When Is Expenditure "Exogenous" in Separable Demand Models?

Jeffrey T. LaFrance

The separability hypothesis and expenditure as an exogenous variable in a system of conditional demands are analyzed. Expenditure cannot be weakly exogenous in a system of conditional demands specified as functions of the prices of the separable goods and total expenditure on those goods. Furthermore, expenditure is uncorrelated with the residuals of the conditional demand equations only when severe restrictions are satisfied. Therefore, expenditure will seldom be strictly exogenous. Econometric methods are presented for the consistent and efficient estimation of the unknown parameters when expenditure is correlated with the residuals and when it is not.

Key words: conditional demand models, exogeneity, weak separability.

In applied demand analysis, incomplete information is the rule and not the exception. We are always concerned with a subset of the total number of commodities that are purchased by consumers. Data limitations, finite computer memory, and the increased complexity and time required for numerical computations in large models make it necessary to abstract from a completely specified system of consumer demands containing a different equation for each of the countless goods available in the market.

Three practical solutions have been proposed to deal with this problem. One approach is to aggregate across commodities and estimate a complete system of demand equations with the commodity aggregates (e.g., food, clothing, housing, transportation, entertainment, and all other goods) as functions of the corresponding set of aggregated price indices and income. This approach has at least two drawbacks. First, the conditions are quite restrictive for consistency of consumer preferences with such a high degree of price and quantity aggregation. Second, considerable information is lost concerning the demands for individual commodities.

A second approach specifies an incomplete system of demand equations as functions of the prices of the goods of interest, the prices of related goods, and income. This approach has been criticized for being "ad hoc" and ignoring the issue of consistency with the underlying theory (Richardson). However, LaFrance and Hanemann demonstrate that coherent applications of this approach can be consistent with consumer choice theory with little loss in generality.

A third approach is the focus of this article. A common empirical practice is to assume that consumer preferences are separable and estimate a set of conditional demands (Pollak 1969) for the goods of interest as functions of this subset of prices and total expenditure on these goods (expenditure, for short).[1] This ap-

The author is a professor in the Department of Agricultural Economics at the University of Arizona. This research was completed while he was an associate professor of agricultural economics in the Department of Agricultural Economics and Economics at Montana State University.

Contribution of the Montana Agricultural Experiment Station, Journal Series No. J-2540. A previous version of this paper was presented at the Western Agricultural Economics Association Meetings in Vancouver, British Columbia, in August 1990.

The comments and suggestions of Bruce Beattie and three anonymous reviewers are greatly appreciated. The author is responsible for any errors that remain.

[1] Throughout the article, separability means asymmetric weak separability—one group of goods is weakly separable from all other goods, but the latter group of goods is not necessarily separable from the former. This eliminates the distinctions among the concepts of weak, strong, strict, and complete separability that arise when the preference ordering is symmetrically separable in two or more sets of goods. Any results that hold for asymmetric weak separability remain true in the more restrictive cases of symmetric separability.

For static models, income here means Becker's full income—nonlabor income plus the labor income that could be earned if all

Western Journal of Agricultural Economics, 16(1): 49–62

proach is based on the fact that weak separability of a subset of goods from all other goods in the consumer's preference relation is necessary and sufficient for the existence of the conditional demand equations (Primont; Gorman 1971; Blackorby, Primont, and Russell).

In this article an important aspect of the separability hypothesis in applied demand is addressed, namely, the possibility of simultaneous equations bias in conditional demand models. Due to the joint determination of the quantities demanded and expenditure on a group of goods, biased parameter estimates will be obtained unless either the joint distribution of the error terms, the functional form for the conditional demands, or both are restricted. The first focus is to identify conditions that ensure that expenditure is uncorrelated with the error terms in the conditional demands. These conditions are important because they are necessary for standard estimation methods to result in consistent and efficient parameter estimates.

One such condition is a singular error covariance matrix for the subset of unconditional demands—demands specified as functions of all prices and income. That is, the unconditional covariance matrix transforms the prices of the goods of interest to the zero vector. But unlike complete systems of demands, this property is not implied by the adding-up condition. The budget identity for unconditional demand functions states that the total expenditure on all goods adds up identically to income. This implies only one source of singularity in the complete system of unconditional demand equations—the price-weighted sum of all unconditional residuals must equal zero. In particular, it does not imply that the price-weighted sum of the unconditional demand residuals for the subset of goods of interest, whether or not those goods are separable from other goods, must equal zero. Furthermore, if the unconditional covariance matrix for the separable goods is singular, then both total expenditure on those goods and total expenditure on all other goods are not random, a contradiction of the randomness of the individual elements that comprise the expenditure totals. Moreover, because expenditure is treated as exogenous, the empirical model is not necessarily consistent with utility maximization. That is, expenditure is treated as predetermined without any requirement that the structure of consumer choices for group expenditure is consistent with utility maximization or even with the structure of the conditional demands. Usually, no equation explaining how group expenditure is chosen accompanies the conditional demand equation estimates. Without such a structure, the conditional demand model is not consistent with the joint maximization of utility with respect to the separable goods and group expenditure, much less the overall maximization of utility with respect to all goods that enter the consumer's budget.

A second set of conditions applies to situations where the unconditional covariance matrix is nonsingular. First, the conditional demand equations and the equation that explains total expenditure on the goods of interest will have error terms with zero means if and only if the conditional demands are linear in expenditure. Therefore, whether or not expenditure is exogenous, the conditional demand model must be a Gorman Polar Form (Gorman 1959, 1961) if the residuals of the empirical model have zero means. If the demand model is nonlinear in expenditure, then the expected values for the conditional error terms depend on the functional form of the demand equations and the specific values of the explanatory variables. Therefore, the distribution of the error terms, the moments of that distribution, and consequently the distribution of the parameter estimates obtained by the application of standard estimation methods to nonlinear conditional demand models, including the bias and mean-square error of the parameter estimates, are unique to the model's structure and the specific values of the explanatory variables.

If the unconditional covariance matrix is nonsingular and constant with respect to prices and income, then expenditure is uncorrelated with the conditional demand residuals if and only if the conditional demand model arises from an extremely restrictive special case. That is, the unconditional covariance matrix completely determines the effects of changes in expenditure on the quantities demanded. This strong conclusion is important because most empirical applications are based on the as-

available time is spent in the labor force. For dynamic models, income here means initial wealth plus the discounted present value of the full income stream. If leisure is included in the list of commodities, this eliminates any difference between income and total expenditure on all goods and permits income to be treated as an exogenous variable.

sumption that the residual covariance matrix is constant.

Finally, Theil's rational random errors hypothesis (Theil 1971, 1975, 1976)—the error covariance matrix is proportional to the matrix of compensated substitution terms—implies that expenditure is uncorrelated with the conditional errors to a first-order Taylor-series approximation for all distributions of the error terms and to a second-order approximation for symmetric distributions. This result is exact—expenditure is uncorrelated with the conditional errors—if and only if the conditional demand model is linear in expenditure.

The second focus of this article is to provide a bridge between the theory of separability and the estimation of separable demand systems. Methods to consistently estimate the unknown parameters are presented. If the conditional demands are nonlinear in expenditure, appropriate estimation procedures differ from the standard instrumental variables method of nonlinear two-stage least squares (Goldfield and Quandt; Amemiya 1974, 1975, 1983, 1985). Instrumental variables do not give consistent estimates of the parameters because the error terms in the model have nonzero means that depend on the model's structure and the specific values of the explanatory variables. However, the budget identity between the quantities demanded and expenditure holds at both the expected and the observed values for these variables. Therefore, a consistent estimate of the expected value of expenditure at each observation combined with Theil's (1953) interpretation of two-stage least squares results in consistent estimates. This aspect of the empirical problem also sheds light on appropriate estimation procedures in other econometric problems that are nonlinear in the variables.

Separability in Demand Analysis

We begin with some definitions and notation. Let $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]' \in \mathbb{R}^n_+$ be the vector of commodities of interest and $\boldsymbol{p} = [p_1, p_2, \ldots, p_n]' \in \mathbb{R}^n_+$ be the corresponding price vector; let $\boldsymbol{z} = [z_1, z_2, \ldots, z_m]' \in \mathbb{R}^m_+$ be the vector of consumption levels for all other goods and $\boldsymbol{q} = [q_1, q_2, \ldots, q_m]' \in \mathbb{R}^m_+$ be the corresponding price vector; let income be y; and let total expenditure on the goods $\boldsymbol{x}$ be $y_x \equiv \boldsymbol{p}'\boldsymbol{x}$. Our focus is on whether or not y_x is exogenous in the demands for $\boldsymbol{x}$. It is assumed, therefore, that the variables $(\boldsymbol{p}, \boldsymbol{q}, y)$ are strictly exogenous as defined by Engle, Hendry, and Richard. That is, the variables $(\boldsymbol{p}, \boldsymbol{q}, y)$ are assumed to be stochastically independent of all of the error terms in the empirical model. Specifically, y is interpreted as full income (Becker) in static models and as the discounted present value of the full income stream plus initial wealth in dynamic models.

