entER
for
omic Research

Discussion paper

Tilburg University

Center
for
Economic Research

go

No. 9659

REVEALED LIKELIHOOD AND KNIGHTIAN UNCERTAINTY

By Rakesh Sarin and Peter Wakker

R 20

June 1996

t uncertainty
v expected utility theory
v revealed preference

ISSN 0924-7815

REVEALED LIKELIHOOD AND KNIGHTIAN UNCERTAINTY*

Rakesh Sarin and Peter Wakker

The Anderson Graduate School of Management
University of California, Los Angeles
Los Angeles, CA
and
CentER for Economic Research
University of Tilburg
Tilburg, The Netherlands
March, 1996

Nonadditive expected utility models were developed for explaining preferences in settings where probabilities cannot be assigned to events. In the absence of probabilities, difficulties arise in the interpretation of likelihoods of events. In this paper we introduce a notion of revealed likelihood that is defined entirely in terms of preferences and that does not require the existence of (subjective) probabilities. Our proposal is that decision weights rather than capacities are more suitable measures of revealed likelihood in rank-dependent expected utility models and prospect theory. Applications of our proposal to the updating of beliefs, to the description of attitudes towards ambiguity, and to game theory are presented.

Keywords: Ellsberg paradox, nonadditive probability, Choquet-expected utility, rank-dependent utility, prospect theory, updating of beliefs

* The support for this research was provided in part by the Decision, Risk, and Management Science branch of the National Science Foundation.

I. INTRODUCTION

It has long been recognized that there is a distinction between risk, where probabilities are known, and uncertainty, where probabilities are unknown (Keynes, 1921; Knight, 1921). In a seminal work, Savage (1954) argued that for a rational agent such a distinction is not relevant. In his framework, probabilities measure the likelihood of events. A key idea in Savage's theory is that probabilities are revealed from preferences rather than from introspection or verbal reports.

There is, however, a large body of empirical evidence that contradicts Savage's subjective expected utility model (Camerer & Weber, 1992). In particular, Ellsberg (1961) showed that Savage's method for revealing probability leads to inconsistencies, i.e. probabilities cannot be assigned to events. In the absence of probability the question arises what, if any, meaning can be given to likelihood.

In this paper, we propose a notion of revealed likelihood that is derived from preferences and that is consistent with Ellsberg's findings. Our measure of revealed likelihood resolves a duality paradox in nonexpected utility and clarifies the definition of null events that are relevant for Nash equilibria. It leads to a new rule for updating that resolves some ambiguities in rules proposed in the literature, such as the Dempster-Shafer update rule. Finally, our measure gives a natural description for several phenomena regarding decision under uncertainty, such as ambiguity aversion (pessimism) and the simultaneous buying of insurance and gambling.

Our analysis is based on rank-dependent ("nonadditive") expected utility for uncertainty, hereafter called Choquet expected utility (CEU) (Schmeidler, 1989; Gilboa, 1987). For the context of risk, similar models were proposed by Quiggin (1982) and Allais (1988). The primary motivation for the development of CEU was to model the distinction between risk and uncertainty that was suggested by Keynes and Knight. CEU is able to accommodate the preference patterns of the Ellsberg examples.

Our results also apply to cumulative prospect theory (Tversky & Kahneman, 1992). Cumulative prospect theory generalizes CEU by permitting decision weights for gains to be different than decision weights for losses, and has a number of empirical advantages. For example, Benartzi & Thaler (1995) explain the equity premium puzzle by loss aversion. Our measure of revealed likelihood can be applied to gains and losses separately, and thus can elicit decision weights in cumulative prospect theory.

In Section II, we review the notion of likelihood in subjective expected utility theory. Section III discusses the discrepancy between likelihood revealed from bets on and bets against events that is commonly found in the Ellsberg examples. In Section IV, we argue that in CEU, one needs to distinguish revealed likelihoods derived from bets on events from revealed likelihoods derived from bets against events. Section V shows that in the derivation of CEU one may use preference conditions in a consistent way so long as one employs the appropriate notion of revealed likelihood. This solves a duality paradox noted in the literature. In Section VI, we generalize revealed likelihood to the multiple consequences case. We argue that, if revealed likelihood should "tell you where to put your money," then decision weights are the proper measure of revealed likelihood under CEU. Section VI sheds new light on axiom P2 of Gilboa (1987). It shows how that axiom can be used to empirically elicit orderings of decision weights.

An attractive property of expected utility is independence of beliefs from tastes. In Section VII, we argue that to some degree independence of revealed likelihood from consequences can be maintained in CEU so long as one specifies a "dominating event." In Section VIII, we argue that decision weights have some distinct advantages over capacities in measuring revealed likelihood. Section IX illustrates an application of our measure of revealed likelihood in defining null events which is an important issue for updating and for the definition of Nash equilibrium in game theory. Several other properties of decision weights as measure of revealed likelihood are discussed. For example, a new interpretation is provided for the case of probabilistic sophistication (Machina & Schmeidler, 1992).

Section X discusses updating if new information is gathered. Several proposals for updating in the literature are explained as different choices of the dominating events introduced in Section VII. In Section XI, we discuss the interpretation of revealed likelihood as a measure of belief. Revealed likelihood may depend both on beliefs and on decision attitudes. Finally, Section XII presents conclusions. Proofs are presented in the appendix.

II. SUBJECTIVE EXPECTED UTILITY

In subjective expected utility (SEU), the likelihood of an event is measured by its subjective probability. Thus,

Event A is more likely than event B

if and only if

the probability of A is greater than the probability of B.

In the above statement the likelihood judgments are quantified by a probability measure. Thus, we write

(1) $$A \succ B \text{ if and only if } P(A) > P(B).$$

Subjective probabilities are often interpreted as a measure of degree of belief, reflecting the state of information of the decision maker. It is however erroneous to assume that directly elicited probability through verbal report (e.g. my probability that it will rain tomorrow is 0.4) will necessarily coincide with the subjective probability that is based on preferences over bets. Savage rejects the approach of eliciting likelihood from direct interrogation. He anticipates "Perhaps the first way that suggests itself to find out which of two events is more probable is simply to ask him." He then goes on to provide counter-arguments for such an approach. Instead, he strongly argues for inferring the likelihood comparison from decision behavior. As an illustration of the latter approach he uses an example "If under these circumstances the person stakes his chance for the dollar on the brown egg, it seems to me to correspond well with ordinary usage to say that it is more probable to him that the brown one is a better one than the white one is." To Savage the theory of personal probability is "a code of consistency for the person applying it, not a system of predictions about the world around him." Thus Savage takes preferences over bets as the observable primitive, and subjective probabilities represent preferences. Any other interpretation of subjective probabilities is speculative. This approach is in line with the revealed preference approach of Samuelson (1938) and others for inferring utilities from choices. For empirical studies of problems for likelihood elicitation under SEU, see Erev, Bornstein, & Wallsten (1992) and Liberman & Tversky (1993).

For the two consequence case, the likelihood relation can be operationalized in either of the following two equivalent ways:

A is more likely than B if one prefers a bet on A to a bet on B (Figure 1a).

A is more likely than B if one disprefers a bet against A to a bet against B (Figure 1b).

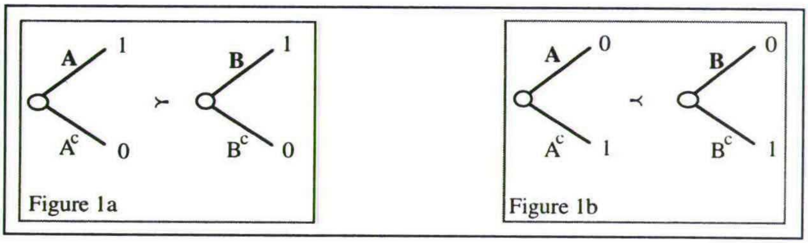

Figure 1a

Figure 1b

Thus, "more likely than" judgments are not elicited through verbal statements; instead they are revealed through preference comparisons between various "win - lose" bets.

Implicit in the betting method for revealing likelihood is the assumption that the likelihood comparison of events A and B is independent of the pairs of consequences used. This independence is ensured through Savage's axiom P4.

It is easy to verify that for an SEU maximizer either preferences in Figure 1a or Figure 1b would reveal the same likelihood relation. The preferences in Figure 1a reveal $P(A) > P(B)$ and those in Figure 1b reveal $P(A^c) = 1-P(A) < P(B^c) = 1-P(B)$, each leading to the conclusion that A is revealed to be more likely than B. Figure 1a illustrates that one prefers to win on the more likely event A, and Figure 1b illustrates that one disprefers to lose on the more likely event A. An example of a bet against an event is as follows. If one does not like (disprefers) radiation therapy as compared to surgery then this may reveal that the likelihood of recurrence is higher under radiation than under surgery.

The desirability of eliciting likelihoods using bets on events or bets against events depends on the decision context. In theoretical analyses, likelihoods have mostly been inferred using bets on events, as in Figure 1a. They can, however, just as well be elicited using bets against events as in Figure 1b. There is no prior reason to prefer one method over the other, though in practical applications one of the two methods may be more convenient. A large part of our risky decisions concerns avoidance of unfavorable events, in which case it is natural to think in terms of bets against events. Examples are health care, safety measures, and insurance. In the next section the choice of method will be more than a matter of practical convenience and will lead to conceptual differences.

III. Revealed Qualitative Likelihood

Ellsberg (1961) showed that empirically the two ways of operationalizing likelihood as in Figure 1 do not lead to the same result for some events. To illustrate this violation of SEU, consider two urns, one containing 50 white and 50 black balls, and the other containing a total of 100 white and black balls in unknown proportion (see Figure 2). From each of the two urns a ball is randomly drawn. People often prefer a bet on event K (white from known urn) to a bet on event U (white from unknown urn), while preferring a bet on K^c (black from known urn) to a bet on U^c (black from unknown urn). In the same issue of the Quarterly Journal of Economics where Ellsberg published his classic article, Fellner (1961) and Raiffa (1961) provided their reactions to Ellsberg's observations. Raiffa (1961) gave normative arguments against Ellsberg's finding, while acknowledging its descriptive validity. Fellner (1961) agreed with Ellsberg and suggested that people distort probabilities when dealing with decisions under uncertainty. Earlier, Keynes (1921) and Knight (1921) had also made a distinction between situations where probabilities are known and where they are unknown. It was precisely this distinction that motivated Schmeidler to propose CEU theory as an alternative to SEU.

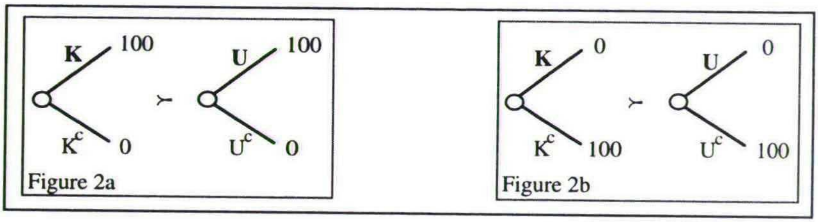

Figure 2a

Figure 2b

The above pattern of preference implies that event K is *revealed more likely* than event U when one derives the likelihood relation from bets on events (Figure 2a). Such an inference is guided by the intuition that one should prefer the more likely gain. Event K is *revealed less likely* than event U when one derives it from bets against events (Figure 2b). In Figure 2b, one loses on events K and U and the inference that K is less likely than U is guided by the intuition that one should prefer to lose on the less likely event.

The above example demonstrates that a revealed likelihood relation derived from bets on events may differ from that derived from bets against events. To distinguish these two notions of revealed likelihood we introduce the following notation. We write $\succcurlyeq^{\uparrow}$ for revealed likelihood derived from bets on events. That is, $A \succcurlyeq^{\uparrow} B$ if there exist consequences $x > y$ such that

(2) $$(A,x;\ A^c,y) \succcurlyeq (B,x;\ B^c,y)$$

where $\succcurlyeq$ denotes weak preference and $(A,x;\ A^c,y)$ denotes the act yielding x if A occurs and y otherwise. We write $\succ^{\uparrow}$ instead of $\succcurlyeq^{\uparrow}$ if $(A,x;\ A^c,y) \succ (B,x;\ B^c,y)$, i.e. the preference in (2) is strict.

