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REFINEMENT OF SOLUTIONS TO THE LINEAR COMPLEMENTARITY PROBLEM

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Refinement of solutions to the Linear complementarity problem

S. R. Mohan and A. J. J. Talman

Abstract

In this paper we study the notion of refinement of solutions to a linear complementarity problem analogous to such a notion in the theory of noncooperative games. We give sufficient conditions under which these refinements exist. In particular we show that if the underlying matrix of the linear complementarity problem is a 2×2 P-matrix, the unique solution to it must be a proper solution. The concept of perfectness is much weaker. We prove that if the underlying matrix is a Q-matrix the problem has at least one perfect solution although not all solutions to it may be perfect. The notion of weak properness is between perfectness and properness and existence of such a solution is guaranteed if the underlying matrix is a P-matrix. We also show that if the underlying matrix of the linear complementarity problem is induced by a bimatrix game as in the formulation similar to the one given by Miller and Zucker for finding an equlibrium of a polymatrix game, at least one of its solutions must be proper corresponding to a proper Nash equilibrium for the game itself. In general this matrix is not a P-matrix.

Key words. Linear complementarity problem, perfectness, properness, weak properness, P-matrices, Q-matrices.

1 Introduction

The notion of a Nash equilibrium point as a solution concept in noncooperative game theory has been refined in various ways mainly to eliminate certain undesirable properties of a Nash equilibrium. Selten ([8]) introduced the concept of a *perfect equilibrium* while Myerson ([5]) developed the stronger notion of a *proper equilibrium*. The concept of a *weakly proper equilibrium* also has been introduced, see [9].

Since the problem of computing a Nash equilibrium point for a bimatrix game (see Lemke and Howson [3]) and for a polymatrix game (see Howson [2]) have been formulated as linear complementarity problems, one question that naturally arises is whether the notion of a perfect or proper equilibrium can be reformulated to refine the solution of a linear complementarity problem. The refinement for properness has already been introduced for stationary points of continuous maps over polytopes. If $f : S \to R^n$ where S is a polytope in R^n, the existence of a robust stationary point and its computation have been studied by van der Laan, Talman and Yang [10], where robust stationarity is the refinement for properness of stationarity. Note that given a square matrix M of order n, and a vector $q \in R^n$, since a stationary point of the map $f : R^n_+ \to R^n$ where $f(x) = Mx + q$, is a solution to the linear complementarity problem $LCP(q, M)$, refinement of solutions to LCP for properness generalizes also the notion of robust stationarity to maps defined on unbounded polyhedral sets. Such refinements have not yet been considered.

In this paper we study the notions of perfect, proper and weakly proper solutions for a linear complementarity problem. We present some sufficient conditions for various refinements to exist. In particular we show that if the matrix M is a 2×2 P-matrix, the unique solution to the $LCP(q, M)$ is a proper solution. This result is presented in Section 4. A weaker concept is the concept of perfectness which we define in Section 3. We derive a necessary and sufficient condition for the $LCP(q, M)$ to have a perfect solution. Further, we show that if the matrix M is a Q-matrix, then for each $q \in R^n$, the $LCP(q, M)$ has a perfect solution, although every solution to it may not be perfect. These results are presented in Section 3. The notion of weak properness is between perfectness and properness. This notion is also studied in Section 4. Finally, in Section 5 we show that the LCP(q, M) arising from the Miller-Zucker [4] type formulation of the problem of finding a Nash equilibrium point of a given bimatrix game always has a proper solution corresponding to a proper equilibrium point of the game. It may be noted that this matrix is not a P-matrix.

2 Mathemetical Preliminaries

Let R^n denote the n-dimensional Euclidean space over the reals and let A be a matrix of order $m \times n$ containing m rows and n columns. We use the notation $A_{.j}$ to denote the jth column of A, for $j = 1, 2, \ldots, n$ and $A_{i.}$, to denote the ith row of A, for $i = 1, 2, \ldots, m$. In particular we consider matrices of order $n \times n$, also called square matrices. Let M be a square matrix of order n. For J and K, being nonempty subsets of the set $\{1, 2, \ldots, n\}$, the symbol M_{JK} is used to denote the submatrix of M containing only those rows and columns of M whose indices are in the sets J and K, respectively, arranged in their natural order. In particular, the symbol M_{JJ} denotes the principal submatrix of M containing only those rows and columns whose indices are in the set J. Given a matrix A, the set of all nonnegative linear combinations of the columns of A is denoted by $Pos(A)$. Note that this set is a polyhedral convex cone.

Given a square matrix M of order n, the linear complementarity problem is the problem of determining vectors $w \in R^n$ and $z \in R^n$ such that

$$w - Mz = q, w \geq 0, z \geq 0$$

and

$$w^t z = 0,$$

where w^t is the transpose of the vector w. This problem is denoted as $LCP(q, M)$.

The class of all square matrices M of order n for which the $LCP(q, M)$ has a solution for each $q \in R^n$ is called a *Q-matrix*. A square matrix M all of whose principal minors are positive is called a *P-matrix*. A matrix M all of whose principal minors are nonzero is called a *nondegenerate matrix*. Note that a P-matrix is a nondegenerate matrix. A Q-matrix need not be nondegenerate. A well known theorem in linear complementarity theory states that for a square matrix M, the $LCP(q, M)$ has a unique solution for each $q \in R^n$ if and only if it is a P-matrix. See [1].

Suppose $(\bar{w}, \bar{z})$ is a solution to $LCP(q, M)$. Let C be an $n \times n$ submatrix of $(I, -M)$ containing for each j either $I_{.j}$ or $-M_{.j}$, such that it contains the columns $I_{.j}$ of I corresponding to $\bar{w}_j > 0$ and the columns $-M_{.j}$ of $-M$ corresponding to $\bar{z}_j > 0$. Such a matrix is called a *complementary matrix induced by the solution* $(\bar{w}, \bar{z})$. The cone generated by a complementary matrix is called a *complementary cone.* A complementary matrix or cone induced by a solution $(\bar{w}, \bar{z})$ need not be unique.

Let M be a given square matrix of order n. We say that M is a *copositive matrix* if $x^t M x \geq 0$, $\forall x \geq 0$. A copositive matrix M is called *copositive plus* if $x^t M x = 0, x \geq 0 \Rightarrow (M + M^t)x = 0$. A square matrix M is called a *Q_0-matrix* if $LCP(q, M)$ has a solution for each $q \in Pos((I, -M))$ or, equivalently, if the union of all complementary cones of $(I, -M)$ is convex. We also note here that if M is a Q-matrix, then the union of all nondegenerate complementary cones of $(I, -M)$ covers R^n.