Separability and Structural Recursivity

The utility function, u: $\mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}$, is assumed to be twice continuously differentiable ($u \in \mathbb{C}^2$), strictly increasing, and strictly quasiconcave for all $(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$. The goods $\boldsymbol{x}$ are separable from the goods $\boldsymbol{z}$ if and only if two functions, u_x: $\mathbb{R}^n_+ \to \mathbb{R}$ and $\tilde{u}$: $\mathbb{R} \times \mathbb{R}^m_+ \to \mathbb{R}$, exist such that $u_x, \tilde{u} \in \mathbb{C}^2$ are strictly increasing and strictly quasiconcave, and

$$(1) \qquad u(\boldsymbol{x}, \boldsymbol{z}) \equiv \tilde{u}(u_x(\boldsymbol{x}), \boldsymbol{z}),$$

for all $(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$. The separable structure given by (1) is sufficient for all that follows. Note, in particular, that the separability of preferences is represented in the most general manner possible. The goods $\boldsymbol{z}$ may not be separable from $\boldsymbol{x}$. The model may be static with $\boldsymbol{z}$ as the vector of quantities of all other goods consumed in the current period. Or the model may be dynamic, and $\boldsymbol{z}$ is the vector of quantities consumed in all other periods. The results that follow are valid for both cases and for any combination of them.

Economic theory tells us much about the structure of the mean levels of economic variables, but little to nothing about the stochastic part of the econometric model (Theil 1971). Therefore, the model structure is presented in terms of the mean values of the commodities of interest, $\bar{\boldsymbol{x}} \equiv E(\boldsymbol{x} \mid \boldsymbol{p}, \boldsymbol{q}, y)$ and $\bar{y}_x \equiv E(y_x \mid \boldsymbol{p}, \boldsymbol{q}, y)$, followed by an analysis of empirical issues. The unconditional demands for $\boldsymbol{x}$ are the result of maximizing $u(\boldsymbol{x}, \boldsymbol{z})$ with respect to $(\boldsymbol{x}, \boldsymbol{z})$ subject to the budget constraint, $\boldsymbol{p}'\boldsymbol{x} + \boldsymbol{q}'\boldsymbol{z} \leq y$. In general, the unconditional demands are functions of all prices and income,

$$(2) \qquad \bar{\boldsymbol{x}} \equiv \boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y).$$

Without separability or other simplifying assumptions, equation (2) represents the basic point of departure in demand analysis.

The problem with (2) in practice is that it represents n quantities as functions of $n + m + 1$ prices and income, and n is generally a relatively small number while m usually tends to be very large. Indeed, in demand analysis using annual time-series data, m is usually much greater than the number of available data points, and it is impossible to estimate the demands in such a general form. This is where separability plays its most important role. Separability of $\boldsymbol{x}$ from $\boldsymbol{z}$ is necessary and sufficient for decentralization of the unconditional demand functions for $\boldsymbol{x}$,

$$\bar{\boldsymbol{x}} \equiv \boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y) \equiv \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \xi_x(\boldsymbol{p}, \boldsymbol{q}, y)), \tag{3}$$

where $\xi_x: \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies

$$\begin{aligned} \bar{y}_x &\equiv \boldsymbol{p}'\bar{\boldsymbol{x}} \\ &\equiv \boldsymbol{p}'\boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y) \\ &\equiv \boldsymbol{p}'\tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \xi_x(\boldsymbol{p}, \boldsymbol{q}, y)) \\ &\equiv \xi_x(\boldsymbol{p}, \boldsymbol{q}, y), \end{aligned} \tag{4}$$

and $\tilde{\boldsymbol{h}}^x: \mathbb{R}^n_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}^n_+$ is the n-vector of conditional demands for $\boldsymbol{x}$ as functions of the prices, $\boldsymbol{p}$, and the expected value of expenditure, $\bar{y}_x$ (Primont; Gorman 1971; Blackorby, Primont, and Russell; Deaton and Muellbauer; Barten and Boehm). In other words, the unconditional demands in (2) can be written as functions of n prices and expenditure, rather than as functions of $n + m$ prices and income, if and only if $\boldsymbol{x}$ is separable from $\boldsymbol{z}$. The advantage of the reduction in the number of unknown parameters that results from separability is clear.

Note that equation (3) shows that weak separability is equivalent to a structurally recursive demand model. That is, the mean quantities demanded are functions of the group prices and the expected value of group expenditure only. Therefore, once group expenditure has been explained, the quantities demanded for the separable goods can be explained entirely by the prices of the goods of interest and the expected value of expenditure on that group of goods.

Is Expenditure Exogenous?

To obtain a stochastic specification, we append error terms to the unconditional demand equations,[2]

$$\begin{aligned} \boldsymbol{x} &\equiv \boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y) + \epsilon_x \\ &\equiv \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \xi_x(\boldsymbol{p}, \boldsymbol{q}, y)) + \epsilon_x, \end{aligned} \tag{5}$$

where ϵ_x is a vector of unobservable random error terms. Standard assumptions for models that are estimated in the levels of $\boldsymbol{x}$ are that the residuals have zero mean vector, $E(\epsilon_x) = \mathbf{0}$, and that the unconditional covariance matrix, $E(\epsilon_x\epsilon_x') \equiv \boldsymbol{\Sigma}_{xx}$, is constant $\forall$ $(\boldsymbol{p}, \boldsymbol{q}, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+$. However, we will not make the latter assumption, except as part of a particularly strong result below.

From the budget identity for all goods

$$\begin{aligned} y &\equiv \boldsymbol{p}'\boldsymbol{x} + \boldsymbol{q}'\boldsymbol{z} \\ &\equiv \boldsymbol{p}'\boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y) + \boldsymbol{q}'\boldsymbol{h}^z(\boldsymbol{p}, \boldsymbol{q}, y) \\ &\quad + \boldsymbol{p}'\epsilon_x + \boldsymbol{q}'\epsilon_z \\ &\equiv y + \boldsymbol{p}'\epsilon_x + \boldsymbol{q}'\epsilon_z, \end{aligned} \tag{6}$$

where $\boldsymbol{h}^z$ is the vector of unconditional demands for all other goods, and ϵ_z is the vector of unobservable errors for $\boldsymbol{z}$. Two results that follow immediately from (6) are $\boldsymbol{p}'\epsilon_x + \boldsymbol{q}'\epsilon_z \equiv 0$ and $\boldsymbol{\Sigma}\begin{bmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{bmatrix} \equiv \mathbf{0}$, where $\boldsymbol{\Sigma} = E\begin{bmatrix}\epsilon_x\epsilon_x' & \epsilon_x\epsilon_z' \\ \epsilon_z\epsilon_x' & \epsilon_z\epsilon_z'\end{bmatrix}$ is the covariance matrix for the complete system of demand equations (Barten). In words, the sum of all price-weighted demand residuals vanishes and therefore the covariance matrix is singular.

Note, however, that this does not imply that $\boldsymbol{p}'\epsilon_x = 0$ nor that $\boldsymbol{\Sigma}_{xx}\boldsymbol{p} = \mathbf{0}$. The importance of this can be seen by combining (4) and (5),

$$\begin{aligned} y_x &\equiv \boldsymbol{p}'\boldsymbol{x} \\ &\equiv \boldsymbol{p}'\boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y) + \boldsymbol{p}'\epsilon_x \\ &\equiv \boldsymbol{p}'\tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \xi_x(\boldsymbol{p}, \boldsymbol{q}, y)) + \boldsymbol{p}'\epsilon_x \\ &\equiv \xi_x(\boldsymbol{p}, \boldsymbol{q}, y) + \boldsymbol{p}'\epsilon_x \\ &\equiv \bar{y}_x + \nu_x, \end{aligned} \tag{7}$$

where $\nu_x \equiv \boldsymbol{p}'\epsilon_x$ is the residual for the expenditure on $\boldsymbol{x}$. Equation (7) shows that the error term in the expenditure equation is an exact linear combination of the unconditional demand residuals. Unless that linear combination is degenerate—that is, $\nu_x \equiv \boldsymbol{p}'\epsilon_x$ is a nonstochastic constant—it is impossible for the expenditure residual to be uncorrelated with the unconditional demand residuals. Zero correlation is a necessary condition for stochastic

[2] All of the results of this research can be shown to follow for the alternative representations in levels of expenditure on the individual goods and in budget shares. The residuals for models in quantities, ϵ_x, are related to the residuals for models in expenditures, ω_x, by $\omega_x = \boldsymbol{P}\epsilon_x$, and to the residuals for models in budget shares, ζ_x, by $\zeta_x = \boldsymbol{P}\epsilon_x/y$, where $\boldsymbol{P}$ is the $n \times n$ diagonal matrix with p_i as the ith diagonal element. Similarly, the residual covariance matrix for models in quantities, $\boldsymbol{\Sigma}_{xx}$, is related to the covariance matrix for models in expenditures, $\boldsymbol{\Omega}_{xx}$, by $\boldsymbol{\Omega}_{xx} = \boldsymbol{P}\boldsymbol{\Sigma}_{xx}\boldsymbol{P}$, and to the covariance matrix for models in budget shares, Ψ_{xx}, by $\Psi_{xx} = \boldsymbol{P}\boldsymbol{\Sigma}_{xx}\boldsymbol{P}/y^2$. With these relationships, any required modifications to the discussion that follows are straightforward.

independence, so that it is also impossible for expenditure to be stochastically independent of the unconditional error terms. For the multivariate normal distribution, for example, independence of the random variables and zero correlation are equivalent. As a result, expenditure generally is a stochastic variable that is neither uncorrelated with nor independent of the unconditional errors.

