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A decomposition formula for option prices in the Heston model and applications to option pricing approximation

Elisa Alòs*
Dpt. d'Economia i Empresa
Universitat Pompeu Fabra
c/Ramon Trias Fargas, 25-27
08005 Barcelona, Spain
elisa.alos@upf.edu

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Abstract

By means of classical Itô's calculus we decompose option prices as the sum of the classical Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility plus a term due by correlation and a term due to the volatility of the volatility. This decomposition allows us to develop first and second-order approximation formulas for option prices and implied volatilities in the Heston volatility framework, as well as to study their accuracy. Numerical examples are given.

Keywords: Stochastic volatility, Heston model, Itô's calculus.

JEL Classification: G13

Mathematics Subject Classification (2000): 91B28, 91B70

1 Introduction

Stochastic volatility models are a natural extension of the classical Black-Scholes model that have been introduced as a way to manage the *skew* and *smiles* observed in real market data (see for example Hull and White (1987), Scott (1987), Stein and Stein (1991), Ball and Roma (1994) and Heston (1993)). The study of these models have introduced new important mathematical and practical challenges, in particular related with the option pricing problem and

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the callibration of the corresponding parameters. In fact, we do not have closed-form option pricing formulas for the majority of the stochastic volatility models and, even in the case when closed-form pricing solutions can be derived (see for example Heston (1993) or Schöbel and Zhu (1999)), they do not allow in general for fast callibration of the parameters.

A recent trend in the literature has been the development of approximate closed-form option pricing formulas. To this end, some authors have presented a perturbation analysis of the corresponding PDE with respect to a specific model parameter, like the volatility (see Hagan, Kumar, Lesniewski and Woodward (2008)), the mean reversion (see Fouque, Papanicolau and Sircar (2000) and Fouque, Papanicolau, Sircar and Solna (2003)) or the correlation (see Antonelli and Scarletti (2008)). In all these techniques, the region of validity of the results is restricted to either short or long maturities.The obtained approximations for option prices allow for fast callibration and give a better understanding of the role of model parameters. More recently, another approach have been proposed by Benhamou, Gobet and Miri (2009a, 2009b and 2009c), where the authors focus directly on the law of the log-stock price at maturity time, given its initial condition. They expand prices with respect o the volatility of the volatility, computing the correction terms using Malliavin calculus. This approach allows the authors to deal with with short and long time maturities, as well as with time-dependent coefficients. Another point of view has been presented in Alòs (2006), where by means of Malliavin calculus the author extends the classical Hull and White formula by decomposing option prices as the sum of the same derivative price if there where no correlation and a correction due by correlation. As an application, the author develop a method to construct first-order option pricing approximation formulas that only needs some regularity conditions (in the Malliavin calculus sense) of the volatility process and that can be applied for a very general class of volatility models, including the case of long-memory volatilities.

Even when the conditions required in Alòs (2006) are satisfied by the majority of stochastic volatility models, they are not trivial in the case of the Heston model. In Alòs and Ewald (2008) the authors studied the Malliavin differentiability of the Heston volatility to adapt the results in Alòs (2006) to the Heston case, but unfortunately the accuracy of the approximation could be proved only in the case when the dimension δ of the underlying Bessel process is greather than 6.

This paper is devoted to obtain a new decomposition formula for option prices, similar to the one presented in Alòs (2006), but valid even when the Malliavin regularity conditions needed in this work are not satisfied. Instead of expanding option prices around the Hull and White term by means of anticipating stochastic calculus (Malliavin calculus), we will use classical Itô formula to expand prices around the classical Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility. This will allow us to describe option prices as the sum of this last term plus a term due to the correlation and a term due to the volatility of the volatility. This method needs only some general integrability conditions that are satisfied by the Heston model and then it allows us to extend the results in Alòs and Ewald (2008) to the case $\delta > 2$ and to prove, in the case $\delta > 3$, a new second-order approximation formula. Even when the paper is focused in the Heston case, the results can be easily extended to other volatility models with good integrability conditions.

The paper is organized as follows. In Section 2 we introduce the main notations and hipotheses and we prove our decomposition formula for option prices. In Section 3 we use the results in Section 2 to obtain a first-order and a second-order option pricing approximation formulas. Some numerical examples are presented in Section 4. The main conclusions are summarized in Section 5.

2 A decomposition formula for option prices

We will consider the Heston model for stock prices in a time interval $[0,T]$ under a risk neutral probability P^* :

$$dS_t = rS_t dt + \sigma_t S_t \left(\rho dW_t^* + \sqrt{1-\rho^2} B_t^*\right),\ t \in [0,T], \tag{1}$$

where

$$d\sigma_t^2 = \kappa\left(m - \sqrt{\sigma_t^2}\right) dt + \nu\sqrt{\sigma_t^2} dW_t^*$$

where r is the instantaneous interest rate (supposed to be constant), W_t^* and B_t^* are independent standard Brownian motions defined in a probability space $(\Omega, \mathcal{F}, P)$ and κ, θ and ν are constants satisfying the Novikov condition $2\kappa\theta > \nu^2$. In the following we will denote by $\mathcal{F}^{W^*}, \mathcal{F}^{B^*}$ the filtrations generated respectively by W^* and B^*. Moreover we define $\mathcal{F} := \mathcal{F}^{W^*} \vee \mathcal{F}^{B^*}$. It will be convenient in the following sections to make the change of variable $X_t = \log(S_t),\ t \in [0,T]$. It is well-known that the price of a contingent claim of the form $h(X_T)$ at time t is given by

$$V_t = e^{-r(T-t)} E^* \left[h(X_T) \middle| \mathcal{F}_t \right], \tag{2}$$

where E^* denotes the expectation with respect to P^*.

We will make use of the following notation

- $v_t^2 = \frac{1}{T-t}\int_t^T E^*\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds$. That is, v_t^2denotes the root-mean square time future average volatility
- $M_t = \int_0^T E^*\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds$. Notice that $v_t^2 = \frac{1}{T-t}\left(M_t - \int_0^t \sigma_s^2 ds\right)$.Moreover, we recall that $dM_t = \nu\sqrt{\sigma_t^2}\left(\int_t^T e^{-\kappa(s-t)} ds\right) dW_t$
- For any $\tau > 0$, $p(x,\tau)$ will denote the centered Gaussian kernel with variance τ^2. If $\tau = 1$ we will write $p(x)$.

- $BS(t,x,\sigma)$ will denote the price of an european call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price K and interest rate r. Remember that in this case

$$BS(t,x,\sigma) = e^x N(d_+) - Ke^{-r(T-t)}N(d_-),$$

where N denotes the cumulative probability function of the standard normal law and

$$d_{\pm} := \frac{x - x_t^*}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t},$$

with $x_t^* := \ln K - r(T-t)$.

- $\mathcal{L}_{BS}(\sigma)$ will denote the Black-Scholes differential operator (in the log variable) with volatility σ :

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} - r\cdot$$

It is well known that $\mathcal{L}_{BS}(\sigma)\,BS(\cdot,\cdot;\sigma) = 0$.

- $G(t,x,\sigma) := (\partial_{xx}^2 - \partial_x)BS(t,x,\sigma)$.

