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PAIRED COMPARISONS ANALYSIS: AN AXIOMATIC APPROACH TO RANKINGS IN TOURNAMENTS

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Paired comparisons analysis: an axiomatic approach to rankings in tournaments*

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Abstract

In this paper we present an axiomatic analysis of several ranking methods for tournaments. We find that two of them exhibit a very good behaviour with respect to the set of properties under consideration. One of them is the maximum likelihood ranking, the most common method in statistics and psychology. The other one is a new ranking method introduced in this paper: recursive Buchholz. One of the most widely studied methods in social choice, the fair bets ranking, also performs quite well, but fails to satisfy some arguably important properties.

JEL Classification Codes: D71, D63, D61.

Keywords: Tournament, ranking, paired comparisons, fair bets, maximum likelihood.

1 Introduction

In a world full of choices and alternatives, rankings are becoming an increasingly important tool to help individuals and institutions make decisions. In this paper we study, from an axiomatic point of view, the classic problem of ranking a series of alternatives when we have information about paired comparisons between them. The set of alternatives an coresponding matrix containing this information are referred to as a *tournament*. Tournaments and rankings appear in a wide variety of situations such as sports, product testing, valuation of

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political candidates and policies to be chosen. Because of this, the issue of defining rankings for tournaments has been studied in various fields and ranking methods based on different motivations have been defined. Sport events and in particular chess motivated the seminal work on rankings by Zermelo (1929). Later on, sparked by Arrow's Impossibility Theorem (see Arrow (1963)), this topic emerged in the context of social choice and voting theory. The theory of rankings has also attracted statisticians and psychologists, who have studied it under the name of paired comparisons analysis.

In this paper a *tournament* is represented by a set of alternatives N and a nonnegative $N \times N$ matrix A with zeros on the diagonal, where A_{ij} is the total score of alternative i against alternative j after their (possibly many) pairwise comparisons. This approach to tournaments is the usual one in fields such as statistics, psychology and applications to sports. On the other hand, in voting theory, a field that has devoted considerable attention to this topic, tournaments are typically defined through complete and asymmetric binary relations, *i.e.*, for each pair of alternatives we know which is the preferred one (and nothing else); there is no measure of intensity of preference. These binary tournaments are a particular case[1] of our more general setting which is able to accommodate the following extra features: i) incomplete tournaments, in which we may not have information about direct confrontations between pairs of alternatives ($A_{ij} + A_{ji}$ may be zero), ii) tournaments in which alternatives may have been compared with each other more than once ($A_{ij} + A_{ji} > 1$),[2] iii) tournaments with ties ($A_{ij} = A_{ji}$), and iv) tournaments in which intensities of preference are present.

Although ideally we would like to work with complete tournaments, there are many situations where it is unfeasible to obtain direct information about each possible pairwise comparison of alternatives. This may be because there is a high number of alternatives to be ranked or just because it is costly to undergo each pairwise comparison and it is preferable to base the ranking on an incomplete set of comparisons. From a conceptual point of view, whether or not tournaments are restricted to be binary has important implications in defining ranking methods. In a binary tournament all the alternatives have "faced" each other exactly once and simple rules that look at the number of "victories" of each alternative may have good properties. However, in more general tournaments it does not suffice to know how well an alternative has scored. We need to take into account the quality of the "opponents".

Our goal in this paper is to take some of the ranking methods considered in the different fields and compare them by looking at their performance with respect to a set of properties. Axiomatic approaches to ranking theory have been taken before in the literature, especially in social choice and voting theory. However, most of these contributions mainly deal with binary tournaments. Laslier (1997) presents a deep analysis of the problem of choosing a *set of winners* for a given binary tournament, discussing a wide number of solutions and

[1] In matrix form, a binary tournament corresponds with a binary matrix $A \in \{0,1\}^{N \times N}$ such that for each pair of alternatives i and j, $A_{ij} + A_{ji} = 1$.

[2] This is specially important in testing objects, where each pair of objects may be tested by several experts.

properties.[3] These properties are stated with respect to the chosen set of winners, whereas in our setting they take into account not only the winners, but the whole ranking over the set of alternatives. Also within the axiomatic approach, Bouyssou (2004) revisits the main ranking methods in Laslier (1997) and studies their monotonicity properties (responsiveness to the beating relation).[4]

Because of the large amount of ranking methods and properties that have been discussed in the different fields, some selection is needed. Our analysis mainly concentrates on the ranking methods listed below.

- **Scores:** A natural choice for binary tournaments (see Rubinstein (1980) for an axiomatic characterisation).
- **Fair bets:** A ranking widely studied in social choice theory and economics (see, for instance, Daniels (1969), Moon and Pullman (1970), Slutzki and Volij (2005) and Slutzki and Volij (2006)).
- **Maximum likelihood:** The most common choice in statistics and psychology (see, for instance, Zermelo (1929) and Bradley and Terry (1952)).
- **Recursive performance and recursive Buchholz:** These ranking methods are the result of a new approach developed in Brozos-Vázquez et al. (2008).

The main contribution of this paper is to study how the above ranking methods perform with respect to a set of properties. This analysis is important not only to get a better understanding of the different ranking methods, but also to learn about the strength and implications of the different properties. Interestingly, maximum likelihood stands up as one of the ranking methods that does well with respect to the chosen properties. This is somewhat surprising since, because of its nature, one would expect maximum likelihood to have good statistical properties (for instance, in terms of asymptotic behaviour), but there is no reason to expect good behaviour with respect to some of the properties we work with. The other ranking method that stands up from our approach is recursive Buchholz, which is defined in this paper. Therefore, not only do we conduct a detailed analysis of the properties of several well known ranking methods, but we also define a new one, recursive Buchholz, which performs well on the discussed properties.

The rest of the paper is structured as follows. In Section 2 we present the main definitions and ranking methods. In Sections 3-6 we define and discuss several families of properties. Finally, we discuss the results of our analysis in Section 7.

[3] Dutta and Laslier (1999) consider a more general setting where they allow for intensities, but completeness is still a requirement.

[4] In recent years, the related issue of ranking scientific journals has received a lot of attention in economics (see, for instance, Liebowitz and Palmer (1984) and Palacios-Huerta and Volij (2004)). In this setting, the rankings are defined on the basis of citation matrices, which contain information regarding the number of times each journal has been cited by any other journal. There is a fundamental difference between the two settings. In our setting, a victory of i over j should be seen as something good for i and bad for j. However, when looking at scientific journals, a citation from journal j to journal i should be good for journal i, but not necessarily bad for journal j. Clearly, this cannot be ignored when defining properties of a ranking method and, therefore, it would be inappropriate to include in our axiomatic analysis ranking methods that are based on citation matrices.

2 Tournaments and ranking methods

As we have already argued, our analysis applies to any situation where there is a set of alternatives to be ranked on the basis of pairwise comparisons among them. For the sake of exposition, in the remainder we adopt the sports' terminology and refer to the alternatives as players and talk about matches, victories, losses, beating, scores, *etc.*

A *tournament* is a pair (N, A), where N is a finite set of players and $A \in \mathbb{R}^{N\times N}$ is a tournament matrix, where A_{ij} represents the aggregate score of player i against j. We assume $A_{ij} \geq 0$ for all $i, j \in N$ and $A_{ii} = 0$ for all $i \in N$.[5] We say that i has *scored* against j if $A_{ij} > 0$ and that i has *beaten* j if $A_{ij} > A_{ji}$. We make the standard assumption that the matrix A is irreducible.[6] This means that for every pair of players $i, j \in N$, $i \neq j$, there is a sequence of players $(i = k_0, k_1, \ldots, k_n = j)$ such that, for each $\ell \in \{0, \ldots, n-1\}$, k_ℓ has scored against $k_{\ell+1}$.

To each tournament (N, A) we associate a (symmetric) *matches matrix* $M(A) = A+A^\top$, where $M_{ij}(A)$ is interpreted as the number of matches between i and j. When no confusion can arise, we denote $M(A)$ by M. For each player $i \in N$, define $m_i = \sum_{j\in N} M_{ij}$ to be the total number of matches played by i, so $m = Me$, where $e \in \mathbb{R}^N$ is the vector $e = (1, \ldots, 1)$. For $i, j \in N$, define $\bar{M}_{ij} = M_{ij}/m_i$ to be the proportion of player i's matches that he plays against j. A tournament is called *round-robin* if $M_{ij} = 1$ for all $i, j \in N, i \neq j$; round-robin tournaments are complete tournaments in which each player has played once against any other player.[7]

A *ranking method* φ assigns to each tournament (N, A) a weak order on N (transitive and complete). For $i, j \in N$, we write $i\ R^\varphi_A\ j$ if i is ranked weakly above j according to ranking method φ in tournament A; a strict ranking is denoted by $i\ P^\varphi_A\ j$ and indifference is denoted by $i\ I^\varphi_A\ j$. A ranking φ is called *flat* on A if, for each pair $i, j \in N$, $i\ I^\varphi_A\ j$.