Similarly, we write $\succcurlyeq^{\downarrow}$ for revealed likelihood derived from bets *against* events. That is, $A \succcurlyeq^{\downarrow} B$ if there exist consequences $x > y$ such that

$$(A^c,x;\ A,y) \preccurlyeq (B^c,x;\ B,y)$$

where $\preccurlyeq$ denotes reversed preference ($f \preccurlyeq g$ meaning $g \succcurlyeq f$). Again, $\succ^{\downarrow}$ denotes strict preference. Note that $\succcurlyeq^{\uparrow}$ and $\succcurlyeq^{\downarrow}$ coincide for SEU. In the elicitation of $\succcurlyeq^{\uparrow}$ a superior consequence is associated with events A and B, that is, A and B play the role of "good-news events." In contrast, in the elicitation of $\succcurlyeq^{\downarrow}$ an inferior consequence is associated with events A and B, hence these events play the role of "bad-news events." The preference pattern observed in the Ellsberg paradox implies $K \succ^{\uparrow} U$ but $U \succ^{\downarrow} K$ and thus constitutes a violation of SEU.

In $K \succ^{\uparrow} U$, one wins if events K or U occur. A person who is pessimistic with respect to unknown probabilities (ambiguity averse) considers winning on the unknown urn less likely. In $U \succ^{\downarrow} K$, one loses if events U and K occur. Now, a pessimist considers losing on the unknown urn more likely.Thus, for a pessimist the bad news looms larger than the good news. Suppose for simplicity that in the known urn, the probability of winning or losing is 0.5 each. A pessimist behaves as if in the unknown urn the probability of winning is less than 0.5 and the probability of losing is more than 0.5. Thus a pessimist downplays the likelihood of winning and exaggerates the likelihood of losing. This behavior is highlighted in Murphy's law: "If something bad can happen, it will." Of course, there may be reasons other than optimism or pessimism for the above preferences.

It is useful to note the following duality between $\succcurlyeq^{\uparrow}$ and $\succcurlyeq^{\downarrow}$:

(3) $$A \succcurlyeq^{\uparrow} B \Leftrightarrow B^c \succcurlyeq^{\downarrow} A^c.$$

The left-hand side says that a bet on A is preferred to a bet on B. As a bet on A is a bet against A^c and a bet on B is a bet against B^c, this means that a bet against B^c is dispreferred to a bet against A^c, which is the right-hand side. In other words, both the left-hand side and the right-hand side describe the preference in Figure 1a. Hence, one relation can be inferred from the other, and they both describe the same information.

IV. CHOQUET-EXPECTED UTILITY

To distinguish Knightian uncertainty from risk and to accommodate the Ellsberg paradox, Schmeidler (1989) proposed nonadditive capacities defined on events. We assume that consequences are amounts of money and that preferences satisfy monotonicity, i.e. higher amounts are preferred to lower amounts. Events A, B, etc. are subsets of the state space S that can be infinite. We do restrict our attention to simple acts (i.e., acts that take only finitely many different consequences) throughout the paper. In Choquet-expected utility, a "capacity" v is used instead of the additive probability measure P of SEU. It is assumed that v assigns value 0 to the impossible event, value 1 to the universal event S, and $A \supset B$ implies $v(A) \geq v(B)$. Then the CEU value of an act $(A_1,x_1;\cdots;A_n,x_n)$ where $x_1 \geq \cdots \geq x_n$, is given by

(4) $$\sum_{i=1}^{n} \pi_i U(x_i)$$

where U is the utility function as in SEU, and the π_i denote decision weights, defined by

(5) $$\pi_i = v(A_1 \cup \cdots \cup A_i) - v(A_1 \cup \cdots \cup A_{i-1}).$$

Similar formulas are used for cumulative prospect theory, except that the capacity for gains can be different than the capacity for losses. CEU permits the preference patterns observed in the Ellsberg paradox by setting $v(K) > v(U)$ and $v(K^c) > v(U^c)$. Under CEU the following results hold:

(i) $A \succcurlyeq^{\uparrow} B$ if and only if $v(A) \geq v(B)$.

(ii) $A \succcurlyeq^{\downarrow} B$ if and only if $1 - v(A^c) \geq 1 - v(B^c)$.

Thus $v(A)$ represents the $\succcurlyeq^{\uparrow}$ ordering, derived from bets on events, and its dual $1-v(A^c)$ represents the $\succcurlyeq^{\downarrow}$ ordering, derived from bets against events. For this reason, we write $v^{\uparrow}(A)$ for $v(A)$, and $v^{\downarrow}(A)$ for $1-v(A^c)$. $v^{\uparrow}$ is the capacity for events in the role of good-news events, and $v^{\downarrow}$ is the capacity for events in the role of bad-news events. In SEU, $v^{\uparrow} = v^{\downarrow} = P$. In CEU, however, $v^{\uparrow}$ and $v^{\downarrow}$ need not be identical.

The discussion above is based on a duality between good- and bad-news events. As there has been confusion about this duality, and it is central for our measure of likelihood, we discuss it in some detail. The duality has also been discussed for Choquet integration. In the literature, an alternative way for defining Choquet integrals that is dual to Formula (5) has been used. This dual Choquet integral is obtained by defining

(6) $$\pi_i = v(A_i \cup \cdots \cup A_n) - v(A_{i+1} \cup \cdots \cup A_n)$$

instead of (5) in Formula (4).[1] Note that the decision weight π_1 in (6) now is equal to $1 - v(A_2 \cup \cdots \cup A_n)$ instead of $v(A_1)$ in (5). The method of integration through (6) is called

[1]Equivalently, one can order consequences alternatively by $x_1 \leq \cdots \leq x_n$ and then use Formula (5). Reversing the rank-ordering of consequences and using Formula (5) gives the same results as keeping the rank-ordering of this paper and using Formula (6).

the *lower Choquet integral*, and similarly the method of integration through (5) is called the *upper Choquet integral.* Clearly, these may yield different orderings of acts. Thus the question arises which formula for computing CEU is the "right" one, and how the seeming inconsistency between (5) and (6) can be resolved. There is no inconsistency, however, between (5) and (6) if the relevance of the role of events is recognized. That is, (5) entails good-news events $A_1 \cup \cdots \cup A_i$ (receive x_i or more) and therefore $v^{\uparrow}$ should be used there. Formula (6) entails bad-news events $A_i \cup \cdots \cup A_n$ (receive x_i or less) and therefore $v^{\downarrow}$ should be used. In this manner, the two methods for computing CEU yield identical results. Note that this consistency is obtained in general and it does not impose restrictions on capacities such as symmetry.

Imagine now that a person uses the capacity $v^{\uparrow}$, elicited from bets on events, but uses Formula (6) to calculate CEU. Note that in this case the capacity $v^{\uparrow}$ for good-news events is applied to bad-news events in (6). For *symmetric* capacities ($v(A) = 1 - v(A^c)$, i.e. $v^{\uparrow} = v^{\downarrow}$), the above scheme results in the correct CEU values after all. For non-symmetric capacities, this mis-matching of capacity and integration will produce wrong results (Gilboa, 1989a). The question of which capacity to use, $v^{\uparrow}$ or $v^{\downarrow}$, and the question of which method of integration to use, (5) or (6), in isolation are not meaningful. They must be considered jointly and applied consistently.

The following linguistic example may illustrate the idea of mis-matching. It is now well-accepted that an author may use male-specific pronouns (he/his/him) or female-specific pronouns (she/her) to designate an abstract person (decision maker, agent, defendant). There is no reason to prefer a choice of "he" to a choice of "she," and there is no reason to prefer a choice of "him" to a choice of "her." These two choices, however, are intertwined and cannot be made independently. An argument to the effect that "he" could be replaced by "she" without recognizing the interdependence of the he/she choice with the his/her choice would lead to anomalies such as "he maximizes her utility." Clearly a mis-match of the pronouns along the way yields an unintended implication of altruism. The sentences "he maximizes his utility" and "she maximizes her utility" are truly dual to each other and either one is acceptable.

Our main point in the above discussion has been that the revealed likelihood ordering ($\succcurlyeq^{\uparrow}$ or $\succcurlyeq^{\downarrow}$), the capacity ($v^{\uparrow}$ or $v^{\downarrow}$), and the manner of integration (upper or lower) should be consistent with the role of events. For the good-news events $\succcurlyeq^{\uparrow}$, $v^{\uparrow}$ and upper integration should be used, and for the bad-news events $\succcurlyeq^{\downarrow}$, $v^{\downarrow}$, and lower integration should be used. Good-news or bad-news events are dual in the same way as the male or female gender are in the linguistic example. There is a complete freedom to choose the role of events in CEU and the gender in the linguistic example, as long as consistency is maintained throughout.

V. Cumulative Dominance

In Sarin & Wakker (1992), CEU is characterized by using a cumulative dominance condition. Cumulative dominance states that act f is weakly preferred to act g whenever, for all consequences x, the good-news event of receiving x or more under f is revealed at least as likely as the good-news event of receiving x or more under g. As this formulation employs good-news events, the revealed likelihood for good-news events ($\succcurlyeq^{\uparrow}$) should be adopted. We display the condition:

(7) $$f \succcurlyeq g \text{ whenever, for all consequences } x, [f \geq x] \succcurlyeq^{\uparrow} [g \geq x].$$

An equivalent dual formulation is given in the observation below. The dual formulation is in terms of bad-news events. Because the proof illustrates the duality between good- and bad-news event, it is presented in the main text.

OBSERVATION 1. Cumulative dominance holds if and only if

(8) $\quad f \succcurlyeq g$ whenever, for all consequences x, $[g \le x] \succcurlyeq^{\downarrow} [f \le x]$.

PROOF. In the dual formulation, for every consequence x a bet against event: $[g \le x]$ is dispreferred to a bet against event $[f \le x]$. Thus, $[f \le x]^c \succcurlyeq^{\uparrow} [g \le x]^c$ for all x, which implies, similarly to (3), that $[f \ge y] \succcurlyeq^{\uparrow} [g \ge y]$ for all consequences y (let $\{z: z \ge y\} = \{z: z > x\}$, using finite ranges of f and g). □

The condition in the observation states that act f is weakly preferred to act g whenever, for all consequences x, the bad-news event of receiving x or less under g is revealed at least as likely as the bad-news event of receiving x or less under f. As this formulation employs bad-news events, the revealed likelihood relation for bad-news events ($\succcurlyeq^{\downarrow}$) is adopted. Thus cumulative dominance can be formulated in two equivalent dual ways: either it is formulated in terms of good-news events, or in terms of bad-news events. In the former case, the revealed likelihood-relation $\succcurlyeq^{\uparrow}$ for good-news events is to be employed, and in the latter case the revealed likelihood relation $\succcurlyeq^{\downarrow}$ for bad-news events. The two statements of the cumulative dominance are then truly dual, i.e. describe the same empirical restriction, and result in the same CEU representation. The important point to note is that the revealed likelihood relation should be consistent with the role of the events.

Cumulative dominance has a resemblance to stochastic dominance when probabilities are given. Although this resemblance makes this condition transparent, it should be understood that cumulative dominance does not have the normative appeal of stochastic dominance. This is because, unlike stochastic dominance, cumulative dominance cannot be derived from a statewise monotonicity condition.

We next study the implications of a variation of the cumulative dominance axiom where the preference condition involves bad-news events, but the revealed likelihood-relation adopted is the one for good-news events. In other words:

(9) $\qquad f \succcurlyeq g$ whenever, for all consequences x, $[g \leq x] \succcurlyeq^{\uparrow} [f \leq x]$.

Consider two-consequence acts $f=(A,x;\ A^c,y)$ and $g=(B,x;\ B^c,y)$, $x>y$. Clearly, $f \succcurlyeq g$ if and only if $A \succcurlyeq^{\uparrow} B$. Condition (9), however, would require that $f \succcurlyeq g$ if $B^c \succcurlyeq^{\uparrow} A^c$, i.e. (by Formula 3) if $A \succcurlyeq^{\downarrow} B$. Thus, $A \succcurlyeq^{\downarrow} B$ would imply $A \succcurlyeq^{\uparrow} B$ which was precisely the restriction we wished to relax to accommodate the Ellsberg paradox. In other words, the mismatch of (bad-news) events and the (good-news) likelihood relation in (9) leads to unwarranted implications.