The next result is similar to Lemma 4.1 in Alòs, León and Vives (2007):

Lemma 1 *Let $0 \le t \le s \le T$ and $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^{W^*}$. Then for every $n \ge 0$, there exists $C = C(n,\rho)$ such that*

$$\left|E^*\left(\left.\partial_x^n G(s,X_s,v_s)\right|\mathcal{G}_t\right)\right| \le C\left(\int_s^T E\left(\left.\sigma_\theta^2\right|\mathcal{F}_s\right)d\theta\right)^{-\frac{1}{2}(n+1)}.$$

Proof. A simple calculation gives us that

$$G(s,X_s,v_s) = Ke^{-r((T-t)}p\left(X_s - \mu, v_s\sqrt{T-s}\right),$$

where $\mu = \ln K - \left(r - v_s^2/2\right)(T-s)$.This allows us to write

$$E(\partial_x^n G(s,X_s,v_s)\big|\mathcal{G}_t) = (-1)^n\, Ke^{-r(T-s)}\partial_\mu^n E(p(X_s-\mu, v_s\sqrt{T-s})\big|\mathcal{G}_t). \quad (3)$$

Being X_s conditioned by $\mathcal{G}_t$ a normal random variable with mean equal to

$$\phi = X_t + \int_t^s \left(r - \sigma_\theta^2/2\right)d\theta + \rho\int_t^s \sigma_\theta dW_\theta$$

and variance equal to $\left(1-\rho^2\right)\int_t^s \sigma_\theta^2 d\theta$, and using the semigroup property of the Gaussian density function it follows that

$$
\begin{aligned}
& E\left(p\left(X_s-\mu, v_s\sqrt{T-s}\right) \middle| \mathcal{G}_t\right) \\
= & \int_R p\left(y-\mu, v_s\sqrt{T-s}\right) p\left(y-\phi, \sqrt{\left(1-\rho^2\right)\int_t^s \sigma_\theta^2 d\theta \pm}\right) dy \\
= & p\left(\phi-\mu, \sqrt{\int_s^T E\left(\sigma_r^2 \middle| \mathcal{F}_s\right) dr + \left(1-\rho^2\right)\int_t^s \sigma_\theta^2 d\theta}\right)
\end{aligned}
$$

Putting this result in (3), we have

$$
\begin{aligned}
& E(\partial_x^n G(s, X_s, v_s) \middle| \mathcal{G}_t) \\
= & (-1)^n K e^{-r(T-s)} \partial_\mu^n p\left(\phi-\mu, \sqrt{\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s\right) d\theta + \left(1-\rho^2\right)\int_t^s \sigma_\theta^2 d\theta}\right)
\end{aligned}
$$

A simple calculation and the fact that, for every positive constants c, d the function $x^c e^{-dx}$ is bounded, give us that

$$
\begin{aligned}
& |\partial_\mu^n p\left(\phi-\mu, \sqrt{\left(1-\rho^2\right)\int_t^T \sigma_s^2 ds + \rho^2 \int_s^T \sigma_s^2 ds}\right)| \\
\leq & C\left(\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s\right) d\theta + \left(1-\rho^2\right)\int_t^s \sigma_\theta^2 d\theta\right)^{-\frac{1}{2}(n+1)} \\
\leq & C\left(\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s\right) d\theta\right)^{-\frac{1}{2}(n+1)},
\end{aligned}
$$

as we wanted to prove. ■

Now we are in a position to prove the main result of this section.

Theorem 2 *(Decomposition formula) Assume the model (1), where the volatility process $\sigma = \{\sigma_s, s \in [0,T]\}$ satisfies the Novikov condition $2\kappa\theta > \nu^2$. Then, for all $t \in [0,T]$*

$$
\begin{aligned}
V_t = & BS\left(t, X_t; v_t\right) \\
& + \frac{1}{2} E^*\left(\int_t^T e^{-r(s-t)} H\left(s, X_s, v_s\right) \sigma_s d\left\langle M, W^*\right\rangle_s \middle| \mathcal{F}_t\right) \\
& + \frac{1}{2} E^*\left(\int_t^T e^{-r(s-t)} K\left(s, X_s, v_s\right) d\left\langle M, M\right\rangle_s \middle| \mathcal{F}_t\right), \qquad (4)
\end{aligned}
$$

where

$$H(s, X_s, v_s) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS(s, X_s, v_s)$$

and

$$K(s, X_s, v_s) := \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) BS(s, X_s, v_s).$$

Proof. Notice that $BS(T, X_T; v_T) = V_T$. As $e^{-rt}V_t$ is a P^*-martingale we can then write

$$e^{-rt}V_t = E^*\left(e^{-rT}V_T \middle| \mathcal{F}_t\right) = E^*\left(e^{-rT}BS(T, X_T; v_T) \middle| \mathcal{F}_t\right). \tag{5}$$

Now our idea is to apply Itô's formula to the process $e^{-rt}BS(t, X_t; v_t)$. As the derivatives of $BS(t, x; y)$ are not bounded we will make use of an approximating argument. Take $\delta > 0$ and consider the process

$$e^{-rt}BS\left(t, X_t; v_t^\delta\right),$$

where $v_t^\delta := \sqrt{\frac{1}{T-t}\left(\delta + \int_t^T E^*\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds\right)}$. Notice that

$$v_t^\delta = \sqrt{\frac{1}{T-t}\left(\delta + M_t - \int_0^t \sigma_s^2 ds\right)}.$$

Applying classical Itô's formula and the relationship between the *Gamma*, the *Vega* and the *Delta* we deduce that

$$\begin{aligned}
& e^{-rT}C_{BS}\left(T, X_T; v_T^\delta\right) \\
= \; & e^{-rt}C_{BS}\left(t, X_t; v_t^\delta\right) \\
& + \int_t^T e^{-rs}\left(\mathcal{L}_{BS}(v_s) + \frac{1}{2}\left(\sigma_s^2 - \left(v_s^\delta\right)^2\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\right) BS\left(s, X_s, v_s^\delta\right) ds \\
& + \int_t^T e^{-rs}\left(\frac{\partial BS}{\partial x}\right)\left(s, X_s, v_s^\delta\right)\sigma_s\left(\rho dW_t^* + \sqrt{1-\rho^2}Z_t^*\right) \\
& + \frac{1}{2}\int_t^T e^{-rs}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(s, X_s, v_s^\delta\right) dM_s \\
& + \frac{1}{2}\int_t^T e^{-rs}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS(s, X_s, v_s)\,\sigma_s d\langle M, W^*\rangle_s \\
& + \frac{1}{2}\int_t^T e^{-rs}\left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) BS(s, X_s, v_s)\, d\langle M, M\rangle_s \\
& - \frac{1}{2}\int_t^T e^{-rs}\frac{\partial BS}{\partial \sigma}\left(s, X_s, v_s^\delta\right)\frac{\left(\sigma_s^2 - \left(v_s^\delta\right)^2\right)}{v_s^\delta (T-s)} ds,
\end{aligned}$$

that is,

$$
\begin{aligned}
& e^{-rT} BS\left(T, X_T; v_T^{\delta}\right) \\
= \; & e^{-rt} BS\left(t, X_t; v_t^{\delta}\right) \\
& + \int_t^T e^{-rs} \left(\frac{\partial BS}{\partial x}\right) \left(s, X_s, v_s^{\delta}\right) \sigma_s \left(\rho dW_t^* + \sqrt{1-\rho^2} Z_t^*\right) \\
& + \frac{1}{2} \int_t^T e^{-rs} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(s, X_s, v_s^{\delta}\right) dM_s \\
& + \frac{1}{2} \int_t^T e^{-rs} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS\left(s, X_s, v_s\right) \sigma_s d\left\langle M, W^* \right\rangle_s \\
& + \frac{1}{2} \int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) BS\left(s, X_s, v_s\right) d\left\langle M, M \right\rangle_s
\end{aligned}
$$