Given a tournament (N, A), a vector $r \in \mathbb{R}^N$ is a *rating vector*, where each r_i is a measure of the performance of player $i \in N$ in the tournament. The ranking methods considered in this paper are all induced by rating vectors: for each ranking method φ there is an underlying rating vector r^φ such that the players are ranked according to it, *i.e.*, $i\ R^\varphi_A\ j$ if and only if $r^\varphi_i \geq r^\varphi_j$. Note that ratings that are used to define a ranking method are ordinal, so for convenience sometimes a normalisation is imposed to guarantee uniqueness of the ratings vector.

Below, we define the ranking methods considered in this paper. Two of them, the Neustadtl and Buchholz ranking methods are mainly defined as a starting point for the fair bets and recursive Buchholz methods, respectively.

Scores. The *vector of average scores*, r^{s}, is defined by $r^{\mathrm{s}}_i = \sum_{j\in N} A_{ij}/m_i$ for all $i \in N$. It follows from the assumption that A is irreducible that $s_i \in (0, 1)$ for all $i \in N$. The corresponding ranking method is denoted by φ^{s}. In the remainder we interchangeably use

[5] We do not restrict the non-zero entries in A to be natural numbers as in, *e.g.*, Slutzki and Volij (2005).

[6] This ensures that all the rankings methods we discuss in this paper are well defined.

[7] Despite being very special, round-robin tournaments are still more general than binary tournaments, since they allow for intensities (A_{ij} needs not be 0 or 1) and ties (A_{ij} may equal A_{ji}).

r^{s} and s to refer to the scores.

Neustadtl. Let $\hat{A}$ be defined, for each pair $i, j \in N$, by $\hat{A}_{ij} = A_{ij}/m_i$. Then, the Neustadtl rating vector is given by $r^{\mathrm{n}} = \hat{A}s$. Neustadtl ranking, which is widely used as a tie-breaker in round-robin tournaments, computes a weighted average of the individual scores of each player i, where the weight of his score against player j is proportional to the score of player j.[8] Thus, the idea behind Neustadtl is to reward a win against a player with a high score more than a win against a player with a lower score.

Fair bets. Let $L_A = \mathrm{diag}(A^\top e)$. So, for every $i \in N$, $(L_A)_i$ represents how much other players have scored against player i, *i.e.*, i's total number of "losses". Consider the system of linear equations given by $L_A^{-1}Ax = x$ or, equivalently, $\sum_{j \in N} A_{ij}x_j = \sum_{j \in N} A_{ji}x_i$ for all $i \in N$. The rating vector r^{fb} is defined to be the unique positive vector such that $(r^{\mathrm{fb}})^\top e = 1$. Fair bets is a ranking method that was originally defined for round-robin tournaments and that has been studied in social choice and voting theory under different names and interpretations: from the classic papers by Daniels (1969) and Moon and Pullman (1970), to more recent references such as Slutzki and Volij (2005) and Slutzki and Volij (2006).[9] In Laslier (1992) this ranking method is called the "ping-pong winners" because of the following interpretation in round-robin tournaments. Suppose several players are waiting to play table tennis. The first two players i and j are randomly chosen and play. Player i wins with probability A_{ij}/M_{ij} and player j with probability A_{ji}/M_{ij}. The winner stays, a new opponent is randomly chosen, and the likelihood of each of them being the winner is derived again from matrix A. If we rank the players according to the amount of time they would play under the above rules, we would get the fair bets method.

The above interpretation suffices to uncover an important property that the fair bets ranking shares with the Neustadtl ranking. Both reward victories against players with high scores. Note that, in the table tennis example, when a player is chosen to play, he will most likely face a player with a high score, and so having good results against good players is more important than having good results against bad players.

Finally, with respect to the Neustadtl ranking, fair bets adds depth to the idea of rewarding results against good players. It is not only important to have beaten players who have high scores, but also that they have achieved this high scores beating players with high scores. In the table tennis example, given two players with the same average score, it is better to have beaten the one who has beaten stronger players. This reasoning can be given further levels of depth and the system of equations defining the fair bets ranking method captures them.

Maximum Likelihood. This is also a classic ranking method whose origins can be traced as far back as Zermelo (1929). It has been studied in several fields, but it is specially popular in statistics, through the literature on paired comparisons (see, for instance, Bradley

[8]This tie-breaking rule is commonly known as Sonneborn-Berger, but it was originally proposed by Hermann Neustadtl. Actually, this ranking method was defined just for round-robin tournaments and what we present here is a natural extension to our more general setting.

[9]Similar ideas have been also used in slightly different settings in papers such as Borm et al. (2002), Herings et al. (2005) and Slikker et al. (2010).

and Terry (1952), Moon and Pullman (1970) and David (1988)). When viewed from the point of view of statistics, maximum likelihood looks for the ratings vector that maximise the probability of the matrix A being realised when all the matches given in matrix M take place.

Formally, this ranking method assumes that each player $i \in N$ has a rating r_i and that, given two players $i, j \in N$, $i \neq j$, the probability that player i beats player j is given by a *rating function* $F(r_i, r_j)$. Although there are several possible choices for the function F, here we follow the classic approach, used already in the early works by Zermelo (1929) and rediscovered by Bradley and Terry (1952). Under this approach, F is based on the (standard) logistic distribution $F_L(x) = 1/(1 + \exp(-x))$, so that $F(r_i, r_j) = F_L(r_i - r_j) = \exp(r_i)/\exp(r_i + r_j)$. Under this specification, the maximum likelihood estimate of the rating of each player i corresponds with $r_i^{\text{ml}} = \log(\pi_i)$, where $\pi \in \mathbb{R}^N$ is the unique positive solution of the system of non-linear equations given by $\pi^\top e = 1$ and, for each $i \in N$,[10]

$$\pi_i = \frac{m_i s_i}{\sum_{j \in N \setminus \{i\}} \frac{M_{ij}}{(\pi_i + \pi_j)}}.$$

Recursive performance. This ranking method, defined in Brozos-Vázquez et al. (2008), also builds upon rating functions. Ideally, given a tournament (N, A), one would like to associate with it a rating of the players that explains all the observed results, that is, for each pair of players $i, j \in N$, $i \neq j$, $F(r_i, r_j) = \frac{A_{ij}}{M_{ij}}$. So, the observed proportion of victories of i against j is exactly what one would predict using F and the ratings r_i and r_j. Unfortunately, finding such ratings amounts to solving a system with far more equations than variables which, for most tournaments, will have no solution. As we said above, what maximum likelihood does is to find the ratings under which the probability of the observed results is maximised, whereas the recursive performance finds the rating that explains the "average" result of each player.

Given a tournament (N, A), a rating vector $r \in \mathbb{R}^N$, and a player i, the *average opponent* of i in the tournament is $(\bar{M}r)_i$, *i.e.*, the average rating of the opponents of i (weighted by the number of matches played against each of them). The recursive performance looks for a rating such that for each player $i \in N$, $F(r_i, (\bar{M}r)_i) = \sum_{j \in N} A_{ij}/m_i = s_i$. Again, we stick to the approach of using the logistic distribution to define F. Formally, if we let $c \in \mathbb{R}^N$ be defined, for each $i \in N$, by $c_i = F_L^{-1}(s_i)$ and $\hat{c} = c - \frac{m^\top c}{m^\top e} e$, the recursive performance rating vector, r^{rp}, is the solution of the system of linear equations given by $x^\top e = 0$ and $\bar{M}x + \hat{c} = x$.

Hence, for each player i, r_i^{rp} takes into account the average strength of i's opponents ($\bar{M}r^{\text{rp}}$) and his own score in the tournament ($\hat{c}_i$ is increasing in s_i).

Buchholz. The Buchholz rating vector is given by $r^{\text{b}} = \bar{M}s + s$. Just as the recursive performance, the Buchholz combines the average strength of i's opponents ($\bar{M}s$) and his own score (s) but in a much simpler way.[11]

[10] Refer, for instance, to Ford (1957) or David (1988).

[11] Actually, Buchholz is commonly used as a tie-breaker in non round-robin tournaments and is computed

Recursive Buchholz. This is a new ranking method that combines the ideas of Buchholz and recursive performance by adding to the Buchholz ranking method the same kind of depth that the fair bets added to Neustadtl. Not only the average score of your opponents ($\bar{M}s$) should be important, but also whether your opponents have achieved this average score against weak or strong opponents. All else equal, having faced opponents with a high score who have themselves played against strong opponents should be better than having faced opponents with a high score who have played against weak opponents. Again, further depth can be given to this argument and recursive Buchholz captures this idea.

The recursive Buchholz rating vector, r^{rb}, is the unique solution of the system of linear equations given by $x^\top e = 0$ and $\bar{M}x + \hat{s} = x$, where $\hat{s} = s - \frac{e}{2}$.[12]

In the following sections we discuss several properties and study whether the above ranking methods satisfy them. Most of the properties we discuss have been studied before. We will be explicit when defining properties that we have not found in the literature. For an overview of the results, see Table 1 in Section 7.

3 Basic properties

In this section we start our analysis by presenting three elementary properties that a ranking method φ should satisfy. In addition, we present a property that deals with a situation in which one can identify two subtournaments that are connected by just a single player.

Anonymity (ANO): Let $i, j \in N$ and let A' be the tournament obtained from A by permuting columns i and j and rows i and j. Then, the rankings $\varphi(A)$ and $\varphi(A')$ are the same but with players i and j interchanged.

