should ω be the future state of nature of the economy. It is the set of agents that prefer, given their private information, to receive a' instead of keeping a:

$$D(a,\, a',\, \omega) = \{t \in T\, : V_t(a'_t | P_t(\omega)) > V_t(a_t | P_t(\omega))\}.$$

Notice that by measurability assumption of the mapping $(t, x) \mapsto u_t(\omega, x)$, the set of deviators is always measurable.

Definition 4.1. ([4]) *An allocation a is said to be blocked by a' in the type-agent sense, or, equivalently, a' is said to be strictly feasible when proposed against a, if for all $\omega \in \Omega$*

$$\int_{D(a,\, a',\, \omega)} a'_t(\omega)\, d\mu \leq \int_{D(a,\, a',\, \omega)} e_t(\omega)\, d\mu,$$

with a strictly inequality in at least one state of nature.

Definition 4.2. *An allocation a is said to belong to the type-agent core of E, denoted by $\mathcal{C}_{type}(E)$, if it is feasible and if it is not blocked in the type-agent sense.*

Requiring that the inequality contained in Definition 4.1 is strict in at least one state, it is ensured that in the state the set of deviators has positive measure. Hence the type-agent core does not coincides with the set of all feasible allocations. Moreover, it should be noticed that coalitions proposing an alternative allocation in the type blocking mechanism vary over the set of states of nature and for each state $\omega \in \Omega$ it collects the set of all deviators. The type-agent core coincides with the usual notion of core when there is no uncertainty.

We remark that the terminology adopted in Definition 4.1 and Definition 4.2 is justified by de Clippel's proposal of adapting the idea of Harsanyi. Harsanyi defines a Bayesian equilibrium to be any Nash equilibrium of the type-agent representation of the original Bayesian game (see [10]). Thus, as in [4], in the next Section we suggest a type-agent representation of the mixed economy with asymmetrically informed agents.

Let us now give main definitions. We adapt the veto mechanism proposed by [4] to the coalitional envy introduced in [7] for markets with perfect information. Notice that the adopted notion of coalitional fairness, differently from the one given by Varian, provides positive existence and convergence result even without assuming the equal distribution of initial endowment among traders.

Definition 4.3. *An allocation a is said to be blocked in the c-type fair sense by a' if for all $\omega \in \Omega$ there exists a coalition $S(a,\, a',\, \omega) \subseteq T$ such that*

$$(1) \qquad D(a,\, a',\, \omega) \cap S(a,\, a',\, \omega) = \emptyset$$

$$(2) \qquad \int_{D(a,\, a',\, \omega)} (a'_t(\omega) - e_t(\omega))\; d\mu \leq \int_{S(a,\, a',\, \omega)} (a_t(\omega) - e_t(\omega))\; d\mu,$$

with a strictly inequality in at least one state of nature.

We say that there is coalitional envy in state ω under the allocation a, if there exist an allocation a' and a coalition $S(a,\, a',\, \omega)$ for which the set of deviators $D(a,\, a',\, \omega)$ satisfies (1) *and* (2) *above.*

Notice that we do not require that $S(a,\, a',\, \omega)$ has positive measure.

The interpretation goes as follows: a is blocked in the c-type fair sense by a' if in each state ω the set of deviators can redistribute among its members the net trade of another disjoint coalition $S(a, a', \omega)$. In this case, we say that for each state ω, the deviators $t \in D(a, a', \omega)$ envy the net trade of coalition $S(a, a', \omega)$.

Definition 4.4. *An allocation a is c-type fair if it is feasible and it is not blocked in the c-type fair sense. We denote by $\mathcal{C}^{fair}_{type}(E)$ the set of c-type fair allocations for the economy E.*

An allocation is qualified *c-type fair* if there is zero probability of a set of deviators interim envying the net trade of any other coalition. Under such distribution of resources, there does not exist an alternative allocation and a state at which potentially deviators are treated in a discriminatory way by the market. According to the c-type fair criterion, agents in a potentially envious coalition make comparisons given their private information and not after the observation of the state.

It is easy to show that any c-type fair allocation is interim efficient. We just need to put for all ω, the set of deviators equal to the whole set of agents and the disjoint coalition equal to the empty set.