Taking now conditional expectations and multiplying by e^{rt} we obtain that

$$
\begin{aligned}
& E^*\left[BS\left(T, X_T; v_T^{\delta}\right) \middle| \mathcal{F}_t \right] \\
= \; & BS\left(t, X_t; v_t^{\delta}\right) \\
& + \frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} H\left(s, X_s, v_s^{\delta}\right) \sigma_s d\left\langle M, W^* \right\rangle_s \middle| \mathcal{F}_t \right) \\
& + \frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} K\left(s, X_s, v_s^{\delta}\right) d\left\langle M, M \right\rangle_s \middle| \mathcal{F}_t \right)
\end{aligned}
$$

Letting now $\delta \to 0$, using the facts that $d\left\langle M, W^* \right\rangle_s = \nu\rho\sigma_s \left(\int_s^T e^{-\kappa(r-s)} dr\right) ds$, $d\left\langle M, M \right\rangle_s = \nu^2 \sigma_s^2 \left(\int_s^T e^{-\kappa(r-s)} dr\right)^2 ds$, Lemma 1 and the dominated convergence theorem the result follows. ■

Remark 3 *The proof of the above Theorem only uses some integrability and regularity conditions on the volatility process and then it can be extended to other volatility models, even non-Markovian or non-continuous volatilities.*

Remark 4 *Formula (4) gives us a tool to describe the impact of the correlacion and he volatility of the volatility on option prices. Notice that the second term in the right-hand side of (4) becomes zero in the uncorrelated case $\rho = 0$.*

3 Approximate option pricing formulas

This section is devoted to present a first-order and a second-order approximation for option prices in the Heston volatility framework and to study their accuracy. For the sake of simplicity we will assume the maturity time $T - t < 1$, which is a reasonable assumption from the financial point of view, as market parameters are usually denoted on a year scale and maturity times are mostly less than one year.

The following Lemma is proved in Bossy (2004).

Lemma 5 *(Bossy, Lemma A.1) Let* $\delta := \frac{4\kappa\theta}{\nu^2} \geq 4$. *Take* $n \leq \delta - 2$. *Then, for all* $(s,t) \in [0,T]$ *with* $s < t$

$$E\left(\frac{1}{\sigma_s^n}\middle|\mathcal{F}_t\right) \leq C_n(T,\sigma_t),$$

where $C_n(T,\sigma_t)$ *is a positive constant non-decreasing as a function of* T.

We will need a similar result in the case $\delta < 4$.

Lemma 6 *Assume the Novikov condition* $2\kappa\theta > \nu^2$. *Assume* $\delta := \frac{4\kappa\theta}{\nu^2} < 4$. *Then, for all* $(s,t) \in [0,T]$ *with* $s < t$ *and for all* $p < \frac{2}{4-\delta}$

$$E\left(\frac{1}{\sigma_s^2}\middle|\mathcal{F}_t\right) \leq \frac{C(T,\sigma_t)}{\left[(s-t)^2\,\nu^2\left[p\left(\delta/2-2\right)+1\right]\right]^{\frac{1}{p}}},$$

where $C(T,\sigma_t)$ *is a positive constant non-decreasing as a function of* T.

Proof. For the sake of simplicity we can take $t = 0$. From he proof of Lemma A.1 in Bossy (2004) we know that

$$E\left(\frac{1}{\sigma_s^2}\right) \leq \frac{C}{L(s)}\int_0^1 (1-u)^{2\kappa\theta/\nu^2-2}\exp\left(-\frac{\sigma_0 e^{-\kappa s}u}{2L(s)}\right)du$$

: .where $L(s) := \frac{\nu^2}{4\kappa}(1-e^{-\kappa s})$.Now, by Hölder inequality we know that, for all $\frac{1}{p}+\frac{1}{q} = 1$ such that $\frac{1}{p} > 2 - 2\kappa\theta/\nu^2 = 2 - \delta/2$

$$\begin{aligned}
E\left(\frac{1}{\sigma_s^2}\right) &\leq \frac{C}{L(s)}\left(\int_0^1 (1-u)^{p(\delta/2-2)}du\right)^{\frac{1}{2}}\left(\int_0^1 \exp\left(-\frac{\sigma_0 e^{-\kappa s}u}{L(s)}\right)du\right)^{\frac{1}{2}} \\
&\leq \frac{C}{L(s)\left[p\left(\delta/2-2\right)+1\right]^{\frac{1}{p}}}\left(\frac{q\sigma_0 e^{-\kappa s}}{L(s)}\right)^{-\frac{1}{q}} \\
&\leq \frac{C(T,\sigma_0)}{L(s)^{1-1/q}\left[p\left(\delta/2-2\right)+1\right]^{\frac{1}{p}}} \\
&\leq \frac{C(T,\sigma_0)}{\nu^{2(1-1/q)}(1-e^{-\kappa s})^{1-1/q}\left[p\left(\delta/2-2\right)+1\right]^{\frac{1}{p}}}.
\end{aligned}$$

Now, using that $(1-e^{-\kappa s}) \geq ske^{-\kappa s}$ it follows that

$$E\left(\frac{1}{\sigma_s^2}\right) \leq \frac{C(T,\sigma_0)}{s^{1-1/q}\nu^{2(1-1/q)}\left[p\left(\delta/2-2\right)+1\right]^{\frac{1}{p}}},$$

and now the proof is complete. ■

Now we are in a position to prove our first approximation result.

Theorem 7 *(First-order approximation formula). Assume the model (1), where the volatility process* $\sigma = \{\sigma_s, s \in [0,T]\}$ *satisfies the Novikov condition* $2\kappa\theta > \nu^2$*. Then, if* $\delta \geq 4$*, for all* $t \in [0,T]$ *such that* $T - t < 1$

$$\begin{aligned} & \left| V_t - BS\left(t, X_t; v_t\right) - \frac{1}{2} H\left(t, X_t, v_t\right) E^* \left(\int_t^T \sigma_s d \left\langle M, W^* \right\rangle_s \middle| \mathcal{F}_t \right) \right| \\ \leq \quad & C(T, \sigma_t) \nu^2 (T-t)^{\frac{3}{2}} \end{aligned} \tag{6}$$

Moreover, if $\delta < 4$

$$\begin{aligned} & \left| V_t - BS\left(t, X_t; v_t\right) - \frac{1}{2} H\left(t, X_t, v_t\right) E^* \left(\int_t^T \sigma_s d \left\langle M, W^* \right\rangle_s \middle| \mathcal{F}_t \right) \right| \\ \leq \quad & C(T, \sigma_t) \nu^{2-2\sqrt{2-\delta/2}} \left(\frac{1}{1-\sqrt{2-\delta/2}} \right)^{1+\sqrt{2-\delta/2}} \\ & \times \left[(T-t)^{\frac{1}{2}\left(3-\sqrt{2-\delta/2}\right)} + (T-t)^{2\left(1-\sqrt{2-\delta/2}\right)} \right], \end{aligned} \tag{7}$$

where $C(T, \sigma_t)$ *is a positive constant non-decreasing as a function of* T*.*