3 Perfectness

Associated with a given $LCP(q, M)$ we consider the following perturbed problem $P(\epsilon)$, where $\epsilon > 0$ is a given vector in R^n. The perturbed problem is: Find $(\bar{w}(\epsilon), \bar{z}(\epsilon))$ such that

$$\bar{w}(\epsilon) - M\bar{z}(\epsilon) = q + M\epsilon$$

$$(\bar{w}(\epsilon), \bar{z}(\epsilon)) \geq 0$$

$$\bar{w}(\epsilon)^t \bar{z}(\epsilon) = 0.$$

Definition 3.1 *Let $(\bar{w}, \bar{z})$ be a solution to the $LCP(q, M)$. Then $(\bar{w}, \bar{z})$ is a perfect solution to the $LCP(q, M)$ if there exists a sequence $\{\epsilon^h, h = 1, 2, \ldots\}$ with $\epsilon^h \in R^n$, $\epsilon^h > 0$, $\epsilon^h \to 0$ as $h \to \infty$ and a solution $(\bar{w}(\epsilon^h), \bar{z}(\epsilon^h))$ to the perturbed problem $P(\epsilon^h)$ for each $h = 1, 2, \ldots$ such that as $h \to \infty$, $\bar{z}(\epsilon^h) \to \bar{z}$.*

Note that the problem $P(\epsilon)$ is equivalent to the following problem: Find $(w^*(\epsilon), z^*(\epsilon))$ satisfying

$$w^*(\epsilon) - Mz^*(\epsilon) = q$$

$$z^*(\epsilon) \geq \epsilon, w^*(\epsilon) \geq 0$$

$$w_i^*(\epsilon) > 0 \Rightarrow z_i^*(\epsilon) = \epsilon_i.$$

We may call the latter problem the ϵ-complementarity problem associated with the $LCP(q, M)$. We note that for the perfectness of a Nash equilibrium in a noncooperative game, the definition given by Selten [8] also requires that the sequence of optimal solutions to a perturbed problem converges to the given equilibrium.

The following example shows that there may be matrices M and vectors q such that no solution to $LCP(q, M)$ is perfect.

EXAMPLE 3.1 *Let* $M = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$. *This is a matrix all of whose entries are negative. Let* $q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *Note that* $LCP(q, M)$ *has the unique solution* $(\bar{w}, \bar{z}) = (0, 0)$. *This solution is not perfect since for any* $\epsilon > 0, \epsilon \in R^2$, $LCP(q + M\epsilon, M)$ *does not have a solution.*

Let $Pos^+(M)$ denote the cone $\{b | b = My, \ for\ some\ y\ \in R^n,\ y > 0\}$. In case M is nonsingular, $Pos^+(M)$ is an open cone. We now have the following theorem.

THEOREM 3.1 *The* $LCP(q, M)$ *has a perfect solution if and only if there exists a complementary cone* $Pos(C)$ *containing* q *such that*

$$(\{q\} + Pos^+(M)) \cap Pos(C) \neq \emptyset.$$

Proof: First suppose that there is a complementary cone $Pos(C)$ associated with $(I, -M)$ and containing the vector q such that

$$(\{q\} + Pos^+(M)) \cap Pos(C) \neq \emptyset.$$

Since $q \in Pos(C)$ there exists a $\beta \in R^n, \beta \geq 0$, such that

$$q = C\beta. \tag{3.1}$$

Further there exists an $\epsilon \in R^n$, $\epsilon > 0$, such that $q + M\epsilon \in Pos(C)$. Hence there exists an $\alpha \in R^n$, $\alpha \geq 0$, such that

$$q + M\epsilon = C\alpha. \tag{3.2}$$

Multiplying (3.1) by $1 - \lambda$ and (3.2) by λ we obtain for any $\lambda \in (0,\ 1)$

$$q + M\lambda\epsilon = C((1 - \lambda)\beta + \lambda\alpha). \tag{3.3}$$

Taking $\lambda = \frac{1}{k}$ and $\epsilon^k = \frac{\epsilon}{k}$, we obtain

$$q + M\epsilon^k = C[(1 - \frac{1}{k})\beta + \frac{\alpha}{k}], k = 2, 3, \ldots.$$

Note that it is now easy to construct a solution (w^k, z^k) to the perturbed problem $LCP(q + M\epsilon^k, M)$ that corresponds to the complementary matrix C and as $k \to \infty$ this solution sequence $\{(w^k, z^k)\}$ converges to $(\bar{w}, \bar{z})$ where $(\bar{w}, \bar{z})$ is a solution to $LCP(q, M)$ that induces the complementary cone $Pos(C)$. Thus we have a perfect solution to $LCP(q, M)$.

Conversely suppose $LCP(q, M)$ has a perfect solution $(\bar{w}, \bar{z})$. By definition there exists a sequence $\{\epsilon^h\}, \epsilon^h \in R^n, \epsilon^h > 0 \ \forall\ h = 1, 2, \ldots,\ \epsilon^h \to 0$ as $h \to \infty$, and a corresponding sequence of solutions $\{(\bar{w}(\epsilon^h), \bar{z}(\epsilon^h)\}$ to $LCP(q + M\epsilon^h, M)$ such that $\bar{w}(\epsilon^h) \to \bar{w}$ and $\bar{z}(\epsilon^h) \to \bar{z}$ as $h \to \infty$. Let $Pos(C^h)$ be a complementary cone induced by the solution $(\bar{w}(\epsilon^h), \bar{z}(\epsilon^h))$. This cone contains $q + M\epsilon^h$. Since there are only finitely many (2^n) complementary cones, it follows that there is a subsequence $\{\epsilon^{h_\nu}\}$ of $\{\epsilon^h\}$ such that $q + M\epsilon^{h_\nu} \in Pos(C)$, $\forall\ \nu$, for some complementary cone C. By the closure property of complementary cones and by the fact that $\{\bar{w}(\epsilon^{h_\nu}), \bar{z}(\epsilon^{h_\nu})\} \to (\bar{w}, \bar{z})$ it follows that $q \in Pos(C)$. Thus $(\{q\} + Pos^+(M)) \cap Pos(C) \neq \emptyset$ and $q \in Pos(C)$. This concludes the proof of the theorem. ∎

COROLLARY 3.1 *Given an $LCP(q, M)$, suppose q is contained in the interior of a nondegenerate complementary cone $Pos(C)$ of $(I, -M)$. Then there is a perfect solution to $LCP(q, M)$.*

Proof: If $q \in int(Pos(C))$, it follows that there exists a $\delta > 0$ such that $q + M\epsilon \in int(Pos(C))\ \forall\, \epsilon > 0, \epsilon \in R^n$ with $\| \epsilon \| < \delta$.