Now, suppose that a system of conditional demands is specified with the observable level of expenditure, y_x, included as a right-hand-side variable,

$$(8) \qquad \boldsymbol{x} = \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, y_x) + \tilde{\epsilon}_x.$$

Then the conditional errors, $\tilde{\epsilon}_x$, and the unconditional errors, ϵ_x, satisfy the identity

$$(9) \qquad \tilde{\epsilon}_x \equiv \epsilon_x + \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x) - \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x + \boldsymbol{p}'\epsilon_x).$$

The implication of (9) is that, although the deterministic structure of the conditional demands is recursive, it does not follow that expenditure is exogenous in the conditional demand equations.

Expenditure is weakly exogenous for the conditional demand parameters if and only if the marginal probability density function for expenditure does not depend on the parameters in the conditional demand equations and there are no cross-equation restrictions between the conditional demand parameters and the expenditure parameters (Engle, Hendry, and Richard). There is no loss of information or efficiency due to estimating the conditional demands with expenditure as a conditioning variable when and only when this condition is satisfied. However, by the budget identity, the structure of the expenditure equation always depends on a nonempty subset of the parameters of the conditional demands. Therefore, expenditure is never weakly exogenous—there is always a loss of information associated with estimating only the conditional demands. Specifically, no information concerning the substitution of $\boldsymbol{x}$ for $\boldsymbol{z}$, or equivalently, information about the change in expenditure due to changes in prices and income, can be recovered from the conditional demands.

These ideas are illustrated best with a simple example. Let there be three goods, (x_1, x_2, z), with a Cobb-Douglas utility function,

$$(10) \qquad u(x_1, x_2, z) = x_1^{\alpha_1} x_2^{\alpha_2} z^{1-\alpha_1-\alpha_2}.$$

Then the mean levels of the unconditional demands for x_1 and x_2 are given by

$$(11) \quad \bar{x}_i = E(x_i | p_1, p_2, q, y) = \alpha_i y/p_i, \qquad i = 1, 2,$$

and the conditional means of the demands for x_1 and x_2 given $y_x = p_1x_1 + p_2x_2$ are given by

$$(12) \quad E(x_i | p_1, p_2, y_x) = [\alpha_i/(\alpha_1 + \alpha_2)]y_x/p_i, \qquad i = 1, 2.$$

Note that the right-hand side of (12) is homogeneous of degree zero in (α_1, α_2), while $\alpha_1/(\alpha_1 + \alpha_2) + \alpha_2/(\alpha_1 + \alpha_2) \equiv 1$, so that only $\beta_1 \equiv \alpha_1/(\alpha_1 + \alpha_2)$ is identified, regardless of the stochastic properties of y_x. Even if β_1 can be estimated consistently or even efficiently from (12), all utility functions of the form

$$(13) \qquad u(x_1, x_2, z) = \tilde{u}(x_1^{\beta_1} x_2^{1-\beta_1}, z)$$

are consistent with the conditional demand model given by (12). The precise information that is not transmitted by the conditional demand model is the structure of the expenditure on (x_1, x_2),

$$(14) \qquad E(y_x | p_1, p_2, q, y) = (\alpha_2 + \alpha_2)y.$$

Consequently, y_x is not and cannot be weakly exogenous in this or any other conditional demand model if the parameters of interest include the income effects or any substitution effects between the goods of interest and the other goods. As shown by LaFrance and Hanemann, all of these elements are necessary for the coherent completion of tasks such as welfare analysis with an incomplete demand model.

Most empirical applications tacitly assume that expenditure is uncorrelated with the conditional demand residuals (Alston and Chalfant 1987, 1991; Alston et al.; Blackorby, Boyce, and Russell; Brown and Heien; Capps and Schmitz; Choi and Sosin; Clements and Selvanathan; Deaton 1975; Heien; Heien and Pompelli; Murray; Safyurtlu, Johnson, and Hassan; Theil 1975, 1976; Van Kooten). This is a weak version of a property that Engle, Hendry, and Richard call strictly exogenous—expenditure is stochastically independent of the conditional residuals. The usual empirical practice is to estimate the equations in (8) as a complete system of demands with a singular covariance matrix. The reason for this latter condition can be clearly understood as follows. Since the conditional demands satisfy the adding-up condition identically in expenditure, the identities

$$(15) \qquad \boldsymbol{p}'\tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x) \equiv \bar{y}_x,$$

and

$$(16) \qquad p'\tilde{h}^x(p, \bar{y}_x + \nu_x) \equiv \bar{y}_x + \nu_x$$

both hold for all possible values of p, $\bar{y}_x$, and $\nu_x \equiv p'\epsilon_x$. Taking the vector product of (9) with p gives

$$(17) \qquad p'\tilde{\epsilon}_x \equiv p'\epsilon_x + p'\tilde{h}^x(p, \bar{y}_x) - p'\tilde{h}^x(p, \bar{y}_x + p'\epsilon_x) \equiv p'\epsilon_x + \bar{y}_x - (\bar{y}_x + p'\epsilon_x) \equiv 0.$$

That is, the sum of the price-weighted conditional demand residuals vanishes and the conditional covariance matrix, $\tilde{\Sigma}_{xx}$, is singular, since

$$(18) \qquad \tilde{\Sigma}_{xx}p = E\{[\tilde{\epsilon}_x - E(\tilde{\epsilon}_x)]\,[\tilde{\epsilon}_x - E(\tilde{\epsilon}_x)]'p\} = E\{[\tilde{\epsilon}_x - E(\tilde{\epsilon}_x)]\,\tilde{\epsilon}_x'p\} = \mathbf{0}.$$

However, neither (17) nor (18) imply that y_x and $\tilde{\epsilon}_x$ are uncorrelated. Combining (7) and (9), the covariance between y_x and $\tilde{\epsilon}_x$ is given by

$$(19) \qquad \mathrm{Cov}(y_x, \tilde{\epsilon}_x) = E[\tilde{\epsilon}_x\nu_x] = E\{[\epsilon_x + \tilde{h}^x(p, \bar{y}_x) - \tilde{h}^x(p, \bar{y}_x + p'\epsilon_x)]\epsilon_x'p\} = \Sigma_{xx}p - E[\tilde{h}^x(p, \bar{y}_x + p'\epsilon_x)\epsilon_x'p].$$

In general, neither of the terms on the right-hand side of (19) need vanish. That this is an important issue in applied demand analysis is obvious. Appropriate estimation procedures and the properties of the resulting parameter estimates depend critically on the exogeneity or endogeneity of expenditure in the empirical equations. Moreover, it is clear from (19) that the covariance between expenditure and the conditional errors depends on the structure of the conditional demand model in a potentially very complex fashion. Therefore, it is essential to understand the issues involved and possibilities for their resolution.

Previous discussions largely contradict one another. Pollak (1971) noted the possibility for simultaneity between the quantities demanded and expenditure and referred to a brief exchange between Prais and Summers on the issue but did not analyze the question further. Brown and Heien, and later Blackorby, Boyce, and Russell and Blackorby, Primont, and Russell, argued that the required conditions for expenditure to be exogenous are that expenditure is nonstochastic and the residuals in the demands for x are uncorrelated with the residuals in the demands for z. Theil (1971, 1975, 1976) showed that if the utility function is quadratic and the covariance matrix is proportional to the Slutsky substitution matrix, then although expenditure is stochastic, it is uncorrelated with the conditional demand residuals. Deaton (1975, 1986) argued that except where Theil's result holds, there will be bias due to the inclusion of y_x as a regressor in the conditional demands. In Deaton's (1975) study the conditional demands were specified in terms of $\bar{y}_x$ rather than y_x. He argued, however, that the bias due to using y_x is small if the equations fit well. He also asserted that using the predicted value of expenditure as an instrument will result in biased parameter estimates.

Anderson claimed that Theil's approach with y_x as a regressor and the approach implied by Deaton's specification with $\bar{y}_x$ as the correct variable are both coherent since $p'\tilde{\epsilon}_x \equiv 0$, while $y_x \equiv \bar{y}_x + \nu_x$ implies $\nu_x \equiv p'\epsilon_x$. He proposed an iterative estimator using predicted values for $\bar{y}_x$ in the conditional demands and argued that Malinvaud's results on minimum distance estimators implied that these estimates are consistent.