4.2 An auxiliary economy associated to a mixed market

In order to prove the existence result and the characterizations of c-type fair allocations we need to introduce a fictitious economy related to E. More precisely, we consider a type-agent representation of the economy described in the previous section. It is a fictitious exchange economy with uncertainty and symmetric information modeled by the following collection:

$$E^* = \left\{(\Omega, \mathcal{F}, \pi);\; (T^*, \mathcal{T}^*, \mu^*);\; \mathbb{R}^\ell_+;\; (u_{(t,\mathcal{E})}, e_{(t,\mathcal{E})})_{(t,\mathcal{E}) \in T \times \mathcal{F}}\right\}$$

where:

1. $(\Omega, \mathcal{F}, \pi)$ the measurable space describing uncertainty has been defined before.

2. T^* is the set of the type agents. More precisely, T^* coincides with the graph of the correspondence $\Pi : T \to 2^{\mathcal{F}}$ defined by $\Pi(t) = \Pi_t$, i.e. T^* is the set of couple $(t, \mathcal{E})$, where t is an agent and $\mathcal{E}$ is an atom of his information partition.
   $\mathcal{T}^*$ is the family of coalitions: a coalition is a measurable subset S^* of T^*, i.e. a subset S^* of T^* that belongs to $\mathcal{T} \otimes \mathcal{B}(2^{\mathcal{F}})$, where $\mathcal{B}(2^{\mathcal{F}})$ denotes the Borel σ-algebra on $2^{\mathcal{F}}$ and $\otimes$ denotes the product σ-algebra. Finally, the measure μ^* on $\mathcal{T}^*$ is defined by $\mu^*(S^*) = \mu(Proj_T S^*)$[5].

3. The couple $(u_{(t,\mathcal{E})}, e_{(t,\mathcal{E})})$ characterizes the type-agent $(t, \mathcal{E})$. That is, given a type-agent $(t, \mathcal{E})$, $u_{(t,\mathcal{E})}$ describes his state-dependent utility function, while $e_{(t,\mathcal{E})}$ represents his initial endowment of physical resources. Formally, the couple $(u_{(t,\mathcal{E})}, e_{(t,\mathcal{E})})$ is defined as follows

$$e_{(t,\mathcal{E})}(\omega) = \begin{cases} e_t(\omega) & \text{if } \omega \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

$$u_{(t,\mathcal{E})}(\omega, \alpha_{(t,\mathcal{E})}(\omega)) = \begin{cases} u_t(\omega, a_t(\omega)) & \text{if } \omega \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

The type-agents decide today about the way to redistribute their endowments when the state will be common knowledge. The presence of uncertainty in this economy, as well as the possibility of writing contracts that are contingent on the future state of the economy, allows us to apply the standard notion of competitive, core and c-fair allocations. We shall rewrite, for reader convenience, the main equilibrium notions in the economy E^*.

An *allocation* in the fictitious economy is a function $\alpha : \Omega \times T^* \to I\!R^\ell_+$ and, for each $(t, \mathcal{E}) \in T^*$, $\alpha_{(t,\mathcal{E})} : \Omega \to I\!R^\ell_+$ represents the bundle that the type-agent $(t, \mathcal{E})$ receives under the allocation α.
An allocation α is feasible for the coalition S^* if

$$\int_{S^*} \alpha_{(t,\mathcal{E})}(\omega) d\mu^* \leq \int_{S^*} e_{(t,\mathcal{E})}(\omega) d\mu^* \qquad \text{for all } \omega \in \Omega.$$

Definition 4.5. *Let S^* be a coalition with positive measure, i.e. $\mu^*(S^*) > 0$. An allocation α' Pareto dominates α for S^* if for almost all $(t, \mathcal{E}) \in S^*$*

$$V_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})}) > V_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}).$$

[5]Since the measure space of agents is assumed to be finite and complete, the measurability of the projection $Proj_T S^*$ follows by the Projection Theorem (see [1, Theorem 14.84])

Definition 4.6. *The core of the economy E^* is the set of feasible allocations for T^* such that there do not exist a coalition S^* and an allocation α' feasible for S^* that Pareto dominates α for S^*.*

Definition 4.7. *An allocation α is c-fair blocked by an assignment α' if there exist two disjoint coalitions S_1^* and S_2^*, such that*

$$\begin{array}{ll} (1) & \mu^*(S_1^*) > 0 \\ (2) & \displaystyle\sum_{\omega\in\Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})}(\omega),\omega) > \sum_{\omega\in\Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}(\omega),\omega) \quad \textit{for a. a. } (t,\mathcal{E}) \in S_1^* \\ (3) & \displaystyle\int_{S_1^*} (\alpha'_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega))\, d\mu^* \leq \int_{S_2^*} (\alpha_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega))\, d\mu^* \quad \textit{for all } \omega \in \Omega. \end{array}$$