Next we demonstrate that cumulative dominance and dual cumulative dominance are necessary conditions for CEU. We present the result here because the, elementary, proof (given in the appendix) further clarifies the duality between the above two dominance conditions, and shows that this duality is the qualitative analog of the duality between upper and lower Choquet integration.

OBSERVATION 2. Cumulative dominance and dual cumulative dominance are necessary conditions for CEU. □

We have emphasized above that the $\succcurlyeq^{\uparrow}$ relation refers to events in the role of good-news events, and the $\succcurlyeq^{\downarrow}$ relation refers to events in the role of bad-news events. Therefore we used $\succcurlyeq^{\uparrow}$ in (7) and $\succcurlyeq^{\downarrow}$ in (8) to avoid mixing and we obtained a consistent characterization of CEU. In (9), the $\succcurlyeq^{\uparrow}$ relation for good-news events is applied to a preference condition defined in terms of bad-news events. This constitutes the same mis-matching as described at the end of Section IV, and illustrated there by the linguistic example. In SEU, (9) will not produce a contradiction because the revealed likelihood relation is independent of the role of events, i.e. $\succcurlyeq^{\uparrow} = \succcurlyeq^{\downarrow}$. In CEU, however, such an identity imposes an unwarranted symmetry of the capacity, i.e. $v^{\uparrow} = v^{\downarrow}$ (Nehring, 1994). The following example illustrates our point further.

EXAMPLE 3. Assume that there is a "known" urn that contains red (R_k), yellow (Y_k), and white (W_k) balls in equal proportion. There is another, "unknown," urn that also contains red (R_u), yellow (Y_u), and white (W_u) balls, but in an unknown proportion. A ball will be drawn at random from each urn. Consider the acts $f = (R_k,100; Y_k,50; W_k,0)$ and $g = (R_u,100; Y_u,50; W_u,0)$. Thus f is related to the known urn, and g to the unknown urn. We assume the most commonly found preference for betting on known urns. Thus $(R_k,100; Y_k,0; W_k,0) \succcurlyeq (R_u,100; Y_u,0; W_u,0)$, i.e. $R_k \succcurlyeq^{\uparrow} R_u$, and $(R_k,100; Y_k,100; W_k,0) \succcurlyeq (R_u,100; Y_u,100; W_u,0)$, i.e. $R_k \cup Y_k \succcurlyeq^{\uparrow} R_u \cup Y_u$. Of course, we trivially have $R_k \cup Y_k \cup W_k \succcurlyeq^{\uparrow} R_u \cup Y_u \cup W_u$. Thus all good-news events under f are at least as likely (by the $\succcurlyeq^{\uparrow}$ relation) as under g, and by cumulative dominance, $f \succcurlyeq g$. This agrees with what is commonly observed.

Next we consider the implications of condition (9). We have $(R_k,0; Y_k,0; W_k,100) \succcurlyeq (R_u,0; Y_u,0; W_u,100)$, i.e. $W_k \succcurlyeq^{\uparrow} W_u$, and $(R_k,0; Y_k,100; W_k,100) \succcurlyeq (R_u,0; Y_u,100; W_u,100)$, i.e. $Y_k \cup W_k \succcurlyeq^{\uparrow} Y_u \cup W_u$, and, trivially, $R_k \cup Y_k \cup W_k \succcurlyeq^{\uparrow} R_u \cup Y_u \cup W_u$. Thus all bad-news events under f are at least as likely (by the $\succcurlyeq^{\uparrow}$ relation) as under g, and by condition (9) (with g and f interchanged), $g \succcurlyeq f$. The implied preference, however, disagrees with what is commonly observed. This counterintuitive prediction of (9) occurs because the events for which likelihood orderings are elicited are bad-news events (yielding consequence x or less) for the acts. In the likelihood elicitations, however, these events play the role of good-news events. Therefore the elicited likelihood orderings give misleading information concerning the preference between the acts f and g. Such a mismatch of roles of events does not occur in cumulative dominance (7). $\square$

Let us summarize the discussion in Sections III, IV, and V. Section III discusses the duality between "good-news" and "bad-news" events in CEU. In a quantitative setting, this duality was discussed by Gilboa (1989a), and in a qualitative setting it was discussed by

Nehring (1994). Our discussion starts in the qualitative context of revealed likelihood orderings, on the issue whether these orderings should be inferred from bets on or bets against events. In Section IV, the same issue is discussed in its quantitative version, i.e. whether a capacity or its dual should be used to measure revealed likelihood. The same duality also underlies the discussion whether one should do Choquet-integration in the "upper" version or in the dual, "lower," version. In Section V, we present a preference condition, cumulative dominance, that was used to characterize CEU by Sarin & Wakker (1992). The distinction between cumulative dominance and its dual is analogous to the distinction between upper and lower integration. Again, the good-news likelihood ordering should be used for cumulative dominance and the bad-news likelihood ordering should be used for dual cumulative dominance. Our approach developed in Sections III, IV, and V boils down to a simple prescription: When defining revealed likelihood and capacities and applying these to preference conditions and Choquet integration, one should be consistent regarding the role of events.

VI. Events with Intermediate Consequences

So far we have discussed revealed likelihood of events when they are associated with best or worst consequences. In the more general multiple-consequence case, some events have intermediate consequences. We examine revealed likelihoods for such events. From now on, in the rest of the paper, we assume CEU.

It has been empirically observed that intermediate consequences have less impact than extreme (best or worst) consequences. Thus the revealed likelihood of an event is lower when it is associated with intermediate consequences than when it is associated with extreme consequences. This phenomenon is described by "bounded subadditivity" for the uncertainty case (Tversky & Kahneman, 1992; Tversky & Fox, 1995; Tversky & Wakker, 1995), and by S-shaped probability transformation for the risk case (Karni & Safra, 1990; Kachelmeier & Shehata, 1992; Tversky & Kahneman, 1992; Bernasconi, 1994; Camerer & Ho, 1994; Wu & Gonzalez, 1994; Tversky & Fox, 1995). Bounded subadditivity underlies the coexistence of insurance and gambling.

This section considers "connected" events. An event is *connected* if each state outside the event either is lower in rank-ordering than all states of the event, or higher in rank-ordering than all states of the event, but never in between the states of the event. For example, for a given act f the event $\{s \in S: x \leq f(s) \leq y]$ is connected. Every event that has a constant consequence is connected.

To illustrate the general idea of revealing likelihood for intermediate events, assume an indifference

$$(A_1,10; \mathbf{A_2,2}; A_3,1) \sim (B_1,12; \mathbf{B_2,2}; B_3,0).$$

In this case, events A_2 and B_2 are associated with an intermediate consequence and our interest is in comparing the revealed likelihoods of A_2 and B_2 . Suppose we ask the question what is preferred, receiving an additional dollar under A_2 or under B_2. That is, what is the preference between

$$(A_1,10; \mathbf{A_2,3}; A_3,1) \text{ and } (B_1,12; \mathbf{B_2,3}; B_3,0)?$$

An intuitive reply may be that the additional dollar is preferred for the "more likely" event. Thus, if the left act is preferred then A_2 is "more likely" than B_2. In this context, A_2 and B_2 are neither good-news events nor bad-news events as they are associated with intermediate consequences. An SEU maximizer will prefer the left act if and only if $P(A_2) > P(B_2)$. The initial indifference and the preference for the left act together imply that the SEU increment for the left act, $P(A_2)(U(3)-U(2))$, is higher than $P(B_2)(U(3)-U(2))$, the SEU increment for the right act. A CEU maximizer will prefer the left act if and only if $\pi(A_2) > \pi(B_2)$, where $\pi(A_2)$ denotes the decision weight of A_2 and $\pi(B_2)$ the decision weight of B_2. This is because the CEU increment for the left act, $\pi(A_2)(U(3) - U(2))$ is higher than $\pi(B_2)(U(3) - U(2))$, the CEU increment for the right act. Since decision weights reflect where one would stake the bet, they can be a plausible measure of revealed likelihood. We further discuss the issue of interpretation after stating a preference condition for comparing revealed likelihoods through decision weights. The condition is based on Gilboa's (1987) condition P2* (see also Gilboa, 1989a)[2] that contains an intuitive and empirically valuable idea for CEU: It shows a way for comparing decision weights.

Suppose that $\beta > \alpha$ and

$$(A_1,x_1; \cdots; A_{i-1},x_{i-1}; \mathbf{A_i},\alpha; A_{i+1}, x_{i+1}; \cdots; A_n,x_n)$$

$$\sim$$

$$(B_1,y_1; \cdots; B_{j-1},y_{j-1}; \mathbf{B_j},\alpha; B_{j+1},y_{j+1}; \cdots; B_m,y_m)$$

where $x_1 \geq \cdots \geq x_{i-1} \geq \beta > \alpha \geq x_{i+1} \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_{j-1} \geq \beta > \alpha \geq y_{j+1} \geq \cdots \geq y_m$.

Then under CEU,

$$(A_1,x_1; \cdots; A_{i-1},x_{i-1}; \mathbf{A_i},\beta; A_{i+1}, x_{i+1}; \cdots; A_n,x_n)$$

$$\succcurlyeq$$

$$(B_1,y_1; \cdots; B_{j-1},y_{j-1}; \mathbf{B_j},\beta; B_{j+1},y_{j+1}; \cdots; B_m,y_m)$$

if and only if the decision weights satisfy $\pi(A_i) \geq \pi(B_j)$.

In the above condition, the incremental impact of A_i is equal to $\pi(A_i)(U(\beta)-U(\alpha))$ whereas the incremental impact of B_j is $\pi(B_j)(U(\beta)-U(\alpha))$. One therefore prefers to stake an additional amount of money on the event with the higher decision weight. It is in this sense that one could interpret that the revealed likelihood of A_i is higher than that of B_j in

---

[2]This condition was called to our attention by Alain Chateauneuf (1991, personal communication).

this decision context. In the next two sections we elaborate on measuring revealed likelihood through decision weights.

VII. Independence of Beliefs from Tastes

A well-known property of SEU is the "independence of beliefs from tastes." It means that the likelihood of an event, i.e. the probability, is independent of the consequences that are associated with the event, thus is independent of the particular acts. If one defines revealed likelihood of an event through its decision weight, as we propose, then the revealed likelihood of the event depends on the acts. More precisely, a revealed likelihood is only relevant in the evaluation of acts that generate, through their consequences, a given rank-ordering over the state space. Such a subset of the act space is called *comonotonic*. Some degree of independence from tastes is achieved here because the decision weights do not depend on the exact magnitudes of consequences so long as the rank-ordering of the consequences remains constant. In particular, we note that CEU satisfies Savage's P4. That is, if $A \succcurlyeq^{\uparrow} B$ is revealed through $(x,A;\ y,A^c) \succcurlyeq (x,B;\ y,B^c)$ for some $x > y$, then for all $x' > y'$, $(x',A;\ y',A^c) \succcurlyeq (x',B;\ y',B^c)$, confirming $A \succcurlyeq^{\uparrow} B$. Nevertheless, dependence of decision weights on the rank-ordering of consequences may be considered excessively flexible. It entails a considerable degree of dependence on tastes, and makes it hard to think of revealed likelihood as a property of events.