Proof. Consider the process $e^{-rt} H\left(t, X_t; v_t\right) U_t$, where

$$U_t := E^* \left(\int_t^T \sigma_s d \left\langle M, W^* \right\rangle_s \middle| \mathcal{F}_t \right).$$

It is easy to check that

$$e^{-rT} H\left(T, X_T; v_T\right) U_T = 0.$$

Then, the same arguments as in the proof of Theorem 1 allow us to write

$$\begin{aligned} 0 = & \, H\left(t, X_t; v_t\right) U_t \\ & - \frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} H\left(s, X_s, v_s\right) \sigma_s d \left\langle M, W^* \right\rangle_s \middle| \mathcal{F}_t \right) \\ & + \frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s^\delta\right) U_s \sigma_s d \left\langle M, W^* \right\rangle_s \middle| \mathcal{F}_t \right) \\ & + \frac{1}{2} E^* \left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s^\delta\right) U_s d \left\langle M, M \right\rangle_s \middle| \mathcal{F}_t \right). \end{aligned}$$

This, together with (4), gives us that

$$
\begin{aligned}
V_t &= BS\left(t, X_t; v_t\right) + H\left(t, X_t; v_t\right) U_t \\
&+ \frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s^\delta\right) U_s \sigma_s d\left\langle M, W^* \right\rangle_s \middle| \mathcal{F}_t \right) \\
&+ \frac{1}{2} E^* \left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s^\delta\right) U_s d\left\langle M, M \right\rangle_s \middle| \mathcal{F}_t \right) \\
&+ \frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} K\left(s, X_s, v_s\right) d\left\langle M, M \right\rangle_s \middle| \mathcal{F}_t \right) \\
&= BS\left(t, X_t; v_t\right) + H\left(t, X_t; v_t\right) U_t + T_1 + T_2 + T_3.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\left|U_s\right| &\leq \nu\rho E^* \left(\int_s^T \sigma_r^2 \left(\int_r^T e^{-\kappa(u-r)} du \right) dr \middle| \mathcal{F}_t \right) \\
&= \nu\rho \int_s^T E^*\left(\sigma_r^2 \middle| \mathcal{F}_s \right) \left(\int_r^T e^{-\kappa(u-r)} du \right) dr
\end{aligned}
$$

Then, Lemma 1 gives us that

$$
\begin{aligned}
T_1 &\leq \frac{\nu^2\rho^2}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s \right) d\theta \right)^{-\frac{5}{2}} \right. \\
&\quad \left. \times \left(\int_s^T E^*\left(\sigma_r^2 \middle| \mathcal{F}_s \right) \left(\int_r^T e^{-\kappa(u-r)} du \right) dr \right) \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} du \right) ds \middle| \mathcal{F}_t \right) \\
&\leq \frac{\nu^2\rho^2}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s \right) d\theta \right)^{-\frac{3}{2}} \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} du \right)^2 ds \middle| \mathcal{F}_t \right)
\end{aligned}
$$

Taking into account that $\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s \right) d\theta \geq \sigma_s^2 \int_s^T e^{-\kappa(r-s)} dr$ it follows that

$$
\begin{aligned}
T_1 &\leq \frac{\nu^2\rho^2}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\sigma_s^2 \int_s^T e^{-\kappa(r-s)} dr \right)^{-\frac{3}{2}} \right. \\
&\quad \left. \times \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} du \right)^2 ds \middle| \mathcal{F}_t \right) \\
&\leq \frac{\nu^2\rho^2}{2} \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-1} \middle| \mathcal{F}_t \right) \left(\int_s^T e^{-\kappa(u-s)} du \right)^{\frac{1}{2}} ds \\
&\leq \frac{\nu^2\rho^2}{2} \int_t^T e^{-r(s-t)} \sqrt{E^*\left(\sigma_s^{-2} \middle| \mathcal{F}_t \right)} \left(\int_s^T e^{-\kappa(u-s)} du \right)^{\frac{1}{2}} ds.
\end{aligned}
$$

Now, Lemma 5 gives us that, if $\delta \geq 4$,

$$\begin{aligned} T_1 &\leq C(T,\sigma_t)\nu^2\rho^2 \int_t^T e^{-r(s-t)} \left(\int_s^T e^{-\kappa(u-s)} du \right)^{\frac{1}{2}} ds \\ &\leq C(\sigma_t)\nu^2\rho^2 (T-t)^{\frac{3}{2}}. \end{aligned}$$

And, if $\delta < 4$,by Lemma 6

$$\begin{aligned} T_1 &\leq \frac{C(T,\sigma_t)\nu^2\rho^2}{\nu^{1/p}\left[p\left(\delta/2-2\right)+1\right]^{\frac{1}{2p}}} \int_t^T \frac{e^{-r(s-t)}}{(s-t)^{1/2p}} \left(\int_s^T e^{-\kappa(u-s)} du \right)^{\frac{1}{2}} ds \\ &\leq \frac{C(\sigma_t)\nu^{2-1/p}\rho^2}{\left[p\left(\delta/2-2\right)+1\right]^{\frac{1}{2p}}} (T-t)^{\frac{3}{2}-1/2p}. \end{aligned}$$

Then, taking $p = \frac{1}{\sqrt{2-\delta/2}}$ we obtain

$$T_1 \leq \frac{C(T,\sigma_t)\rho^2\nu^{2-\sqrt{2-\delta/2}}}{\left[\sqrt{2-\delta/2}+1\right]^{\frac{\sqrt{2-\delta/2}}{2}}} (T-t)^{\frac{1}{2}\left(3-\sqrt{2-\delta/2}\right)}$$

On the other hand, the same arguments gives us that

$$\begin{aligned} T_2 &\leq \frac{\nu^2\rho^2}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s \right) d\theta \right)^{-3} \right. \\ &\quad \times \left. \left(\int_s^T E^*\left(\sigma_r^2 \middle| \mathcal{F}_s \right) \left(\int_r^T e^{-\kappa(u-r)} du \right) dr \right) \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} du \right)^2 ds \middle| \mathcal{F}_t \right) \\ &\leq \frac{\nu^2\rho^2}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\sigma_s^2 \int_s^T e^{-\kappa(r-s)} dr \right)^{-2} \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} du \right)^3 ds \middle| \mathcal{F}_t \right) \\ &\leq \frac{\nu^2\rho^2}{2} \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-2} \middle| \mathcal{F}_t \right) \left(\int_s^T e^{-\kappa(r-s)} dr \right) ds. \end{aligned}$$