Attfield showed that, for the Rotterdam model, tests of zero-degree homogeneity maintaining the hypothesis that expenditure is strictly exogenous are equivalent to tests of the strict exogeneity of expenditure maintaining zero-degree homogeneity of the demands. Finally, Blundell (1986, 1988) argued in a pair of recent surveys that it is most important to allow for endogeneity of total expenditure in demand systems estimated with cross-section data. He also asserted that constructing an instrument for total expenditure is completely straightforward for any functional form for the demand equations since there is no shortage of theories of the consumption function.

Clearly, Theil's result is a counter example to Brown and Heien's argument. Similarly, Anderson's claim that Theil's and Deaton's approaches are both coherent contradicts the arguments of Deaton, Attfield, and Blundell. Furthermore, the claim by Anderson that using an estimate of the mean level of expenditure as an instrument in the conditional demands will produce consistent estimates is a contradiction to Deaton's assertion to the contrary.

This problem can be understood best by recognizing that there are two separate, but not mutually exclusive, aspects involved in the joint determination of quantities consumed and expenditure. One aspect involves the joint distribution of the error terms in the unconditional demand model. The other involves the functional form of the conditional demand model. One approach to the question of correlation between expenditure and the conditional demand residuals is to restrict the class of joint density functions for the unconditional error terms to those with a singular unconditional covariance matrix (Brown and Heien; Blackorby, Boyce, and Russell; and Blackorby, Primont, and Russell). This approach is illustrated by the following result.[3]

Lemma 1. Suppose that $E(\epsilon_x) = \mathbf{0}$. Then $\mathbf{\Sigma}_{xx}\boldsymbol{p} = \mathbf{0}$ if and only if $\boldsymbol{p}'\epsilon_x = 0$. Furthermore, $\boldsymbol{p}'\epsilon_x = 0$ implies $\text{Cov}(y_x, \tilde{\epsilon}_x) = \mathbf{0}$.

The intuition behind lemma 1 is as follows. If the covariance matrix is singular (that is, it always transforms the prices $\boldsymbol{p}$ to the zero vector), then the sum of the price-weighted residuals must be a nonstochastic constant (conditional on $(\boldsymbol{p}, \boldsymbol{q}, y)$, of course). But because prices are exogenous, this constant is zero because the unconditional residuals have zero means. Conversely, if the price-weighted sum of the residuals vanishes, then the covariance matrix must be singular. Moreover, the only source of stochastic variation in expenditure is due to the sum of the price-weighted demand residuals. Therefore, if this sum is zero, then expenditure is a deterministic function of the exogenous variables only and is uncorrelated with the conditional demand residuals.

Since $y_x = \bar{y}_x + \boldsymbol{p}'\epsilon_x$, lemma 1 states that the necessary and sufficient condition for expenditure to be nonstochastic is that the unconditional covariance matrix is singular. If the unconditional covariance matrix is singular, then standard systems estimation methods can be applied to the conditional demand model with expenditure included as a regressor. However, this approach has at least two weaknesses. First, it contradicts the structural simultaneity between $\boldsymbol{x}$ and y_x as reflected by (7). Separability does not imply that expenditure is fixed when quantities are chosen. By definition, the opposite is true. A specific structure for the simultaneous determination of quantities and expenditure is given by the separability hypothesis. Second, if the assumption that $\boldsymbol{p}'\epsilon_x = 0$ is false and the fact that it is false goes undetected, then the empirical results will not have any of the optimal properties of efficient estimators. In particular, the parameter estimates will be biased and inconsistent, and in general, the asymptotic distribution for the parameter estimates will not be multivariate normal. In principle the hypothesis that $\boldsymbol{p}'\epsilon_x = 0$ is testable and definitely should be tested in separable demand studies.

Suppose that the unconditional covariance matrix is nonsingular and the conditional demand residuals and the residual in the expenditure equation have means that equal zero. To see that the conditional demand model is linear in expenditure, note first that $E(\nu_x) = E(\boldsymbol{p}'\epsilon_x) = \boldsymbol{p}'E(\epsilon_x) = 0$ for all $\boldsymbol{p} \in \mathbb{R}^n_+$ if and only if $E(\epsilon_x) = \mathbf{0}$, and second that $E(\tilde{\epsilon}_x) = E(\epsilon_x) + \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x) - E[\tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x + \nu_x)]$. The next result follows immediately from these two facts and Jensen's inequality.

Lemma 2. If $|\mathbf{\Sigma}_{xx}| \neq 0$, then both $E(\nu_x) = 0$ and $E(\tilde{\epsilon}_x) = \mathbf{0}$ if and only if $\tilde{\boldsymbol{h}}^x(\boldsymbol{p}, y_x) \equiv \boldsymbol{\alpha}(\boldsymbol{p}) + \boldsymbol{\beta}(\boldsymbol{p})y_x$.

In other words, whether or not expenditure is exogenous (for any definition of the word), if the econometric model defined by (7) and (8) has the usual properties for the error terms, then the functional form of the demand model is restricted to Gorman Polar Forms for the conditional indirect preferences (Gorman 1959, 1961),

$$(20) \qquad v_x(\boldsymbol{p}, y_x) = (y_x - \Lambda(\boldsymbol{p}))/\Pi(\boldsymbol{p}),$$

where $v_x(\boldsymbol{p}, y_x)$ is the conditional indirect utility function,

$$(21) \qquad v_x(\boldsymbol{p}, y_x) \equiv \max\{u_x(\boldsymbol{x}): \boldsymbol{p}'\boldsymbol{x} \leq y_x\},$$

and $\Lambda(\boldsymbol{p})$ and $\Pi(\boldsymbol{p})$ are linearly homogeneous functions of the prices $\boldsymbol{p}$.

By itself, this result represents an important aspect of empirical demand analysis. Virtually all previous analyses of the endogeneity of expenditure used models that were linear in expenditure. However, the arguments were presented as if they were also true for any functional forms of the conditional demand

[3] Detailed proofs of the main results of this section are contained in a longer paper that is available from the author upon request. The proofs are not included here in order to focus on the interpretation of the results and their implications rather than on the technical arguments required to demonstrate their validity.

model.[4] Lemma 2 shows that functional forms that are nonlinear in expenditure imply that either the conditional demand residuals will have nonzero means or the residual in the expenditure equation will have a nonzero mean. If this aspect of the empirical problem is not accounted for explicitly, then there is no logical basis for any claims regarding the relative bias due to the endogeneity of expenditure.

For the moment, let us ignore this problem and consider both linear and nonlinear expenditure systems for the conditional demand model. It is clear from (19) that no generally valid conclusions can be reached without additional assumptions. Most empirical studies employ functional forms for the demand equations that are smooth in expenditure—possess continuous partial derivatives of all orders—and rely on the assumption that the error terms are multivariate normal. The former hypothesis and an assumption that is weaker than the latter are useful in an argument that expenditure is an endogenous explanatory variable. Specifically, we will assume that $\partial^i \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)/\partial y_x^i$ and $E(\nu_x^i)$ exist and are finite for each $i \geq 1$.

The first assumption implies that the conditional demands can be written as an infinite Taylor-series expansion about the mean value of expenditure,

$$(22)\quad \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, y_x) = \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x) + \sum_{i=1}^{\infty} \frac{1}{i!}\left[\frac{\partial^i \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)}{\partial y_x^i}\right]\nu_x^i.$$

Combining (22) with the second assumption then results in an infinite series expression for the covariance between expenditure and the conditional demand residuals,

$$(23)\quad \mathrm{Cov}(y_x, \tilde{\epsilon}_x) = \boldsymbol{\Sigma}_{xx}\boldsymbol{p} - \sum_{i=1}^{\infty} \frac{1}{i!}\left[\frac{\partial^i \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)}{\partial y_x^i}\right] E(\nu_x^{i+1}).$$

This expression for the covariance between expenditure and the conditional demand residuals now permits the straightforward development of some useful approximations. For a first-order approximation (and second order if ν_x is distributed symmetrically), we have

$$(24)\quad \mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \boldsymbol{\Sigma}_{xx}\boldsymbol{p} - \left[\frac{\partial \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)}{\partial y_x}\right]\boldsymbol{p}'\boldsymbol{\Sigma}_{xx}\boldsymbol{p}.$$

This leads immediately to a very strong result. If the unconditional covariance matrix is nonsingular and does not depend on prices or income, then even with a first-order approximation, expenditure is uncorrelated with the conditional residuals if and only if the demand model arises from a generalized quadratic conditional indirect utility function.