Thus it follows that

$$(\{q\} + Pos^{+}(M)) \cap Pos(C) \neq \emptyset.$$

■

The next theorem shows that if M is a Q-matrix and $q \in R^n$ then the $LCP(q, M)$ has a perfect solution.

THEOREM 3.2 *Let M be a Q-matrix. Then given any $q \in R^n$, there is a perfect solution to $LCP(q, M)$.*

Proof: Since M is a Q- matrix, for any $q \in R^n$ and $\epsilon \in R^n, \epsilon > 0$, $LCP(q + M\epsilon, M)$ has a solution. Take any sequence $\{\epsilon^h\}, \epsilon^h \in R^n, \epsilon^h > 0 \; \forall \; h = 1, 2 \ldots$, such that $\epsilon^h \to 0$ as $h \to \infty$. Let $Pos(C^h)$ be a complementary cone induced by a solution to $LCP(q + M\epsilon^h, M)$. Since there are only finitely many complementary cones of $(I, -M)$, it follows that there is a subsequence $\{\epsilon^{h_\nu}\}$ of $\{\epsilon^h\}$ such that $Pos(C^{h_\nu}) = Pos(C)$, $\forall \; \nu$, for some complementary cone C of $(I, -M)$. By a result of Murty (Exercise 3.85 in [6]) we can assume without loss of generality that $int(Pos(C)) \neq \emptyset$. Thus for each ϵ^{h_ν} there exists a $\beta^{h_\nu} \geq 0$ satisfying $q + M\epsilon^{h_\nu} = C\beta^{h_\nu}$. Since C is nonsingular it follows that as $h_\nu \to \infty, \beta^{h_\nu} \to \bar{\beta}$ for some $\bar{\beta} \geq 0$, and hence $q \in Pos(C)$. This concludes the proof. ■

Since a P-matrix is a Q-matrix we have the following corollary.

COROLLARY 3.2 *Let M be a P-matrix. The unique solution to $LCP(q, M)$, for any $q \in R^n$, is perfect.*

The following theorem presents two other equivalent formulations for the notion of perfectness of a solution to the LCP. In what follows let e denote the n-dimensional vector of ones.

THEOREM 3.3 *Let the $LCP(q, M)$ be given. The following assertions are equivalent: (i) $(\bar{w}, \bar{z})$ is a perfect solution for $LCP(q, M)$;*
(ii) $\bar{z}$ is a limit point of a sequence of positive vectors $\{z(\alpha)\}$ for positive real numbers α going to zero where, with $w(\alpha) = q + Mz(\alpha), w(\alpha) \geq 0$, $w_i(\alpha) > 0 \Rightarrow z_i(\alpha) \leq \alpha$, for all $i = 1, 2 \ldots n$;
(iii) $\bar{z}$ is a limit point of a sequence of positive vectors $\{z(\alpha)\}$ for positive real numbers α going to zero, where, with $w(\alpha) = q + Mz(\alpha), w(\alpha) \geq 0$ and $\bar{z}^t w(\alpha) = 0$.

Proof:. (i) $\Rightarrow$ (ii): There exists a sequence $\{\epsilon^h\}, \epsilon^h \in R^n, \epsilon^h > 0, \epsilon^h \to 0$ as $h \to \infty$, and a sequence $\{\bar{z}(\epsilon^h)\}$ such that with $\bar{w}(\epsilon^h) = q + M\epsilon^h + M\bar{z}(\epsilon^h)$, we have $\bar{z}(\epsilon^h) \geq 0, \bar{w}(\epsilon^h) \geq 0$, $\bar{w}(\epsilon^h)^t \bar{z}(\epsilon^h) = 0$ and $\{\bar{z}(\epsilon^h)\} \to \bar{z}$. Now take $\alpha_h = max_{1 \leq i \leq n}[\epsilon_i^h]$. Also take $z(\alpha_h) = \bar{z}(\epsilon^h) + \epsilon^h > 0$. This sequence satisfies all the conditions of (ii).
(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Then there is a sequence $\{z(\alpha)\}$ with limit $\bar{z}$ as $\alpha \to 0$ such that, $z(\alpha) > 0, w(\alpha) = q + Mz(\alpha) \geq 0$ and $w_i(\alpha) > 0 \Rightarrow z_i(\alpha) \leq \alpha$. Let $\bar{w} = \lim_{\alpha \to 0} w(\alpha) = \lim_{\alpha \to 0}[q + Mz(\alpha)] = q + M\bar{z}$. If $\bar{w}_i > 0$, then $w_i(\alpha) > 0$ for all sufficiently small α and hence $z_i(\alpha) \leq \alpha$ for all sufficiently small α. This implies that $\bar{z}_i = 0$. Thus we can obtain a subsequence of $\{z(\alpha)\}$ that satisfies the requirement of (iii).
(iii)$\Rightarrow$ (i): Given a sequence $\{z(\alpha)\}$ as in (iii) define the sequence of vectors $\{\epsilon(\alpha)\}_{\alpha > 0}$ in R^n as follows:

$$\epsilon_i(\alpha) = \begin{cases} z_i(\alpha) & \text{if } \bar{z}_i = 0 \\ \alpha & \text{if } \bar{z}_i > 0. \end{cases}$$

Define the vector $\bar{z}(\epsilon(\alpha))$ by $\bar{z}(\epsilon(\alpha)) = z(\alpha) - \epsilon(\alpha)$. Note that as $\alpha \to 0, \epsilon(\alpha) \to 0$, and that for α sufficiently small $\bar{z}(\epsilon(\alpha))$ is nonnegative. Further $\{\bar{z}(\epsilon(\alpha))\}$ tends to $\bar{z}$. Note also that by the complementarity condition of (iii) it is easy to verify that with

$$\bar{w}(\alpha) = q + Mz(\alpha) = q + M(\bar{z}(\epsilon(\alpha)) + \epsilon(\alpha)),$$

$(\bar{w}(\alpha), \bar{z}(\epsilon(\alpha)))$ solves the $LCP(q + M\epsilon(\alpha), M)$, for a sequence of vectors $\epsilon(\alpha)$ going to zero. Thus (i) follows. This completes the proof of the theorem. ■

The next example shows that Theorem 3.2 does not hold for the class of copositive plus matrices.