Definition 4.8. *An allocation α is c-fair for E^* if it is feasible and it is not c-fair blocked by any other allocation.*

Definition 4.9. *An allocation α is an Arrow-Debreu equilibrium in the type-agent representation of the economy E^* if it is feasible for T^* and there exists a price system $p : \Omega \to I\!R_+^\ell$ such that*

$$\alpha_{(t,\mathcal{E})} \in \arg \max_{\alpha'_{(t,\mathcal{E})\in B_{(t,\mathcal{E})}(p)}} V_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})})$$

where for all $(t,\mathcal{E}) \in T^*$

$$B_{(t,\mathcal{E})}(p) = \left\{ \alpha'_{(t,\mathcal{E})} \in I\!R_+^{\ell k} | \sum_{\omega\in\Omega} p(\omega) \cdot \alpha'_{(t,\mathcal{E})}(\omega) \leq \sum_{\omega\in\Omega} p(\omega) \cdot e_{(t,\mathcal{E})}(\omega) \right\}.$$

5 One to one correspondence between E and E^*

In this section we construct a natural isomorphism between equilibrium allocations of the economy E and of the type-agent representation E^*. We observe that the following one-to-one correspondence is an extension to the general framework of mixed markets of the isomorphism used in de Clippel (2007).

Given an allocation $a \in E$, its type-agent representation is the allocation α such that for each $(t,\mathcal{E})$ in T^*

$$\alpha(t,\mathcal{E})(\omega) = a_t(\omega)\chi_{\mathcal{E}}(\omega).$$

Given an allocation $\alpha \in E^*$, its associated allocation a in the original economy E is such that for each t in T and each ω in Ω

$$a_t(\omega) = \alpha_{(t,P_t(\omega))}(\omega).$$

What we prove in the next subsections is the invariance, under the above defined correspondence, of the main allocations properties we are investigating.

5.1 Feasibility invariance

Proposition 5.1. *If a is a feasible allocation for E, then the corresponding allocation α is feasible.*
Conversely, if α is a feasible allocation for E^*, then the corresponding allocation a is feasible for E.

PROOF: Trivially,

$$\int_T a_t(\bar{\omega})\, d\,\mu \leq \int_T e_t(\bar{\omega})\, d\,\mu$$

is equivalent to

$$\int_T \alpha_{(t,P_t(\bar{\omega}))}(\bar{\omega})\, d\,\mu \leq \int_T e_{(t,P_t(\bar{\omega}))}(\bar{\omega})\, d\,\mu$$

which, in turn, it is equivalent to

$$\int_{T^*} \alpha_{(t,P_t(\bar{\omega}))}(\bar{\omega})\, d\,\mu^* \leq \int_{T^*} e_{(t,P_t(\bar{\omega}))}(\bar{\omega})\, d\,\mu^*$$

that is

$$\int_{T^*} \alpha_{(t,\mathcal{E})}(\bar{\omega})\, d\,\mu^* \leq \int_{T^*} e_{(t,\mathcal{E})}(\bar{\omega})\, d\,\mu^*.$$

$\square$

5.2 Core correspondence

Theorem 5.2. *If an allocation a belongs to the type-agent core of the economy E, then the corresponding allocation α is in the core of the economy E^*.*
Conversely, if an allocation α belongs to the core of E^*, then the associated allocation a is a type-core allocation for E.

PROOF: Let a be a type-agent core allocation for E, then it is feasible as the corresponding allocation α in E^* (see Proposition 5.1). Assume contrary to the statement that α is not in the core of E^*. This means that there exist an assignment $\alpha^{'}$ and a coalition S^*, with $\mu^*(S^*) > 0$, such that for almost all $(t, \mathcal{E}) \in S^*$

$$(1)\quad \sum_{\omega\in\Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha^{'}_{(t,\mathcal{E})}(\omega), \omega) > \sum_{\omega\in\Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}(\omega), \omega),\ \text{for a.a.}\ (t,\mathcal{E}) \in S^*$$

and

$$(2)\quad \int_{S^*} \alpha'_{(t,\mathcal{E})}(\omega)\, d\mu^* \leq \int_{S^*} e_{(t,\mathcal{E})}(\omega)\, d\mu^*,\ \text{for all}\ \omega \in \Omega.$$

Without loss of generality we can assume that, for all $(t, \mathcal{E}) \in T^*$, $\alpha'_{(t,\mathcal{E})}(\omega) = 0$ for all $\omega \notin \mathcal{E}$.