Lemma 3. If $\boldsymbol{\Sigma}_{xx}$ does not depend on $(\boldsymbol{p}, \boldsymbol{q}, y)$ and $|\boldsymbol{\Sigma}_{xx}| \neq 0$, then $\mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \boldsymbol{0}\ \forall\ (\boldsymbol{p}, \boldsymbol{q}, \underset{\sim}{y}) \in \mathbb{R}_+^n \times \mathbb{R}_+^m \times \mathbb{R}_+$ if and only if $\tilde{\boldsymbol{h}}^x(\boldsymbol{p}, y_x) \equiv \boldsymbol{\alpha}(\boldsymbol{p}) + \boldsymbol{\beta}(\boldsymbol{p})y_x$ and $\boldsymbol{\beta}(\boldsymbol{p}) \equiv (\boldsymbol{p}'\boldsymbol{\Sigma}_{xx}\boldsymbol{p})^{-1}\boldsymbol{\Sigma}_{xx}\boldsymbol{p}$.

In other words, if the unconditional covariance matrix is nonsingular and is not a function of prices or income, then expenditure is uncorrelated with the conditional errors if and only if all of the expenditure effects on the quantities demanded are completely determined by the unconditional covariance matrix. This occurs whether or not the conditional error terms have vanishing means.

The conditional indirect utility function in lemma 3 is given by

$$(25)\quad v_x(\boldsymbol{p}, y_x) = (y_x - \Lambda(\boldsymbol{p}))/\sqrt{\boldsymbol{p}'\boldsymbol{\Sigma}_{xx}\boldsymbol{p}}.$$

The quadratic subutility function,

$$(26)\quad u_x(\boldsymbol{x}) = \tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{\delta})'\boldsymbol{B}(\boldsymbol{x} - \boldsymbol{\delta}),$$

where $\boldsymbol{\delta} \gg \boldsymbol{0}$ and $\boldsymbol{B}$ is negative definite, has a Gorman Polar Form representation given by (25) with $\Lambda(\boldsymbol{p}) = \boldsymbol{\delta}'\boldsymbol{p}$ and $\boldsymbol{\Sigma}_{xx} = -\sigma\boldsymbol{B}^{-1}$ for arbitrary $\sigma > 0$. This is the rationale for calling

[4] Blundell's (1986, 1988) arguments were presented for models that are linear in expenditure, although he suggested that they hold with equal weight for models that are nonlinear in expenditure. That the demand model is linear in expenditure is clear for the S-branch utility model of Brown and Heien; the Gorman Polar Form of Blackorby, Boyce, and Russell and of Browning, Deaton, and Irish; the linear expenditure system of Deaton (1975); and the quadratic utility function of Theil (1971, 1975, 1976). This is not quite as clear for the Rotterdam model, which is employed in the arguments of Theil (1975, 1976, 1980); Clements; Attfield and Browning; and Attfield. However, the Rotterdam model in log-differential form,

$$w_i\, d\log(x_i) = \mu_i\, d\log(y_x) + \sum_{j=1}^{n} \pi_{ij}\, d\log(p_j), \qquad i = 1, \ldots, n,$$

where $w_i = p_i x_i / y_x$, is equivalent to the total differential equation

$$dx_i = (\mu_i/p_i)dy_x + y_x \sum_{j=1}^{n} (\pi_{ij}/p_i p_j)dp_j, \qquad i = 1, \ldots, n.$$

With constant coefficients, this total differential equation has a solution (when one exists) of the form

$$x_i = \theta_i + \left[\mu_i + \log\left(\prod_{j=1}^{n} p_j^{\pi_{ij}}\right)\right]\frac{y_x}{p_i}, \qquad i = 1, \ldots, n,$$

which is linear in expenditure. Of course, Frobenius' theorem requires that $\theta_i = \pi_{ij} = 0$ for all $i, j = 1, \ldots, n$ for a solution to exist.

this class of conditional demand models generalized quadratic.

The implications of lemma 3 are illustrated with several examples from the set of models that have a single nonlinear function of expenditure,

$$(27)\quad \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, y_x) = \boldsymbol{\alpha}(\boldsymbol{p}) + \boldsymbol{\beta}(\boldsymbol{p})y_x + \boldsymbol{\gamma}(\mathbf{p})\psi(\boldsymbol{p}, y_x).$$

This class of models includes all of the nominal income Gorman Engel curve demand models (Gorman 1981) that were analyzed by Lewbel (1987, 1989a) as well as several deflated income demand models (Lewbel 1989b, 1990). The latter group of demand models encompasses most existing empirical demand specifications. The examples also provide first-order approximations (second-order approximations if the distribution of ν_x is symmetric) to the covariance between expenditure and the conditional error terms for these demand models.

(*a*) *Linear Expenditure System:* $\psi(\boldsymbol{p}, y_x) \equiv 0$

$$\mathrm{Cov}(y_x, \tilde{\epsilon}_x) = [\boldsymbol{I} - \boldsymbol{\beta}(\boldsymbol{p})\boldsymbol{p}']\boldsymbol{\Sigma}_{xx}\boldsymbol{p}.$$

(*b*) *Quadratic Expenditure System:* $\psi(\boldsymbol{p}, y_x) = y_x^2$

$$\mathrm{Cov}(y_x, \tilde{\epsilon}_x) = \{\boldsymbol{I} - [\boldsymbol{\beta}(\boldsymbol{p}) + 2\bar{y}_x\boldsymbol{\gamma}(\boldsymbol{p})]\boldsymbol{p}'\}\boldsymbol{\Sigma}_{xx}\boldsymbol{p}.$$

(*c*) *Extended PIGL:* $\psi(\boldsymbol{p}, y_x) = y_x^k$, where $k \neq 0, 1, 2$

$$\mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \{\boldsymbol{I} - [\boldsymbol{\beta}(\boldsymbol{p}) + k\bar{y}_x^{k-1}\boldsymbol{\gamma}(\boldsymbol{p})]\boldsymbol{p}'\}\boldsymbol{\Sigma}_{xx}\boldsymbol{p}.$$

(*d*) *Extended PIGLOG:* $\psi(\boldsymbol{p}, y_x) = y_x \log(y_x)$

$$\mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \{\boldsymbol{I} - [\boldsymbol{\beta}(\boldsymbol{p}) + [1 + \log(\bar{y}_x)]\boldsymbol{\gamma}(\boldsymbol{p})]\boldsymbol{p}'\}\boldsymbol{\Sigma}_{xx}\boldsymbol{p}.$$

(*e*) *LINLOG:* $\psi(\boldsymbol{p}, y_x) = \log(y_x)$

$$\mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \{\boldsymbol{I} - [\boldsymbol{\beta}(\boldsymbol{p}) + \bar{y}_x^{-1}\boldsymbol{\gamma}(\boldsymbol{p})]\boldsymbol{p}'\}\boldsymbol{\Sigma}_{xx}\boldsymbol{p}.$$

(*f*) *LINEXP:* $\psi(\boldsymbol{p}, y_x) = \exp\{y_x/\theta(\boldsymbol{p})\}$, with $\theta(\boldsymbol{p})$ linearly homogeneous

$$\mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \{\boldsymbol{I} - [\boldsymbol{\beta}(\boldsymbol{p}) + \exp\{\bar{y}_x/\theta(\boldsymbol{p})\}\boldsymbol{\gamma}(\boldsymbol{p})/\theta(\boldsymbol{p})]\boldsymbol{p}'\}\boldsymbol{\Sigma}_{xx}\boldsymbol{p}$$

$$\mathrm{Cov}(y_x, \tilde{\epsilon}_x) = \{\boldsymbol{I} - [\boldsymbol{\beta}(\boldsymbol{p}) + \exp[\bar{y}_x/\theta(\boldsymbol{p}) + \tfrac{1}{2}\boldsymbol{p}'\boldsymbol{\Sigma}_{xx}\boldsymbol{p}/\theta(\boldsymbol{p})^2]\boldsymbol{\gamma}(\boldsymbol{p})/\theta(\boldsymbol{p})]\boldsymbol{p}'\}\boldsymbol{\Sigma}_{xx}\boldsymbol{p}$$

if $\nu_x \sim n(0, \boldsymbol{p}'\boldsymbol{\Sigma}_{xx}\boldsymbol{p})$.

In each example, a $\boldsymbol{\Sigma}_{xx}$ that depends on $(\boldsymbol{p}, \boldsymbol{q}, y)$ can be found such that y_x is approximately uncorrelated with $\tilde{\epsilon}_x$. In every case, however, the specification of $\boldsymbol{\Sigma}_{xx}$ is model specific and depends on the latent variable $\bar{y}_x \equiv \xi_x(\boldsymbol{p}, \boldsymbol{q}, y)$, which is not a part of the conditional demand model. We now consider this issue in more detail.