Then, if $\delta \geq 4$

$$T_2 \leq C(T,\sigma_t)\nu^2\rho^2(T-t)^2,$$

and assuming that $T-t<1$,

$$T_2 \leq C(T,\sigma_t)\nu^2\rho^2(T-t)^{\frac{3}{2}}.$$

If $\delta < 4$

$$\begin{aligned} T_2 &\leq C(T,\sigma_t)\nu^2\rho^2 \int_t^T e^{-r(s-t)} E^* \left(\sigma_s^{-2} \middle| \mathcal{F}_t \right) \left(\int_s^T e^{-\kappa(r-s)} dr \right) ds \\ &\leq \frac{C(T,\sigma_t)\nu^2\rho^2}{\nu^{2/p} \left[p\left(\delta/2-2\right)+1 \right]^{\frac{1}{p}}} \int_t^T \frac{e^{-r(s-t)}}{(s-t)^{1/p}} \left(\int_s^T e^{-\kappa(u-s)} du \right) ds \\ &\leq \frac{C(T,\sigma_t)p\nu^2\rho^2}{(p-1)\nu^{2/p} \left[p\left(\delta/2-2\right)+1 \right]^{\frac{1}{p}}} (T-t)^{2-1/p} \\ &= C(T,\sigma_t)\nu^2\rho^2(T-t)^2 \left(\frac{p}{p-1} \right) \left(\frac{1}{\nu^2(T-t)\left[p\left(\delta/2-2\right)+1 \right]} \right)^{\frac{1}{p}}, \end{aligned}$$

and then, taking $p = \frac{1}{\sqrt{2-\delta/2}}$ it follows that

$$T_2 \leq C(T,\sigma_t)\rho^2 \left[\nu(T-t) \right]^{2-2\sqrt{2-\delta/2}} \left(\frac{1}{1-\sqrt{2-\delta/2}} \right)^{1+\sqrt{2-\delta/2}}$$

Finally,

$$\begin{aligned} T_3 &\leq \frac{\nu^2}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \middle| \mathcal{F}_s \right) d\theta \right)^{-\frac{3}{2}} \right. \\ &\quad \left. \times\ \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} du \right)^2 ds \middle| \mathcal{F}_t \right) \\ &\leq C\nu^2 E^* \left(\int_t^T e^{-r(s-t)} \left(\sigma_s^2 \int_s^T e^{-\kappa(r-s)} dr \right)^{-\frac{3}{2}} \right. \\ &\quad \left. \times\ \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} du \right)^2 ds \middle| \mathcal{F}_t \right) \\ &\leq C\nu^2 \int_t^T E^* \left(\sigma_s^{-1} \middle| \mathcal{F}_t \right) \left(\int_s^T e^{-\kappa(r-s)} dr \right)^{\frac{1}{2}} ds \end{aligned}$$

Then, using the same arguments as for T_1 it follows that, if $\delta \geq 4$

$$T_3 \leq C(T,\sigma_t)\nu^2(T-t)^{\frac{3}{2}},$$

and, if $\delta < 4$

$$T_3 \leq C(T,\sigma_t) \frac{\nu^{2-\sqrt{2-\delta/2}}}{\left[\sqrt{2-\delta/2}+1 \right]^{\frac{\sqrt{2-\delta/2}}{2}}} (T-t)^{\frac{1}{2}\left(3-\sqrt{2-\delta/2}\right)},$$

and this allows us to complete the proof. ■

Remark 8 *Notice that, when* $\delta = 4$*, formula (6) coincides with (7). On the other hand, the accuracy of the approximation given in (7) becomes worse as* δ *tends to* 2.

Remark 9 *Formulas (6) and (7) show us that, fixed* δ*, the accuracy of this first-order approximation becomes good when the volatility of the volatility or the time to maturity are small enough.*

The decomposition formula (4) suggests us we can obtain a second-order approximation formula by approximating its last term. To this end, we will need the following lemma.

Lemma 10 *Assume* $\delta := \frac{4\kappa\theta}{\nu^2} \in (3,4)$*. Then, for all* $t \in [0,T]$ *and for all* $p < \frac{2}{5-\delta}$

$$E\left(\frac{1}{\sigma_s^3}\bigg|\mathcal{F}_t\right) \leq \frac{C(T,\sigma_t)}{\nu^{2(1-1/q)}s^{1-1/q}\left[\frac{p}{2}(\delta-5)+1\right]^{\frac{1}{p}}},$$

where $C(T,\sigma_t)$ *is a positive constant non-decreasing as a function of* T*.*

Proof. For the sake of simplicity we can take $t = 0$. From he proof of Lemma A.1 of Bossy (2004) we know that

$$E\left(\frac{1}{\sigma_s^3}\right) \leq \frac{C}{L(s)}\int_0^1 (1-u)^{2\kappa\theta/\nu^2-\frac{5}{2}} \exp\left(-\frac{\sigma_0 e^{-\kappa s}u}{2L(s)}\right) du$$

where $L(s) := \frac{\nu^2}{4\kappa}(1-e^{-\kappa s})$.Now, by Hölder inequality we know that, for all $\frac{1}{p}+\frac{1}{q}=1$ such that $p < \frac{2}{5-\delta}$

$$\begin{aligned}
E\left(\frac{1}{\sigma_s^3}\right) &\leq \frac{C}{L(s)}\left(\int_0^1 (1-u)^{\frac{p}{2}(\delta-5)}du\right)^{\frac{1}{2}}\left(\int_0^1 \exp\left(-\frac{\sigma_0 e^{-\kappa s}u}{L(s)}\right) du\right)^{\frac{1}{2}} \\
&\leq \frac{C}{L(s)\left[\frac{p}{2}(\delta-5)+1\right]^{\frac{1}{p}}}\left(\frac{q\sigma_0 e^{-\kappa s}}{L(s)}\right)^{-\frac{1}{q}} \\
&\leq \frac{C(T,\sigma_0)}{L(s)^{1-1/q}\left[\frac{p}{2}(\delta-5)+1\right]^{\frac{1}{p}}} \\
&\leq \frac{C(T,\sigma_0)}{\nu^{2(1-1/q)}(1-e^{-\kappa s})^{1-1/q}\left[\frac{p}{2}(\delta-5)+1\right]^{\frac{1}{p}}}.
\end{aligned}$$

Now, using that $(1-e^{-\kappa s}) \geq ske^{-\kappa s}$ it follows that

$$E\left(\frac{1}{\sigma_s^3}\right) \leq \frac{C(T,\sigma_0)}{\nu^{2(1-1/q)}s^{1-1/q}\left[\frac{p}{2}(\delta-5)+1\right]^{\frac{1}{p}}},$$

as we wanted to prove. ■

Theorem 11 *(Second-order approximation formula) Assume the model (1), where the volatility process* $\sigma = \{\sigma_s, s \in [0,T]\}$ *satisfies the Novikov condition* $2\kappa\theta > \nu^2$. *Then, if* $\delta \geq 5$, *for all* $t \in [0,T]$ *such that* $T - t < 1$

$$\begin{aligned}
&\left| V_t - BS\left(t, X_t; v_t\right) - \frac{1}{2} H\left(t, X_t, v_t\right) E^*\left(\int_t^T \sigma_s d\left\langle M, W^*\right\rangle_s \middle| \mathcal{F}_t \right) \right. \\
&\left. -\frac{1}{2} K\left(t, X_t, v_t\right) E^*\left(\int_t^T d\left\langle M, M\right\rangle_s \middle| \mathcal{F}_t \right) \right| \\
\leq\ & C(T, \sigma_t)\left(\nu^2\rho^2 (T-t)^{\frac{3}{2}} + \nu^3\rho(T-t)^2 + \nu^4 (T-t)^{5/2} \right) \qquad (8)
\end{aligned}$$