Let then a' be the allocation rule defined as follows: for all $t \in T$ and for all $\omega \in \Omega$,

$$a'_t(\omega) = \begin{cases} \alpha'_{(t,P_t(\omega))}(\omega) & \text{if } (t, P_t(\omega)) \in S^* \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for all $\omega \in \Omega$,

$$t \in D(a,\, a',\, \omega) \text{ if and only if } (t, P_t(\omega)) \in S^*.$$

Hence for each ω in Ω

$$\int_{D(a,\, a',\, \omega)} a'_t(\omega) d\,\mu = \int_{D(a,\, a',\, \omega)} a'_{(t,P_t(\omega))}(\omega)\, d\,\mu = \int_{S^*} \alpha'_{(t,P_t(\omega))}(\omega)\, d\,\mu^* \leq$$

$$\leq \int_{S^*} e_{(t,P_t(\omega))}(\omega)\, d\,\mu^* = \int_{D(a,\, a',\, \omega)} e_{(t,P_t(\omega))}(\omega)\, d\,\mu = \int_{D(a,\, a',\, \omega)} e_t(\omega)\, d\,\mu.$$

We can observe that if a' is not strictly feasible when proposed against a, then one can slightly modify a as follows.

From continuity of utility function there exist a state $\bar{\omega}$, $\varepsilon \in (0, 1)$ and $S(a, a', \bar{\omega}) \subseteq D(a, a', \bar{\omega})$ such that for almost all $t \in S(a, a', \bar{\omega})$,

$$V_t(\varepsilon a'_t | P_t(\bar{\omega})) > V_t(a_t | P_t(\bar{\omega})).$$

Let us consider the allocation a'' defined by the following law

$$a''_t(\omega) = \begin{cases} \varepsilon a'_t(\bar{\omega}) & \text{if } \omega = \bar{\omega} \text{ and } t \in S(a, a', \bar{\omega}) \\ a'_t(\omega) & \text{otherwise.} \end{cases}$$

Then it is easy to see that for each ω in Ω

$$D(a,\, a'',\, \omega) = D(a,\, a',\, \omega).$$

Thus, for every ω,

$$\int_{D(a,\, a'',\, \omega)} a''_t(\omega)\, d\mu \leq \int_{D(a,\, a'',\, \omega)} e_t(\omega)\, d\mu.$$

Moreover,

$$\int_{D(a,\, a''\,, \bar{\omega})} a''_t(\bar{\omega})\, d\mu = \int_{D(a,\, a',\, \bar{\omega})} a''_t(\bar{\omega})\, d\mu =$$

$$= \int_{D(a,\, a',\, \bar{\omega}) \setminus S(a,\, a',\, \bar{\omega})} a'_t(\bar{\omega})\, d\mu + \int_{S(a,\, a',\, \bar{\omega})} \varepsilon a'_t(\bar{\omega})\, d\mu <$$

$$< \int_{D(a,\, a',\, \bar{\omega})} a'_t(\bar{\omega})\, d\mu \leq \int_{D(a,\, a',\, \bar{\omega})} e_t(\bar{\omega})\, d\mu = \int_{D(a,\, a'',\, \bar{\omega})} e_t(\bar{\omega})\, d\mu.$$

This contradicts the assumption that a is a type-agent core allocation.

Conversely, let α be a core allocation for E^*, then it is feasible as the corresponding allocation a (see Proposition 5.1). Suppose that a is not in the type-agent core. This means that there exists an assignment a' such that for all $\omega \in \Omega$,

$$\int_{D(a,\, a',\, \omega)} a'_t(\omega)\, d\,\mu \leq \int_{D(a,\, a',\, \omega)} e_t(\omega)\, d\,\mu,$$

with a strictly inequality in at least one state of nature. Hence there exists at least one state such that the corresponding set of deviators has positive measure.
Define the subset S^* of T^* as follows

$$S^* = \{(t, \mathcal{E}) \in T^* \text{such that } \; V_t(a'_t|\mathcal{E}) > V_t(a_t|\mathcal{E})\}$$

and observe that

$$S^* = \bigcup_{\omega \in \Omega} \bigcup_{i=1}^{N} \big(D(a,\, a',\, \omega) \cap T_i\big) \times \{\Pi_i(\omega)\}$$

hence S^* is a coalition with positive measure. Moreover, for each state $\bar{\omega}$,

$$(t, P_t(\bar{\omega})) \in S^* \text{ if and only if } t \in D(a, a', \bar{\omega}).$$