We now illustrate how the dependence of the revealed likelihood of events on the rank-order of the consequences can be reduced to such a degree that it becomes possible to consider revealed likelihood as a property of events. To do so, we introduce the following definition. We call an event D a dominating event for event A if, loosely speaking, the consequences under D are superior to the consequences under A, and the remaining consequences under events outside D and A are inferior. More precisely, given an act f, event D is a *dominating event* for event A if $A \cap D = \varnothing$ and $f(t) \geq f(s) \geq f(t')$ for all $t \in D$, $s \in A$, and $t' \in (A \cup D)^c$. Because the rank-ordering of states with equivalent consequences can be chosen arbitrarily, we can choose a rank-ordering of states that is compatible with f and is such that the states in D are ranked higher than the states in A, and the latter are ranked higher than those in $(A \cup D)^c$. As we shall see, for a large class of events, the revealed likelihood of A only depends on what the dominating event D of A is. That is, the revealed likelihood is relevant for the subset of all acts for which D is a dominating event for A.[3] For simplicity, first think of the case where A describes the receipt of a single consequence. Then the decision weight for A is given by

$$(10) \qquad v(A \cup D) - v(D),$$

where D denotes the dominating event. D is disjoint from A. The dependence of the decision weight of an event A on the dominating event D can be expressed in notation by writing $\pi(A,D)$.[4] Implicit in this notation is that A and D are disjoint. The decision weight of an event A can vary depending on whether the dominating event D is $\varnothing$, A^c, or some other intermediate event. Thus decision weight, as a measure of revealed likelihood, is a two-argument-function, depending on two events – the event itself and the dominating event. Interpreted thus, decision weights are to a high degree independent of consequences.

For more general, nonconnected events, decision weights can still be used as an index of revealed likelihood, but their dependency on the rank-ordering of the other events is more complex and cannot be described merely by one dominating event. Of course, the decision weight of a nonconnected event can be derived from the decision weights of the separate connected components through summation. We restrict most of the discussion of revealed likelihood in this paper to the class of connected events. The class is rich enough to cover

---

[3]Of course one could just as well express this dependence in terms of the dominated, "inferior," event I, i.e. the event yielding inferior consequences, by substituting $D = (A \cup I)^c$.

[4]When no confusion can arise, the event D is sometimes suppressed.

the majority of cases where likelihood is relevant. We think that a desirable feature of rank-dependent theories is that it achieves this degree of independence of revealed likelihood from consequences.

Let us next explain the Ellsberg paradox in terms of the dependence of revealed likelihood on a dominating event. Note that in the Ellsberg example presented in Section III, $\pi(K,D) > \pi(U,D)$ when $D = \varnothing$, and $\pi(K,D') < \pi(U,D'')$ when D' and D'' represent complementary events K^c and U^c respectively. The following example explains a variation of the Ellsberg paradox in terms of dependence of revealed likelihood on dominating events.

EXAMPLE 4. Consider an urn containing 30 red (R) balls, and 60 yellow (Y) and white (W) balls in unknown proportion. Two pairs of bets are illustrated in the table.

	R	**Y**	W
bet 1	**90**	0	100
bet 2	0	**90**	100
bet 3	**90**	0	0
bet 4	0	**90**	0

One may prefer bet 2 over bet 1 and bet 3 over bet 4. The first preference shows that Y is revealed more likely than R when the dominating event is W (i.e., $\pi(Y,W) > \pi(R,W)$). The second preference reveals the reverse ordering, i.e. R is revealed more likely than Y when the dominating event is null (i.e., $\pi(R,\varnothing) > \pi(Y,\varnothing)$). □

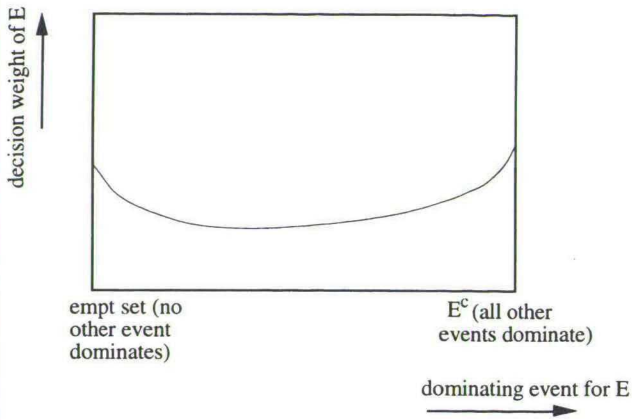

FIGURE 3. Decision weight of an event E as a function of the dominating event.

Figure 3 depicts the decision weight of an event E as a function of the dominating event, for the case of bounded subadditivity. For illustration, the dominating events are depicted as if they lie on one line. The decision weight of an event E is large when the dominating event is maximal (E^c), i.e. all other events are dominating. Then E is associated with the worst consequences and has a salient role as compared to the other events. Similarly, the decision weight of E is also large when the dominating event is minimal ($\varnothing$), i.e. no other events are dominating and E is associated with the best consequences. Then again E has a salient role. The decision weight of E is smaller when the dominating event is neither maximal nor minimal, i.e. when E is associated with intermediate consequences. In this case the role of E in comparison to the other events is less salient.

We concede that decision weights as measures of revealed likelihood in the CEU model are not as elegant as probabilities in the SEU model. For a comonotic class (fixed rank-ordering), however, decision weights share some common features with probabilities. For example, decision weights sum to one. As a result, if in an n-fold partition of the universal event all decision weights are the same then one immediately concludes they are all 1/n. For a subset of acts for which the dominating event D associated with an event A remains the same, the decision weight for A does not change. Clearly, in comparison to SEU, where the probability of A is independent of what goes on outside of A, revealed likelihood in CEU is more complicated. In CEU willingness to bet on an event A depends on the dominating event. Since revealed likelihood is elicited from preferences, there seems to be no escape from revealed likelihood to depend on the dominating event as well.

VIII. CAPACITIES VERSUS DECISION WEIGHTS

In this section we argue that decision weights have some distinct advantages over capacities in measuring revealed likelihood. We begin with a simple Ellsberg example given in Section III to illustrate our viewpoint.

EXAMPLE 5. The capacity-interpretation and the decision-weight interpretation agree that the preference in Figure 2a suggests a higher revealed likelihood for K than for U. However, the conclusion that K be revealed more likely than U cannot be made in general, and is not appropriate in Figure 2b. The preference in Figure 2b illustrates that one prefers to lose on event K rather than on event U. Therefore, event K is revealed less likely than event U. The decision weight for K in Figure 2b is indeed smaller than the decision weight for U. The capacity for K is, however, larger than the capacity for U. Therefore the decision weight seems a better measure of revealed likelihood than the capacity. From our perspective, capacities measure revealed likelihood only for events in the role of good-news events. □

In Figure 2b one could use the dual capacity to compare the revealed likelihood of K and U because it so turns out that the dual capacity is indeed the decision weight. In the multiple-consequence case, however, neither the capacity nor its dual will suffice as an index of revealed likelihood. This is illustrated by the following example.

EXAMPLE 6.

	A_1	$\mathbf{A_2}$	A_3		B_1	$\mathbf{B_2}$	B_3
pref. 1	10	**2**	1	$\sim$	12	**2**	0
pref. 2	10	**3**	1	$\prec$	12	**3**	0
pref. 3	0	**1**	0	$\succ$	0	**1**	0

Consider the preferences in the table. Suppose in the first indifference situation the person is asked if he prefers to receive an additional dollar on event A_2 or on event B_2. Suppose the person prefers the extra dollar on B_2, as shown in preference 2. Such a preference reveals that the person considers B_2 to be more likely than A_2. Indeed, the decision weight for B_2 is higher than that for A_2. The capacity, however, produces the reverse ordering of revealed likelihood as shown in preference 3. If the decision situations in preferences 1 and 2 are relevant to us, where A_2 and B_2 play the role of intermediate event, then we think that B_2 is revealed more likely than A_2. For such multiple-consequence cases the capacity is not an appropriate index of revealed likelihood. It may be noted that in this example the dual capacity may not be an appropriate index of revealed likelihood either. This would be the case, for example, if $(A_1,10; \mathbf{A_2,1}; A_3,10) \prec (B_1,10; \mathbf{B_2,1}; B_3,10)$. Then one disprefers losing on A_2 to losing on B_2 which implies that A_2 is revealed more likely than B_2 when they are both bad-news events.

Preferences as above are commonly found when the phenomenon of overestimating low likelihoods and underestimating high likelihoods is more pronounced for the A-events than for the B-events. Then the A-events receive relatively more decision weights than the B-events when they are associated with highest or lowest consequences, and they receive relatively less decision weights when they are associated with intermediate consequences. This phenomenon was characterized by Tversky & Wakker (1995), empirically found by Tversky & Fox (1995), and it becomes salient by the finding of Fox & Tversky (1995). □

The next example considers null events.

EXAMPLE 7.

	R	**Y**	W		R	**Y**	W
pref. 1	0	**1**	0	$\sim$	0	**0**	0
pref. 2	0	**1**	10	$\succ$	0	**0**	10

Here the capacity-interpretation of revealed likelihood suggests, according to the first indifference, that Y is null, which agrees with the decision-weight interpretation for good-news events. We think, however, that the claim that Y be null cannot be accepted in the second preference, where the person strictly prefers receiving an additional dollar on Y if it is an intermediate event. The decision weight for Y is indeed larger in this case. The preferences in the table above result from the example in Gilboa & Schmeidler (1993, introduction). □

Capacities resemble probabilities because they preserve independence of beliefs from tastes. In CEU, however, using capacities as a measure of likelihood introduces arbitrariness. From our perspective, it means that events are implicitly assumed to be good-news events. The capacity has a seductive appeal as a measure of likelihood since it does not depend on the rank-ordering of consequences. In CEU, insisting on a measure of revealed likelihood that is entirely independent of the rank-ordering of consequences (or a dominating event) is akin to throwing out the baby with the bath water. This is because, in

CEU, preferences depend on the rank-ordering of consequences and if the revealed likelihood is derived solely through preferences, there seems to be no escape from revealed likelihood to depend on the rank-ordering as well. Our three examples have demonstrated this dependence of revealed likelihood on the dominating event.

To further underscore the analogy between decision weights and probabilities, consider an act $(A_1,x_1; \cdots; A_n,x_n)$ with $x_1 > \cdots > x_n$ and define $U(x_i) = u_i$. Then the act can be represented as $(A_1,u_1; \cdots; A_n,u_n)$, $u_1 > \cdots > u_n$. In SEU, $\delta U/\delta u_i = p_i = P(A_i)$, where U is the SEU value of an act. In a similar manner, in CEU, $\delta U/\delta u_i = \pi_i = \pi(A_i)$, where U represents the CEU value of an act. This observation illustrates once more that in many respects decision weights are the analogs of probabilities in CEU.

Capacities measure revealed likelihood of cumulative events (x or more), whereas decision weights measure revealed likelihoods of separate events. Thus, capacities are special cases of decision weights in the same way as cumulative probabilities are special cases of probabilities.

IX. Restrictions on Decision Weights

We have observed that, under CEU, revealed likelihood of an event measured by its decision weight depends on the dominating event, whereas under SEU, the revealed likelihood of an event is entirely independent of the dominating event. This section describes a number of cases that are intermediate between CEU and SEU in restricting the dependence of revealed likelihood on the dominating event.

We first demonstrate the application of decision weights as measure of revealed likelihood in defining null events. Loosely speaking, a null event is equally likely as the impossible event. In our interpretation it means that an event is null if its decision weight is 0. Null events are important for updating (Gilboa, 1989a) and for the definition of the support of a distribution, which is central in some problems in game theory (Dow & Werlang, 1994; Eichberger & Kelsey, 1994; Hendon, Jacobsen, Sloth, & Tranaes, 1995); Haller (1995) proposed three different definitions of support, depending on how null events are interpreted.

Whether an event is null can be inferred from preferences as follows.

Suppose that $\alpha > \beta$. Then

(11) $\pi(A,D) = 0$ if and only if $(D,\alpha; A,\alpha; I,\beta) \sim (D,\alpha; A,\beta; I,\beta)$,

where $I = (A \cup D)^c$. Substitution of CEU shows that the above indifference holds if and only if

$$(A_1,x_1; \cdots; A_{i-1},x_{i-1}; A_i,\alpha; A_{i+1},x_{i+1}; \cdots; A_n,x_n) \sim$$
$$(A_1,x_1; \cdots; A_{i-1},x_{i-1}; A_i,\beta; A_{i+1},x_{i+1}; \cdots; A_n,x_n)$$

for any $\alpha > \beta$, $A_i = A$, $D = A_1 \cup \cdots \cup A_{i-1}$, $I = A_{i+1} \cup \cdots \cup A_n$, and where $x_1 \geq \cdots \geq x_{i-1} \geq \alpha > \beta \geq x_{i+1} \geq \cdots \geq x_n$. We use the simpler condition (11) in the analysis below.