The conditional errors are (approximately) uncorrelated with expenditure in one other important case. Theil's (1971, 1975, 1976) rational random errors hypothesis states that the unconditional errors are "rationally random" if they have vanishing means and covariance matrix given by $\boldsymbol{\Sigma}_{xx} = -\varphi(\boldsymbol{p}, \boldsymbol{q}, y)\boldsymbol{S}_x$, where $\varphi(\boldsymbol{p}, \boldsymbol{q}, y) > 0$ and $\boldsymbol{S}_x$ is the $n \times n$ matrix of unconditional substitution terms,

$$(28)\quad \boldsymbol{S}_x \equiv \frac{\partial \boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y)}{\partial \boldsymbol{p}} + \frac{\partial \boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y)}{\partial y}\boldsymbol{h}^x(\boldsymbol{p}, \boldsymbol{q}, y)'.$$

Theil showed that if the utility function is quadratic, then the conditional covariance matrix is given by $\tilde{\boldsymbol{\Sigma}}_{xx} = -\varphi(\boldsymbol{p}, \boldsymbol{q}, y)\tilde{\boldsymbol{S}}_x$, where $\tilde{\boldsymbol{S}}_x$ is the $n \times n$ matrix of conditional substitution terms,

$$(29)\quad \tilde{\boldsymbol{S}}_x \equiv \frac{\partial \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \xi_x(\boldsymbol{p}, \boldsymbol{q}, y))}{\partial \boldsymbol{p}'} + \frac{\partial \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \xi_x(\boldsymbol{p}, \boldsymbol{q}, y))}{\partial y_x} \cdot \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \xi_x(\boldsymbol{p}, \boldsymbol{q}, y))'.$$

He also proved that expenditure is uncorrelated with the conditional demand residuals under these conditions.

The conditional demands are homogeneous of degree zero in group prices and expenditure, so that $\tilde{\boldsymbol{S}}_x\boldsymbol{p} \equiv \boldsymbol{0}$. Also, $\tilde{\boldsymbol{S}}_x \equiv [\boldsymbol{I} - \partial\tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)/\partial y_x\boldsymbol{p}']\boldsymbol{S}_x$ follows from the budget identity (4) and the zero-degree homogeneity for $\tilde{\boldsymbol{h}}^x$ for any functional form for the demands for $\boldsymbol{x}$ (LaFrance, theorem 4). Combining these two facts, we obtain the identity

$$(30)\quad \tilde{\boldsymbol{S}}_x\boldsymbol{p} \equiv \left[\boldsymbol{I} - \left[\frac{\partial \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)}{\partial y_x}\right]\boldsymbol{p}'\right]\boldsymbol{S}_x\boldsymbol{p} \equiv \boldsymbol{0},$$

whenever the goods $\boldsymbol{x}$ are separable from all other goods. Now, it follows from (24) that to a first-order approximation (again, second order if ν_x has a symmetric distribution) the covariance between expenditure and the conditional errors can be written in a form that is related to (30), specifically

$$(31)\quad \mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \left[\boldsymbol{I} - \left[\frac{\partial \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)}{\partial y_x}\right]\boldsymbol{p}'\right]\boldsymbol{\Sigma}_{xx}\boldsymbol{p}.$$

These two expressions lead to another strong result. If the unconditional covariance matrix is nonsingular, the rational random errors hypothesis is a sufficient condition for zero correlation between expenditure and the conditional demand residuals to first order. Furthermore, a necessary and sufficient condition for expenditure to be uncorrelated with the conditional error terms to first order is that the unconditional covariance and substitution matrices transform the price vector $\boldsymbol{p}$ into the same space. We call this the generalized rational random errors hypothesis.

Lemma 4. If $|\boldsymbol{\Sigma}_{xx}| \neq 0$, then $\mathrm{Cov}(y_x, \tilde{\epsilon}_x) \approx \mathbf{0}$ $\forall$ $(\boldsymbol{p}, \boldsymbol{q}, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^m \times \mathbb{R}_+$ if and only if $\boldsymbol{\Sigma}_{xx}\boldsymbol{p} \equiv -\varphi \boldsymbol{S}_x \boldsymbol{p}$ for some φ: $\mathbb{R}_+^n \times \mathbb{R}_+^m \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Sufficiency in lemma 1 follows immediately from (30) and (31). Necessity follows from substituting $\partial \tilde{\boldsymbol{h}}^x(\boldsymbol{p}, \bar{y}_x)/\partial y_x \equiv (\boldsymbol{p}'\boldsymbol{S}_x\boldsymbol{p})^{-1}\boldsymbol{S}_x\boldsymbol{p}$ from (30) into (31), setting (31) equal to $\mathbf{0}$, solving for $\boldsymbol{\Sigma}_{xx}\boldsymbol{p}$, and defining $-\varphi(\boldsymbol{p}, \boldsymbol{q}, y) \equiv \boldsymbol{p}'\boldsymbol{\Sigma}_{xx}\boldsymbol{p}/\boldsymbol{p}'\boldsymbol{S}_x\boldsymbol{p}$. Note, however, that it does not necessarily follow that $\boldsymbol{\Sigma}_{xx} \equiv -\varphi \boldsymbol{S}_x$. The generalized quadratic demand model of lemma 3 is one counter example, and $\boldsymbol{\Sigma}_{xx} \equiv -\varphi \boldsymbol{S}_x + \partial^2\Omega(\boldsymbol{p})/\partial \boldsymbol{p} \partial \boldsymbol{p}'$, Ω homogeneous of degree one in $\boldsymbol{p}$, is another. Also note that lemma 4 is exact if and only if the conditional demand model is linear in expenditure.

It is useful to summarize our results up to this point. The structure of demand models for a set of separable goods is given by equation (3). Whether the stochastic part of the empirical model is specified in terms of the unconditional demands or the conditional demands, the unconditional and conditional error terms are related by the identity (9), and the choices for $\boldsymbol{x}$ determine the choices for y_x through equation (7). If the optimization errors and measurement errors for $\boldsymbol{x}$ are not systematic, then the unconditional errors will have zero means. This property is necessary and sufficient for the mean of the residual in the expenditure equation to vanish. Moreover, the adding-up condition implies that the structure of the demands for the separable goods can always be written with the means of the quantities demanded as functions of the group prices and the mean level of expenditure. It is also desirable to be able to test if the data are consistent with the assumption that expenditure is uncorrelated with the conditional demand residuals. For this a mean value of zero for ν_x is necessary. Finally, unless the parameters of the error process are incorporated directly into the structure of the conditional demands during estimation, it is desirable for $\tilde{\epsilon}_x$ to have a zero mean vector. In combination, these factors imply the following.

First, it is impossible for expenditure to be weakly exogenous in a set of conditional demands. Second, there are only three cases in which expenditure is uncorrelated with the conditional errors (a weak form of strict exogeneity): (*a*) if the unconditional covariance matrix is singular and transforms $\boldsymbol{p}$ identically to $\mathbf{0}$, (*b*) if the unconditional covariance matrix is nonsingular and does not depend on prices or income and the conditional indirect utility function is a generalized quadratic Gorman Polar Form, or (*c*) if the conditional demand model is linear in expenditure and the error terms satisfy the generalized rational random errors hypothesis. In each of these cases standard systems methods of estimation can be applied to the conditional demands with the observed level of expenditure included as one of the explanatory variables. However, each case is restrictive, and the necessary restrictions can be tested against the data. The Wu-Hausman specification test (Wu; Hausman) can be used to test for correlation between expenditure and the conditional errors in the latter two cases. In the first case, the solution is not nearly so simple if the model is nonlinear in expenditure.

Estimation when Expenditure Is Endogenous

We now turn to the problem of estimating the separable set of demands when the conditional demands are not necessarily linear in expenditure and expenditure is not necessarily uncorrelated with the conditional error terms. Clearly, the results to this point demonstrate that this is the generic situation, and therefore the most important. The main thrust of the discussion is that, when it converges, the iterative two-stage estimation procedure proposed by Anderson produces consistent estimates of the model parameters whether or not expenditure is correlated with the conditional error terms (LaFrance, theorem 5). This iterative procedure is complex and computationally intensive and does not generate consistent estimates of the covariance matrix for the parameter estimates. Also, there can be some difficulty with convergence of the iterative procedure. However, it offers a feasible solution to the simultaneity problem in conditional demand models and can be used to obtain good starting values for full-information maximum-likelihood or two-step linearized maximum-likelihood estimation procedures.

Combining equations (4), (5), and (7), the system of equations for the unconditional demands and expenditure can be written as

$$x = \tilde{h}^x(p, \bar{y}_x) + \epsilon_x, \tag{32}$$

$$y_x = \xi_x(p, q, y) + \nu_x. \tag{33}$$

The identity $\nu_x \equiv p'\epsilon_x$ implies that the system of $n + 1$ equations has a singular covariance matrix and cannot be estimated jointly. However, if $|\Sigma_{xx}| \neq 0$ and if a consistent set of estimates for the mean levels of expenditure is available, then it is clear that the unknown parameters in (32) can be estimated consistently by substituting these estimates for the latent variable $\bar{y}_x$. In essence, this is the basis for the two-stage iterative procedure proposed by Anderson.