Homogeneity (HOM): For all $k > 0$, $\varphi(kA) = \varphi(A)$. Note that homogeneity is an ordinal property. It relates to the ordering of the players and not necessarily to the underlying rating vector.

Symmetry (SYM): φ is flat on any symmetric tournament ($A = A^\top$). So if everyone has a 50% score against all opponents, not necessarily with the same number of matches, all players end up equally ranked.

These three properties require no motivation. Further, it is readily verified that all our ranking methods satisfy them.

For the next property we need to introduce the notion of *bridge player*. Given a tournament (N, A), a player $b \in N$ is a *bridge player* if there exist $N^1, N^2 \subseteq N$ with $|N^1| \geq 2$ and $|N^2| \geq 2$ such that $N^1 \cup N^2 = N$, $N^1 \cap N^2 = \{b\}$ and $M_{ij} = 0$ for all $i \in N^1 \backslash \{b\}, j \in N^2 \backslash \{b\}$. Since no player of $N^1 \backslash \{b\}$ has played against any player in $N^2 \backslash \{b\}$, the connectedness of

as the average of the scores of the opponents of each player, *i.e.*, $\bar{M}s$. For players with an equal score, this definition results in the same relative ranking as the definition we present here.

[12] Since the recursive Buchholz can be seen as a variation of the recursive performance where F_L is taken to be the identity, the existence and uniqueness of r^{rb} follows from Theorem 2 in Brozos-Vázquez et al. (2008).

irreducible tournaments with bridge players depends crucially on them, in the sense that the tournament obtained after removing a bridge player would not be connected. We denote by (N^1, A^1) and (N^2, A^2) the subtournaments obtained from (N, A) by reducing A to the player sets N^1 and N^2, respectively. Further, note that the irreducibility of A implies that a bridge player has scored against at least one player in each of the subtournaments, and that at least one player in each of the subtournaments has scored against him.

Bridge player independence (BPI): Let b be a bridge player with corresponding subtournaments (N^1, A^1) and (N^2, A^2). Then $iR^{\varphi}_{A}j$ if and only if $iR^{\varphi}_{A^1}j$ for all $i, j \in N^1$.

The idea of this property is fairly intuitive. Since, regarding the bridge player, the results in tournament (N^2, A^2) only convey information about how strong he is in comparison to the players in N^2, these results should be irrelevant to decide the relative ranking of the players in N^1. Think, for instance, of a player who plays a tournament (N^1, A^1) with a very bad result and then goes on to play a new tournament (N^2, A^2) with completely new opponents (possibly very weak) achieving an excellent result. What BPI implies is that, when looking at the complete tournament (N, A), the second tournament should have no impact in the relative ranking of the players in N^1 (including player b).

Proposition 3.1. *φ^{fb}, φ^{ml}, and φ^{rb} satisfy* BPI.

Proof. Let b be a bridge player with respect to the subtournaments (N^1, A^1) and (N^2, A^2). We start with the proof for φ^{fb}.

Take $x^1 = r^{\mathrm{fb}}(N^1, A^1)$ and $x^2 = r^{\mathrm{fb}}(N^2, A^2)$ and define, for all $i \in N$,

$$y_i = \begin{cases} \frac{x_b^2}{x_b^1} x_i^1 & \text{if } i \in N^1, \\ x_i^2 & \text{if } i \in N^2. \end{cases}$$

Since the fair bets rating vector associated with an irreducible tournament is positive, the vector y is well defined. Then, for $i \in N^1 \backslash \{b\}$ we have that, for all $j \in N^2$, $A_{ij} = A_{ji} = 0$ and hence

$$\sum_{j \in N} A_{ij} y_j = \sum_{j \in N^1} A^1_{ij} \frac{x_b^2}{x_b^1} x_j^1 = \frac{x_b^2}{x_b^1} \sum_{j \in N^1} A^1_{ji} x_i^1 = \sum_{j \in N} A_{ji} y_i.$$

Similarly, for $i \in N^2 \backslash \{b\}$ we have

$$\sum_{j \in N} A_{ij} y_j = \sum_{j \in N^2} A^2_{ij} x_j^2 = \sum_{j \in N^2} A^2_{ji} x_i^2 = \sum_{j \in N} A_{ji} y_i.$$

Finally,

$$\sum_{j \in N} A_{bj} y_j = \sum_{j \in N^1} A^1_{bj} \frac{x_b^2}{x_b^1} x_j^1 + \sum_{j \in N^2} A^2_{bj} x_j^2 = \frac{x_b^2}{x_b^1} \sum_{j \in N^1} A^1_{jb} x_b^1 + \sum_{j \in N^2} A^2_{jb} x_b^2 = \sum_{j \in N} A_{jb} y_b.$$

Since the system given by the $\sum_{j \in N} A_{ji} y_j = \sum_{j \in N} A_{ij} y_i$ equations has a unique solution up to a positive scalar multiplication, $r^{\mathrm{fb}}(N, A)$ and y induce the same rankings. From this, BPI follows.

The proof for φ^{ml} is analogous, but we use the vector π that solves the system of non-linear equations that are used to compute r^{ml}. Since r^{ml} is a strictly monotone transformation of π, they induce the same ranking.

Finally, the proof for φ^{rb} goes along similar lines, but the vector y is defined, for all $i \in N$, by $y_i = \begin{cases} x_i^1 + x_b^2 & \text{if } i \in N^1, \\ x_b^1 + x_i^2 & \text{if } i \in N^2. \end{cases}$ □

Example 3.1. *Consider the tournaments A and A' described below:*

$$\begin{array}{c} A \\ \begin{pmatrix} 0 & 1 & 20 & 20 \\ 1 & 0 & 20 & 0 \\ 20 & 20 & 0 & 0 \\ 20 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

r^s	r^n	r^{fb}	r^{ml}	r^{rp}	r^b	r^{rb}
0.5	0.25	0.25	−1.386	0	1	0
0.5	0.25	0.25	−1.386	0	1	0
0.5	0.25	0.25	−1.386	0	1	0
0.5	0.25	0.25	−1.386	0	1	0

$$\begin{array}{c} A' \\ \begin{pmatrix} 0 & 1 & 20 & 39 \\ 1 & 0 & 20 & 0 \\ 20 & 20 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

r^s	r^n	r^{fb}	r^{ml}	r^{rp}	r^b	r^{rb}
0.732	0.140	0.331	−1.107	0.416	1.000	0.119
0.500	0.256	0.331	−1.107	1.398	1.011	0.119
0.500	0.308	0.331	−1.107	1.170	1.116	0.119
0.025	0.018	0.008	−4.771	−2.984	0.757	−0.356

Player 1 is a bridge player in both A and A', with $N^1 = \{1, 4\}$ and $N^2 = \{1, 2, 3\}$. In tournament A, all players are tied according to all ranking methods. Yet, in A', players 1 and 3 are not tied anymore according to φ^s, φ^n, φ^{rp}, and φ^b. Since the only difference between A and A' is in the subtournament (N^1, A^1), these rules do not satisfy BPI.

4 Response to victories and losses

In this section we consider two types of properties for a ranking method φ. The first type deals with preserving a ranking when two tournaments (N, A) and (N, A') are combined. The second type deals with the (a)symmetric role victories and losses play in a ranking method.

Flatness preservation (FP): If $\varphi(A)$ and $\varphi(A')$ are both flat, then so is $\varphi(A + A')$. This property just says that if all players are regarded as equal in two tournaments, this should not change when we add up the tournaments.

Order preservation (OP): Let $i, j \in N$. If $i \; P_A^{\varphi} \; j$, $i \; P_{A'}^{\varphi} \; j$, and $\frac{m_i}{m_j} = \frac{m_i'}{m_j'}$, then $i \; P_{A+A'}^{\varphi} \; j$. If i is better than j in two tournaments, this should not change when we add them up. The condition on m and m' imposes some balance between the number of matches played in tournaments A and A'. In Example 4.3 below we show that OP without this condition is not even satisfied by the scores.

Symmetry between victories and losses (SVL): Let $i, j \in N$. Then $i \; R_A^{\varphi} \; j$ if and only if $j \; R_{A^{\top}}^{\varphi} \; i$. If we reverse all the results in a tournament, then the ranking should be reversed as well. We have not seen this property in the literature despite it being quite natural. Note that SVL trivially implies SYM.

Negative response to losses (NRL): Let $\lambda \in \mathbb{R}^N$, $\lambda > 0$ and define $\Lambda = \operatorname{diag}((\lambda_i)_{i\in N})$. If $\varphi(A)$ is flat, then $i\ R^{\varphi}_{A\Lambda}\ j$ if and only if $\lambda_i \leq \lambda_j$. This property is introduced in Slutzki and Volij (2005) and is the key ingredient of the characterisation they obtain for the fair bets ranking method. In words of the authors: "Negative responsiveness to losses concerns situations in which all players are equally ranked and the problem is irreducible. If a new problem is obtained by multiplying each player's losses by some positive constant (which may be different for each player), then the players should be ranked in the new problem in a way that is inversely related to these constants".