Moreover, if $\delta \in [4, 5)$

$$\begin{aligned}
&\left| V_t - BS\left(t, X_t; v_t\right) - \frac{1}{2} H\left(t, X_t, v_t\right) E^*\left(\int_t^T \sigma_s d\left\langle M, W^*\right\rangle_s \middle| \mathcal{F}_t \right) \right. \\
&\left. -\frac{1}{2} K\left(t, X_t, v_t\right) E^*\left(\int_t^T d\left\langle M, M\right\rangle_s \middle| \mathcal{F}_t \right) \right| \\
\leq\ & C(T, \sigma_t)\left\{ \nu^2\rho^2 (T-t)^{\frac{3}{2}} + \nu^3\rho(T-t)^2 \right. \qquad (9) \\
&\left. + \nu^{4-2\sqrt{5/2-\delta/2}} (T-t)^{5/2-2\sqrt{5/2-\delta/2}} \left(\frac{1}{1-\sqrt{5/2-\delta/2}} \right)^{1+\sqrt{5/2-\delta/2}} \right\}.
\end{aligned}$$

Finally, if $\delta \in [3, 4)$

$$\begin{aligned}
&\left| V_t - BS\left(t, X_t; v_t\right) - \frac{1}{2} H\left(t, X_t, v_t\right) E^*\left(\int_t^T \sigma_s d\left\langle M, W^*\right\rangle_s \middle| \mathcal{F}_t \right) \right. \\
&\left. -\frac{1}{2} K\left(t, X_t, v_t\right) E^*\left(\int_t^T d\left\langle M, M\right\rangle_s \middle| \mathcal{F}_t \right) \right| \\
\leq\ & C(T, \sigma_t)\left\{ \nu^{2-2\sqrt{2-\delta/2}} \left(\frac{1}{1-\sqrt{2-\delta/2}} \right)^{1+\sqrt{2-\delta/2}} \right| \\
&\times \left[(T-t)^{\frac{1}{2}\left(3-\sqrt{2-\delta/2}\right)} + (T-t)^{2\left(1-\sqrt{2-\delta/2}\right)} \right] \qquad (10) \\
&\left. + \nu^{4-2\sqrt{5/2-\delta/2}} (T-t)^{5/2-2\sqrt{5/2-\delta/2}} \left(\frac{1}{1-\sqrt{5/2-\delta/2}} \right)^{1+\sqrt{5/2-\delta/2}} \right\},
\end{aligned}$$

where $C(T, \sigma_t)$ *is a positive constant non-decreasing as a function of* T

Proof. Consider the process $e^{-rt} K\left(t, X_t; v_t\right) R_t$, where $R_t := E^*\left(\int_t^T d\left\langle M, M\right\rangle_s \middle| \mathcal{F}_t \right)$.It is easy to check that

$$e^{-rT} H\left(T, X_T; v_T\right) R_T = 0.$$

Again, the same arguments as in the proof of Theorem 1 give us that

$$
\begin{aligned}
0 &= H(t, X_t; v_t) R_t \\
&\quad -\frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} K(s, X_s, v_s) \, d\langle M, M\rangle_s \middle| \mathcal{F}_t \right) \\
&\quad +\frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) K\left(s, X_s, v_s^\delta\right) R_s \sigma_s d\langle M, W^*\rangle_s \middle| \mathcal{F}_t \right) \\
&\quad +\frac{1}{2} E^* \left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) K\left(s, X_s, v_s^\delta\right) R_s d\langle M, M\rangle_s \middle| \mathcal{F}_t \right).
\end{aligned}
$$

This, together with (4), allows us to write

$$
\begin{aligned}
V_t &= BS(t, X_t; v_t) + H(t, X_t; v_t) U_t + K(t, X_t; v_t) R_t \\
&\quad +\frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s^\delta\right) U_s \sigma_s d\langle M, W^*\rangle_s \middle| \mathcal{F}_t \right) \\
&\quad +\frac{1}{2} E^* \left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s^\delta\right) U_s d\langle M, M\rangle_s \middle| \mathcal{F}_t \right) \\
&\quad +\frac{1}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) K\left(s, X_s, v_s^\delta\right) R_s \sigma_s d\langle M, W^*\rangle_s \middle| \mathcal{F}_t \right) \\
&\quad +\frac{1}{2} E^* \left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) K\left(s, X_s, v_s^\delta\right) R_s d\langle M, M\rangle_s \middle| \mathcal{F}_t \right) \\
&= BS(t, X_t; v_t) + H(t, X_t; v_t) U_t + K(t, X_t; v_t) R_t + T_1 + T_2 + T_3 + T_4.
\end{aligned}
$$

Then, by the proof of Theorem 1 we know that, if $\delta \geq 4$,

$$
T_1 + T_2 \leq C(T, \sigma_t) \nu^2 \rho^2 (T-t)^{\frac{3}{2}},
$$

and, if $\delta < 4$

$$
\begin{aligned}
T_1 + T_2 &\leq C(T, \sigma_t) \rho^2 \nu^{2-\sqrt{2-\delta/2}} \\
&\quad \times \left[(T-t)^{\frac{1}{2}\left(3-\sqrt{2-\delta/2}\right)} + (T-t)^{2-2\sqrt{2-\delta/2}} \right]
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|R_s| &\leq \nu^2 E^* \left(\int_s^T \sigma_r^2 \left(\int_r^T e^{-\kappa(u-r)} du \right)^2 dr \middle| \mathcal{F}_t \right) \\
&= \nu^2 \int_s^T E^* \left(\sigma_r^2 \middle| \mathcal{F}_s \right) \left(\int_r^T e^{-\kappa(u-r)} du \right)^2 dr
\end{aligned}
$$

Then, using Lemma 1 as in the proof of Theorem 1 we obtain

$$\begin{aligned} T_3 &\leq \frac{\nu^3\rho}{2}E^*\left(\left(\left(\int_t^T e^{-r(s-t)}\left(\int_s^T E\left(\sigma_\theta^2\middle|\mathcal{F}_s\right)d\theta\right)^{-3}\right.\right.\right.\\ &\quad\times\left(\int_s^T E^*\left(\sigma_r^2\middle|\mathcal{F}_s\right)\left(\int_r^T e^{-\kappa(u-r)}du\right)^2 dr\right)\\ &\quad\times\left.\left.\sigma_s^2\left(\int_s^T e^{-\kappa(u-s)}du\right)ds\right|\mathcal{F}_t\right)\\ &\leq C(T,\sigma_t)\nu^3\rho\int_t^T e^{-r(s-t)}E^*\left(\sigma_s^{-2}\middle|\mathcal{F}_t\right)\left(\int_s^T e^{-\kappa(r-s)}dr\right)ds \end{aligned}$$

Now, by Lemma 5, if $\delta \geq 4$

$$T_3 \leq C(T,\sigma_t)\nu^3\rho(T-t)^2$$

and, if $\delta < 4$, by Lemma 6

$$T_3 \leq C(T,\sigma_t)\rho\nu\left[\nu(T-t)\right]^{2-2\sqrt{2-\delta/2}}\left(\frac{1}{1-\sqrt{2-\delta/2}}\right)^{1+\sqrt{2-\delta/2}}$$