Before discussing this approach in detail, note that a conditional estimation scheme such as this is of interest in a larger econometric context. Consider the general problem of estimating any simultaneous equations model that is nonlinear in the endogenous right-hand-side variables. Suppose that the structural model has a representation where the error terms have zero means when the expected values of the endogenous variables are included on the right-hand side of the regression equations. For example, when economic agents are assumed to make choices on the basis of expectations that are consistent with the model structure, this is the proper way to formulate the econometric problem. In such circumstances, consistent estimates of the means of the endogenous right-hand-side variables can be used directly to replace the unobservable true means, and consistent estimates of the remaining structural parameters can be obtained with standard least squares methods.

To see the nature of Anderson's two-stage estimation procedure, a change in notation is helpful. Let $\bar{x}_t \equiv f_t(\alpha, g_t(\alpha, \beta)) \equiv h^x(p_t, q_t, y_t)$ be the vector of the expected values of the quantities demanded at observation t, let $\bar{y}_{xt} \equiv g_t(\alpha, \beta) \equiv \xi_x(p_t, q_t, y_t)$ be the expected value of the expenditure on x at observation t, $t = 1, \ldots, T$, and let α and β be unknown parameters to be estimated. Then the econometric model can be rewritten as

$$x_t = f_t(\alpha, g_t(\alpha, \beta)) + \epsilon_{xt} \tag{32'}$$

and

$$y_{xt} = g_t(\alpha, \beta) + \nu_{xt}. \tag{33'}$$

Anderson's iterative procedure begins with an initial guess for α and Σ_{xx}, and then proceeds with the following steps:

(*a*) Given estimates of α and Σ_{xx}, estimate β with generalized least squares on (33'),

$$R_y(\hat{\beta} \mid \hat{\alpha}, \hat{\Sigma}_{xx}) \equiv \min_{\beta} \sum_{t=1}^{T} [y_{xt} - g_t(\hat{\alpha}, \beta)]^2 / p_t'\hat{\Sigma}_{xx}p_t.$$

(*b*) Given estimates of α, β, and Σ_{xx}, predict $\bar{y}_{xt}$ with $\hat{g}_t \equiv g_t(\hat{\alpha}, \hat{\beta})$ and reestimate α with generalized least squares on (32'),

$$R_x(\hat{\alpha} \mid \hat{g}, \hat{\Sigma}_{xx}) \equiv \min_{\alpha} \sum_{t=1}^{T} [x_t - f_t(\alpha, \hat{g}_t)]'\hat{\Sigma}_{xx}^{-1}[x_t - f_t(\alpha, \hat{g}_t)].$$

(*c*) Update the estimate of Σ_{xx} with

$$\hat{\Sigma}_{xx} = \frac{1}{T} \sum_{t=1}^{T} [x_t - f_t(\hat{\alpha}, \hat{g}_t)][x_t - f_t(\hat{\alpha}, \hat{g}_t)]'.$$

(*d*) Repeat (*a*) through (*c*) until convergence is obtained.

Anderson claimed that the solution to this iterative procedure is consistent due to the results on minimum distance estimators by Malinvaud. However, α is estimated at each stage conditional on the fixed previous estimates of α, β, and Σ_{xx}, while β is estimated from the auxiliary sum-of-squares criterion, R_y, conditional on the fixed previous estimates of α and Σ_{xx}. Therefore, the residual sum-of-squares criterion, R_x, is not minimized with respect to either α or β, the resulting estimates are not true minimum distance estimators, and Malinvaud's results do not have any bearing on this procedure. But, if the iterative process converges, then it can be shown that the final estimates of α and β are consistent and asymptotically normal (LaFrance, theorem 5).

One problem, however, is that the estimated covariance matrices for $\hat{\alpha}$ and $\hat{\beta}$ that result from Anderson's iterative procedure are inconsistent, and the bias is likely to be substantial. This problem can be overcome by adding a third stage to the end of the two-stage estimation procedure. The third stage employs the consistent estimates of α, β, and Σ_{xx} obtained at the end of the two-stage procedure as initial values for a linearized maximum-likelihood procedure applied to (32′) estimating α and β jointly. A single iteration ensures first-order asymptotic efficiency (Rothenberg and Leenders), while two iterations guarantee second-order efficiency (Rothenberg). Alternatively, the expressions in theorem 5 of LaFrance can be used to construct consistent estimates of the asymptotic covariances of $\hat{\alpha}$ and $\hat{\beta}$. Then the estimates $(\hat{\alpha}, \hat{\beta}, \hat{\Sigma}_{\hat{\alpha}}, \hat{\Sigma}_{\hat{\beta}})$, which are consistent under both the null and the alternative hypothesis, can be used to construct a Wu-Hausman test of zero correlation between y_x and $\tilde{\epsilon}_x$. A third alternative is to apply full-information maximum-likelihood procedures to (32′) directly by estimating α and β jointly with iterative nonlinear generalized least squares. However, when both iterative processes converge, the three-stage estimators and the iterative generalized nonlinear least squares estimators are asymptotically equivalent, fully efficient among the class of minimum distance estimators, and robust against the possibility that expenditure is endogenous.

The procedure proposed by Anderson is a nonlinear analogue to Theil's (1953) interpretation of two-stage least squares. That is, predicted values for the endogenous right-hand-side variables are formed in the first-stage regression and the predicted values replace their observed values in the second-stage regression. Also, if the conditional demands are nonlinear in expenditure, then Amemiya's (1974, 1975, 1983, 1985) instrumental variables interpretation of nonlinear two-stage least squares does not give consistent parameter estimates. Once again, the reason is that the adding-up condition implies that the structure of the demands can always be written with the means of the quantities demanded as functions of the mean level of expenditure. This implies that the conditional errors do not have zero means whenever the conditional demands are nonlinear in expenditure. Moreover, the mean vector of the conditional demand residuals depends on the model structure and the precise values of the explanatory variables. Therefore, simply absorbing the means of the conditional error terms into the intercepts of the demand equations will not solve this problem.

Conclusions

When is expenditure "exogenous" in separable demand models? Given the definitions of exogeneity of Engle, Hendry, and Richard the answer is never! Does this preclude the meaningful application of separability assumptions in empirical demand analysis? My reaction is equally emphatic—No it does not! What then, have we learned from the analysis in this article?

Some relevant information is always lost when econometric models are developed conditional on a subset of the choice variables. Given current data limitations, we may have to live with this as an unavoidable cost of feasible empirical work. But a lack of consideration for the interactions among the model's structure, the properties of the residuals, the appropriate estimation techniques, and a reasonable choice of the conditioning variables can lead to serious problems in empirical work.

The necessary conditions for expenditure even to be approximately uncorrelated with the error terms in a set of conditional demand equations are too restrictive to be plausible. But if one is willing to adopt a subset of these conditions as an initial point of departure, then standard simultaneous equations estimation methods for complete demand systems can be applied. But it is at least advisable to check whether or not the data are consistent with the assumptions. The appropriate test is the Wu-Hausman specification test.

The Wu-Hausman specification test requires a set of consistent estimates for the model parameters under the alternative hypothesis—i.e., expenditure is correlated with the conditional demand residuals. If the demand model is nonlinear in expenditure, the instrumental variables interpretation of nonlinear two-stage least squares does not produce consistent parameter estimates and an alternative approach to estimation is necessary.

Anderson's iterative nonlinear two-stage estimation procedure—a nonlinear application of Theil's (1953) interpretation of linear two-stage least squares—produces consistent estimates of the structural parameters. Anderson's method does not produce consistent estimates of the asymptotic covariance matrix for the parameter estimates, but the correct expressions are presented in theorem 5 of LaFrance. The iterative procedure is complex and computationally intensive, and there can be some difficulty with convergence of the iterative process. However, it offers a feasible solution to the simultaneity problem in conditional demand models and can be used to obtain good starting values for full-information maximum-likelihood or two-step linearized maximum-likelihood estimation procedures. A one-step linearized maximum-likelihood procedure that uses the final estimates from Anderson's iterative two-stage procedure as starting values produces consistent estimates of the structural parameters and their asymptotic covariance matrix that are first-order efficient. Two steps guarantee second-order efficiency of the estimates. An added attraction of this procedure is the fact that the parameter estimates are fully efficient whether or not expenditure is correlated with the conditional error terms. Although relatively complex and computationally intensive, this is a general and feasible solution to a complicated econometric problem.

Perhaps the most important implication of this article is the fact that there is a subtle but significant difference between structural recursivity and the concept of exogeneity in econometric models. As we have seen, weak separability of consumer preferences is equivalent to structural recursivity of the econometric demand model for the separable goods. But we have also found that expenditure is neither exogenous nor predetermined in conditional demand models. This conclusion applies with equal weight and for precisely the same reasons to, for example, the level of output in a set of conditional factor demand equations. Once the mean level of output has been explained, it is always possible to explain the mean levels of the factor inputs as functions of the input prices and the mean level of output. But this does not imply that output is predetermined or exogenous in a system of conditional factor demand equations. Notwithstanding the influence of weather and other uncontrollable random factors, the production technology requires that the actual level of output is determined by the actual choices for the inputs to the production process. Therefore, for precisely the same reasons as were identified for separable demand models, output will not be exogenous in conditional factor demand models.