It is rather straightforward that φ^{s} satisfies FP. The following proposition relates flatness of φ^{rp}, φ^{rb} and φ^{ml} to flatness of φ^{s}.

Proposition 4.1. *$\varphi^{rp}(A)$, $\varphi^{rb}(A)$ and $\varphi^{ml}(A)$ are flat if and only if $\varphi^{s}(A)$ is flat.*

Proof. We provide the proof for φ^{rp}. The proofs for $\varphi^{\mathrm{rb}}(A)$ and $\varphi^{\mathrm{ml}}(A)$ are analogous.

"$\Rightarrow$": Assume that $\varphi^{\mathrm{rp}}(A)$ is flat, so there is $k \in \mathbb{R}$ such that $r^{\mathrm{rp}} = ke$. Recall that r^{rp} is a solution of $(I - \bar{M})r^{\mathrm{rp}} = \hat{c}$, where $\hat{c}_i$ is strictly increasing in s_i. Then, $(I - \bar{M})r^{\mathrm{rp}} = ke - k\bar{M}e = 0$. Hence, $\hat{c} = 0$ and therefore, $\varphi^{\mathrm{s}}(A)$ is flat.
"$\Leftarrow$": Assume that $\varphi^{\mathrm{s}}(A)$ is flat, so $s = \frac{1}{2}e$. Then, $c = 0$ and $\hat{c} = 0$. So, a particular solution of $(I - \bar{M})x = \hat{c}$ is 0 and the solution set is $\operatorname{span}\{e\}$. Hence, $\varphi^{\mathrm{rp}}(A)$ is flat. $\square$

By FP of φ^{s} we obtain the following corollary.

Corollary 4.2. *φ^{rp}, φ^{rb} and φ^{ml} satisfy* FP.

Slutzki and Volij (2005) show that φ^{fb} satisfies FP. The following example shows that φ^{n} and φ^{b} do not satisfy FP.

Example 4.1. *Consider the tournaments A and A' described below:*

$$\begin{array}{c} A \\ \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{ccccccc} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.667 & 0.222 & 0.333 & -1.099 & 0.347 & 1 & 0.083 \\ 0.667 & 0.222 & 0.333 & -1.099 & 0.347 & 1 & 0.083 \\ 0.333 & 0.222 & 0.167 & -1.792 & -0.347 & 1 & -0.083 \\ 0.333 & 0.222 & 0.167 & -1.792 & -0.347 & 1 & -0.083 \end{array}$$

$$\begin{array}{c} A' \\ \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{ccccccc} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{c} A + A' \\ \begin{pmatrix} 0 & 1 & 4 & 3 \\ 1 & 0 & 4 & 3 \\ 3 & 3 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{ccccccc} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.571 & 0.255 & 0.292 & -1.229 & 0.171 & 1.021 & 0.043 \\ 0.571 & 0.255 & 0.292 & -1.229 & 0.171 & 1.021 & 0.043 \\ 0.438 & 0.240 & 0.217 & -1.530 & -0.128 & 0.990 & -0.032 \\ 0.417 & 0.227 & 0.198 & -1.617 & -0.215 & 0.966 & -0.053 \end{array}$$

Then $\varphi^n(A)$, $\varphi^n(A')$, $\varphi^b(A)$ and $\varphi^b(A')$ are all flat, but $\varphi^n(A+A')$ and $\varphi^b(A+A')$ are not.

Proposition 4.3. *φ^s satisfies* OP.

Proof. Let (N, A), (N, A') and $i, j \in N$ be such that $s_i > s_j$, $s'_i > s'_j$ and $\frac{m_i}{m_j} = \frac{m'_i}{m'_j}$. Note that $\frac{m_i + m'_i}{m_i} = \frac{m_j + m'_j}{m_j}$. Hence, $\frac{m_i}{m_i + m'_i} = \frac{m_j}{m_j + m'_j}$ and, clearly, $\frac{m'_i}{m_i + m'_i} = \frac{m'_j}{m_j + m'_j}$ as well. It is straightforward to check that the score of player i in the combined tournament $A + A'$ equals $\frac{m_i}{m_i + m'_i} s_i + \frac{m'_i}{m_i + m'_i} s'_i$. Then,

$$\frac{m_i}{m_i + m'_i} s_i + \frac{m'_i}{m_i + m'_i} s'_i = \frac{m_j}{m_j + m'_j} s_i + \frac{m'_j}{m_j + m'_j} s'_i > \frac{m_j}{m_j + m'_j} s_j + \frac{m'_j}{m_j + m'_j} s'_j,$$

which coincides with the score of player j in the combined tournament, so we have established OP. ☐

The following example shows that the other ranking methods do not satisfy OP.

Example 4.2. *Consider the tournaments A and A' described below:*

$$\begin{array}{c} A \\ \begin{pmatrix} 0 & 1 & 60 & 3 \\ 0 & 0 & 4 & 40 \\ 1 & 1 & 0 & 0 \\ 2 & 21 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{rrrrrrr} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.955 & 0.053 & 0.520 & -0.567 & 1.025 & 1.019 & 0.198 \\ 0.657 & 0.216 & 0.297 & -1.322 & 0.948 & 0.999 & 0.112 \\ 0.030 & 0.024 & 0.013 & -4.162 & -2.383 & 0.954 & -0.282 \\ 0.358 & 0.235 & 0.170 & -1.892 & 0.410 & 1.028 & -0.029 \end{array}$$

$$\begin{array}{c} A' \\ \begin{pmatrix} 0 & 2 & 5 & 30 \\ 3 & 0 & 55 & 2 \\ 0 & 2 & 0 & 0 \\ 27 & 3 & 5 & 0 \end{pmatrix} \end{array} \quad \begin{array}{rrrrrrr} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.552 & 0.263 & 0.347 & -1.040 & 1.399 & 1.066 & 0.133 \\ 0.896 & 0.065 & 0.319 & -1.183 & 0.154 & 1.001 & 0.109 \\ 0.030 & 0.027 & 0.010 & -4.648 & -2.888 & 0.872 & -0.359 \\ 0.522 & 0.265 & 0.324 & -1.107 & 1.334 & 1.061 & 0.117 \end{array}$$

$$\begin{array}{c} A + A' \\ \begin{pmatrix} 0 & 3 & 65 & 33 \\ 3 & 0 & 59 & 42 \\ 1 & 3 & 0 & 0 \\ 29 & 24 & 6 & 0 \end{pmatrix} \end{array} \quad \begin{array}{rrrrrrr} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.754 & 0.140 & 0.318 & -1.126 & 0.692 & 1.007 & 0.122 \\ 0.776 & 0.168 & 0.414 & -0.902 & 0.900 & 1.041 & 0.155 \\ 0.030 & 0.023 & 0.012 & -4.522 & -2.350 & 0.780 & -0.336 \\ 0.440 & 0.303 & 0.256 & -1.351 & 0.758 & 1.173 & 0.058 \end{array}$$

According to φ^{fb}, φ^{ml}, φ^{rp}, φ^{b}, and φ^{rb}, player 1 is better than player 2 in both A and A'. However, all of them rank player 2 on top of player 1 in tournament $A + A'$. Hence, none of these ranking methods satisfies OP. *Consider now the tournament A'' given below. According to φ^n, player 1 is better than player 2 in both A' and A''. However, φ^n ranks player 2 on top of player 1 in tournament $A + A'$.*

$$\begin{array}{c} A'' \\ \begin{pmatrix} 0 & 0 & 12 & 0 \\ 25 & 0 & 0 & 10 \\ 8 & 22 & 0 & 25 \\ 22 & 10 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{r} r^n \\ 0.147 \\ 0.138 \\ 0.371 \\ 0.137 \end{array} \quad \textit{and} \quad \begin{array}{c} A' + A'' \\ \begin{pmatrix} 0 & 2 & 17 & 30 \\ 28 & 0 & 55 & 12 \\ 8 & 24 & 0 & 25 \\ 49 & 13 & 5 & 0 \end{pmatrix} \end{array} \begin{array}{r} r^n \\ 0.176 \\ 0.296. \\ 0.242 \\ 0.218 \end{array}$$

Moreover, note that all players have played the same number of matches in A, A', and A'', whereas this was not required in the definition of OP. *Hence, a weakening of* OP *in this direction would also be violated by all these ranking methods.*

The following example shows that a stronger version of OP without requiring the balance between m and m' is not even satisfied by φ^{s}.

Example 4.3. *Consider the tournaments A and A' described below:*

$$\begin{array}{cc} A & r^s \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 99 \\ 0.01 & 1 & 0 \end{pmatrix} & \begin{array}{c} 0.9901 \\ 0.9900 \\ 0.0100 \end{array} \end{array} \qquad \begin{array}{cc} A' & r^s \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0.01 \\ 99 & 1 & 0 \end{pmatrix} & \begin{array}{c} 0.0100 \\ 0.0099 \\ 0.9900 \end{array} \end{array} \qquad \begin{array}{cc} A+A' & r^s \\ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 99.01 \\ 99.01 & 2 & 0 \end{pmatrix} & \begin{array}{c} 0.0198 \\ 0.9802 \\ 0.5000 \end{array} \end{array}$$

Then, according to the score method φ^s, player 1 is ranked above player 2 in both A and A' and yet, when adding them up, player 2 has a higher score.