EXAMPLE 3.2 *Let* $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and* $q = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *Then* $(\bar{w}, \bar{z})$ *with* $\bar{z} = (0, \lambda)^t$ *and* $\bar{w} = (\lambda, 0)^t$ *is a solution to* $LCP(q, M)$, *for each* $\lambda \geq 0$, *none of them being perfect, whereas* M *is a copositive plus matrix.*

REMARK 3.1 *The notion of perfectness of solution to a linear complementarity problem is also related to the weak upper Lipchitzian property of the solution map. It is known that given an* $LCP(\bar{q}, M)$ *if* $S(q)$ *denotes the set of solutions to the* $LCP(q, M)$ *then*

$$S(q) \subseteq S(\bar{q}) + c||q - \bar{q}||\mathcal{B}$$

where c *is a positive real constant and* $\mathcal{B}$ *denotes the unit ball of radius 1 in* R^n. *This holds for all* $\bar{q}$ *in* R^n. *See [1] for a proof. In particular if* M *is a* Q*-matrix this implies that for any* $\bar{q} \in R^n$ *there is a perfect solution.*

4 Properness

In this section we develop the concept of properness of a solution to a linear complementarity problem. Properness is a much stronger refinement of solutions than perfectness as defined in the previous section. We also introduce weak properness being a weaker concept than properness but stronger than perfectness.

We discuss some classes of LCPs satisfying the condition that for every q there is a proper or weakly proper solution.

Properness of a solution to an LCP is motivated by a strategy being a proper Nash equilibrium in a noncooperative game. See Myerson [5].

To define properness we introduce the concept of an α-proper solution to a linear complementarity problem for some positive number α.

Definition 4.1 *Let the $LCP(q, M)$ be given. Then for $\alpha > 0, \alpha \in R$, $(z(\alpha), w(\alpha))$ is an α- proper solution to $LCP(q, M)$ if (i) $z_i(\alpha) > 0 \ \forall \ i = 1, 2, \ldots n$, (ii) $w(\alpha) = q + Mz(\alpha) \geq 0$, (iii) $w_i(\alpha) > 0 \Rightarrow z_i(\alpha) \leq \alpha$, (iv) $w_i(\alpha) < w_j(\alpha) \Rightarrow z_i(\alpha) \leq \alpha z_j(\alpha)$.*

Definition 4.2 *Let the $LCP(q, M)$ be given and let $(\bar{w}, \bar{z})$ be a solution to it. Then $(\bar{w}, \bar{z})$ is a proper solution to $LCP(q, M)$ if there exists a sequence $\{\alpha_h\}, \alpha_h \in R, \alpha_h > 0$, and a sequence $\{(z(\alpha_h), w(\alpha_h))\}$ of α_h- proper solutions to $LCP(q, M)$ such that (i) $lim_{h\rightarrow\infty}\alpha_h = 0$ and (ii) $lim_{h\rightarrow\infty} z(\alpha_h) = \bar{z}$.*

THEOREM 4.1 *Let $(\bar{w}, \bar{z})$ be a solution to $LCP(q, M)$. Then $(\bar{w}, \bar{z})$ is a proper solution if the following conditions hold:*
(i) The solution $(\bar{w}, \bar{z})$ is nondegenerate; (ii) The complementary matrix C induced by the solution is nonsingular; (iii) The positive coordinates of $\bar{w}$ are distinct (i.e., $\bar{w}_i \neq 0 \Rightarrow \bar{w}_i \neq \bar{w}_j$ for any $i \neq j$).

Proof. Let $(\bar{w}, \bar{z})$ be a solution to the $LCP(q, M)$ satisfying (i), (ii) and (iii). Now we construct a sequence $(w(\alpha), z(\alpha))$ of α-proper solutions for sufficiently small α as follows. Let $(\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the elements of the set $\{1, 2, \ldots, n\}$ such that

$$\bar{w}_{\pi(i)} = 0, \ for \ i = 1, 2, \ldots, \ell,$$

where $1 \leq \ell \leq n$ is the number of zero coordinates of $\bar{w}$, and

$$0 < \bar{w}_{\pi(i)} < \bar{w}_{\pi(j)}, \; for \; \ell < i < j.$$

Take

$$z_{\pi(i)}(\alpha) = [C^{-1}(q + \sum_{j=\ell+1}^{n} M_{.\pi(j)}\alpha^j)]_{\pi(i)}, \; for \; i = 1, 2 \ldots, \ell$$

and

$$z_{\pi(i)}(\alpha) = \alpha^i \; for \; i = \ell + 1, \ell + 2, \ldots, n.$$

For sufficiently small α it is clear that $z_j(\alpha) > 0, \forall j = 1, 2, \ldots n$, and $w(\alpha) = q + Mz(\alpha) \geq 0$. Also note that $w_{\pi(i)}(\alpha) = 0, \; i = 1, 2 \ldots \ell, z_i(\alpha) < \alpha$ if $w_i(\alpha) > 0$, and that

$$w_i(\alpha) < w_j(\alpha) \Rightarrow z_j(\alpha) < \alpha z_i(\alpha).$$

Therefore $(w(\alpha), z(\alpha))$ is α-proper for small enough α. Moreover $z(\alpha)$ converges to $\bar{z}$ as $\alpha \to 0$, since the matrix C is nonsingular. ■

In our next theorem we show that the unique solution to an $LCP(q, M)$ when the matrix M is a 2×2 P-matrix is a proper solution. At present we do not have a proof to show that a general P-matrix induces a proper solution.