Let α' be an allocation such that for all $(t, \mathcal{E}) \in T^*$ and for all $\omega \in \Omega$

$$\alpha'_{(t,\mathcal{E})}(\omega) = \begin{cases} a'_t(\omega) & \text{if } \omega \in \mathcal{E} \text{ and } (t, \mathcal{E}) \in S^* \\ 0 & \text{otherwise} \end{cases}$$

Thus, for almost all $(t, \mathcal{E}) \in S^*$

$$\sum_{\omega \in \Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})}(\omega), \omega) > \sum_{\omega \in \Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}(\omega), \omega)$$

and for each ω

$$\int_{S^*} \alpha'_{(t,\mathcal{E})}(\omega)\, d\,\mu^* = \int_{D(a,\, a',\, \omega)} a'_t(\omega)\, d\,\mu \leq \int_{D(a,\, a',\, \omega)} e_t(\omega)\, d\,\mu = \int_{S^*} e_{(t,\mathcal{E})}(\omega)\, d\,\mu^*.$$

This contradicts the assumption that α is a core allocation for E^*. $\square$

5.3 Type fairness correspondence

Theorem 5.3. *If a is a c-type fair allocation for E, then the corresponding allocation α is c-fair for E^**
Conversely, if α is a c-fair allocation for E^*, then the corresponding allocation a is c-type fair for E.

PROOF: Let a be a c-type fair allocation for E and assume on the contrary that the corresponding allocation $\alpha \in E^*$ is not c-fair. This means that there exist an assignment α' and two disjoint coalitions S_1^* and S_2^* such that

(1) $\mu^*(S_1^*) > 0$

(2) $\sum_{\omega\in\Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})}(\omega), \omega) > \sum_{\omega\in\Omega} \pi(\omega) u_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}(\omega), \omega)$ for almost all $(t, \mathcal{E}) \in S_1^*$

(3) $\int_{S_1^*} (\alpha'_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega)) \; d\mu^* \leq \int_{S_2^*} (\alpha_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega)) \; d\mu^*$ for all $\omega \in \Omega$.

Notice that without loss of generality we can assume that for all $(t, \mathcal{E}) \in T^*$, $\alpha'_{(t,\mathcal{E})}(\omega) = 0$ for all $\omega \notin \mathcal{E}$.
Let us consider the allocation a' such that

$$a'_t(\omega) = \begin{cases} \alpha'_{(t,P_t(\omega))}(\omega) & \text{if } (t, P_t(\omega)) \in S_1^* \\ 0 & \text{otherwise.} \end{cases}$$

We can notice that for each $\omega \in \Omega$,

$$t \in D(a,\, a',\, \omega) \;\text{ if and only if }\; (t, P_t(\omega)) \in S_1^*.$$

Define for each fixed state $\omega \in \Omega$, the set

$$S_2(a,\, a',\, \omega) = \{t \in T \;:\; (t, P_t(\omega)) \in S_2^*\}.$$

Then $S_2(a,\, a',\, \omega)$ is a measurable subset of T, since it coincides with the projection over T of the measurable subset of T^* defined by $S_2^* \cap \{(t, \Pi_t(\omega) : t \in T\}$. Moreover, for all $\omega \in \Omega$, $S_2(a,\, a',\, \omega) \cap D(a,\, a',\, \omega) = \emptyset$. Then we get

$$\int_{D(a,\, a',\, \omega)} (a'_t(\omega) - e_t(\omega)) \; d\mu = \int_{S_1^*} (\alpha'_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega)) \; d\mu^* \leq$$

$$\leq \int_{S_2^*} (\alpha_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega)) \; d\mu^* = \int_{S_2(a,\, a',\, \omega)} (a_t(\omega) - e_t(\omega)) \; d\mu.$$

As in Theorem 5.2, we observe that if a' is such that there is no ω for which the previous inequality is strict we can consider a new allocation a''.

From continuity of utility function there exist a state $\bar{\omega}$, $\varepsilon \in (0,1)$ and $S_1(a,a',\bar{\omega}) \subseteq D(a,a',\bar{\omega})$ such that for almost all $t \in S_1(a,a',\bar{\omega})$,

$$V_t(\varepsilon a'_t|P_t(\bar{\omega})) > V_t(a_t|P_t(\bar{\omega})).$$

Let us consider the allocation a''

$$a''_t(\omega) = \begin{cases} \varepsilon a'_t(\omega) & \text{if } t \in S_1(a,a',\bar{\omega}) \text{ and } \omega = \bar{\omega} \\ a'_t(\omega) & \text{otherwise} \end{cases}$$

Then, for each ω, $D(a,\, a'',\, \omega) = D(a,\, a',\, \omega)$ and

$$\int_{D(a,\, a'',\, \omega)} (a''_t(\omega) - e_t(\omega))\, d\,\mu \leq \int_{S_2(a,\, a'',\, \omega)} (a_t(\omega) - e_t(\omega))\, d\,\mu,$$

where we can define, for each ω, $S_2(a,\, a'',\, \omega) = S_2(a,\, a',\, \omega)$.