The score method φ^{s} trivially satisfies SVL. For various other ranking methods, SVL can be shown by explicitly transforming the rating vector.

Proposition 4.4. *φ^{ml} satisfies* SVL.

Proof. Recall that φ^{ml} orders the players according to r^{ml} where, for each $i \in N$, $r_i^{\mathrm{ml}} = \log(\pi_i)$ and vector π is such that $(\pi)^\top e = 1$ and, for each $i \in N$,

$$\pi_i = \frac{m_i s_i}{\sum_{j \in N \setminus \{i\}} \frac{M_{ij}}{\pi_i + \pi_j}}.$$

Let $\bar{x}$ be defined, for each $i \in N$, by $\bar{x}_i = r_i^{\mathrm{ml}} - \frac{1}{|N|} \sum_{j \in N} r_j^{\mathrm{ml}}$. Then, $\sum_{i \in N} \bar{x}_i = 0$ and, for each $i \in N$, $\pi_i = \alpha \exp(\bar{x}_i)$ with $\alpha = (\prod_{j \in N} \pi_j)^{1/|N|}$. Hence, for each $i \in N$, we have

$$\exp(\bar{x}_i) = \frac{m_i s_i}{\sum_{j \in N \setminus \{i\}} M_{ij} \frac{1}{\exp(\bar{x}_i) + \exp(\bar{x}_j)}}. \tag{4.1}$$

Now consider the following system of equations in $y \in \mathbb{R}^N$: $\sum_{i \in N} y_i = 0$ and, for each $i \in N$,

$$\exp(y_i) = \frac{m_i(1 - s_i)}{\sum_{j \in N \setminus \{i\}} M_{ij} \frac{1}{\exp(y_i) + \exp(y_j)}}. \tag{4.2}$$

If we show that $y = -\bar{x}$ solves this system, then (because the transformation from π to $\bar{x}$ is monotonic) we are done since player i's score in $A^\top$ is $1 - s_i$ and M is the same in both tournaments. Filling in $y = -\bar{x}$ in the right hand side of (4.2) yields

$$\begin{aligned} \frac{m_i(1 - s_i)}{\sum_{j \in N \setminus \{i\}} M_{ij} \frac{1}{\exp(-\bar{x}_i) + \exp(-\bar{x}_j)}} &= \frac{m_i(1 - s_i)}{\sum_{j \in N \setminus \{i\}} M_{ij} \frac{\exp(\bar{x}_i)\exp(\bar{x}_j)}{\exp(\bar{x}_i) + \exp(\bar{x}_j)}} \\ &= \frac{1}{\exp(\bar{x}_i)} \frac{m_i(1 - s_i)}{\sum_{j \in N \setminus \{i\}} M_{ij} \frac{\exp(\bar{x}_j)}{\exp(\bar{x}_i) + \exp(\bar{x}_j)}} \\ &= \frac{1}{\exp(\bar{x}_i)} \frac{m_i(1 - s_i)}{\sum_{j \in N \setminus \{i\}} M_{ij} (1 - \frac{\exp(\bar{x}_i)}{\exp(\bar{x}_i) + \exp(\bar{x}_j)})} \\ &= \frac{1}{\exp(\bar{x}_i)} \frac{m_i(1 - s_i)}{m_i - \exp(\bar{x}_i) \sum_{j \in N \setminus \{i\}} \frac{M_{ij}}{\exp(\bar{x}_i) + \exp(\bar{x}_j)}}, \end{aligned}$$

which, by (4.1), reduces to

$$\frac{1}{\exp(\bar{x}_i)} \frac{m_i(1 - s_i)}{m_i - \exp(\bar{x}_i) \frac{m_i s_i}{\exp(\bar{x}_i)}} = \frac{1}{\exp(\bar{x}_i)} = \exp(y_i).$$

So $y = -\bar{x}$ solves the system for $A^\top$ and therefore, φ^{ml} satisfies SVL. □

Proposition 4.5. *φ^{rp}, φ^{rb}, and φ^{b} satisfy* SVL.

Proof. To show that φ^{rp} satisfies SVL, observe that if r^{rp} solves $\bar{M}x + \hat{c} = x$, then $-r^{\mathrm{rp}}$ solves the corresponding equation for $A^\top$, because $\bar{M} = \bar{M}^\top$ and $\hat{c}(A^\top) = -\hat{c}(A)$ as a result of F^{-1} being symmetric around $\frac{1}{2}$. The argument for φ^{rb} is analogous. For φ^{b}, observe that $s(A^\top) = e - s(A)$, from which it readily follows that $r^{\mathrm{b}}(A^\top) = 2e - (\bar{M}s(A) + s(A)) = 2e - r^{\mathrm{b}}(A)$ and so φ^{b} satisfies SVL as well. $\square$

Not all ranking methods satisfy SVL, as is shown in the following example.

Example 4.4. *Consider the following tournaments:*

$$\overset{A}{\begin{pmatrix} 0 & 0.5 & 0.2 & 1 \\ 0.5 & 0 & 0.3 & 0.8 \\ 0.8 & 0.7 & 0 & 0.9 \\ 0 & 0.2 & 1 & 0 \end{pmatrix}} \begin{array}{cc} r^n & r^{fb} \\ 0.176 & 0.195 \\ 0.201 & 0.210 \\ 0.306 & 0.559 \\ 0.062 & 0.036 \end{array} \quad \textit{and} \quad \overset{A^\top}{\begin{pmatrix} 0 & 0.5 & 0.8 & 0 \\ 0.5 & 0 & 0.7 & 0.2 \\ 0.2 & 0.3 & 0 & 0.1 \\ 1 & 0.8 & 0.9 & 0 \end{pmatrix}} \begin{array}{cc} r^n & r^{fb} \\ 0.131 & 0.065 \\ 0.179 & 0.137 \\ 0.106 & 0.054 \\ 0.329 & 0.744 \end{array}.$$

Since player 2 is ranked above player 1 in both A and $A^\top$ and for both φ^n and φ^{fb}, these ranking methods do not satisfy SVL. *Note that A is a round-robin tournament. Hence, φ^n and φ^{fb} do not satisfy* SVL *even if we restrict to round-robin tournaments.*

Our analysis of NRL builds upon Slutzki and Volij (2005), though some care is needed. On the one hand, they develop their characterisation of φ^{fb} for a larger class that allows for reducible tournaments. On the other hand, they restrict to tournament matrices with integer entries.

A tournament A is called *balanced* if $Ae = A^\top e$, *i.e.*, if each player has the same number of victories and losses. It is strongly balanced if, moreover, there is a constant k such that $Ae = ke$, so the number of victories (and losses) is equal across all players. The next result is an adaptation of Lemmas 3 and 4 in Slutzki and Volij (2005).

Lemma 4.6. *Let φ be a ranking method satisfying* ANO, HOM, SYM *and* FP. *Then φ is flat on balanced tournaments.*

Proof. First, suppose that A is strongly balanced with $Ae = ke$. Then by Birkhoff's theorem (Birkhoff (1946)), matrix A can be written as k times a convex combination of permutation matrices. By ANO, φ is flat on permutation matrices. By HOM, φ is also flat on the tournaments that result after the multiplication of the permutation matrices by positive numbers. Finally, by FP and HOM again, φ is flat also on matrix A.

If A is not strongly balanced, then A can be decomposed as the sum of a strongly balanced tournament, in which we have just seen that φ is flat, and a symmetric tournament (see the proof of Lemma 4 in Slutzki and Volij (2005)). By SYM, φ is flat on the symmetric tournament as well, and by FP it is then flat on the original tournament A. $\square$

Most of the ranking methods we consider in this paper satisfy ANO, HOM, SYM, and FP, and, therefore, all of them coincide (and are flat) for balanced tournaments. The next result, which is the adaptation of the main result in Slutzki and Volij (2005) to our setting, illustrates the strength of the NRL property.

Proposition 4.7. *The fair bets ranking method, φ^{fb}, is the unique ranking method satisfying* ANO*,* HOM*,* SYM*,* FP*, and* NRL*.*

Proof. φ^{fb} was already shown to satisfy ANO, HOM, SYM and FP. NRL follows from Slutzki and Volij (2005).

To show the converse, let φ be a ranking method satisfying ANO, HOM, SYM, FP, and NRL. Given an irreducible tournament A and corresponding fair bets rating vector, r^{fb}, the tournament $A' = A\,\text{diag}((r_i^{\text{fb}})_{i\in N})$ is a balanced (and irreducible) tournament because, by definition, for all $i \in N$,

$$\sum_{j\in N} A_{ij} r_j^{\text{fb}} = \sum_{j\in N} A_{ji} r_i^{\text{fb}}.$$

Then, $A = A'(\text{diag}((r_i^{\text{fb}})_{i\in N}))^{-1}$. Since φ satisfies ANO, HOM, SYM, and FP, by Lemma 4.6, $\varphi(A')$ is flat. Then, by NRL, $i\ R_A^{\varphi}\ j \iff 1/r_i^{\text{fb}} \leq 1/r_j^{\text{fb}} \iff i\ R_A^{\varphi^{\text{rb}}}\ j$. $\square$

As a result of Proposition 4.7, φ^{s}, φ^{ml}, φ^{rp} and φ^{rb} do not satisfy NRL because they satisfy all other properties in the characterisation. The following example shows that φ^{n} and φ^{b} do not satisfy NRL either.