THEOREM 4.2 *Let $LCP(q, M)$ be given. Suppose M is a 2×2 P-matrix. Then the unique solution $(\bar{w}, \bar{z})$ to the $LCP(q, M)$ is proper.*

Proof: In case $\bar{w}_1 \neq \bar{w}_2$ and $(\bar{w}, \bar{z})$ is a nondegenerate solution, the result follows from Theorem 4.1. Suppose now that $\bar{w}_1 \neq \bar{w}_2$ and $(\bar{w}, \bar{z})$ is a degenerate solution. Without loss of generality, assume that $\bar{w}_1 > 0$. This implies that $\bar{w}_2 = 0; \bar{z}_1 = \bar{z}_2 = 0$. Choose $z(\alpha) = (\alpha^2, \alpha)^t$ and let $w(\alpha) = q + Mz(\alpha)$. Then

$$w_1(\alpha) = q_1 + m_{11}\alpha^2 + m_{12}\alpha \text{ and } w_2(\alpha) = q_2 + m_{21}\alpha^2 + m_{22}\alpha.$$

Since $m_{22} > 0$, for sufficiently small α we have that $w_1(\alpha) > w_2(\alpha) \geq 0$, $z_1(\alpha) \leq \alpha z_2(\alpha)$ and $z_2(\alpha) \leq \alpha$. Hence $(w(\alpha), z(\alpha))$ is an α-proper solution for α sufficiently small. Clearly, $\{z(\alpha)\}$ converges to $\bar{z}$ as α goes to zero.

Next suppose that $\bar{w}_1 = \bar{w}_2 > 0$. In this case $\bar{z} = 0$ and $q = (c, c)^t$ for some real number $c > 0$. Let $z(\epsilon) = (\epsilon_1, \epsilon_2)$ for any $\epsilon = (\epsilon_1, \epsilon_2) > 0$. We have

$$w_1(\epsilon) = c + m_{11}\epsilon_1 + m_{12}\epsilon_2$$

$$w_2(\epsilon) = c + m_{21}\epsilon_1 + m_{22}\epsilon_2.$$

If $m_{21} < m_{11}$ and $m_{12} < m_{22}$ or if $m_{21} > m_{11}$ and $m_{12} > m_{22}$, then choose ϵ_1 and ϵ_2 such that

$$\epsilon_2 = (m_{11} - m_{21})\epsilon_1/(m_{22} - m_{12}).$$

For sufficiently small ϵ_1 we have $w_1(\epsilon) = w_2(\epsilon) > 0$, and therefore $(w(\epsilon), z(\epsilon))$ is an α-proper solution with $\alpha = max\{\epsilon_1, \epsilon_2\}$ when sufficiently small. If $m_{21} \leq m_{11}$ and $m_{12} \geq m_{22}$ then choose $z_1(\alpha) = \alpha^2$ and $z_2(\alpha) = \alpha$. Note that

$$w_1(\alpha) = c + m_{11}\alpha^2 + m_{12}\alpha \text{ and } w_2(\alpha) = c + m_{21}\alpha^2 + m_{22}\alpha.$$

For sufficiently small α we have $w_1(\alpha) \geq w_2(\alpha) \geq 0$ and therefore $(w(\alpha), z(\alpha))$ is an α-proper solution. If $m_{21} \geq m_{11}$ and $m_{12} \leq m_{22}$, choose $z_1(\alpha) = \alpha$ and $z_2(\alpha) = \alpha^2$. Then for sufficiently small α we have again that $(w(\alpha), z(\alpha))$ is an α-proper solution.

Finally suppose that $\bar{w}_1 = \bar{w}_2 = 0$. Without loss of generality assume that $\bar{z}_2 = 0$. If $\bar{z}_1 > 0$, we have $q = (-m_{11}\bar{z}_1, -m_{21}\bar{z}_1)^t$. Let us consider the sequence $\{z(\alpha)\}$ where $z_2(\alpha) = \alpha$ and $z_1(\alpha)$ is such that $w_1(\alpha) = 0$. Since $w_1(\alpha) = m_{11}(-\bar{z}_1 + z_1(\alpha)) + m_{12}\alpha$ it follows that $z_1(\alpha) = \bar{z}_1 - \frac{m_{12}\alpha}{m_{11}}$. Then we get

$$w_2(\alpha) = \frac{\alpha(m_{11}m_{22} - m_{12}m_{21})}{m_{11}} > 0$$

as $\alpha > 0$ and M is a P-matrix. Hence $(w(\alpha), z(\alpha))$ is an α-proper solution for sufficiently small α and $z(\alpha)$ converges to $\bar{z}$ as α goes to zero. If $\bar{z}_1 = 0$ we have $q = (0,0)^t$. Let $z(\epsilon) = (\epsilon_1, \epsilon_2)$ for any $\epsilon = (\epsilon_1, \epsilon_2)$. Then

$$w_1(\epsilon) = m_{11}\epsilon_1 + m_{12}\epsilon_2$$

$$w_2(\epsilon) = m_{21}\epsilon_1 + m_{22}\epsilon_2.$$

If $m_{21} < m_{11}$ and $m_{12} < m_{22}$ then choose ϵ_1 and ϵ_2 such that

$$\epsilon_2 = \frac{(m_{11} - m_{21})\epsilon_1}{m_{22} - m_{12}}.$$

For sufficiently small ϵ_1 we have that $w_1(\epsilon) \geq w_2(\epsilon) \geq 0$ and therefore $(w(\epsilon), z(\epsilon))$ is an α-proper solution with $\alpha = \epsilon_1$. The case $m_{21} \geq m_{11}$ and $m_{12} \geq m_{22}$ cannot arise because M is a P-matrix. If $m_{21} < m_{11}$ and $m_{12} > m_{22}$ then choose $\epsilon_1 = \alpha^2$ and $\epsilon_2 = \alpha$, so that $(w(\epsilon), z(\epsilon))$ is an α-proper solution for α sufficiently small. If $m_{21} \geq m_{11}$ and $m_{12} \leq m_{22}$, but $M_{.1} \neq M_{.2}$, then choose $\epsilon_1 = \alpha$ and $\epsilon_2 = \alpha^2$, so that $(w(\epsilon), z(\epsilon))$ is an α-proper solution for α sufficiently small. This concludes the proof. ∎

Condition (ii) of properness seems to be too strong a requirement in case $\bar{w}_i = \bar{w}_j > 0$ for some indices i and j. To relax this condition we introduce a weaker concept, to be called weak properness. See Van Damme [9].

Definition 4.3 *A solution $(\bar{w}, \bar{z})$ to a given $LCP(q, M)$ is weakly proper if there exists a sequence $\{z(\alpha_h)\}$ for some sequence $\{\alpha_h\}, \alpha_h \in R, \alpha_h > 0,\ \forall\, h$ such that*
(i) $z(\alpha_h) > 0\ \forall\, h$ and $lim_{h \to \infty} z(\alpha_h) = \bar{z}$;
(ii) $\bar{z}^t w(\alpha_h) = 0$ where $w(\alpha_h) = q + Mz(\alpha_h) \geq 0\ \forall\, h$;
(iii) $\bar{w}_i < \bar{w}_j \Rightarrow z_j(\alpha_h) \leq \alpha_h z_i(\alpha_h)\ \forall\, h$.