Finally, by Lemma 1 and using the same arguments as before we can write

$$\begin{aligned} T_4 &\leq \frac{\nu^4}{2}E^*\left(\int_t^T e^{-r(s-t)}\left(\int_s^T E\left(\sigma_\theta^2\middle|\mathcal{F}_s\right)d\theta\right)^{-\frac{7}{2}}\right.\\ &\quad\times\left(\int_s^T E^*\left(\sigma_r^2\middle|\mathcal{F}_s\right)\left(\int_r^T e^{-\kappa(u-r)}du\right)^2 dr\right)\\ &\quad\times\left.\sigma_s^2\left(\int_s^T e^{-\kappa(u-s)}du\right)^2 ds\right|\mathcal{F}_t\Bigg)\\ &\leq \frac{\nu^4}{2}E^*\left(\int_t^T e^{-r(s-t)}\sigma_s^{-3}\left(\int_s^T e^{-\kappa(r-s)}dr\right)^{\frac{3}{2}}ds\middle|\mathcal{F}_t\right)\\ &\leq \frac{\nu^4}{2}\int_t^T e^{-r(s-t)}E^*\left(\sigma_s^{-3}\middle|\mathcal{F}_t\right)\left(\int_s^T e^{-\kappa(r-s)}dr\right)^{\frac{3}{2}}ds. \end{aligned}$$

Then, Lemma 5 gives us that, if $\delta \geq 5$

$$T_4 \leq C(T,\sigma_t)\nu^4(T-t)^{5/2},$$

and applying Lemma 10 we obtain that, if $\delta \in (3,5)$

$$
\begin{aligned}
T_4 &\leq C(T,\sigma_t)\nu^4 \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-3}\middle| \mathcal{F}_t\right)\left(\int_s^T e^{-\kappa(r-s)}dr\right)^{\frac{3}{2}} ds \\
&\leq C(T,\sigma_t)\frac{\nu^4}{\nu^{2/p}\left[p\left(\delta/2-5/2\right)+1\right]^{\frac{1}{p}}}\int_t^T \frac{e^{-r(s-t)}}{(s-t)^{1/p}}\left(\int_s^T e^{-\kappa(u-s)}du\right)^{\frac{3}{2}} ds \\
&\leq C(T,\sigma_t)\frac{p\nu^4}{(p-1)\nu^{2/p}\left[p\left(\delta/2-5/2\right)+1\right]^{\frac{1}{p}}}(T-t)^{5/2-1/p} \\
&= C(T,\sigma_t)\nu^4(T-t)^{5/2}\left(\frac{p}{p-1}\right)\left(\frac{1}{\nu^2(T-t)\left[p\left(\delta/2-5/2\right)+1\right]}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Now, taking $p=\sqrt{\frac{2}{5-\delta}}$ it follows that

$$
T_4 \leq C(T,\sigma_t)\nu^{4-2\sqrt{5/2-\delta/2}}(T-t)^{5/2-2\sqrt{5/2-\delta/2}}\left(\frac{1}{1-\sqrt{5/2-\delta/2}}\right)^{1+\sqrt{5/2-\delta/2}},
$$

and this allows us to complete the proof. ■

Remark 12 *For an european call option, it is easy to check that*

$$
H\left(t,x,\sigma\right) := \frac{e^x}{\sigma\sqrt{2\pi\left(T-t\right)}}\exp\left(-\frac{d_+^2}{2}\right)\left(1-\frac{d_+}{\sigma\sqrt{T-t}}\right)
$$

and

$$
K\left(t,x,\sigma\right) = \frac{e^x}{\sigma\sqrt{2\pi\left(T-t\right)}}\exp\left(-\frac{d_+^2}{2}\right)\left[\left(-\frac{d_+}{\sigma\sqrt{T-t}}+\frac{d_+^2}{\sigma^2\left(T-t\right)}\right)-\frac{1}{\sigma^2\left(T-t\right)}\right].
$$

Moreover, in the case of the Heston volatility we can easily see that

$$
E^*\left(\int_t^T \sigma_s^2 ds\middle|\mathcal{F}_t\right) = \theta\left(T-t\right)+\frac{\left(\sigma_t^2-\theta\right)}{\kappa}\left(1-e^{-\kappa(T-t)}\right),
$$

$$
\begin{aligned}
&E^*\left(\int_t^T \sigma_s d\left\langle M,W^*\right\rangle_s\middle|\mathcal{F}_t\right) \\
&= \frac{\nu\rho}{\kappa^2}\left(\theta\kappa\left(T-t\right)-2\theta+\sigma_t^2+e^{-\kappa(T-t)}\left(2\theta-\sigma_t^2\right)-\kappa\left(T-t\right)e^{-\kappa(T-t)}\left(\sigma_t^2-\theta\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& E^*\left(\left.\int_t^T d\langle M, M\rangle_s \right| \mathcal{F}_t\right) \\
= \; & \frac{\nu^2}{\kappa^2}\left\{\theta(T-t)+\frac{\left(\sigma_t^2-\theta\right)}{\kappa}\left(1-e^{-k(T-t)}\right)\right. \\
& -\frac{2\theta}{\kappa}\left(1-e^{-k(T-t)}\right)-2\left(\sigma_t^2-\theta\right)(T-t)\, e^{-k(T-t)} \\
& \left.+\frac{\theta}{2\kappa}\left(1-e^{-2k(T-t)}\right)+\frac{\left(\sigma_t^2-\theta\right)}{\kappa}\left(e^{-k(T-t)}-e^{-2k(T-t)}\right)\right\}.
\end{aligned}
$$

Then we can easily obtain explicit first-order and second-order approximations formulas by substituting the above quantities in the approximation expressions proposed in Theorems 7 and 11.

Remark 13 *(Approximations for the implied volatility). It is easy to deduce from the expressions in Theorems 7 and 11, by using Taylor expansions as in Fouque, Papanicolau and Sircar (2000), the following first-order and second-order approximations for the implied volatility*

$$
\hat{I}_1 := v_t + \frac{\rho}{2v_t(T-t)}\left(1-\frac{d_+}{v_t\sqrt{T-t}}\right)E^*\left(\int_t^T \sigma_s d\langle M, W^*\rangle_s\right)
$$

$$
\begin{aligned}
\hat{I}_2 \; : \; & = v_t + \frac{\rho}{2v_t(T-t)}\left(1-\frac{d_+}{v_t\sqrt{T-t}}\right)E^*\left(\int_0^T \sigma_s d\langle M, W^*\rangle_s\right) \\
& + \frac{1}{2v_t T}\left[\left(-\frac{d_+}{v_t\sqrt{(T-t)}}+\frac{d_+^2}{v_t^2(T-t)}\right)-\frac{1}{v_t^2(T-t)}\right]E^*\left(\int_t^T d\langle M, M\rangle_s\right)
\end{aligned}
$$

Notice that, as

$$
d_+ = \frac{x - x_t^*}{v_t\sqrt{T-t}}
$$

the first expression is linear in the initial log-stock price x, and the second one is quadratic in x. Then we deduce that the first-order approximation formula will help us to describe the skew effect, while the second one will be necesary if we try to describe a smile.

4 Numerical examples

This section is devoted to exemplify the results in the previous section. For the sake of simplicity we will take $t = 0$.