Example 4.5. *Let $\lambda = (0.99, 2, 1, 1)$ and $\Lambda = \text{diag}((\lambda_i)_{i\in N})$. Let A and $A\Lambda$ be as follows:*

$$\begin{array}{cc} A & \begin{array}{cc} r^n & r^b \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} & \begin{array}{cc} 0.25 & 1 \\ 0.25 & 1 \\ 0.25 & 1 \\ 0.25 & 1 \end{array} \end{array} \quad \text{and} \quad \begin{array}{cc} A\Lambda & \begin{array}{cc} r^n & r^b \end{array} \\ \begin{pmatrix} 0 & 4 & 1 & 1 \\ 1.98 & 0 & 1 & 1 \\ 0.99 & 2 & 0 & 2 \\ 0.99 & 2 & 2 & 0 \end{pmatrix} & \begin{array}{cc} 0.245 & 1.024 \\ 0.192 & 0.911 \\ 0.264 & 1.046 \\ 0.264 & 1.046 \end{array} \end{array}$$

Note that both φ^n and φ^b are flat on A but, despite $\lambda_1 \leq \lambda_3$, $r_3^n(A\Lambda) > r_1^n(A\Lambda)$ and $r_3^b(A\Lambda) > r_1^b(A\Lambda)$.

5 Score consistency

In this section we investigate to what extent a ranking method φ preserve some of the features of the score ranking method, making it appealing for round-robin tournaments.

Score consistency (SCC): If A is a round-robin tournament, then $\varphi(A) = \varphi^s(A)$.

Homogeneous treatment of victories (HTV): Let $i, j \in N$. If $M_{ik} = M_{jk}$ for all $k \in N\backslash\{i,j\}$, then $i\ R_A^{\varphi}\ j$ if and only if $i\ R_A^{\varphi^{\text{s}}}\ j$. Roughly speaking, if i and j play the same number of matches against the other players, then they should be ranked according to their aggregate scores. Note that HTV trivially implies SCC.

Note that both SCC and HTV relate to the ranking of the players, not necessarily to the underlying rating vectors. It follows from the tournament A in Example 4.4 that φ^{n} and φ^{fb} do not satisfy SCC.

The remaining ranking methods all satisfy HTV, and hence SCC.

Proposition 5.1. *φ^{ml} satisfies* HTV*.*

Proof. Let (N, A) and $i, j \in N$ be such that $M_{ik} = M_{jk}$ for all $k \in N \backslash \{i, j\}$. Rewriting the equations used to define φ^{ml} we have

$$s_i = \frac{1}{m_i} \sum_{k \in N \backslash \{i\}} M_{ik} \frac{\pi_i}{\pi_i + \pi_k}.$$

Since $\frac{\pi_i}{\pi_i+\pi_k}$ is increasing in π_i, the right hand side of the equation is increasing in π_i. Then, because $M_{ik} = M_{jk}$ for all $k \neq i, j$ and therefore $m_i = m_j$, we have that $s_i \geq s_j$ if and only if $\pi_i \geq \pi_j$. Hence, φ^{ml} satisfies HTV. □

If $|N| = 2$ we have that $Ms+s = (s_1+s_2, s_1+s_2)$, so φ^{b} is flat in two-player tournaments and therefore satisfies neither HTV nor SCC.

Proposition 5.2. *If $|N| > 2$, then φ^b satisfies* HTV.

Proof. Let (N, A) and $i, j \in N$ be such that $M_{ik} = M_{jk}$ for all $k \in N \backslash \{i, j\}$. Given $i, j \in N$, since $M_{ik} = M_{jk}$ for all $k \neq i, j$, we have that $m_i = m_j$ and, hence, $\bar{M}_{ij} = \bar{M}_{ji}$. Then,

$$r_i^{\mathrm{b}} - r_j^{\mathrm{b}} = (\bar{M}s + s)_i - (\bar{M}s + s)_j = (1 - \bar{M}_{ij})(s_i - s_j).$$

Since A is irreducible and $|N| > 2$, it cannot be the case that $\bar{M}_{ij} = 1$. Then, $(1 - \bar{M}_{ij}) > 0$ and φ^{b} and φ^{s} produce the same ranking. □

Proposition 5.3. *φ^{rb} and φ^{rp} satisfy* HTV.

Proof. Recall that φ^{rb} solves $(I - \bar{M})x = \hat{s}$. So, in particular

$$x_i - \bar{M}_{ij}x_j - \sum_{k \in N \backslash \{i,j\}} \bar{M}_{ik}x_k = \hat{s}_i \quad \text{and} \quad -\bar{M}_{ji}x_i + x_j - \sum_{k \in N \backslash \{i,j\}} \bar{M}_{jk}x_k = \hat{s}_j.$$

Substracting the two equations and using that $\bar{M}_{ij} = \bar{M}_{ji}$ and $\bar{M}_{ik} = \bar{M}_{jk}$ for all other k yields

$$(1 + \bar{M}_{ij})(x_i - x_j) = \hat{s}_i - \hat{s}_j.$$

Therefore, $x_i - x_j$ and $\hat{s}_i - \hat{s}_j$ have the same sign. Hence, φ^{rb} satisfies HTV.

The proof for φ^{rp} is analogous, but with $\hat{c}$ in the right hand side. Since $\hat{c}$ and $\hat{s}$ induce the same ranking, the same argument works. □

6 Monotonicity

In this section we present three properties that deal with changes in the tournament matrix. If an existing result is changed or a new one is added, how should the rankings change? The first property states that your relative ranking compared to a particular other player cannot depend on any result involving neither of you. The other two properties state that winning a game should always be beneficial to your ranking.

Independence of irrelevant matches (IIM): We follow the definition introduced in Rubinstein (1980): take four different players $i, j, k, \ell \in N$. Suppose that A and A' are identical, except for the results between k and ℓ. Then $i\ R_A^{\varphi}\ j$ if and only if $i\ R_{A'}^{\varphi}\ j$.

Positive responsiveness to the beating relation (PRB): Let A be such that $i\ R_A^{\varphi}\ j$. Let A' be a tournament identical to A, except that there is $k \in N \backslash \{i\}$ such that $M'_{ik} = M_{ik}$ and $A'_{ik} > A_{ik}$. Then, $i\ P_A^{\varphi}\ j$. Note that this should hold in particular for $k = j$.

Nonnegative responsiveness to the beating relation (NNRB): Let A be such that $i\ R_A^{\varphi}\ j$. Let A' be a tournament identical to A, except that there is $k \in N \backslash \{i\}$ such that $M'_{ik} = M_{ik}$ and $A'_{ik} > A_{ik}$. Then, $i\ R_A^{\varphi}\ j$. Of course, PRB trivially implies NNRB.

Rubinstein (1980) uses ANO, IIM and PRB to characterise φ^{s} on the class of round-robin tournaments. Clearly, in our wider class of tournaments φ^{s} also satisfies IIM and PRB. We show below that all the other ranking methods violate IIM.

Example 6.1. *Consider the tournaments A and A' described below:*

$$\begin{array}{c} A \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{ccccccc} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.429 & 0.224 & 0.2 & -1.609 & -0.201 & 0.959 & -0.05 \\ 0.571 & 0.265 & 0.3 & -1.204 & 0.201 & 1.041 & 0.05 \\ 0.571 & 0.265 & 0.3 & -1.204 & 0.201 & 1.041 & 0.05 \\ 0.429 & 0.224 & 0.2 & -1.609 & -0.201 & 0.959 & -0.05 \end{array}$$

$$\begin{array}{c} A' \\ \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{ccccccc} r^s & r^n & r^{fb} & r^{ml} & r^{rp} & r^b & r^{rb} \\ 0.500 & 0.237 & 0.233 & -1.430 & -0.015 & 0.998 & -0.004 \\ 0.571 & 0.278 & 0.308 & -1.189 & 0.224 & 1.056 & 0.055 \\ 0.571 & 0.260 & 0.292 & -1.230 & 0.182 & 1.038 & 0.045 \\ 0.375 & 0.205 & 0.167 & -1.807 & -0.390 & 0.920 & -0.096 \end{array}$$

In tournament A, all ranking methods rank players 2 and 3 equally. In tournament A', except for φ^s, all ranking methods rank player 2 on top of player 3, violating IIM.

Note that whereas IIM is a very natural property in round-robin tournaments, it is questionable in our more general setting. Indeed, we argue in section 7 that when players face different opponents, IIM is a property not to be desired.

The score ranking method φ^{s} also turns out to be the only one satisfying PRB. Consider the tournaments A and A' in Example 3.1. According to all methods under consideration, players 1, 2 and 3 are equally ranked in A. In A', player 1 has a better result against player 4 than in A, but only r^{s} ranks him above players 2 and 3. Hence, all of them but r^{s} violate PRB. Moreover, φ^{n}, φ^{rp} and φ^{b} actually rank player 1 lower than 2 and 3 in A', so these three methods do not satisfy NNRB either.