Under general CEU, the above conditions depend on the event D, and A can be null for some dominating event D but nonnull for another. As an example, for maximin behavior ($v(A) = 0$ whenever A is not the universal event), $\pi(A,D)$ is 1 if $D = A^c$ and A is nonempty, but $\pi(A,D)$ is 0 whenever $D \neq A^c$. Therefore we call an event A *D-null* if $\pi(A,D) = 0$. We next discuss invariance of null events with respect to the dominating event D. In order to ensure that an event is null regardless of its rank-ordering we need an additional preference condition that, under CEU, turns out to be equivalent to Savage's (1954) P3.

If one assumes that events should be null only if they are logically impossible, then it is unsatisfactory that the logical (im)possibility of an event would depend on which other event were to yield better consequences in an act. One will want to ensure that an event A is null regardless of the dominating event. It must then be required that the decision weight

$\pi(A,D)$ be zero for all dominating events D as soon as it is for one, i.e. $\pi(A,D)=0$ for one event D if and only if $\pi(A,D)=0$ for all events D. In terms of preferences, this means that

(12) $(D,\alpha;\ A,\alpha;\ I,\beta) \sim (D,\alpha;\ A,\beta;\ I,\beta) \ \Rightarrow\ (D',\alpha;\ A,\alpha;\ I',\beta) \sim (D',\alpha;\ A,\beta;\ I',\beta)$

for all $\alpha > \beta$, I, D, I', D'. We call this condition *invariance of impossible events*. It rules out phenomena such as in Example 7. Next we demonstrate that invariance of impossible events is equivalent to Savage's P3 condition. To define Savage's P3, first we define his notion of null events. We use here a somewhat simplified formulation, that is motivated in Lemma 12 in the appendix. An event A is *S-null* if $f \sim g$ whenever f and g coincide outside of A; here "S" abbreviates Savage.

P3. If A is S-nonnull, acts f and g coincide outside of A, and $f=\alpha$ on A, $g=\beta$ on A, then $f \succcurlyeq g$ if and only if $\alpha \succcurlyeq \beta$.

THEOREM 8. Under CEU, invariance of impossible events holds if and only if Savage's postulate P3 holds. □

The formulation in terms of dependence of decision weights on dominating events gives clarifying alternative interpretations of several properties of capacities that have been studied in the literature. We list a number of them, leaving the proofs to the reader.

(13) v is symmetric if and only if $\pi(A,\varnothing)=\pi(A,A^c)$ for all events A.

(14) v is convex $(v(A) + v(B) \le v(A \cup B) + v(A \cap B))$ if and only if $\pi(A,D)$ is increasing in D.

(15) v is concave $(v(A) + v(B) \ge v(A \cup B) + v(A \cap B))$ if and only if $\pi(A,D)$ is decreasing in D.

Condition (14) illustrates pessimism, where a higher decision weight is assigned to an event as the event is lower in the rank-ordering. Similarly, condition (15) illustrates optimism. Condition (14) and (15) are reminiscent of the characterization of convex functions through increasing derivatives and concave functions through decreasing derivatives. Note here that the decision weight $\pi(A,D)$ describes the increase of v if A is added to D.

Tversky & Wakker (1995) proposed the following conditions to reflect bounded subadditivity, stated here in a somewhat informal manner. v satisfies *bounded subadditivity* if

(i) $\pi(A,\varnothing) \geq \pi(A,B)$ whenever $A \cup B$ is "sufficiently remote" from certainty.

(ii) $\pi(A,A^c) \geq \pi(A,B)$ whenever B is "sufficiently remote" from impossibility.

The conditions imply that decision weights with respect to intermediate dominating events are less than with respect to the extreme dominating events and have been illustrated in Figure 3.

We finally turn to the characterization of probabilistic sophistication for the context of CEU. In a general setting, probabilistic sophistication was characterized by Machina & Schmeidler (1992); they argued for a normative status of probabilistic sophistication. In the case of probabilistic sophistication, the ordering of revealed likelihoods of events remains invariant with the dominating event D. The characterizing condition for probabilistic sophistication is:

(16) $$\pi(A,D) \geq \pi(B,D) \Rightarrow \pi(A,D') \geq \pi(B,D')$$

for all events A, B, D, D'. In the theorem below, *solvability* of v (introduced by Gilboa (1987) under the name convex-rangedness) means that for all events $A \subset C$ and $v(A) \leq \beta \leq v(C)$ there exists an event B such that $A \subset B \subset C$ and $v(B) = \beta$. We now state a theorem that uses condition (16) to relate capacities to probabilities.

THEOREM 9. Let the collection of events be a sigma-algebra, and let CEU hold. There exists a countably additive atomless probability measure P and a strictly increasing continuous transformation ϕ such that $v = \phi \circ P$ if and only if the following conditions hold:

(i) v satisfies solvability;

(ii) (set-continuity) If $A_n \uparrow A$ (i.e., $A_{n+1} \supset A_n$ and $\cup A_n = A$) then $\lim_{j \to \infty} v(A_j) = v(A)$.

(iii) Condition (16) holds.

□

X. UPDATING REVEALED LIKELIHOOD

In the approach proposed in this paper, decision weights are taken as indices of revealed likelihood. Then the definition of revealed conditional likelihood is straightforward. Consider two events A and B, and assume that the rank-ordering of the state space has been fixed. The revealed conditional likelihood of A given B is simply defined by

$$\pi(A|B) = \frac{\pi(A \cap B)}{\pi(B)} . \tag{17}$$

Obviously, the resulting number is always between 0 and 1. The above definition of revealed conditional likelihood essentially requires a complete rank-ordering of states. This is in line with the observation of Eichberger & Kelsey (1993), that with CEU preferences it is not possible to update beliefs independently of consequences. For two cases, that are sufficiently general to cover most cases of interest, revealed conditional likelihood requires only partial information on the ranking of events. In the first case, $A \cap B$ and B are connected events; this case is discussed in most of this section. In the end we briefly discuss a second case, where $A \cap B$ and B\A are connected.

Let us consider now the first case, where $A \cap B$ and B are connected. In contrast to the additive probability case, we require the specification of a dominating event D for $A \cap B$ and D' for B. Thus revealed conditional likelihood is written as $\pi(A,D|B,D')$. For consistency of rank-ordering, $D \supset D'$. We propose the following definition of revealed conditional likelihood.

$$(18) \qquad \pi(A,D|B,D') = \frac{\pi(A \cap B,D)}{\pi(B,D')} = \frac{v((A \cap B) \cup D) - v(D)}{v(B \cup D') - v(D')} .$$

In this definition we assume further that $\pi(B,D') \neq 0$. Note that consequences outside the conditioning event B are relevant in this formula because the dominating events depend on them. This relevance of foregone consequences is the price one has to pay for giving up separability of disjoint events that is characteristic for SEU (Machina, 1989). Some definitions of revealed conditional likelihood have been proposed in the literature. Gilboa (1989a) proposed the following rule.

$$(19) \qquad \pi(A|B) = \frac{v(A \cap B)}{v(B)} = \frac{\pi(A \cap B,\varnothing)}{\pi(B,\varnothing)} .$$

Gilboa & Schmeidler (1993) pointed out that this rule corresponds with the optimistic decision maker who assumes that the event B, of which s/he has been informed, corresponds with the "best of all possible worlds," which in our terminology means that B is taken as a good-news event. In addition, given the information B, $A \cap B$ is in turn treated as a good-news event. That is, both $D' = \varnothing$ and $D = \varnothing$. In updating, the case of null-conditioning events is usually excluded. The conditioning event here is taken as a good-news event, i.e. the dominating event D' is empty. Therefore it seems appropriate that the conditioning event should not be D'-null for $D' = \varnothing$. This was indeed the definition adopted by Gilboa (1989a).

The following updating rule was proposed by Dempster (1967) and Shafer (1976) for belief functions (a special case of capacities). It was characterized and advocated by Gilboa

& Schmeidler (1993), and used by Dow & Werlang (1992). This rule was also used by Gilboa (1989b) for nonadditive measures that are not directly related to decisions.

$$\text{(20)} \qquad \pi(A|B) = \frac{v((A \cap B) \cup B^c) - v(B^c)}{1 - v(B^c)} = \frac{\pi(A \cap B, B^c)}{\pi(B, B^c)} .$$

As pointed out by Gilboa & Schmeidler (1993), this rule corresponds to a pessimistic decision maker. Indeed, it results from our proposal if $D' = B^c$ is taken. Thus the received information is taken as bad news. In addition, $A \cap B$ is assigned the highest-possible rank-ordering within B. Thus, D is also taken as B^c. The following example was discussed by Gilboa & Schmeidler (1993).

EXAMPLE 10. Assume the Ellsberg example with an urn containing 90 balls, 30 of which are red and 60 are either white or yellow. A ball is drawn and R describes the event that red is drawn, and W and Y designate white and yellow. A person who deviates from SEU because of unknown probabilities and who is maximally pessimistic regarding unknown probabilities, can be modeled through CEU with a capacity $v(\varnothing) = 0$, $v(R) = 1/3$, $v(W) = v(Y) = 0$, $v(R \cup W) = v(R \cup Y) = 1/3$, $v(W \cup Y) = 2/3$, $v(R \cup W \cup Y) = 1$. Assume now that the information is obtained that the color is not Y, i.e. the conditioning event is $B = R \cup W$. What would then be a reasonable revealed likelihood for event R, given this information? Our reply is that first dominating events D for R and D' for $R \cup W$ should be specified. Assume $D = Y = D'$. In this situation, event W is rank-ordered lowest and the pessimist assigns decision weight 2/3 to W. Then

$$\pi(R, Y | R \cup W, Y) = \frac{\pi(R, Y)}{\pi(R \cup W, Y)} = \frac{1/3}{1} = 1/3$$

follows. As $R \cup Y$ is the dominating event for W, we have

$$\pi(W, R \cup Y | R \cup W, Y) = \frac{\pi(W, R \cup Y)}{\pi(R \cup W, Y)} = \frac{2/3}{1} = 2/3.$$

If the dominating events D and D' are $\varnothing$, then it first of all follows that the information was good news. Now the pessimist assigns decision weight 0 to W, and

$$\pi(R,\varnothing|R \cup W,\varnothing) = \frac{\pi(R,\varnothing)}{\pi(R \cup W,\varnothing)} = \frac{1/3}{1/3} = 1$$

follows. Here R is the dominating event for W so that

$$\pi(W,R|R \cup W,\varnothing) = \frac{\pi(W,R)}{\pi(R \cup W,\varnothing)} = \frac{0}{1/3} = 0.$$

These two methods of calculating the revealed conditional likelihoods given $R \cup W$ agree with the optimistic and pessimistic methods considered by Gilboa & Schmeidler (1993). Let us next consider $\pi(R,W|R \cup W,\varnothing)$, where W is taken as dominating event for R and R $\cup$ W is a good-news event. Then again the pessimist assigns decision weight 0 to W, so that

$$\pi(R,W|R \cup W,\varnothing) = \frac{\pi(R,W)}{\pi(R \cup W,\varnothing)} = \frac{1/3}{1/3} = 1.$$

EXAMPLE 11. Assume that a die with six numbered sides yields j if side j shows up, $j = 1, \ldots, 6$. Our interest is in computing the revealed conditional likelihood of receiving 5 given the information that the prize is 3 or more. Note that the dominating event D for $\{5\}$ is $\{6\}$, and the dominating event D' for 3 or more is the empty set. Then

$$\pi(\{5\},\{6\}|j \geq 3,\varnothing) = \frac{\pi(\{5\},\{6\})}{\pi(\{3,4,5,6\},\varnothing)} = \frac{v(5,6) - v(6)}{v(3,4,5,6)}.$$

This is an example where the event for which the revealed conditional likelihood is to be determined is neither the best nor the worst event given the conditioning event, which is a case that has not been considered in the literature yet.

We assume that v(A) depends only on the number of elements in A, and is given by Table 1 below.