The next theorem relates the various refinements introduced.

THEOREM 4.3 *Every proper solution to a given $LCP(q, M)$ is weakly proper and every weakly proper solution to it is perfect.*

Proof: Suppose $(\bar{w}, \bar{z})$ is a proper solution to $LCP(q, M)$. Then there exists a sequence $\{z(\alpha_h)\}$ with $z(\alpha_h) > 0$ such that $lim_{\alpha_h \to 0} z(\alpha_h) = \bar{z}$, $w(\alpha_h) = q + Mz(\alpha_h) \geq 0$, $w_i(\alpha_h) > 0 \Rightarrow z_i(\alpha_h) \leq \alpha_h$, and $w_i(\alpha_h) \leq w_j(\alpha_h) \Rightarrow z_j(\alpha_h) \leq \alpha_h z_i(\alpha_h)$, $\forall\ h > 0$. Suppose now that there is a subsequence $\{\alpha_{h_\nu}\}$ of $\{\alpha_h\}$ such that $w_i(\alpha_{h_\nu}) > 0\ \forall\ \nu$. Then it follows that $z_i(\alpha_{h_\nu}) \leq \alpha_{h_\nu}$, $\forall\ \nu$, and hence that $\bar{z}_i = lim_{\alpha_h \to 0} z_i(\alpha_h) = 0$. So for h sufficiently large it holds that $\bar{z}^t w(\alpha_h) = 0$. Thus it follows that the solution $(\bar{w}, \bar{z})$ is weakly proper.

Suppose now $(\bar{w}, \bar{z})$ is a weakly proper solution to $LCP(q, M)$. It follows that there is a sequence $\{z(\alpha_h)\}$ with $z(\alpha_h) > 0$ such that $lim_{\alpha_h \to 0} z(\alpha_h) = \bar{z}, w(\alpha_h) = q + Mz(\alpha_h) \geq 0$, $\bar{w}_i \leq \bar{w}_j \Rightarrow z_j(\alpha_h) \leq \alpha_h z_i(\alpha_h)$, and $\bar{z}^t w(\alpha_h) = 0\ \forall\ h > 0$. It follows now, that if $w_i(\alpha_h) > 0$ then $\bar{z}_i = 0$ and hence $z_i(\alpha_h) \to 0$. Thus we can find a sequence θ_h going to zero such that $z_i(\alpha_h) \leq \theta_h$, if $w_i(\alpha_h) > 0$. Thus $(w(\alpha_h), z(\alpha_h))$ is a θ_h-perfect solution converging to $(\bar{w}, \bar{z})$. It follows that $(\bar{w}, \bar{z})$ is a perfect solution. ■

As in the case of perfectness of a solution, weak properness can be characterized in terms of the nonempty intersection of two cones. From this characterization also it follows immediately that weak properness implies perfectness. Let $(\bar{w}, \bar{z})$ be a solution to $LCP(q, M)$. Then define the cone $Pos(M, \bar{w})$ by

$$Pos(M, \bar{w}) = \{y \in R^n | y = Mz, z \geq 0,\ \bar{w}_i < \bar{w}_j \Rightarrow z_j \leq \alpha z_i\ \forall\, i, j\ for\ \alpha > 0\}$$

and let E be the matrix, of which the columns are $I_{.j}$ for j such that $\bar{z}_j = 0$.

THEOREM 4.4 *Let $(\bar{w}, \bar{z})$ be a solution to $LCP(q, M)$ and let $Pos(M, \bar{w})$ and the matrix E be as defined above. Then $(\bar{w}, \bar{z})$ is a weakly proper solution to*

LCP(q, M) if and only if

$$(\{q\} + Pos(M, \bar{w})) \cap Pos(E) \neq \emptyset.$$

Proof: Suppose $(\bar{w}, \bar{z})$ is a weakly proper solution to $LCP(q, M)$. Then there exists a sequence $\{z(\alpha)\}$ with α tending to 0 and $z(\alpha) \to \bar{z}$ such that $w(\alpha) = q + Mz(\alpha) \geq 0, \bar{z}^t w(\alpha) = 0$, and $\bar{w}_i < \bar{w}_j \Rightarrow z_j(\alpha) \leq \alpha z_i(\alpha)$. For any $\alpha > 0$ we have that $w(\alpha) \in (\{q\} + Pos(M, \bar{w})) \cap Pos(E)$.

Now suppose that $(\{q\} + Pos(M, \bar{w})) \cap Pos(E) \neq \emptyset$ and let w^* be a point in this cone. Then there exists a real number $\alpha^* > 0$ such that

$$w^* = q + Mz(\alpha^*)$$

for some $z(\alpha^*)$ for which $\bar{w}_i < \bar{w}_j \Rightarrow z_j(\alpha^*) < \alpha^* z_i(\alpha^*), z_i(\alpha^*) > 0 \ \forall \ i$, and $\bar{z}^t w^* = 0$, because $w^* \in Pos(E)$. For $0 < \alpha < 1$, take $z(\alpha) = (1 - \alpha)\bar{z} + \alpha z(\alpha^*)$. Clearly, $z(\alpha) > 0 \ \forall \ \alpha, \ 0 < \alpha < 1, \ \bar{w}_i < \bar{w}_j \Rightarrow z_j(\alpha) \leq \alpha z_i(\alpha)$, and $w(\alpha) = q + Mz(\alpha) \geq 0$. Further, $\bar{z}^t w(\alpha) = 0 \ \forall \alpha, 0 < \alpha < 1$, and $z(\alpha) \to \bar{z}$. This concludes the proof. ∎

In case M is a P-matrix, the solution to $LCP(q, M)$ is weakly proper.