Moreover,

$$\int_{D(a,\, a'',\, \bar{\omega})} (a''_t(\bar{\omega}) - e_t(\bar{\omega}))\ d\mu = \int_{D(a,\, a',\, \bar{\omega})} (a''_t(\bar{\omega}) - e_t(\bar{\omega}))\ d\mu =$$

$$= \int_{D(a,\, a',\, \bar{\omega})\setminus S_1(a,\, a',\, \bar{\omega})} (a'_t(\bar{\omega}) - e_t(\bar{\omega}))\ d\mu + \int_{S_1(a,\, a',\, \bar{\omega})} (\varepsilon a'_t(\bar{\omega}) - e_t(\bar{\omega}))\ d\mu <$$

$$< \int_{D(a,\, a',\, \bar{\omega})} (a'_t(\bar{\omega}) - e_t(\bar{\omega}))\ d\mu \leq \int_{S_2(a,\, a',\, \bar{\omega})} (a_t(\bar{\omega}) - e_t(\bar{\omega}))\ d\mu = \int_{S_2(a,\, a'',\, \bar{\omega})} (a_t(\bar{\omega}) - e_t(\bar{\omega}))\ d\mu.$$

This contradicts the assumption that a is c-type fair.

We are now ready to prove the converse. Let α be a c-fair allocation for E^* and assume on the contrary that the corresponding allocation a is not c-type fair. This means that there exists an assignment a' such that for all $\omega \in \Omega$, there exists a coalition $S_2(a,\, a',\, \omega)$ for which

(1) $D(a,\, a',\, \omega) \cap S_2(a,\, a',\, \omega) = \emptyset$

(2) $\displaystyle\int_{D(a,\, a',\, \omega)} (a'_t(\omega) - e_t(\omega))\, d\mu \leq \int_{S_2(a,\, a',\, \omega)} (a_t(\omega) - e_t(\omega))\, d\mu,$
with a strictly inequality in at least one state of nature.

Consider the following subsets of T^*

$$S_1^* = \{(t, \mathcal{E}) \in T^* \text{ such that } V_t(a'_t|\mathcal{E}) > V_t(a_t|\mathcal{E})\},$$

$$S_2^* = \{(t, \mathcal{E}) \in T^* \text{ such that } t \in S_2(a,\, a',\, \bar{\omega}) \text{ with } \bar{\omega} \in \mathcal{E}\}.$$

Hence we have that

$$S_1^* = \bigcup_{\omega\in\Omega} \bigcup_{i=1}^{N} \big(D(a,\, a',\, \omega) \cap T_i\big) \times \{\Pi_i(\omega)\}$$

and

$$S_2^* = \bigcup_{\omega\in\Omega}\bigcup_{i=1}^{N}\big(S_2(a,\,a',\,\omega)\cap T_i\big)\times\{\Pi_i(\omega)\}.$$

Hence S_1^* and S_2^* are measurable, $\mu^*(S_1^*) > 0$, while S_2^* may also be the empty coalition. Moreover, we can observe that S_1^* and S_2^* are disjoint and, given a state ω, $(t, P_t(\omega)) \in S_1^*$ if and only if $t \in D(a,\, a',\, \omega)$.
Let us define a new allocation α' as follows: for each $(t,\mathcal{E}) \in T^*$ and for all $\omega \in \Omega$

$$\alpha'_{(t,\mathcal{E})}(\omega) = \begin{cases} a'_t(\omega) & \text{if } \omega \in \mathcal{E} \text{ and } (t,\mathcal{E}) \in S_1^* \\ 0 & \text{otherwise} \end{cases}$$

Thus, for each $(t,\mathcal{E}) \in S_1^*$

$$\sum_{\omega\in\Omega}\pi(\omega)u_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})}(\omega),\omega) > \sum_{\omega\in\Omega}\pi(\omega)u_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}(\omega),\omega)$$

and for each ω in Ω

$$\int_{S_1^*}(\alpha'_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega))\,d\mu^* = \int_{D(a,\,a',\,\omega)}(a'_t(\omega) - e_t(\omega))\;d\mu \le$$

$$\le \int_{S_2(a,\,a',\,\omega)}(a_t(\omega) - e_t(\omega))\;d\mu = \int_{S_2^*}(\alpha_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega))\;d\mu^*.$$