$\|A\|$	0	1	2	3	4	5	6
v	0	0.25	0.40	0.50	0.60	0.75	1

TABLE 1

Thus, for example, $v(1) = \cdots = v(6) = 0.25$, $v(1,2) = v(5,6) = 0.40$, $v(1,2,5) = 0.50$, $v(3,4,5,6) = 0.60$, etc. A singleton event has decision weight 0.25 if it is extreme in the rank-ordering ($0.25 = 0.25-0$ if it is best, $0.25 = 1-0.75$ if it is worst), decision weight 0.15 if it is second-best ($0.15 = 0.40-0.25$) or second-worst ($0.15 = 0.75-0.60$), and decision weight 0.10 if it has a middle position ($0.10 = 0.50-0.40$ if it is third in ranking, $0.10 = 0.60-0.50$ if it is fourth in ranking). This capacity v is symmetric and satisfies bounded subadditivity.

Formula (19) gives

$$\pi(\{5\}|j\geq 3) = \frac{v(5)}{v(3,4,5,6)} = \frac{0.25}{0.60} = 0.42,$$

and the Dempster-Shafer update rule (20) gives

$$\pi(\{5\}|j\geq 3) = \frac{\pi(\{5\},\{1,2\})}{\pi(\{3,4,5,6\},\{1,2\})} = \frac{0.10}{0.60} = 0.17.$$

Our update rule (18) gives

$$\pi(\{5\}|j\geq 3) = \pi(\{5\},\{6\}|j\geq 3,\varnothing) = \frac{\pi(\{5\},\{6\})}{\pi(\{3,4,5,6\},\varnothing)} = \frac{0.15}{0.60} = 0.25.$$

Note that the dominating event for $\{5\}$ is $\{6\}$ and our update rule (18) assigns the weight $v\{5,6\}-v\{6\} = 0.40-0.25 = 0.15$ to event $\{5\}$. Formula (19) assumes that no event dominates $\{5\}$ and thus it overweighs $\{5\}$ by assigning a decision weight $v\{5\}=0.25$. The Dempster-Shafer Formula (20) treats $\{1,2\}$ as the dominating event for $\{5\}$ and thereby underweighs it by assigning a decision weight $v\{1,2,5\}-v\{5\} = 0.50-0.25 =$

0.25 to {5}. Both (18) and (19) treat {3,4,5,6} as a good-news event, i.e. take an empty dominating event and assign decision weight 0.60 to {3,4,5,6}. Formula (20) takes {3,4,5,6} as a bad-news event, which differs from our interpretation but, by symmetry, assigns the same decision weight 0.60 as our method does. □

The central aspect of Bayes theorem is to derive the probability of B given A from the probability of A given B. That is, in our case, $\pi(A,D|B,D')$ is to be related to $\pi(B,\bar{D}|A,\bar{D}')$, where the dominating events are discussed next. There cannot be expected to be a simple relation between the two conditional likelihoods if $D \neq \bar{D}$, i.e. if in one case the event $A \cap B$ has a different dominating event than in the other. However, as soon as $D = \bar{D}$, then we obtain the following extension of Bayesian calculation:

$$\pi(A,D|B,D')\pi(B,D') = \pi(A \cap B,D) = \pi(A \cap B,\bar{D}) = \pi(B,\bar{D}|A,\bar{D}')\pi(A,\bar{D}')$$

whenever $\pi(B,D')$ and $\pi(A,\bar{D}')$ are nonzero. This analysis shows that the inverse relation for revealed conditional likelihood also holds for (19), because here all dominating events in the conditionalization are chosen empty, but the inverse relation will not hold for the Dempster-Shafer update rule (20) because in $\pi(A|B)$ the dominating event for $A \cap B$ is B^c, in $\pi(B|A)$ it is A^c.

We briefly mention a second case in which Formula (17) also yields a tractable result that requires only a partial specification of the rank-ordering of many events. It concerns the case where $A \cap B$ and $B \backslash A$ are connected. Then again we only need to specify two dominating events, D for $A \cap B$ and D' for $B \backslash A$, and we obtain

$$\pi(A|B) = \frac{\pi(A \cap B,D)}{\pi(A \cap B,D) + \pi(B \backslash A,D')} . \tag{21}$$

The case where $D = \varnothing$ and $D' = (B \backslash A)^c$ has received much attention in the literature (Jaffray, 1992; Denneberg, 1994). In this case, $A \cap B$ is a good-news event but the other part of the conditioning event, $B \backslash A$, is a bad-news event, and B is not connected. B^c is connected and contains the intermediate states. If the capacity is convex ("pessimism"), for instance if it is a Dempster-Shafer belief function, then the decision weight of $A \cap B$ is minimal if the event is good news, and the decision weight of $B \setminus A$ is maximal if it is bad news. Hence these choices of D and D' then minimize $\pi(A|B)$. The formula also results if one identifies the capacity with the set of dominating probability distributions, and applies conditionalization to each dominating probability measure separately.

The alternative notions of revealed conditional likelihood that have been discussed can be tested empirically. Some work along this line has begun (Cohen, Gilboa, Jaffray, & Schmeidler, in preparation). Specifically, it will be interesting to examine the role of dominating events in the revision of beliefs.

XI. Revealed Likelihood and Beliefs

In SEU, revealed likelihood can be interpreted as a measure of belief. This interpretation is appealing because in this model probabilities that measure revealed likelihoods are independent of tastes. In nonadditive models, decision weights measure revealed likelihoods. Decision weights are, however, not independent of the rank-ordering of consequences. Therefore, if decision weights are interpreted as measures of belief then independence of beliefs from tastes cannot be entirely maintained. It is quite possible that capacities and decision weights reflect not only beliefs, but also decision attitudes (e.g., ambiguity aversion).

Capacities and decision weights may be different than likelihood elicited through introspection. Some may regard that beliefs are best captured by an extraneous notion (introspection, verbal report) of likelihood that precedes preferences. In this view, beliefs depend only on the degree and extent of information that a decision maker possesses. For example, Kadane & Winkler (1988) and Karni (1995) note that even under SEU, revealed likelihood through bets may not represent beliefs. Shafer's (1976) belief functions provide an example of beliefs that precede preferences. Future studies may be able to disentangle beliefs and decision attitudes in the analysis of decision weights. A step toward this direction has been provided by (Jaffray, 1989; Hendon, Jacobsen, Sloth, & Tranaes, 1994; Tversky & Fox, 1995).

XII. Summary and Conclusion

In decision under uncertainty, there is often a difficulty in assigning probabilities to events. Ellsbergs examples demonstrated these difficulties convincingly. In recent years, Choquet expected utility (CEU) has been introduced to describe the observed violations of expected utility as in the Ellsberg examples. In the context of CEU, we propose that decision weights be interpreted as a measure of revealed likelihood. Under this interpretation, the revealed likelihood of an event depends on the dominating event.

Several applications of our measure of revealed likelihood are illustrated. The definition of null events and supports is clarified, new interpretations are given for convexity, concavity, bounded subadditivity, and probabilistic sophistication. We define revealed conditional likelihood in the context of CEU and show several implications for existing rules for updating if new information is gathered.

In CEU, capacities resemble probabilities and therefore are often treated as measures of belief. Two objections can be raised against this customary interpretation of capacities. First, this interpretation, arbitrarily, considers events only in the role of good-news events. Events may as well play the role of bad-news events, in which case the dual capacity should be considered. Indeed, a number of papers have pointed out that the dual capacity is just as valid a measures of belief as the capacity or, similarly but in qualitative terms, that bets against events provide as valid an ordering of likelihood as bets on events (Gilboa, 1989; Nehring, 1994). We have argued that, more generally, events may also play the role of intermediate events and that in many respects (such as the study of bounded subadditivity) decision weights are relevant, rather than capacities or their duals. In our framework, capacities measure revealed likelihoods of cumulative events (receive x or more), of course because for such events capacities coincide with decision weights.

Second, capacities, their duals, and decision weights, all may comprise not only a belief component, but may also be affected by decision attitudes. To avoid commitment to a pure belief-interpretation, we used the term revealed likelihood rather than likelihood throughout the paper.

We realize that our interpretations are subject to counter viewpoints and that better arguments for (or against) defining revealed likelihood in CEU may yet emerge.

APPENDIX. PROOFS

PROOF OF OBSERVATION 2. First we derive cumulative dominance for CEU. To do so, we use the following formula, where we write $v^{\uparrow}$ for v.

$$\mathrm{CEU}(f) \;=\; \int_{\mathbb{R}^+} v^{\uparrow}[U{\circ}f \geq t]\,dt \;+\; \int_{\mathbb{R}^-} (v^{\uparrow}[U{\circ}f \geq t]-1)\,dt$$

It is well-known, and can be derived by partial integration, that this formula provides an alternative manner for writing the upper Choquet integral of $U{\circ}f$ with respect to the capacity $v^{\uparrow}$, i.e. for calculating CEU(f). To prove cumulative dominance, assume that $[f \geq x] \succcurlyeq^{\uparrow} [g \geq x]$ for all x. Because $v^{\uparrow}$ represents $\succcurlyeq^{\uparrow}$, for all t the integrand in the above formula is at least as large as the integrand when g is substituted for f. Therefore the CEU value of f exceeds that of g, and $f \succcurlyeq g$ follows. That is, cumulative dominance has been shown.

The derivation of dual cumulative dominance from Formula 3.3 is perfectly dual. We use the following formula, where $v^{\downarrow}$ denotes the dual of v.

$$\mathrm{CEU(f)} \;=\; \int_{\mathrm{I\!R}^+}(1-v^{\downarrow}[\mathrm{U{\circ}f} \le t])dt \;-\; \int_{\mathrm{I\!R}^-} v^{\downarrow}[\mathrm{U{\circ}f} \le t]dt$$

It is also well-known, and can be derived by partial integration, that this formula provides an alternative manner for writing the lower integral of U∘f with respect to the capacity $v^{\downarrow}$; this also yields CEU(f) (Gilboa, 1989a). To prove dual cumulative dominance, assume that $[f \le x] \preccurlyeq^{\downarrow} [g \le x]$ for all x. Because $v^{\downarrow}$ represents $\succcurlyeq^{\downarrow}$, $v^{\downarrow}[\mathrm{U{\circ}f} \le t]$ in the above formula is less than or equal to $v^{\downarrow}[\mathrm{U{\circ}g} \le t]$, for all t. Therefore the CEU value of f exceeds that of g, and $f \succcurlyeq g$ follows. That is, dual cumulative dominance has been shown.

To further clarify the duality between good-news and bad-news events in the two displayed formulas, that can result by simply replacing events by their complements, we note that the lower formula is equal to

$$\mathrm{CEU(f)} \;=\; \int_{\mathrm{I\!R}^+}(1-v^{\downarrow}[\mathrm{U{\circ}f} < t])dt \;-\; \int_{\mathrm{I\!R}^-} v^{\downarrow}[\mathrm{U{\circ}f} < t]dt\ .$$

The reason is that the nondecreasing integrands in the last two formulas can have at most countable many discontinuities, and therefore differ at countably many t at most. Those t provide a Lebesgue 0 set and do not contribute to the integrals. □

The following lemma prepares for the proof of Theorem 8.

LEMMA 12. B is null by Savage's (1954) definition if and only if $f \sim g$ whenever f and g coincide outside of B.

PROOF. Savage mentions, without proof, that his definition of a null event is equivalent to our formulation in his Theorem 1 in Section 2.7. There, however, the sure-thing principle is assumed in the presence of which the claim is trivial indeed. We show now that the result holds as soon as weak ordering holds, and does not need the sure-thing principle. The case is somewhat subtle as in the second edition of his book, Savage changed one definition.

f_Bh denotes the act that agrees with f on B and with h outside of B. In the second edition of his book, Savage defines $f \succcurlyeq g$ given B as meaning that $f_Bh \succcurlyeq g_Bh$ for all h and either $g_Bh \succcurlyeq f_Bh$ for all h or for none. The last clause, concerning g_Bh and f_Bh, was not there in the first edition. In each edition, Savage defines an event B as null if and only if $f \succcurlyeq g$ given B for every f,g.