THEOREM 4.5 *Let the $LCP(q, M)$ be given. If the matrix M is a P-matrix then the unique solution to $LCP(q, M)$ is weakly proper.*

Proof: Let $(\bar{w}, \bar{z})$ be the unique solution to $LCP(q, M)$. Let

$$L = \{l \in \{1, 2, \ldots n\} | \bar{z}_l > 0\}$$

and $J = \{j \in \{1, 2, \ldots, n\} | \bar{w}_j = \bar{z}_j = 0\}$. From the proof of Theorem 4.1 and Theorem 4.3 it follows that $(\bar{w}, \bar{z})$ is weakly proper if $J = \emptyset$. Suppose therefore that $J \neq \emptyset$. Since M is a P-matrix, the matrix $M_{JJ} - M_{JL}M_{LL}^{-1}M_{LJ}$ is also a P-matrix. Hence there exists a vector $d \in R^{|J|}, d > 0$, such that

$$(M_{JJ} - M_{JL}M_{LL}^{-1}M_{LJ})d > 0.$$

Now we construct a sequence $\{z(\alpha)\}_{\alpha>0}$ by choosing $z_J(\alpha) = \alpha d$. For the coordinates of $z_{\bar{J}}(\alpha)$ we proceed as follows. First we arrange the elements of $\overline{L \cup J}$ in increasing order of the coordinates of $\bar{w}_{\overline{L \cup J}}$. Let $\pi(\overline{L \cup J}) = (\pi(1), \pi(2), \dots \pi(|\overline{L \cup J}|))$ be such an order. Then we define $z_{\pi(k)}(\alpha) = \alpha^k,\ for\ k = 1, 2, \dots |\overline{L \cup J}|$. Finally, for L we define $z_L(\alpha) = -M_{LL}^{-1}(q_L + M_{L\bar{L}} z_{\bar{L}}(\alpha))$. Clearly, $z(\alpha) > 0$ for sufficiently small $\alpha > 0$. Moreover $z_j(\alpha)$ converges to 0 for $j \in \bar{L}$ and $z_L(\alpha)$ converges to $\bar{z}_L$ if α goes to 0. Hence $\lim_{\alpha \downarrow 0} z(\alpha) = \bar{z}$.

Next we show that $w(\alpha) = q + Mz(\alpha) \geq 0$ and $\bar{z}^t w(\alpha) = 0$ for sufficiently small α. For the set L we obtain

$$w_L(\alpha) = q_L + M_{LL} z_L(\alpha) + M_{L\bar{L}} z_{\bar{L}}(\alpha),$$

so $w_L(\alpha) = q_L + M_{LL}(-M_{LL}^{-1}(q_L + M_{L\bar{L}} z_{\bar{L}}(\alpha))) + M_{L\bar{L}} z_{\bar{L}}(\alpha) = 0$. For sufficiently small α we also have that $w_{\overline{L \cup J}}(\alpha) \geq 0$ because $\bar{w}_{\overline{L \cup J}} > 0$. To prove that $w_J(\alpha) \geq 0$, note that

$$w_J(\alpha) = (q_J - M_{JL} M_{LL}^{-1} q_L) + (M_{JJ} M_{JL} M_{LL}^{-1} M_{LJ})(\alpha d) + terms\ of\ O(\alpha^2).$$

The first term is $\bar{w}_J$ and therefore equal to 0. The second term is strictly positive by the choice of d and dominates the third term which only contains terms of order higher than or equal to α^2, for sufficiently small α. Therefore for small enough α we have $w_J(\alpha) \geq 0$. Moreover, since $w_L(\alpha) = 0$ for all α and $\bar{z}_{\bar{L}} = 0$ we also have $\bar{z}^t w(\alpha) = 0$ for any α. This concludes the proof that $(\bar{w}, \bar{z})$ is a weakly proper solution to $LCP(q, M)$. ∎

5 The bimatrix case

In this section we consider the LCP arising from the problem of computing a Nash equilibrium for a bimatrix game. A bimatrix game is specified by (n_1, n_2, A, B) where n_1 is the number of actions available to Player 1 and n_2 is the number of actions available to Player 2. The $n_1 \times n_2$ matrix A (B) is the pay-off matrix for Player 1 (Player 2) i.e., a_{ij} (b_{ij}) is the payoff for Player 1 (Player 2) if Player 1 chooses his i-th action and Player 2 chooses his j-th action. See Van Damme [9]. Let $S^1 = \{x \in R^{n_1} | x \geq 0, \sum_{i=1}^{n_1} x_i = 1\}$. Any $x \in S^1$ is called a mixed strategy for Player 1. Similarly let $S^2 = \{x \in R^{n_2} | x \geq 0, \sum_{i=1}^{n_1} x_i = 1\}$ be the set of mixed strategies for Player 2. We say that (x^*, y^*) is a Nash equilibrium strategy for the bimatrix game (n_1, n_2, A, B) if $x^* \in S^1, y^* \in S^2$, $(x^*)^t A y^* \geq x^t A y^* \; \forall \; x \in S^1$, and $(x^*)^t B y^* \geq (x^*)^t B y \; \forall \; y \in S^2$. The problem of finding a Nash equilibrium strategy and the corresponding equilibrium payoffs has been formulated as a linear complementarity problem by Lemke and Howson. See [3]. There have also been other formulations of this problem as a linear complementarity problem. In what follows we shall use the following formulation which is similar to the one presented by Miller and Zucker [4]. We have the following result.

LEMMA 5.1 *Given a bimatrix game* (n_1, n_2, A, B) *with* $A > 0, B > 0$, (x^*, y^*) *is a Nash equilibrium with equilibrium pay-offs* $\beta_1^* = (x^*)^t A y^*$ *and* $\beta_2^* = (x^*)^t B y^*$ *if and only if* $(x^*, y^*, \beta_1^* + 1, \beta_2^* + 1)$ *is a solution to the* $LCP(q, M)$ *with*

$$M = \begin{bmatrix} E_1 & A & -e_1 & 0 \\ B^t & E_2 & 0 & -e_2 \\ e_1^t & 0 & 0 & 0 \\ 0 & e_2^t & 0 & 0 \end{bmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix},$$

where E_1 *is a* $n_1 \times n_1$ *matrix whose entries are all equal to 1 and* E_2 *is a matrix of order* $n_2 \times n_2$ *whose entries are all equal to 1.*

THEOREM 5.1 *The* $LCP(q, M)$ *as introduced in Theorem 5.1 induced by a bimatrix game* (n_1, n_2, A, B) *has a proper solution which yields a proper Nash equilibrium to that game.*

Proof: Let $k = n_1 + n_2$. Let α be a real number such that $0 < \alpha < 1$. Let $\epsilon \in R^k$ be defined by taking its ith coordinate as $\epsilon_i = \alpha^k, 1 \leq i \leq k$.