This contradicts the assumption that α is a c-fair allocation for E^*. □

5.4 Equilibria correspondence

Theorem 5.4. *If a is a constrained market equilibrium for E, then the associated allocation α is an Arrow Debreu equilibrium for E^*.
Conversely, if α is an Arrow-Debreu equilibrium for E^*, then the associated allocation a is a constrained market equilibrium for E.*

PROOF: Let a be a constrained market equilibrium. Then α is feasible.
Let p be the price system supporting a as constrained market equilibrium.
Let $(t,\mathcal{E}) \in T\times\mathcal{F}$. Since $a_t \in B_t(p,\omega)$ for each ω in $\mathcal{E}$, then $\alpha_{(t,\mathcal{E})} \in B_{(t,\mathcal{E})}(p)$.
Let $\alpha'_{(t,\mathcal{E})} \in B_{(t,\mathcal{E})}(p)$. Given the preferences of type-agent $(t,\mathcal{E})$, we may assume that $\alpha'_{(t,\mathcal{E})}(\omega) = 0$ for each ω in $\Omega\setminus\mathcal{E}$. Let $\omega \in \mathcal{E}$ and let a'_t be defined as follows $a'_t(\bar\omega) = \alpha'_{(t,P_t(\omega))}(\bar\omega)$ for each $\bar\omega \in \Omega$. Then $a'_t \in B_t(p,\omega)$ and hence $V_t(a'_t|P_t(\omega)) \leqslant V_t(a_t|P_t(\omega))$. Hence, it is easy to conclude that $V_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})}) \leqslant V_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})})$.

Conversely, let α be a Arrow Debreu equilibrium. Then a is feasible. Let p be the price system supporting α as Arrow Debreu equilibrium. Let $t \in T$ and let $\omega \in \Omega$. Since $\alpha_{(t,P_t(\omega))} \in B_{(t,P_t(\omega))}(p)$, then $a_t \in B_t(p,\omega)$.

Let $a'_t \in B_t(p,\omega)$ and $\alpha'_{(t,P_t(\omega))}$ defined as $\alpha'_{(t,P_t(\omega))}(\bar{\omega}) = a'_t(\bar{\omega})$ for each $\bar{\omega} \in P_t(\omega)$ and $\alpha'_{(t,P_t(\omega))}(\bar{\omega}) = 0$ for each $\bar{\omega} \in \Omega \setminus P_t(\omega)$. Then $\alpha'_{(t,P_t(\omega))} \in B_{(t,P_t(\omega))}(p)$. Hence $V_t(a'_t|P_t(\omega)) \leqslant V_t(a_t|P_t(\omega))$. □

Remark 5.5. It should be noted that Theorem 5.4 implies the existence of constrained market equilibria in the general case of mixed markets. Indeed, competitive equilibria of mixed markets are in a one-to-one correspondence with equilibria of suitable atomless economies (see [13] for the case of asymmetric information models). Hence the existence of Arrow-Debreu equilibria of atomless economies joint with Theorem 5.4 implies that a constrained market equilibrium always exists even in mixed markets with asymmetric information. This result also guarantees that Theorem 3.8 can be extended to the case of atomic economies.

5.5 Main results

The objective of this final Section is to study relations among type-allocations and, in particular, the relation between constrained market equilibria and c-type fair allocations. We will show that in a mixed market in which agents are asymmetrically informed at the time of contracting, the set of constrained market allocations is contained in the set of c-type fair allocations which itself is contained in the type-core. These inclusions may be strict. This is the case of economies in which, despite of the existence of an atomless sector, the type-core does not converge to the set of constrained market equilibria.

Theorem 5.6. *Any c-type fair allocation is in the type-agent core, i.e.,*

$$C^{fair}_{type} \subseteq C_{type}.$$

PROOF: Let a be an allocation which is blocked in the type-core sense by an assignment a'. Putting for all $\omega \in \Omega$, the coalition $S(a, a', \omega)$ equal to the empty set, we get the conclusion. □

The correspondence between the economy E and its type representation implies the following result.