[Received March 1990; final revision received November 1990.]

References

Alston, J. M., and J. A. Chalfant. "Can We Take the Con Out of Meat Demand Studies?" *West. J. Agr. Econ.* 16(1991):36–48.

———. "Weak Separability and a Test for the Specification of Income in Demand Models with an Application to the Demand for Meat in Australia." *Aust. J. Agr. Econ.* 31(1987):1–15.

Alston, J. M., C. A. Carter, R. Green, and D. Pick. "Wither Armington Trade Models?" *Amer. J. Agr. Econ.* 72(1990):455–67.

Amemiya, T. *Advanced Econometrics.* Cambridge: Harvard University Press, 1985.

———. "Nonlinear Regression Models." In *Handbook of Econometrics: Vol. I,* eds., Z. Griliches and M. Intriligator, pp. 333–97. New York: North-Holland, 1983.

———. "The Nonlinear Limited-Information Maximum-Likelihood Estimator and the Modified Nonlinear Two-Stage Least Squares Estimator." *J. Econometrics* 3(1975):375–86.

———. "The Nonlinear Two-Stage Least Squares Estimator." *J. Econometrics* 2(1974):105–10.

Anderson, R. W. "Perfect Price Aggregation and Empirical Demand Analysis." *Econometrica* 47(1979):1209–30.

Attfield, C. "Homogeneity and Endogeneity in Systems of Demand Equations." *J. Econometrics* 27(1985): 197–209.

Attfield, C. L. F., and M. J. Browning. "A Differential Demand System, Rational Expectations, and the Life Cycle Hypothesis." *Econometrica* 53(1985):31–48.

Barten, A. P. "Estimating Demand Equations." *Econometrica* 36(1968):213–51.

Barten, A., and V. Boehm. "Consumer Theory." In *Handbook of Mathematical Economics, Vol. II,* eds., K. Arrow and M. Intriligator, pp. 381–427. New York: North-Holland, 1983.

Becker, G. S. "A Theory of the Allocation of Time. *Econ. J.* 75(1965):493–517.

Blackorby, C., R. Boyce, and R. R. Russell. "Estimation of Demand Systems Generated by the Gorman Polar Form; A Generalization of the S-Branch Utility Tree." *Econometrica* 46(1978):345–63.

Blackorby, C., D. Primont, and R. Russell. *Duality, Separability, and Functional Structure: Theory and Economic Applications.* New York: North-Holland, 1978.

Blundell, R. "Consumer Behaviour: Theory and Empirical Evidence—A Survey." *The Econ. J.* 98(1988):16–65.

———. "Econometric Approaches to the Specification of Life-Cycle Labour Supply and Commodity Demand Behaviour." *Econometric Rev.* 5(1986):89–146.

Brown, M., and D. Heien. "The S-Branch Utility Tree: A Generalization of the Linear Expenditure System." *Econometrica* 40(1972):737–47.

Browning, M., A. Deaton, and M. Irish. "A Profitable Approach to Labor Supply and Commodity Demands Over the Life Cycle." *Econometrica* 53(1985):503–43.

Capps, O., Jr., and J. D. Schmitz. "A Recognition of Health and Nutrition Factors in Food Demand Analysis." *West. J. Agr. Econ.* 16(1991):21–35.

Choi, S., and K. Sosin. "Testing for Structural Change: The Demand for Meat." *Amer. J. Agr. Econ.* 72(1990): 227–36.

Clements, K. "The Demand for Groups of Goods and Conditional Demand." In *Applied Demand Analysis: Evidence from the Systemwide Approach,* eds., H. Theil and K. Clements, pp. 163–84. New York: North-Holland, 1987.

Clements, K., and E. Selvanathan. "Alcohol Consumption." In *Applied Demand Analysis: Evidence from the Systemwide Approach,* eds., H. Theil and K. Clements, pp. 185–264. New York: North-Holland, 1987.

Deaton, A. "Demand Analysis." In *Handbook of Econometrics, Vol. III,* eds., Z. Griliches and M. Intriligator, pp. 1767–839. New York: North-Holland, 1986.

———. *Models and Projections of Demand in Post War Britain.* London: Chapman and Hall, 1975.

Deaton, A., and J. Muellbauer. *Economics and Consumer Behavior.* Cambridge: Cambridge University Press, 1980.

Engle, R. F., D. F. Hendry, and J.-F. Richard. "Exogeneity." *Econometrica* 51(1983):277–304.

Goldfield, S., and R. Quandt. "Nonlinear Simultaneous Equations: Estimation and Prediction." *Int. Econ. Rev.* 9(1968):113–36.

Gorman, W. "On a Class of Preference Fields." *Metroeconomica* 13(1961):53–56.

———. "Separable Utility and Aggregation." *Econometrica* 27(1959):469–81.

———. "Some Engel Curves." In *Essays in the Theory and Measurement of Consumer Behaviour in Honor of Sir Richard Stone,* ed., A. Deaton, pp. 7–29. Cambridge: Cambridge University Press, 1981.

———. "Two Stage Budgeting." Unpublished paper, London School of Economics and Political Science, 1971.

Hausman, J. "Specification Tests in Econometrics." *Econometrica* 46(1978):1251–71.

Heien, D. "Some Further Results on the Estimation of the S-Branch Utility Tree." Research Discussion Paper No. 10, Research Division, Office of Prices and Living Conditions, U.S. Bureau of Labor Statistics, Washington DC, 1973.

Heien, D., and G. Pompelli. "The Demand for Beef Products: Cross-Section Estimation of Demographic and Economic Effects." *West. J. Agr. Econ.* 13(1988):37–44.

LaFrance, J. T. "Weak Separability and Two-Stage Budgeting: Problems of Estimation and Welfare Management." Paper presented at the Australasian meeting of the Econometric Society, 1989.

LaFrance, J. T., and W. M. Hanemann. "The Dual Structure of Incomplete Demand Systems." *Amer. J. Agr. Econ.* 71(1989):262–74.

Lewbel, A. "A Demand System Rank Theorem." *Econometrica* 57(1989a):701–05.

———. "Characterizing Some Gorman Engel Curves." *Econometrica* 55(1987):1451–59.

———. "Exact Aggregation and a Representative Consumer." *Quart. J. Econ.* 104(1989b):622–33.

———. "Full Rank Demand Systems." *Int. Econ. Rev.* 32(1990):289–300.

Malinvaud, E. *Statistical Methods of Econometrics,* 2nd ed. New York: North-Holland, 1970.

Murray, J. "Retail Demand for Meat in Australia: A Utility Theory Approach." *The Econ. Record* 60(1984): 45–56.

Pollak, R. "Conditional Demand Functions and Consumption Theory." *Quart. J. Econ.* 83(1969):60–78.

———. "Conditional Demand Functions and the Implications of Separable Utility." *S. Econ. J.* 37(1971): 423–33.

Prais, S. "A Comment." *Econometrica* 27(1959):127–29.

Primont, D. *Functional Structure and Economic Decision Making.* Ph.D. dissertation, University of California, Santa Barbara, 1970.

Richardson, R. "Structural Estimates of Domestic Demand for Agricultural Products in Australia: A Review." *Rev. Market. and Agr. Econ.* 44(1976):172–200.

Rothenberg, T. "Second Order Efficiency of Estimators and Tests in Simultaneous Equations Models." Paper presented at the European meeting of the Econometric Society, Athens, 1979.

Rothenberg, T. J., and C. T. Leenders. "Efficient Estimation of Simultaneous Equation Systems." *Econometrica* 32(1964):57–76.

Safyurtlu, A. N., S. R. Johnson, and Z. A. Hassan. "Recent Evidence on Market Demand Systems for Food in Canada." *Can. J. Agr. Econ.* 34(1986):475–93.

Summers, R. "A Note on Least Squares Bias in Household Expenditure Analysis." *Econometrica* 27(1959): 121–26.

Theil, H. "An Economic Theory of the Second Moments of Disturbances of Behavioral Equations." *Amer. Econ. Rev.* 61(March 1971):190–94.

———. "Repeated Least-Squares Applied to Complete Equations Systems." Mimeo. The Hague: Central Planning Bureau, 1953.

———. *The Systemwide Approach to Microeconomics.* Oxford: Basil Blackwell, 1980.

———. "The Theory of Rational Random Behavior and its Application to Demand Analysis." *Eur. Econ. Rev.* 6(1975):217–26.

———. *Theory and Measurement of Consumer Demand, Vol. II.* Amsterdam: North-Holland, 1976.

Van Kooten, G. C. "Economic Impacts of Supply Management: Review and Comparison of Alternative Measures of Consumer Welfare Loss." *Can. J. Agr. Econ.* 36(1988):425–41.

Wu, D.-M. "Alternative Tests of Independence Between Stochastic Regressors and Disturbances." *Econometrica* 41(1973):733–50.