Example 14 *In Fig. 1 we can see the corresponding error of approximation (%) relative to the option price evaluated analytically, for the parameters* $T = 0.5, K = 100$, $\kappa = 2, \theta = 0.04, \sigma_0 = 0.15, \nu = 0.1$ *and* $\rho = -0.5$. *We can observe the error in the second approximation is smaller than the error in the first approximation.*

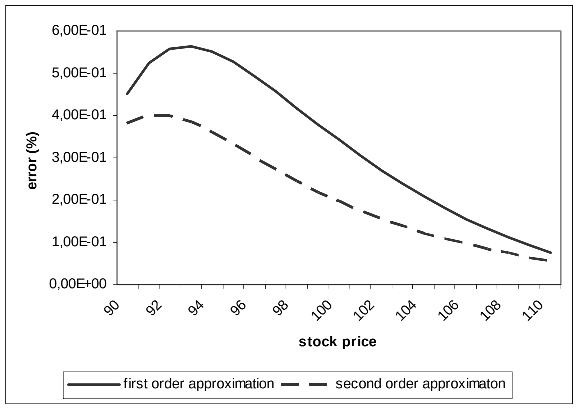

Figure 1: Error of approximation as a function of the log-stock price when $T = 0.5, K = 100$, $\kappa = 2, \theta = 0.04, \sigma_0 = 0.15, \nu = 0.1$ and $\rho = -0.5$.

Example 15 *In Fig. 2 we can see the percentage errors changing the above parameters to* $\kappa = 4, \nu = 0.3$ *and* $\rho = -0.1$. *Then, the last term in (4) becomes more significative and we can observe a bigger difference in the corresponding percentage errors.*

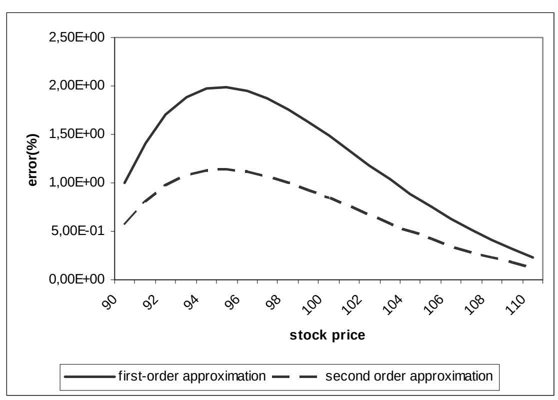

Figure 2: Error of approximation as a function of the log-stock price when $T = 0.5, K = 100$, $\kappa = 4, \theta = 0.04, \sigma_0 = 0.15, \nu = 0.3$ and $\rho = -0.1$.

Example 16 *Finally, we have considered the same parameters as in Fig. 2, but keeping* $X_t = 100$ *and taking* $\kappa \in (1.7, 5)$ *, in such a way that* $\delta = 4\kappa\theta/\nu^2 \in (3, 8.8)$ *.Fig. 3 shows the percentage errors as a function of* δ*.As expected from Theorems 7 and 11, the accuracy of the approximation depends strongly on* δ*. Moreover, the difference between the accuracy of the first and the second approximation becomes not significative when* δ *tends to* 3.

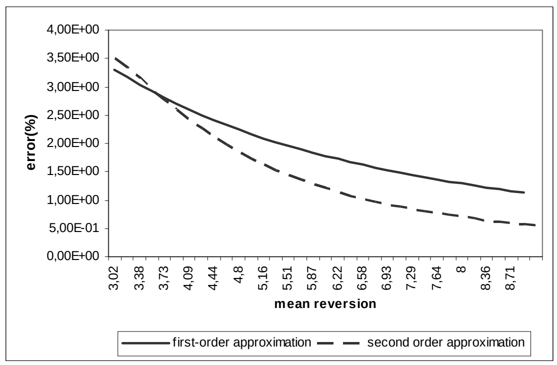

Figure 3: Error of approximation as a function of $\delta = 4\theta\kappa/\nu^2$ when $T = 0.5, K = 100$, $X = 100$, $\theta = 0.04, \sigma_0 = 0.15$ and $\nu = 0.1$.

5 Conclusions

By means of classical Itô's calculus we have decomposed option prices in the Heston volatility framework as the sum of the classical Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility plus a term due to the correlation and a term due to the volatility of the volatility. This decomposition formula allows us to construct first and second-order option pricing approximation formulas that are extremely easy to compute, as well as to study their accuracy. Moreover we have seen the corresponding approximations for the implied volatility are linear (first-order approximation) and quadratic (second-order approximation) in the log-stock price variable. The presented methods need only some general integrability conditions and extend some recent results in Alòs and Ewald (2008).

References

[1] Alòs, E. An extension of the Hull and White formula with applications to option pricing approximation. Finance and Stochastics, volume 10 (3), 353-365 (2006)

[2] Alòs, E. and Ewald, C.O.Malliavin Differentiability of the Heston Volatility and Applications to Option Pricing. Advances in Applied Probability 40 (1), 144-162 (2008)

[3] Antonelli, F. and Scarlatti, S. Pricing options under stochastic volatility: a power series approach. Finance and Stochastics 13 (2), 269-303 (2009)

[4] Ball, C. A. and Roma, A. (1994): Stochastic volatility option pricing. *Journal of Financial and Quantitative Analysis* **29**, 589-607.

[5] Benhamou, E., Gobet, E. and Miri, M. Smart expansion and fast callibration for jump diffusion. Finance and Stochastics 13 (4), 563-589 (2009a)

[6] Benhamou, E., Gobet, E. and Miri, M. Closed forms for European options in a local volatility model. Forthcoming in International Journal of Theoretical and Applied Finance (2009b)

[7] Benhamou, E., Gobet, E. and Miri, M. Time dependent Heston model. Preprint (2009c)

[8] Bossy, M. Diop, A. An efficient discretization sheme for one dimensional SDE's with a diffusion coefficient function of the form $|x|^{\alpha}$,$\alpha \in [1/2, 1)$. Rapport de recherche, Institut Nationale de Recherche en Informatique et en Automatique (INRIA), No. 5396 (2004)

[9] Fouque, J-P., Papanicolau, G. and Sircar, K. R. *Derivatives in Finantial markets with Stochastic Volatility.* Cambridge (2000)

[10] Fouque, J-P., Papanicolau, G., Sircar, K. R. and Solna, K.(2003): Singular Perturbations in Option Pricing. Preprint.

[11] Hagan, P. S., Kumar, D., Lesniewski, A. S. and Woodward, D. E. Managing smile risk. Willmot magazine 15, 84-108 (2002)

[12] Heston, S.L.; A Closed-Form for Options with Stochastic Volatility with Applications to Bond and Currency Options; Review of Financial Studies, Vol.6, Issue 2 (1993)

[13] Hull, J. C. and White, A. The pricing of options on assets with stochastic volatilities. *Journal of Finance* **42**, 281-300 (1987).

[14] Stein, E. M. and Stein, J. C. Stock price distributions with stochastic volatility: An analytic approach. *The review of Finantial Studies* **4**, 727-752 (1991):.

[15] Scott, L. O. Option pricing when the variance changes randomly: Theory, estimacion and application. *Journal of Financial and Quantitative Analysis* **22**, 419-438 (1987)

[16] Schöbel, R. and Zhu, J. Stochastic volatility with an Ornstein-Uhlenbeck process: an extension. *European Finance Review* **3**, 23-46 (1999).