Theorem 5.7. *Any constrained market equilibrium is a c-type fair allocation.*

PROOF: Let a be a constrained market equilibrium, then the associated allocation α is an Arrow-Debreu equilibrium of the economy E^*. Since any Arrow-Debreu equilibrium is a c-fair allocation (see [7, Prposition 1], then

α is c-fair for the economy E^* and hence, from Theorem 5.3, it follows that a is c-type fair for E. □

Theorem 5.8. *Suppose that almost all agents are endowed with a strictly positive amount of each good in each state of the economy. Then the set of c-type fair allocations is non-empty.*

PROOF: It is a direct consequence of Theorem 5.7 and the existence of an Arrow-Debreu equilibrium (see Remark 5.5). □

Theorem 5.8 proves that assuming, contrary to the analysis of Section 3.3, that objections are not bound to emerge from state-independent coalitions, then the notion of coalitional fairness is not vacuous. Coalitional fairness solves the tension that might be between efficiency and the absence of envy that is proved by non existence result due to [5]. First we identify mixed markets as a natural framework to study fair resource distributions. Indeed, such models embody a large number of traders some of which are endowed with an exceptional initial bundle. In this setting, coalitions, rather than individuals, are the primitive entities that are naturally called to compare discriminatory distribution of resources. The idea of replacing potentially envious individuals by coalitions allows us to assume, following the veto mechanism proposed by [4], that coalitions are endogenously determined by comparing different allocation rules.

Remark 5.9. The relation contained in Theorem 5.6 would allow us to state a result similar to [4, Theorem 2]: The set of c-type fair allocations is a subset of the coarse core defined by Wilson (see [16]). This result is not surprising even though we assume no communication and information transmission, as the veto mechanism supporting fairness criterion can be interpreted via competitive screening games.

In the mixed market framework, the set of c-type fair allocations and the type-core may differ. Indeed, in a model with just one state of nature, the type core coincides with the classical core, c-type fair allocations coincide with c-fair allocations. Thus we can use the example given by Gabszewicz ([7, Proposition 2]) to say that there may exists an allocation that is in the type core, which is not c-type fair. This allocation discriminates between coalitions in the class $\mathcal{T}$[6]. The same example shows that c-type fair allocations may differ from the set of constrained market equilibria when the set of atoms is non-empty (and in particular, in the case of finite economies).

[6] In the case of mixed market with perfect information, even though c-fairness cannot be extended to the whole class $\mathcal{T}$ of coalitions, it can be proved for smaller classes. This is possible, as proved by [7, Theorem 2], when any allocation in the core is a restricted competitive equilibrium. In this case, one obtains at least that core allocations do not discriminate between a coalition made by small traders and a coalition containing all large traders and vice versa (see also [9]).

Differently, in atomless economies, applying the Core-Walras equivalence theorem due to Aumann in the type agent representation, we get a characterization of c-fair allocations in terms of type core and constrained market equilibria.

Theorem 5.10. *Let E be an atomless economy. Suppose that almost all agents are endowed with a strictly positive amount of each good in each state of the economy. Then the set of constrained market equilibria and the set of c-type fair allocations coincide.*

PROOF: It has been already proved that any constrained market equilibrium is c-type fair and that c-type fair allocations are in the type core. Let a be a type core allocation for E and assume that a cannot be supported as a constrained market equilibrium. Therefore, by Theorem 5.4, the associated allocation α is not an Arrow Debreu equilibrium for E^*. The Core-Walras equivalence Theorem for atomless economies (see [2]) guarantees that α is not a core allocation and hence it is not a type core allocation (recall Theorem 5.2). □

As a consequence of the above theorem, we can conclude that, whenever T_1 is empty, under the assumptions which guarantee the Core-Walras Equivalence Theorem, the type core coincides with the set of c-type fair allocations. This conclusion confirms the idea that mixed markets are the most natural framework to study the notion of coalitional fairness.

On the other hand, whereas E is a finite economy, if we replicate the economy E infinitely many times, we get again the equivalence between coalitional fairness and the set of constrained market equilibria, as the following theorem illustrates.

The economy E with asymmetric information can be replicated as in [4]. For each agent t, the agent (t, k) in the r-replica of E for each $k = 1, \ldots r$, has the same information and the same utility function of agent t. Observe also that the type-agent representation of each replica of the economy E coincides with the Debreu-Scarf replication of the type-agent representation E^* of E.

Theorem 5.11. *Suppose that each agent's utility function is strictly concave in each state of the economy and that each agent is endowed with a strictly positive amount of each good in each state of the economy. Then the set of c-type fair allocations shrinks to the set of constrained market equilibria as the number of replicas tends to infinity.*

PROOF: The proof follows fron [4, Theorem 6] □

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