In the remainder of this section we show that both φ^{fb} and φ^{rb} satisfy NNRB. Although we conjecture that φ^{ml} also satisfies NNRB, this is still an open question. The result for φ^{fb} below extends the result in Levchenkov (1992) for round-robin tournaments (we build upon the proof in Laslier (1997)). We start with an auxiliary result that will be crucial in the proof for both φ^{fb} and φ^{rb}.

Lemma 6.1. *Let $B \in \mathbb{R}^{n \times n}$ be such that*

(i) *B is invertible,*

(ii) *for all $i \neq j$, $B_{ij} \leq 0$, and*

(iii) $\sum_{j=1}^{n} B_{ji} \geq 0$ *for all* $i \in \{1, \ldots, n\}$.

If $\gamma, \lambda \in \mathbb{R}^n$ *are two vectors such that* λ *is nonnegative and* $B\gamma = \lambda$*, then* γ *is nonnegative.*

Proof. First note that (i)-(iii) imply that $B_{ii} > 0$ for all i since invertibility precludes a zero column. We do the proof by induction on n, the size of the square matrix B. For $n = 1$, the result follows immediately from $B_{11} > 0$. If $n > 1$, suppose the result is true for matrices of size $n - 1$ and let $\lambda \geq 0$ be such that $B\gamma = \lambda$. The last equation of $B\gamma = \lambda$ can be written as

$$B_{nn}\gamma_n = \lambda_n - \sum_{j=1}^{n-1} B_{nj}\gamma_j. \tag{6.1}$$

Now, we substitute γ_n in the other equations and, for each $i \in \{1, \ldots, n-1\}$, the equation $\sum_{j=1}^{n} B_{ij}\gamma_j = \lambda_i$ can be rewritten as

$$\sum_{j=1}^{n-1} (B_{nn}B_{ij} - B_{in}B_{nj})\gamma_j = B_{nn}\lambda_i - B_{in}\lambda_n.$$

Define $\bar{B} \in \mathbb{R}^{(n-1)\times(n-1)}$ by $\bar{B}_{ij} = B_{nn}B_{ij} - B_{in}B_{nj}$ for all $i, j \in \{1, \ldots, n-1\}$. Also, define $\bar{\gamma}, \bar{\lambda} \in \mathbb{R}^{n-1}$ by $\bar{\gamma}_i = \gamma_i$ and $\bar{\lambda}_i = B_{nn}\lambda_i - B_{in}\lambda_n$ for all $i \in \{1, \ldots, n-1\}$. Then the above $n - 1$ equations can be expressed in matrix form as $\bar{B}\bar{\gamma} = \bar{\lambda}$. It is now easy to check that $\bar{\lambda} \geq 0$, $\bar{B}$ is invertible, and for all $i \neq j$, $\bar{B}_{ij} \leq 0$. Thus, in order to apply the induction hypothesis we just need to show that $\sum_{j=1}^{n-1} \bar{B}_{ji} \geq 0$:

$$\begin{aligned}
\sum_{j=1}^{n-1} \bar{B}_{ji} &= \bar{B}_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \bar{B}_{ji} \\
&= B_{nn}B_{ii} - B_{in}B_{ni} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} (B_{nn}B_{ji} - B_{jn}B_{ni}) \\
&= B_{nn}(B_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} B_{ji}) - B_{ni}(B_{in} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} B_{jn}) \\
&= B_{nn}(\sum_{j=1}^{n-1} B_{ji}) - B_{ni}(\sum_{j=1}^{n-1} B_{jn}) \\
&\overset{(iii)}{\geq} B_{nn}(-B_{ni}) - B_{ni}(-B_{nn}) \\
&= 0.
\end{aligned}$$

Therefore, we can apply the induction hypothesis to conclude that $\bar{\gamma}$ is nonnegative. Finally, nonnegativity of γ_n easily follows from (6.1), $B_{nn} > 0$ and nonnegativity of the other components of γ. □

Proposition 6.2. *Let tournament* A *be such that* $i \ R_A^{\varphi^{rb}} \ j$*. Let* A' *be a tournament identical to* A*, except that there is* $k \in N\backslash\{i\}$ *such that* $M'_{ik} = M_{ik}$ *and* $A'_{ik} > A_{ik}$*. Then,* $i \ R_A^{\varphi^{rb}} \ j$ *and, if* $k = j$*,* $i \ P_A^{\varphi^{rb}} \ j$*. In particular,* φ^{rb} *satisfies* NNRB.

Proof. Let $i, j, k \in N$ be as in the statement. Below we explicitly characterise how the recursive Buchholz ranking varies as a function of A_{ik} and A_{ki}, provided that M_{ik} stays constant. Recall that r^{rb} is the unique solution of $\bar{M}x + \hat{s} = x$ such that $(r^{\mathrm{rb}})^\top e = 0$. Hence, $(I - \bar{M})r^{\mathrm{rb}} = \hat{s}$. Define $B = I - \bar{M}$ and $\breve{N} = N \backslash \{i, k\}$. Then, equation ℓ of the system $Br^{\mathrm{rb}} = \hat{s}$ can be written as

$$\sum_{h \in \breve{N}} B_{\ell h} r_h^{\mathrm{rb}} = \hat{s}_\ell - B_{\ell i} r_i^{\mathrm{rb}} - B_{\ell k} r_k^{\mathrm{rb}}. \tag{6.2}$$

with $\ell \in \breve{N}$. Define $\breve{B} \in \mathbb{R}^{(n-2)\times(n-2)}$ to be the matrix obtained from B by deleting the rows and columns corresponding to players i and k.

We prove now that $\breve{B}$ is invertible. Suppose, on the contrary, that there is an $y \in \mathbb{R}^{\breve{N}}$, $y \neq 0$, such that $y^\top \breve{B}^\top = 0$. Let $\ell \in \breve{N}$ be such that $y_\ell = \max_{h \in \breve{N}} y_h$. We assume, without loss of generality, that $y_\ell > 0$. For each $h \neq \ell$, $B_{h\ell}^\top \leq 0$ and, hence, $-y_h B_{h\ell}^\top \leq -y_\ell B_{h\ell}^\top$, with equality only if $y_h = y_\ell$ or $B_{h\ell}^\top = 0$. Since $y^\top \breve{B}^\top = 0$, $\sum_{h \in \breve{N}\backslash\{\ell\}} -y_h B_{h\ell}^\top = y_\ell B_{\ell\ell}^\top$. Further, since $\sum_{h \in N} B_{h\ell}^\top = 0$, we have $\sum_{h \in \breve{N}\backslash\{\ell\}} -B_{h\ell}^\top = B_{\ell\ell}^\top + B_{i\ell}^\top + B_{k\ell}^\top \leq B_{\ell\ell}^\top$, with equality only if $B_{i\ell}^\top = B_{k\ell}^\top = 0$. Then, we have

$$y_\ell B_{\ell\ell}^\top = \sum_{h \in \breve{N}\backslash\{\ell\}} -y_h B_{h\ell}^\top \leq y_\ell \sum_{h \in \breve{N}\backslash\{\ell\}} -B_{h\ell}^\top \leq y_\ell B_{\ell\ell}^\top$$

and, hence, all the inequalities are indeed equalities. Therefore, $B_{i\ell}^\top = B_{k\ell}^\top = 0$ and, for each $h \in \breve{N}\backslash\{\ell\}$, $y_h = y_\ell$ or $B_{h\ell}^\top = 0$. Define $\bar{N} = \{m \in \breve{N} \,|\, y_m = \max_{h \in \breve{N}} y_h\}$. Now, for each $m \in \bar{N}$, we have $B_{im}^\top = B_{km}^\top = 0$ and, further, for each $h \in \breve{N}\backslash\bar{N}$, $B_{hm}^\top = 0$. That is, no player outside $\bar{N}$ has played against players inside $\bar{N}$, which contradicts the irreducibility of A.