First assume that B is null by any of Savage's definitions. Then surely $f_Bh \succcurlyeq g_Bh$ for all f,g,h. This implies that $f_Bh \sim g_Bh$ for all f,g,h, proving one direction of the lemma. For the other direction, assume that $f_Bh \sim g_Bh$ for all f,g,h. Then, first, for every pair f,g, $f_Bh \succcurlyeq g_Bh$ for all h, second, $g_Bh \succcurlyeq f_Bh$ must hold for all h. This implies that B is a null event in each edition of Savage's book. □

PROOF OF THEOREM 8. First assume that P3 holds. Then the antecedent in (12) implies that A must be S-null. From that the consequent indifferent in (12) follows. Next assume that invariance of impossible events holds. To derive P3, let A be S-nonnull.

LEMMA 13. A is D-nonnull for some D.

PROOF. For event A to be S-nonnull, there exist acts that coincide outside of A and are nonindifferent. We may assume that these acts are constant in A. (Replace all consequences of the preferred act in A by their maximum and all consequences of the dispreferred act by their minimum. The maximum and minimum exist because all acts in this paper are simple; in CEU the preferred act becomes better and the dispreferred act becomes worse by these replacements.) Assume now that the acts are

$$(A_1,x_1; \cdots; A_{i-1},x_{i-1}; A_i,\alpha; A_{i+1},x_{i+1}; \cdots; A_n,x_n) \succ$$
$$(A_1,x_1; \cdots; A_{i-1},x_{i-1}; A_i,\beta; A_{i+1},x_{i+1}; \cdots; A_n,x_n)$$

for any $\alpha > \beta$, and $A_i = A$. We can write $\alpha_A f$ for the first act and $\beta_A f$ for the second (e.g., define f as the first act). We may assume that the consequences in the second act are rank-ordered, i.e. $x_1 \geq \cdots \geq x_{i-1} \geq \beta \geq x_{i+1} \geq \cdots \geq x_n$. If now $i = n$ or $\alpha \leq x_{i+1}$ then the two acts are comonotonic, we define $D = A_1 \cup \cdots \cup A_{i-1}$, and it follows that $A_i = A$ is D-nonnull. The general case where $i < n$ and $\alpha > x_{i+1}$ is possible, is more complicated. Therefore, consider the set of n+2 acts containing the above two acts and all acts of the form $(x_j)_A f$ for $j = 1, \cdots n$. This set of acts can be ordered so that every consecutive pair is comonotonic. To this end, $\beta_A f$ is between $(x_{i-1})_A f$ and $(x_i)_A f$, and if $x_{k-1} \leq \alpha \leq x_k$ then $\alpha_A f$ is between $(x_{k-1})_A f$ and $(x_k)_A f$. In this n+2 tuple of acts, there must be a consecutive pair of acts that are nonindifferent. This pair of acts shows that A is D-nonnull, where D is the set of dominating consequences for the two comonotonic acts. QED

Consider now $\alpha_A f$ and $\beta_A f$ for $\alpha > \beta$. We must prove that $\alpha_A f \succ \beta_A f$. The reasoning is similar to the above proof as again comonotonicity complications must be dealt with. Again we write

$$(A_1, x_1; \cdots; A_{i-1}, x_{i-1}; A_i, \alpha; A_{i+1}, x_{i+1}; \cdots; A_n, x_n) = \alpha_A f \text{ and}$$
$$(A_1, x_1; \cdots; A_{i-1}, x_{i-1}; A_i, \beta; A_{i+1}, x_{i+1}; \cdots; A_n, x_n) = \beta_A f$$

for $A_i = A$, and $x_1 \geq \cdots \geq x_{i-1} \geq \beta \geq x_{i+1} \geq \cdots \geq x_n$. If $i = n$ or $\alpha \leq x_{i+1}$, then the two acts are comonotonic, and the CEU difference between the two acts is $\pi(A,D)(U(\alpha) - U(\beta))$ where $D = A_1 \cup \cdots \cup A_{i-1}$. By invariance of impossible events, $\pi(A,D) > 0$, and as the utility difference is also positive, this CEU difference is positive. Hence a strict preference follows. Assume next that $\alpha > x_{i+1}$. As for identical consequences the rank-ordering is arbitrary, we can, by renumbering the events and consequences if necessary, take the index i maximal so that either $x_{i+1} > \beta$ or $i = n$. The case $i = n$ was dealt with before, hence assume $x_{i+1} > \beta$. Then $\alpha_A f \succcurlyeq (x_{i+1})_A f \succ \beta_A f$, where the first preference follows from substitution of CEU and $\alpha > x_{i+1}$ and the second strict preference follows because $x_{i+1} > \beta$, the two acts are comonotonic, and $\pi(A,D) > 0$ with $D = A_1 \cup \cdots \cup A_{i-1}$. This proves again that $\alpha_A f \succ \beta_A f$. □

PROOF OF THEOREM 9. First assume that the conditions (i), (ii), and (iii) hold. We prove the existence of P and ϕ as described in the theorem. We write $A \succcurlyeq B$ for events A, B whenever $v(A) \geq v(B)$. No confusion with the preference relation $\succcurlyeq$ on acts will arise. First we derive the properties of a qualitative probability ordering for $\succcurlyeq$.

Obviously $\succcurlyeq$ is a weak order; the notation $\succ$ is as usual. We have $S \succcurlyeq A \succcurlyeq \varnothing$ for all events A, and $S \succ \varnothing$. Finally, assume that event D is disjoint from events A,B. Then the following six statements are equivalent: (1) $A \succcurlyeq B$; (2) $v(A) \geq v(B)$; (3) $\pi(A,\varnothing) \geq \pi(B,\varnothing)$; (4) $\pi(A,D) \geq \pi(B,D)$; (5) $v(A \cup D) \geq v(B \cup D)$; (6) $A \cup D \succcurlyeq B \cup D$. Here the equivalence between (3) and (4) holds because of (16). The equivalence of (1) and (6) is the well-known additivity condition of qualitative probability theory.

Thus $\succcurlyeq$ is a qualitative probability ordering (see Villegas, 1964). Solvability of v implies that no atoms exist, and Condition (ii) implies monotone continuity of Villegas (1964). Therefore, by Villegas' Theorem 4.3, there exists a unique countably additive atomless probability measure P on the sigma algebra of events that represents the qualitative probability ordering $\succcurlyeq$. Hence there exists a strictly increasing transformation ϕ such that v $= \phi \circ P$. By solvability of v, the range of ϕ is [0,1], hence ϕ must be continuous.

For the reversed implication, solvability of v is implied because P is atomless and ϕ is continuous, Condition (ii) is implied by sigma-additivity of P and continuity of ϕ, and (16) follows from additivity of P.

Our proof has been based on the qualitative probability result of Villegas (1964), which concerns countably additivity measures. Alternatively, one can consider finite additivity and use results such as Savage's (1954). This approach is, however, more complicated, mainly because the fineness condition now is complicated; see Gilboa (1985). □

ACKNOWLEDGMENT. This paper received valuable comments from Alain Chateauneuf, Jean-Yves Jaffray, Hans-Joergen Jacobsen, and Amos Tversky.

References

Allais, M. (1988), "The General Theory of Random Choices in Relation to the Invariant Cardinal Utility Function and the Specific Probability Function." *In* B.R. Munier (Ed.), *Risk, Decision and Rationality*, 233–289, Reidel, Dordrecht.

Benartzi, S. & R.H. Thaler (1995), "Myopic Loss Aversion and the Equity Premium Puzzle," *The Quarterly Journal of Economics* 110, 73–92.

Bernasconi, M. (1994), "Nonlinear Preference and Two-Stage Lotteries: Theories and Evidence," *The Economic Journal* 104, 54–70.

Camerer, C.F. & T.-H. Ho (1994), "Violations of the Betweenness Axiom and Nonlinearity in Probability," *Journal of Risk and Uncertainty* 8, 167–196.

Camerer, C.F. & M. Weber (1992), "Recent Developments in Modelling Preferences: Uncertainty and Ambiguity," *Journal of Risk and Uncertainty* 5, 325–370.

Dempster, A.P. (1967), "Upper and Lower Probabilities Induced by a Multivalued Mapping," *Annals of Mathematical Statistics* 38, 325–339.

Denneberg, D. (1994), "Conditioning (Updating) Non-Additive Measures," *Annals of Operations Research* 52, 21–42.

Dow, J. & S.R.C. Werlang (1992), "Excess Volatility of Stock Prices and Knightian Uncertainty," *European Economic Review* 36, 631–638.

Dow, J. & S.R.C. Werlang (1994), "Nash Equilibrium under Knightian Uncertainty: Breaking Down Backward Induction," *Journal of Economic Theory* 64, 304–324.

Eichberger, J. & D. Kelsey (1993), "Uncertainty Aversion and Dynamic Consistency," *International Economic Review*, forthcoming.

Eichberger, J. & D. Kelsey (1994), "Non-Additive Beliefs and Game Theory," working paper 94-01, Dept. of Economics, University of Birmingham, Birmingham, England.

Ellsberg, D. (1961), "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics* 75, 643–669.

Erev, I, G. Bornstein, & T.S. Wallsten (1992), "The Negative Effect of Probability Assessments on Decision Quality," *Organizational Behavior and Human Decision Processes* 55, 78–94.

Fellner, W. (1961), "Distortion of Subjective Probabilities as a Reaction to Uncertainty," *Quarterly Journal of Economics* 75, 670–690.

Fox, C. & A. Tversky (1995), "Ambiguity Aversion and Comparative Ignorance," *Quarterly Journal of Economics* 110, 585–603.

Gilboa, I. (1985), "Subjective Distortions of Probabilities and Non-Additive Probabilities," Working paper 18–85, Foerder Institute for Economic Research, Tel-Aviv University, Ramat Aviv, Israel.

Gilboa, I. (1987), "Expected Utility with Purely Subjective Non-Additive Probabilities," *Journal of Mathematical Economics* 16, 65–88.

Gilboa, I. (1989a), "Duality in Non-Additive Expected Utility Theory." *In* P.C. Fishburn & I.H. LaValle (Eds.), *Choice under Uncertainty*, Annals of Operations Research, J.C. Baltzer AG., Basel, 405–414.

Gilboa, I. (1989b), "Additivizations of NonAdditive Measures," *Mathematics of Operations Research* 14, 1–17.

Gilboa, I. & D. Schmeidler (1993), "Updating Ambiguous Beliefs," *Journal of Economic Theory* 59, 33–49.

Haller, H. (1995), "Non-Additive Beliefs in Solvable Games," Dept. of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA.

Hendon, E., H.J. Jacobsen, B. Sloth, & T. Tranaes (1994), "Expected Utility with Lower Probabilities," *Journal of Risk and Uncertainty* 8, 197–216.

Hendon, E., H.J. Jacobsen, B. Sloth, & T. Tranaes (1995), "Nash Equilibrium in Lower Probabilities," Institute of Economics, University of Copenhagen.

Jaffray, J.Y. (1989), "Linear Utility Theory for Belief Functions," *Operations Research Letters* 8, 107–112.

Jaffray, J.Y. (1992), "Bayesian Updating and Belief Functions," *IEEE Transactions on Systems, Man, and Cybernetics* 22, 1144–1152.

Kachelmeier, S.J. & M. Shehata (1992), "Examining Risk Preferences under High Monetary Incentives: Experimental Evidence from the People's Republic of China," *American Economic Review* 82, 1120-1141; comment see AER 84, 1994, 1104–1106.

Kadane, J.B. & R.L. Winkler (1988), "Separating Probability Elicitation from Utilities," *Journal of the American Statistical Association* 83, 357–363.

Karni, E. (1995), "Probabilities and Beliefs," Dept. of Economics, The Johns Hopkins University, Baltimore.

Karni, E. & Z. Safra (1990), "Rank-Dependent Probabilities," *Economic Journal* 100, 487–495.

Keynes, J.M. (1921), "*A Treatise on Probability.*" McMillan, London. Second edition 1948.

Knight, F.H. (1921), "*Risk, Uncertainty, and Profit.*" Houghton Mifflin, New York.