Let $S(\alpha) = \{u \in R^k | u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, u^1 \in S_1, u^2 \in S_2, u \geq \epsilon\}$. Note that $S(\alpha)$ is a nonempty compact convex subset of R^k. Define a point to set function F on $S(\alpha)$ as follows: Given $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in S(\alpha)$, let

$$\beta_1 = min_{1 \leq i \leq n_1}[1 + (Au^2)_i] \tag{5.1}$$

$$L_1 = \{i | \beta_1 = 1 + (Au^2)_i\} \tag{5.2}$$

and let

$$\beta_2 = min_{1 \leq i \leq n_2}[1 + (B^t u^1)_i] \tag{5.3}$$

$$L_2 = \{i | \beta_2 = 1 + (B^t u^1)_i\}. \tag{5.4}$$

Let $\xi \in R^{k+2}$ be defined as the vector $\begin{pmatrix} u \\ \beta_1 \\ \beta_2 \end{pmatrix}$. Note that $w(\xi) = q + M\xi \geq 0$. Further note that the set $L = \{i | w_i(\xi) = 0\}$ equals $L_1 \cup (\{n_1\} + L_2) \cup \{k+1, k+2\}$. The image set $F(u)$ is defined as

$$\{z \in S(\alpha) | w_i(\xi) > 0 \Rightarrow z_i \leq \alpha,\ w_i(\xi) < w_j(\xi) \Rightarrow z_j \leq \alpha z_i,\ 1 \leq i, j \leq k\}.$$

Note that this is a convex and compact subset of $S(\alpha)$. To show that the set is nonempty for all α sufficiently small, we note that the point z^* defined as follows is contained in $F(u)$. For any i, $1 \leq i \leq k$, first let $r_i = \#\{j | w_j(\xi) < w_i(\xi),\ 1 \leq j \leq k\}$. Then let

$$(z^{*1})_i = \alpha^{r_i} \quad \text{if } r_i \geq 1, 1 \leq i \leq n_1 \tag{5.5}$$

$$= \frac{1 - \sum_{j \notin L_1} (z^{*1})_j}{\#(L_1)} \quad \text{if } i \in L_1 \tag{5.6}$$

$$(z^{*2})_i = \alpha^{r_{n_1+i}} \quad \text{if } r_{n_1+i} \geq 1, 1 \leq i \leq n_2 \tag{5.7}$$

$$= \frac{1 - \sum_{j \notin L_2} (z^{*2})_j}{\#(L_2)} \quad \text{if } i \in L_2. \tag{5.8}$$

Now take $z^* = \begin{pmatrix} z^{*1} \\ z^{*2} \end{pmatrix}$. Note that $z^* \in S(\alpha)$ for all α sufficiently small and that $w_i(\xi) > 0 \Rightarrow z_i^* \leq \alpha$. For $1 \leq i,j \leq k$ if $w_j(\xi) < w_i(\xi)$, then note that $i \notin L$ and that $r_j < r_i$. Hence it follows that $z_i^* \leq \alpha z_j^*$. Thus $z^* \in F(u)$. We now claim that the map F is a closed map. To see this, suppose we have a sequence $\{u^n\}$, where $u^n \in S(\alpha)$ which converges to some $\bar{u} \in S(\alpha)$. Also suppose that $z^n \in F(u^n)$ and the sequence $\{z^n\}$ converges to some $\bar{z}$. Given u^n, let β_1^n and β_2^n be defined as in (5.1) and (5.3), respectively. It is clear that β_1^n converges to $\bar{\beta}_1$ where

$$\bar{\beta}_1 = min_{1 \leq i \leq n_1}[1 + (A\bar{u}^2)_i]$$

and β_2^n converges to

$$\bar{\beta}_2 = min_{1 \leq i \leq n_2}[1 + (B^t\bar{u^1})_i].$$

Let ξ^n be defined as $\begin{pmatrix} u^n \\ \beta_1^n \\ \beta_2^n \end{pmatrix}$. Note that $\bar{\xi} = lim_{n \to \infty}\xi^n$ is given by $\begin{pmatrix} \bar{u} \\ \bar{\beta}_1 \\ \bar{\beta}_2 \end{pmatrix}$.

Suppose now $w_i(\bar{\xi}) < w_j(\bar{\xi})$. It follows that for all n sufficiently large, $w_i(\xi^n) < w_j(\xi^n)$. Hence it follows that $z_j^n \leq \alpha z_i^n$ and hence $\bar{z}_j \leq \alpha \bar{z}_i$. Similarly it is easy to verify that $w_i(\bar{\xi}) > 0 \Rightarrow \bar{z}_i \leq \alpha$. Thus $\bar{z} \in F(\bar{u})$ and hence F is closed. We now appeal to Kakutani's fixed point theorem (see p.67 in [7]) to conclude that there is a $v(\alpha) \in S(\alpha)$ such that $v(\alpha) \in F(v(\alpha))$. In other words, given any α sufficiently small, there is a $v(\alpha) \in S(\alpha)$ such that with $\beta_1(v(\alpha))$ and $\beta_2(v(\alpha))$ as defined in (5.1) and (5.3), respectively, and with $\eta(\alpha) = \begin{pmatrix} v(\alpha) \\ \beta_1(v(\alpha)) \\ \beta_2(v(\alpha)) \end{pmatrix}$, $\eta(\alpha)$ is an α- proper solution. As α goes to zero, since $v(\alpha)$ and hence also $\beta_1(v(\alpha))$ and $\beta_2(v(\alpha))$ are bounded, it follows that there is a subsequence $\{\alpha_\nu\}$ for which $v(\alpha_\nu)$ converges to a limit v^* and $\beta_1(v(\alpha_\nu))$ and

$\beta_2(v(\alpha_))$ converge to some β_1^* and β_2^*, respectively. Let $\eta^* = \begin{pmatrix} v^* \\ \beta_1^* \\ \beta_2^* \end{pmatrix}$. Let $w(\eta^*) = q + M\eta^*$. Since For $1 \leq i \leq k$, $w_i(\eta(\alpha)) > 0 \rightarrow v_i(\alpha) \leq \alpha$, it follows that $w_i(\eta^*) > 0 \rightarrow \eta_i^* = 0$, $\forall\ i = 1, 2 \ldots k$. Thus $(w(\eta^*), \eta^*)$ is a solution to the $LCP(q, M)$ which is proper. Moreover, for small enough $\alpha, v(\alpha)$ is an α-proper Nash equilibrium and v^* itself is a proper Nash equilibrium to the game.

■

REMARK 5.1 *Theorem 5.2 can also be stated and proved in a similar manner for a polymatrix game (see [2] and [4] for a discussion on polymatrix games).*

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