Define $C = (\breve{B})^{-1}$, $\breve{r}^{\mathrm{rb}} = (r_h^{\mathrm{rb}})_{h \in \breve{N}}$, $B^i = (B_{hi})_{h \in \breve{N}}$ and $B^k = (B_{hk})_{h \in \breve{N}}$. Then, using (6.2) we have $\breve{B}\breve{r}^{\mathrm{rb}} = \breve{s} - B^i r_i^{\mathrm{rb}} - B^k r_k^{\mathrm{rb}}$ and hence, $\breve{r}^{\mathrm{rb}} = C(\breve{s} - B^i r_i^{\mathrm{rb}} - B^k r_k^{\mathrm{rb}})$. So, for all $\ell \in \breve{N}$,

$$r_\ell^{\mathrm{rb}} = \breve{r}_\ell^{\mathrm{rb}} = \sum_{h \in \breve{N}} C_{\ell h}(\breve{s}_h - B_{hi} r_i^{\mathrm{rb}} - B_{hk} r_k^{\mathrm{rb}}).$$

Define $\gamma_\ell^s = \sum_{h \in \breve{N}} C_{\ell h} \breve{s}_h$, $\gamma_\ell^i = -\sum_{h \in \breve{N}} C_{\ell h} B_{hi}$ and $\gamma_\ell^k = -\sum_{h \in \breve{N}} C_{\ell h} B_{hk}$. Then, for each $\ell \in \breve{N}$,

$$r_\ell^{\mathrm{rb}} = \gamma_\ell^s + \gamma_\ell^i r_i^{\mathrm{rb}} + \gamma_\ell^k r_k^{\mathrm{rb}}. \tag{6.3}$$

Furthermore, equation i in $Br^{\mathrm{rb}} = \hat{s}$ is

$$B_{ii} r_i^{\mathrm{rb}} + B_{ik} r_k^{\mathrm{rb}} + \sum_{\ell \in \breve{N}} B_{i\ell} r_\ell^{\mathrm{rb}} = \hat{s}_i. \tag{6.4}$$

Define $\Gamma^{i,i} = -\sum_{\ell \in \breve{N}} B_{i\ell} \gamma_\ell^i$ and $\Gamma^{i,k} = -\sum_{\ell \in \breve{N}} B_{i\ell} \gamma_\ell^k$. Then, plugging in the expression of each r_ℓ^{rb} (6.3) into (6.4) we get

$$(B_{ii} - \Gamma^{i,i}) r_i^{\mathrm{rb}} + (B_{ik} - \Gamma^{i,k}) r_k^{\mathrm{rb}} = \hat{s}_i - \sum_{\ell \in \breve{N}} \gamma_\ell^s. \tag{6.5}$$

Now, adding up (6.3) over all $\ell \in \breve{N}$ and using that $\sum_{h\in N} r_h^{\mathrm{rb}} = 0$,

$$(1 + \sum_{\ell\in\breve{N}} \gamma_\ell^i) r_i^{\mathrm{rb}} + (1 + \sum_{\ell\in\breve{N}} \gamma_\ell^k) r_k^{\mathrm{rb}} = -\sum_{\ell\in\breve{N}} \gamma_\ell^s. \tag{6.6}$$

Define $\sigma_i = \sum_{\ell\in\breve{N}} \gamma_\ell^i$ and $\sigma_k = \sum_{\ell\in\breve{N}} \gamma_\ell^k$. Then, solving equations (6.5) and (6.6), we get

$$r_i^{\mathrm{rb}} = \frac{\hat{s}_i - (1 - \frac{B_{ik} - \Gamma^{i,k}}{1+\sigma_k}) \sum_{\ell\in\breve{N}} \gamma_\ell^s}{(B_{ii} - \Gamma^{i,i}) - (B_{ik} - \Gamma^{i,k}) \frac{1+\sigma_k}{1+\sigma_i}} \quad \text{and} \quad r_k^{\mathrm{rb}} = \frac{-\sum_{\ell\in\breve{N}} \gamma_\ell^s}{1+\sigma_k} - \frac{1+\sigma_i}{1+\sigma_k} r_i^{\mathrm{rb}}. \tag{6.7}$$

To understand how r_i^{rb} and r_k^{rb} vary with $\hat{s}_i$, it is convenient to know the signs of γ^i and γ^k. We claim that both γ^i and γ^k are nonnegative vectors. By definition, $\gamma^i = -CB^i$ and, since $C^{-1} = \breve{B}$, $\breve{B}\gamma = -B^i$. Furthermore, $-B^i \geq 0$. Since matrix $\breve{B}$ and vectors γ^i and $-B^i$ satisfy the conditions of Lemma 6.1, γ^i is nonnegative. The argument for γ^k is analogous using $-B^k$ instead of $-B^i$. The nonnegativity of γ^i and γ^k implies that σ_i and σ_k are also nonnegative. Since γ^k is nonnegative, also $\Gamma^{i,k}$ is nonnegative and $B_{ik} - \Gamma^{i,k}$ is negative. Furthermore,

$$B_{ii} - \Gamma^{i,i} = B_{ii} + \sum_{\ell\in\breve{N}} B_{i\ell}\gamma_\ell^i \geq B_{ii} + \sum_{\ell\in\breve{N}} B_{i\ell} \geq 0.$$

We reexamine now equation (6.7). Note that γ^s, γ^i, γ^k, $\Gamma^{i,i}$, $\Gamma^{i,k}$, B_{ii}, and B_{ik} only depend on $\breve{B}$. Then, the denominator of the expression for r_i^{rb} is positive and so r_i^{rb} is strictly increasing in $\hat{s}_i$. Further, since r_k^{rb} is strictly decreasing in r_i^{rb}, it is strictly decreasing in $\hat{s}_i$.

Now, because of (6.3), r_ℓ^{rb} is weakly increasing in r_i^{rb} and r_k^{rb}. Yet, since r_i^{rb} and r_k^{rb} are strictly increasing and decreasing, respectively, in $\hat{s}_i$, some extra work is needed to understand how r_ℓ^{rb} varies with $\hat{s}_i$. To do so, we first show that all the components of γ^i and γ^k are no larger than 1. We prove it for γ^i, the proof for γ^k being analogous.

$$\breve{B}(e - \gamma^i) = \breve{B}e - \breve{B}\gamma^i = \breve{B}e - B^i,$$

and, for each $\ell \in \breve{N}$,

$$(\breve{B}e - B^i)_\ell = \sum_{h\in\breve{N}} B_{\ell h} + B_{\ell i} \geq \sum_{h\in\breve{N}} B_{\ell h} + B_{\ell i} + B_{ki} = 0.$$

Then, since $\breve{B}e - B^i$ is a nonnegative vector, matrix $\breve{B}$ and vectors $e - \gamma^i$ and $\breve{B}e - B^i$ are in the conditions of Lemma 6.1 and, hence, $e - \gamma^i$ is nonnegative.

Therefore, we know that all the components of γ^i and γ^k are no larger than 1. Looking again at equation (6.3), we have that r_ℓ^{rb} cannot increase with $\hat{s}_i$ faster than r_i^{rb} so $r_\ell^{\mathrm{rb}}/r_i^{\mathrm{rb}}$ is weakly decreasing in $\hat{s}_i$. Similarly, $r_\ell^{\mathrm{rb}}/r_k^{\mathrm{rb}}$ is weakly increasing in $\hat{s}_i$. From this, the statement follows. □

Proposition 6.3. *Let tournament A be such that $i\ R_A^{\varphi^{fb}}\ j$. Let A' be a tournament identical to A, except that there is $k \in N\backslash\{i\}$ such that $M'_{ik} = M_{ik}$ and $A'_{ik} > A_{ik}$. Then, $i\ R_A^{\varphi^{fb}}\ j$ and, if $k = j$, $i\ P_A^{\varphi^{fb}}\ j$. In particular, φ^{fb} satisfies* NNRB.

The proof of this proposition is analogous to the proof of Proposition 6.2, with $B = L_A - A$ instead of $B = I - \bar{M}$.

7 Discussion

Table 1 summarises the results of the ranking methods we have studied with respect to the different properties. The scores ranking method satisfies most of the properties we have studied. However, although this ranking method is very natural when looking at round-robin tournaments, in our more general setting it has the important drawback that it just looks at the aggregate score of each player, ignoring the opponents he has faced to obtain this score. All the other ranking methods we have considered use this information. That is, in one way or another, they are responsive to the strength of the opponents of each player. This is captured by the fact that the scores ranking is the only one satisfying IIM outside the subdomain of round-robin tournaments as well. So when players have different opponents (or face opponents with different intensities), IIM is a property one would rather not have.

	Scores	Neustadtl	Fair bets	Maximum Likelihood	Recursive Performance	Buchholz	Recursive Buchholz
ANO	✓	✓	✓	✓	✓	✓	✓
HOM	✓	✓	✓	✓	✓	✓	✓
SYM	✓	✓	✓	✓	✓	✓	✓
BPI	X	X	✓	✓	X	X	✓
FP	✓	X	✓	✓	✓	X	✓
OP	✓	X	X	X	X	X	X
SVL	✓	X	X	✓	✓	✓	✓
NRL	X	X	✓	X	X	X	X
SCC	✓	X	X	✓	✓	✓*	✓
HTV	✓	X	X	✓	✓	✓*	✓
IIM	✓	X	X	X	X	X	X
PRB	✓	X	X	X	X	X	X
NNRB	✓	X	✓	?	X	X	✓

*Requires $|N| > 2$.

Table 1: Ranking methods and properties.

On the entire domain of tournaments, maximum likelihood and recursive Buchholz are the two ranking methods that look most appealing. One potential advantage of φ^{rb} with respect to φ^{ml} is that, since φ^{ml} requires to solve a system of non-linear equations, it may be very hard to compute in settings where there is a high number of players to be ranked. The difficulties to compute r^{ml} were already studied in Dykstra (1956). Also, recall that recursive Buchholz can be seen as a variation of the recursive performance but, differently from the vector $\hat{c}$ used to define r^{rp}, the vector $\hat{s}$ used for calculating r^{rb} is linear in s; this linearity seems to be behind the good behaviour of this ranking method.

Finally, from our point of view, the major weakness of the fair bets ranking is that it violates SVL, which imposes the natural requirement that if we reverse all the results in the tournament, then the corresponding ranking should be obtained by reverting original ranking as well.

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