Liberman, V. & A. Tversky (1993), "On the Evaluation of Probability Judgments: Calibration, Resolution, and Monotonicity," *Psychological Bulletin* 114, 162–173.

Machina, M.J. (1989), "Dynamic Consistency and Non-Expected Utility Models of Choice under Uncertainty," *Journal of Economic Literature* 27, 1622–1688.

Machina, M.J. & D. Schmeidler (1992), "A More Robust Definition of Subjective Probability," *Econometrica* 60, 745–780.

Nehring, K. (1994), "On the Interpretation of Sarin and Wakker's "A Simple Axiomatization of Nonadditive Expected Utility" ," *Econometrica* 62, 935–938.

Quiggin, J. (1982), "A Theory of Anticipated Utility," *Journal of Economic Behaviour and Organization* 3, 323–343.

Raiffa, H. (1961), "Risk, Uncertainty and the Savage Axioms: Comment," *Quarterly Journal of Economics* 75, 690–695.

Samuelson, P.A. (1938), "A Note on the Pure Theory of Consumer's Behaviour," *Economica*, N.S. 5, 61–71, 353–354.

Sarin, R.K. & P.P. Wakker (1992), "A Simple Axiomatization of Nonadditive Expected Utility," *Econometrica* 60, 1255–1272.

Savage, L.J. (1954), "*The Foundations of Statistics*." Wiley, New York. (Second edition 1972, Dover, New York.)

Schmeidler, D. (1989), "Subjective Probability and Expected Utility without Additivity," *Econometrica* 57, 571–587.

Shafer, G. (1976), "*A Mathematical Theory of Evidence*." Princeton University Press, Princeton NJ.

Tversky, A. & C. Fox (1995), "Weighing Risk and Uncertainty," *Psychological Review* 102, 269-283.

Tversky, A. & D. Kahneman (1992), "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty* 5, 297–323.

Tversky, A. & P.P. Wakker (1995), "Risk Attitudes and Decision Weights," *Econometrica* 63, 1255–1280.

Villegas, C. (1964), "On Quantitative Probability σ-Algebras," *Annals of Mathematical Statistics* 35, 1787–1796.

Wu, G. & R. Gonzalez (1994), "Curvature of the Probability Weighting Function," *Management Science*, forthcoming.

No.	Author(s)	Title
9599	H. Bloemen	The Relation between Wealth and Labour Market Transitions: an Empirical Study for the Netherlands
95100	J. Blanc and L. Lenzini	Analysis of Commmunication Systems with Timed Token Protocols using the Power-Series Algorithm
95101	R. Beetsma and L. Bovenberg	The Interaction of Fiscal and Monetary Policy in a Monetary Union: Balancing Credibility and Flexibility
95102	P. de Bijl	Aftermarkets: The Monopoly Case
95103	F. Kumah	Unanticipated Money and the Demand for Foreign Assets - A Rational Expectations Approach
95104	M. Vázquez-Brage, A. van den Nouweland, I. García-Jurado	Owen's Coalitional Value and Aircraft Landing Fees
95105	Y. Kwan and G. Chow	Estimating Economic Effects of the Great Leap Forward and the Cultural Revolution in China
95106	P. Verheyen	The missing Link in Budget Models of Nonprofit Institutions; Two Practical Dutch Applications
95107	J. Miller	Should we Offer the Unemployment Places on Labour Market Programmes With the Intention That They Reject Them?
95108	C. van Raalte and H. Webers	Statial Competition with Intermediated matching
95109	W. Verkooijen, J. Plasmans and H. Daniels	Long-Run Exchange Rate Determination: A Neutral Network Study
95110	E. van der Heijden, J. Nelissen, J. Potters and H. Verbon	Transfers and Reciprocity in Overlapping-Generations Experiments
95111	A. van den Elzen and D. Talman	An Algorithmic approach towards the Tracing Procedure of Harsanyi and Selten
95112	R. Beetsma and L. Bovenberg	Does Monetary Unification Lead to Excessive Debt Accumulation?
95113	H. Keuzenkamp	Keynes and the Logic of Econometric Method
95114	E. Charlier, B. Melenberg and A. van Soest	Estimation of a Censored Regression Panel Data Model Using Conditional Moment Restrictions Efficiently
95115	N. G. Noorderhaven, B. Nooteboom and H. Berger	Exploring Determinants of Perceived Interfirm Dependence in Industrial Supplier Relations

No.	Author(s)	Title
95116	F. Kleibergen and H. Hoek	Bayesian Analysis of ARMA models using Noninformative Priors
95117	J. Lemmen and S. Eijffinger	The Fundamental Determiniants of Financial Integration in the European Union
95118	B. Meijboom and J. Rongen	Clustering, Logistics, and Spatial Economics
95119	A. de Jong, F. de Roon and C. Veld	An Empirical Analysis of the Hedging Effectiveness of Currency Futures
95120	J. Miller	The Effects of Labour Market Policies When There is a Loss of Skill During Unemployment
95121	S. Eijffinger, M. Hoeberichts and E. Schaling	Optimal Conservativeness in the Rogoff (1985) Model: A Graphical and Closed-Form Solution
95122	W. Ploberger and H. Bierens	Asymptotic Power of the Integrated Conditional Moment Test Against Global and Large Local Alternatives
95123	H. Bierens	Nonparametric Cointegration Analysis
95124	H. Bierens and W. Ploberger	Asymptotic Theory of Integrated Conditional Moment Tests
95125	E. van Damme	Equilibrium Selection in Team Games
95126	J. Potters and F. van Winden	Comparative Statics of a Signaling Game: An Experimental Study
9601	U. Gneezy	Probability Judgements in Multi-Stage Problems: Experimental Evidence of Systematic Biases
9602	C. Fernández and M. Steel	On Bayesian Inference under Sampling from Scale Mixtures of Normals
9603	J. Osiewalski and M. Steel	Numerical Tools for the Bayesian Analysis of Stochastic Frontier Models
9604	J. Potters and J. Wit	Bets and Bids: Favorite-Longshot Bias and Winner's Curse
9605	H. Gremmen and J. Potters	Assessing the Efficacy of Gaming in Economics Educating
9606	J. Potters and F. van Winden	The Performance of Professionals and Students in an Experimantal Study of Lobbying
9607	J.Kleijnen, B. Bettonvil and W. van Groenendaal	Validation of Simulation Models: Regression Analysis Revisited
9608	C. Fershtman and N. Gandal	The Effect of the Arab Boycott on Israel: The Automobile Market
9609	H. Uhlig	Bayesian Vector Autoregressions with Stochastic Volatility

No.	Author(s)	Title
9610	G. Hendrikse	Organizational Change and Vested Interests
9611	F. Janssen, R. Heuts and T. de Kok	On the *(R,s,Q)* Inventory Model when Demand is Modelled as a compound Bernoulli Process
9612	G. Fiestras-Janeiro P. Borm and F. van Megen	Protective Behaviour in Games
9613	F. van Megen, G. Facchini, P. Borm and S. Tijs	Strong Nash Equilibria and the Potential Maximizer
9614	J. Miller	Do Labour Market Programmes Necessarily Crowd out Regular Employment? - A Matching Model Analysis
9615	H. Huizinga	Unemployment Benefits and Redistributive Taxation in the Presence of Labor Quality Externalities
9616	G. Asheim and M. Dufwenberg	Admissibility and Common Knowlegde
9617	C. Fernández, J. Osiewalski and M. Steel	On the Use of Panel Data in Bayesian Stochastic Frontier Models
9618	H. Huizinga	Intrafirm Information Transfers and Wages
9619	B. Melenberg and B. Werker	On the Pricing of Options in Incomplete Markets
9620	F. Kleibergen	Reduced Rank Regression using Generalized Method of Moments Estimators
9621	F. Janssen, R. Heuts and T. de Kok	The Value of Information in an *(R,s,Q)* Inventory Model
9622	F. Groot, C. Withagen and A. de Zeeuw	Strong Time-Consistency in the Cartel-Versus-Fringe Model
9623	R. Euwals and A. van Soest	Desired and Actual Labour Supply of Unmarried Men and Women in the Netherlands
9624	M. Khanman, M. Perry and P.J. Reny	An Ex-Post Envy-Free and Efficient Allocation Mechanism: Imperfect Information Without Common Priors
9625	C. Eaves, G. van der Laan D. Talman and Z. Yang	Balanced Simplices on Polytopes
9626	H. Bloemen and A. Kalwij	Female Employment and Timing of Births Decisions: a Multiple State Transition Model
9627	P. Bolton	Strategic Experimentation: a Revision
9628	C. Fershtman and Y. Weiss	Social Rewards, Externalities and Stable Preferences

No.	Author(s)	Title
9629	P.M. Kort, G. Feichtinger R.F. Hartl and J.L. Haunschmied	Optimal Enforcement Policies (Crackdowns) on a Drug Market
9630	C. Fershtman and A. de Zeeuw	Tradeable Emission Permits in Oligopoly
9631	A. Cukierman	The Economics of Central Banking
9632	A. Cukierman	Targeting Monetary Aggregates and Inflation in Europe
9633	F. de Roon, C. Veld and J. Wei	A Study on the Efficiency of the Market for Dutch Long Term Call Options
9634	B. van Aarle, L. Bovenberg M. Raith	Money, Fiscal Deficits and Government Debt in a Monetary Union
9635	G. van der Laan, D. Talman and Z. Yang	Existence of and Equilibrium in a Competitive Economy with Indivisibilities and Money
9636	I. Brouwer, J. van der Put C. Veld	Contrarian Investment Strategies in a European Context
9637	M. Berg, F. van der Duyn Schouten and J. Jansen	Optimal Service Policies to Remote Customers with Delay-Limits
9638	F.C. Drost and C.A.J. Klaassen	Efficient Estimation in Semiparametric GARCH Models
9639	A.G. de Kok and F.B.S.L.P. Janssen	Demand Management in Multi-Stage Distribution Chain
9640	D.A. Freund, T.J. Kniesner A.T. LoSasso	How Managed Care Affects Medicaid Utilization A Synthetic Difference-in-Differences Zero-Inflated Count Model
9641	M.J. Lee	Instrumental Variable Estimation For Linear Panel Data Models
9642	W. Härdle, E. Mammen and M. Müller	Testing Parametric versus Semiparametric Modelling in Generalized Linear Models
9643	C. Dustmann, N. Rajah and A. van Soest	Part-Time Work, School Success and School Leaving
9644	S.C.W. Eijffinger and M. Hoeberichts	The Trade Off Between Central Bank Independence and Conservativeness
9645	R. Sarin and P. Wakker	A Single-Stage Approach to Anscombe and Aumann's Expected Utility

No.	Author(s)	Title
9646	J.P. Ziliak and T.J. Kniesner	The Importance of Sample Attrition in Life Cycle Labor Supply Estimation
9647	P.M. Kort	Optimal R&D Investments of the Firm
9648	M.P. Berg	Performance Comparisons for Maintained Items
9649	H. Uhlig and Y. Xu	Effort and the Cycle: Cyclical Implications of Efficiency Wages
9650	M. Slikker and A. van den Nouweland	Communication Situations with a Hierarchical Player Partition
9651	H.L.F. de Groot	The Struggle for Rents in a Schumpeterian Economy
9652	R.M. de Jong and J. Davidson	Consistency of Kernel Estimators of heteroscedastic and Autocorrelated Covariance Matrices
9653	J. Suijs, A. De Waegenaere and P. Borm	Stochastic Cooperative Games in Insurance and Reinsurance
9654	A.N. Banerjee and J.R. Magnus	Testing the Sensitivity of OLS when the Variance Matrix is (Partially) Unknown
9655	A. Kalwij	Estimating the Economic Return to Schooling on the basis of Panel Data
9656	M. Lind and F. van Megen	Order Based Cost Allocation Rules
9657	A. van Soest, P. Fontein and Rob Euwals	Earnings Capacity and Labour Market Participation
9658	C. Fernández and M.F.J. Steel	On Bayesian Modelling of Fat Tails and Skewness
9659	R. Sarin and P. Wakker	Revealed Likelihood and Knightian Uncertainty

P.O. BOX 90153, 5000 LE TILBURG, THE NETHERLANDS